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SECOND ORDER SHANNON WAVELET APPROXIMATION OF C^2-FUNCTIONS

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In aceasta lucrare este prezentata aproximatia de ordinul doi a unei functii C^2 bazata pe functiile tip Shannon Wavelets. Aproximatia este comparata cu formula de reconstructie a functiilor tip wavelet, eroarea de aproximare fiind calculata in mod explicit.

In this paper, the second order approximation of a C^2-function, based on Shannon wavelet functions, is given. The approximation is compared with the wavelet reconstruction formula and the error of approximation is explicitly computed.

Keywords: Shannon wavelet, Approximation, Error computation.
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1. **Introduction.**

In this paper it the approximation formula of C^2-function based on the fundamental instances of the wavelet family is given. The main advantages of this approximation are that

(1) locally this reconstruction holds for a broad class of functions than the $L_2(\mathbb{R})$-function family where the wavelet theory is valid
(2) up to the second order this approximation is more efficient than the wavelet theory approximation
(3) any C^2-function can be locally expressed in terms of wavelets.

In recent years wavelets have been successfully applied to the wavelet representation of integro-differential operators [1-4], thus giving rise to the so-called wavelet solutions of PDE and integral equations. However, there are only a few examples of wavelets having some physical meaning [5,6]. In the sense, that there are not so many examples where wavelets mostly coincide with the solution of a physical problem or with special functions which have been shown to be classical solutions of physical problems.

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Wavelets are $L_2(\mathbb{R})$-functions, having a decay to zero, which are used in many different analytical problems, such as operator analysis, PDE solving, integral equations (see e.g. [1-4] and references therein). The main reasons of their successful applications is that they are localized and mostly they enable us to easily and completely reconstruct any $L_2(\mathbb{R})$-function. However, the reconstruction is confined to the $L_2(\mathbb{R})$-functions and it is based on the computation of wavelet coefficients through some integrals on infinite domain (or Fourier integrals).

In the following, it will be shown that Shannon wavelets can well approximate not only C^2-functions, but also some special functions [7,8] (Bessel functions) which define the solution of Weber equation, thus giving a physical meaning to Shannon wavelets, since Bessel function are the classical tool for investigating wave propagation in cylinders. Bessel functions decay to zero as the $L_2(\mathbb{R})$-functions do, and the basic instances of Shannon wavelets, i.e. the scaling and Shannon wavelet, give the best approximation of $L_2(\mathbb{R})$-functions. However, it should be emphasized that this equivalence holds only in an open small interval. Among the many families of wavelets, Shannon wavelets [1-4] offer some more specific advantages, which are often missing in the others. Shannon wavelets are analytically defined, infinitely differentiable, and sharply bounded in the frequency domain. Starting from the approximation of Bessel function, a more general formula will be given for the (local) approximation of any C^2-function. The approximation error will be explicitly computed and a comparison with the wavelet reconstruction will be also given.

2. **Shannon Wavelet**

Shannon scaling function $\varphi(x)$ and wavelet function $\psi(x)$ are defined as

$$\left\{\begin{array}{rcl}\varphi(x) &=& \dfrac{\sin \pi x}{\pi x} = \dfrac{e^{\pi i x} - e^{-\pi i x}}{2\pi i x} \\[2ex] \psi(x) &=& \dfrac{\sin \pi (x-\frac{1}{2}) - \sin 2\pi (x-\frac{1}{2})}{\pi (x-\frac{1}{2})} \\[2ex] &=& \dfrac{e^{-2\,i\,\pi\,x}\,(-i + e^{i\,\pi\,x} + e^{3\,i\,\pi\,x} + i\,e^{4\,i\,\pi\,x})}{(\pi - 2\,\pi\,x)}\end{array}\right. \tag{1}$$

The corresponding families of translated and dilated instances wavelet [1-4], on which is based the multiscale analysis, are

$$
\left\{
\begin{array}{rcl}
\varphi^n_k(x) &=& 2^{n/2}\varphi(2^nx-k) = 2^{n/2}\dfrac{\sin\pi\,(2^nx-k)}{\pi\,(2^nx-k)} \\
&=& 2^{n/2}\dfrac{e^{\pi i(2^nx-k)}-e^{-\pi i(2^nx-k)}}{2\pi i\,(2^nx-k)} \\
\psi^n_k(x) &=& 2^{n/2}\psi(2^nx-k) \\
&=& 2^{n/2}\,\dfrac{\sin\pi(2^nx-k-\frac{1}{2})-\sin 2\pi(2^nx-k-\frac{1}{2})}{\pi(2^nx-k-\frac{1}{2})} \\
&=& \dfrac{2^{n/2}}{2\pi(2^nx-k+\frac{1}{2})}\displaystyle\sum_{s=1}^{2} i^{1+s}e^{s\pi i(2^nx-k)}-i^{1-s}e^{-s\pi i(2^nx-k)}
\end{array}
\right. \tag{2}
$$

being

$$\varphi^0_k(x)=\varphi_k(x), \psi^0_k(x)=\psi_k(x)\ .$$

Both families of Shannon scaling and wavelet are $L_2(\mathbb{R})$-functions, therefore for each $f(x)\in L_2(\mathbb{R})$ and $g(x)\in L_2(\mathbb{R})$ the inner product is defined as

$$< f,g >= \int_{-\infty}^{\infty} f(x)\overline{g(x)}\mathrm{d}\,x,$$

where the bar stands for the complex conjugate.

Shannon wavelets fulfill the following orthogonality properties (for the proof see e.g. [2-4])

$$
\begin{array}{rcl}
\langle \psi^n_k(x)\,,\psi^m_h(x)\rangle &=& \delta^{nm}\delta_{hk} \\
\left\langle \varphi^0_k(x)\,,\varphi^0_h(x)\right\rangle &=& \delta_{kh}, \\
\left\langle \varphi^0_k(x)\,,\psi^m_h(x)\right\rangle &=& 0 \qquad , \qquad m\geq 0,
\end{array}
$$

δ^{nm},δ_{hk} being the Kroenecker symbols.

Shannon wavelets (2.2) can be particularly useful when they are evaluated at some special points. From the definition (2.1) it can be easily seen that [4]

$$\varphi^0_k(h)=\delta_{kh}\quad,\quad(h,k\in\mathbb{Z})\quad,$$

and

$$\psi^n_k(h)=-\frac{2^{1+n/2}}{(2^{n+1}h-2k-1)\pi}\ .$$

For the wavelet functions it is also [4]

$$\lim_{x\to 2^{-n}(h+\frac{1}{2})}\psi^n_k(x)=2^{n/2}\delta_{hk}$$

and

$$\psi_k^n(x) = 0 \quad , \quad x = 2^{-n}\left(k + \frac{1}{2} \pm \frac{1}{3}\right), \quad (n \in \mathbb{N}, k \in \mathbb{Z}) \; . \tag{3}$$

It can be seen that $\psi_k^n(x)$ is $o\,(x)$ in the two sets of points

$$x_\pm = 2^{-n}\left(k + \frac{1}{2} \pm \frac{1}{3}\right)$$

because

$$\lim_{x \to 2^{-n}\left(k+\frac{1}{2}\pm\frac{1}{3}\right)} \frac{\psi_k^n(x)}{x - 2^{-n}\left(k + \frac{1}{2} \pm \frac{1}{3}\right)} = \pm 9 \cdot 2^{-1+3n/2} \; . \tag{4}$$

Since

$$\lim_{x \to 0} \frac{\varphi(x) - 1}{x^2} = -\frac{\pi^2}{6} \; , \tag{5}$$

the scaling function $\varphi(x) - 1$ is $o\,(x^2)$ in $x = 0$.

Both families of scaling and wavelet functions belong to $L_2(\mathbb{R})$, and they have a (slow) decay to zero, in fact, according to their definition (2.2)

$$\lim_{x \to \pm\infty} \varphi_k^n(x) = 0 \quad , \quad \lim_{x \to \pm\infty} \psi_k^n(x) = 0 \; .$$

The maximum value of the scaling function $\varphi_k^0(x)$ can be found in correspondence of $x = k$

$$\max[\varphi_k^0(x_M)] = 1 \quad , \quad x_M = k \; .$$

The min value of $\varphi_k^0(x)$ can be computed only numerically [4] and it is

$$\min[\varphi_k^0(x)] \cong \varphi_k^0(x_m) = \frac{\sin\sqrt{2}\pi}{\sqrt{2}\pi} \quad , \quad x_m = k - 1 \pm \sqrt{2} \; .$$

The minimum of the wavelet $\psi_k^n(x)$ is located in the middle point of the zeroes so that

$$\min[\psi_k^n(x_m)] = -2^{n/2} \quad , \quad x_m = 2^{-n-1}(2k+1)$$

and the max values of $\psi_k^n(x)$ are

$$\max[\psi_k^n(x_M)] = 2^{n/2}\frac{3\sqrt{3}}{\pi} \quad , \quad x_M = \begin{cases} -2^{-n}\left(k + \frac{1}{6}\right) \\ \dfrac{2^{-n-1}}{3}(18k+7). \end{cases}$$

Let $f(x)$ be a $L_2(\mathbb{R})$ function such that

$$\left| \int_{-\infty}^{\infty} f(x)\varphi_k^0(x)\mathrm{d}\,x \right| \le A_k < \infty \quad , \quad \left| \int_0^{\infty} f(x)\psi_k^n(x)\mathrm{d}\,x \right| \le B_k^n < \infty, \qquad \forall n, k \in \mathbb{Z}$$

and $B \subset L_2(\mathbb{R})$ be the Paley-Wiener space, i.e. the space of band limited functions; as a generalization of the Paley-Wiener space, we have the space $B_\psi \supseteq B$ of functions $f(x)$ such that the integrals

$$\begin{cases} \alpha_k &= \ < f(x), \varphi^0_k(x) >= \displaystyle\int_{-\infty}^{\infty} f(x)\overline{\varphi^0_k(x)}dx \\ \beta^n_k &= \ < f(x), \psi^n_k(x) >= \displaystyle\int_{-\infty}^{\infty} f(x)\overline{\psi^n_k(x)}dx \end{cases} \tag{6}$$

exist and have finite values. There follows that, if $f(x) \in B_\psi \subset L_2(\mathbb{R})$ the series

$$f(x) = \sum_{h=-\infty}^{\infty} \alpha_h \, \varphi^0_h(x) + \sum_{n=0}^{\infty} \sum_{k=-\infty}^{\infty} \beta^n_k \psi^n_k(x),$$

on the right side converges to $f(x)$, with α_h and β^n_k given by (2.6). With a fixed upper bound we have the approximation

$$f(x) \cong \sum_{h=-K}^{K} \alpha_h \, \varphi^0_h(x) + \sum_{n=0}^{N} \sum_{k=-S}^{S} \beta^n_k \psi^n_k(x) \ . \tag{7}$$

The error estimate of the approximation (2.7) was given in [4] .

3. **Similarities between Bessel functions and Shannon wavelets**

Bessel functions are some special functions, which are used to define the solution of some fundamental equations like the (homogeneous) Weber equation [7,8]. In particular, the Bessel function $J_n(x)$ of order n is defined as the solution of the Weber equation

$$x^2 y'' + xy' + (x^2 - n^2)y = 0 \quad , \quad n \in \mathbb{C} \ . \tag{8}$$

So that with $n = 1$ the solution of

$$x^2 y'' + xy' + (x^2 - 1)y = 0$$

is

$$y\,(x) = c_1 J_1\,(x) + c_2 J_2\,(x) \ .$$

The Taylor series for Bessel function is

$$J_n(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\Gamma(n+k+1)} \left(\frac{1}{2}x\right)^{2k+n} \quad , \quad x \in (-\varepsilon, \varepsilon)$$

with $\Gamma(n)$ gamma function.

It can be easily seen that $J_{2n}(x), (n \in \mathbb{N})$ are even functions and $J_{2n+1}(x), (n \in \mathbb{N})$ are odd functions, while both have a slow decay to zero (Fig. 1).

Since Bessel functions are $L_2(\mathbb{R})$ they can be easily represented in terms of Shannon wavelets, in particular, around $x = 0$ they nearly coincide with

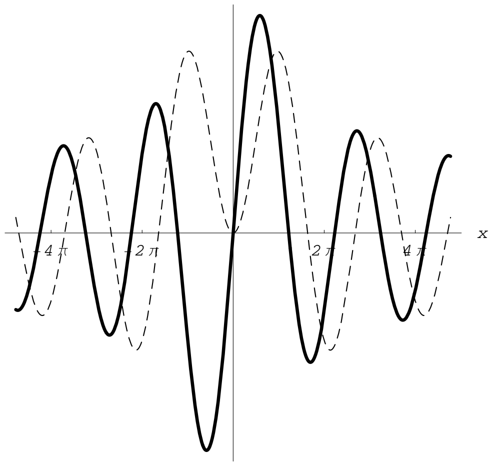

FIGURE 1. *Bessel Functions $J_1(x)$ and $J_2(x)$ (dashed).*

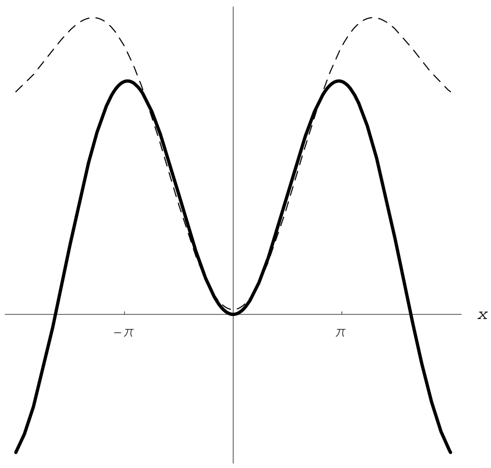

FIGURE 2. *Bessel Functions $J_2(x)$ and (dashed) the Shannon scaling function* $-\dfrac{1}{2}\varphi\left(\dfrac{x}{2\sqrt{2}}\right)+\dfrac{1}{2}$.

the Shannon scaling and wavelet, so that the even $J_{2n}(x), (n \in \mathbb{N})$ can be well approximated by the scaling Shannon functions (Fig. 2) while the odd Bessel functions $J_{2n+1}(x), (n \in)$ can be well approximated by the Shannon wavelets (Fig. 3)

Although this approximation for both is restricted to a small interval, we can assume that in the interval $|\varepsilon| \leq 1$, where the Bessel functions substantially

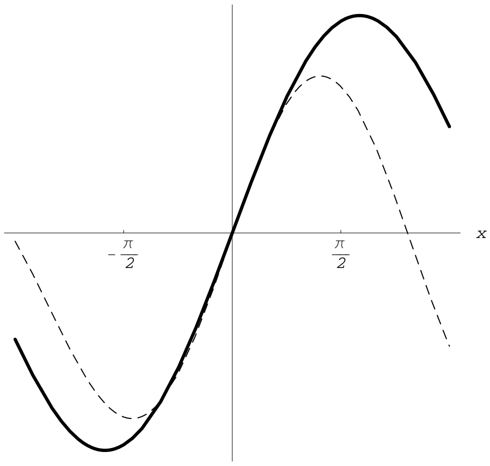

FIGURE 3. *Bessel Functions* $J_2(x)$ *and (dashed) the Shannon scaling function* $-\frac{1}{2}\psi\left(\frac{x}{3\sqrt{2}}+\frac{1}{5}\right)-0.08$.

coincide with the Shannon wavelet families, Shannon scaling functions and Shannon wavelets are solution of the Weber equation in this interval.

The Taylor expansion (in $x=0$) for the scaling wavelet is

$$\varphi\,(x)=\sum_{n=0}^{\infty}(-1)^n\frac{\pi^{2n}x^{2n}}{(2n+1)!}$$

while for the Shannon wavelet $\psi(x)$ in $x=0$, it is

$$\psi\,(x)=\sum_{n=0}^{\infty}a_k x^k$$

$$a_{k+1}=2a_k+b_{k+1}, k\geq 0$$
$$b_k=\left[\left(1+(-1)^k\right)(-1)^{[k/2]}\tfrac{2}{(k-1)!k}+\left(1+(-1)^{k+1}\right)(-1)^{[(k+1)/2]}\tfrac{2^{k+1}}{k!}\right]\pi^{k-1}, k\geq 1$$
$$a_0=\tfrac{2}{\pi}, b_0=0,$$

where $[k/2]$ is the integer quotient of k and 2.

Up to the second order, we have

$$\psi\,(x)\cong\frac{2}{\pi}-4\left(\frac{\pi-1}{\pi}\right)x-\left(\frac{\pi^2+8\pi-8}{\pi}\right)x^2$$

therefore we can assume that

$$\begin{array}{rcll} J_1(x) & = & -\dfrac{1}{2}\psi\left(\dfrac{x}{3\sqrt{2}}+\dfrac{1}{5}\right)-0.08 & ,\quad x\in\left(-\dfrac{\pi}{2},\dfrac{\pi}{2}\right) \\ J_2(x) & = & -\dfrac{1}{2}\varphi\left(\dfrac{x}{2.7}\right)+\dfrac{1}{2} & ,\quad x\in(-\pi,\pi) \end{array}$$

so that the solution of the Weber equation (3.1) is

$$-\frac{1}{2}\psi\left(\frac{x}{3\sqrt{2}}+\frac{1}{5}\right)-0.08 \quad , \quad x\in\left(-\frac{\pi}{2},\frac{\pi}{2}\right) \ .$$

4. Second order approximation by Shannon wavelets

In order to generalize the results of previous section, let us define the 2nd order approximation of a function $f(x)\in C^2$ in x_0 by

$$f(x)\cong f(x_0)+au(x)+bv(x)$$

where $u(x), v(x)$ are chosen in a such a way that

$$\begin{array}{rcl} u(x_0)=0 & , & v(x_0)=0 \\ u'(x_0)\neq 0 & , & u''(x_0)=0 \\ v'(x_0)=0 & , & v''(x_0)=0 \ . \end{array} \tag{9}$$

It can be easily shown, by deriving $f(x)$ up to the second order that

Lemma 4.1. *Under the conditions* (9)*, any $f(x)\in C^2$ can be approximated in x_0 by*

$$f(x)\cong f(x_0)+\frac{f'(x_0)}{u'(x_0)}u(x)+\frac{f''(x_0)}{v''(x_0)}v(x) \ . \tag{10}$$

Theorem 4.1. *Let $f(x)$ be a given function, such that for a fixed n, k, in one of the two points*

$$x_\pm = 2^{-n}\left(k+\frac{1}{2}\pm\frac{1}{3}\right) \tag{11}$$

it is at least C^2 and $f'(x_+)>0, f'(x_-)<0$. In an open interval centered at $x_\pm$, $f(x)$ can be approximated up to the second order by

$$f(x)\cong f(x_\pm)+\frac{2^{1-3n/2}}{9}\left[f'(x_\pm)\,\psi_k^n(x)\right]-\frac{6}{\pi^2}f''(x_\pm)\left[\varphi(x-x_\pm)-1\right] \ . \tag{12}$$

Proof: It follows from the previous lemma, in particular from Eqs. (2.4), (2.5) and the series expansion of $f(x)$.

□

However, in practical approximation problem usually what it is given is $f(x)$ and the point x_0 where we want to have the approximation (12). Therefore we can solve the equation

$$x_0 = 2^{-n}\left(k+\frac{1}{2}\pm\frac{1}{3}\right)$$

by fixing one of the two parameters n, k and computing the other. Indeed since the two parameters must be integers for this reason it is convenient to take

$$k=\left\lfloor x_0-\left(\frac{1}{2}\pm\frac{1}{3}\right)\right\rfloor , n=\left\lfloor \log_2\frac{k+\frac{1}{2}\pm\frac{1}{3}}{x_0}\right\rfloor$$

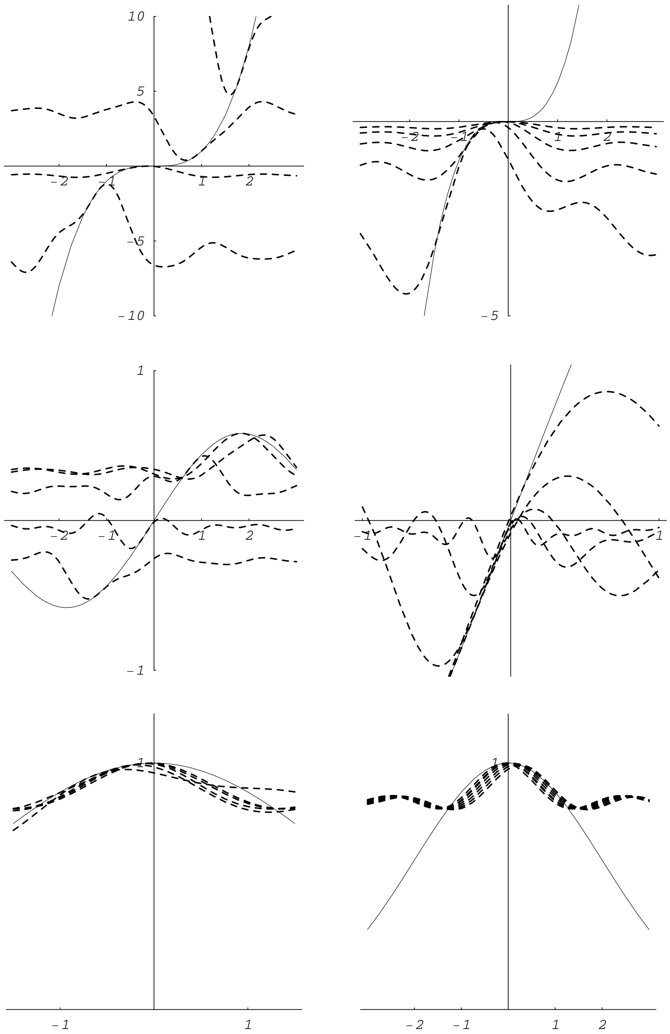

FIGURE 4. *Approximation by Shannon scaling and wavelet function according to (12) of (from top): x^3 with $n = 0, -2 \leq k \leq 2$ (left) and $-2 \leq n \leq 2, k = -1$; $J_1(x)$: $n = 0, -2 \leq k \leq 2$ (left) and $n = 0, -2 \leq k \leq 2$; $e^{-x^2/8}$: $-2 \leq n \leq 2, k = -1$, (left) and $n = 4, -2 \leq k \leq 4$.*

being $\lfloor x \rfloor$ the floor of x, i.e. the greatest integer less than or equal to x.

Under some further hypotheses on the given function $f(x)$ it is possible to estimate the approximation error as follows.

Theorem 4.2. *Let $f(x)$ be a given bounded function, such that, for fixed n , k, in one of the two points (11) it is at least C^2 and $f'(x_+) > 0, f'(x_-) < 0$ with*

$$f(x) < K \quad , \quad x \in I_\pm = (x_\pm - \delta, x_\pm + \delta) \ . \tag{13}$$

In $I_\pm$ *the approximation error of* (4.4) *is*

$$\varepsilon_\pm \leq K + \left| \mp \frac{2^{1-3n/2}}{9} \left[f'\left(x_\pm\right) \psi_k^n \left(x_\pm \mp \delta\right)\right] + \frac{6}{\pi^2} f''\left(x_\pm\right) \left[\varphi\left(\delta\right) - 1\right] \right| \ , \ x \in I_\pm \ . \tag{14}$$

Proof: Let ε be the error, from (4.4) there follows

$$\varepsilon = \max_{x \in I_\pm} \left| f\left(x\right) - f\left(x_\pm\right) - \frac{2^{1-3n/2}}{9} \left[f'\left(x_\pm\right) \psi_k^n \left(x\right)\right] + \frac{6}{\pi^2} f''\left(x_\pm\right) \left[\varphi\left(x - x_\pm\right) - 1\right] \right| \ .$$

It is

$$\left| f\left(x\right) - f\left(x_\pm\right) - \frac{2^{1-3n/2}}{9} \left[f'\left(x_\pm\right) \psi_k^n \left(x\right)\right] + \frac{6}{\pi^2} f''\left(x_\pm\right) \left[\varphi\left(x - x_\pm\right) - 1\right] \right|$$

$$\leq \left| f\left(x\right) - f\left(x_\pm\right) \right| + \left| - \frac{2^{1-3n/2}}{9} \left[f'\left(x_\pm\right) \psi_k^n \left(x\right)\right] + \frac{6}{\pi^2} f''\left(x_\pm\right) \left[\varphi\left(x - x_\pm\right) - 1\right] \right|$$

and, by taking into account (4.5)

$$\varepsilon \leq K + \left| - \frac{2^{1-3n/2}}{9} \left[f'\left(x_\pm\right) \psi_k^n \left(x\right)\right] + \frac{6}{\pi^2} f''\left(x_\pm\right) \left[\varphi\left(x - x_\pm\right) - 1\right] \right| \ .$$

Thus we have to separate the computation in two different intervals.

In I_+ it is $f'\left(x_+\right) > 0$ so that we have

$$\begin{aligned} \varepsilon_+ &\leq K + \left| - \frac{2^{1-3n/2}}{9} \left[f'\left(x_+\right) \psi_k^n \left(x\right)\right] + \frac{6}{\pi^2} f''\left(x_+\right) \left[\varphi\left(x - x_+\right) - 1\right] \right| \\ &\leq K + \left| - \frac{2^{1-3n/2}}{9} \left[f'\left(x_+\right) \min_{x \in I_+} \psi_k^n \left(x\right)\right] + \frac{6}{\pi^2} f''\left(x_+\right) \max_{x \in I_+} \left[\varphi\left(x - x_+\right) - 1\right] \right| \end{aligned}$$

Analogously, in I_- where $f'\left(x_-\right) < 0$ we have

$$\varepsilon_- \leq K + \left| \frac{2^{1-3n/2}}{9} \left[\left| f'\left(x_-\right)\right| \max_{x \in I_-} \psi_k^n \left(x\right)\right] \right| + \left| \frac{6}{\pi^2} f''\left(x_-\right) \max_{x \in I_-} \left[\varphi\left(x - x_-\right) - 1\right] \right|$$

so that, according to the definitions (2.2) it is

$$\min_{x \in I_+} \psi_k^n \left(x\right) = \psi_k^n \left(x_+ - \delta\right) \quad , \quad \max_{x \in I_-} \psi_k^n = \psi_k^n \left(x_- + \delta\right)$$

$$\max_{x \in I_\pm} \left[\varphi\left(x - x_\pm\right) - 1\right] = \varphi\left(-\delta\right) - 1 = \varphi\left(\delta\right) - 1 \ .$$

Therefore we have

$$\varepsilon_+ \leq K + \left| - \frac{2^{1-3n/2}}{9} \left[f'\left(x_+\right) \psi_k^n \left(x_+ - \delta\right)\right] + \frac{6}{\pi^2} f''\left(x_+\right) \left[\varphi\left(\delta\right) - 1\right] \right| , \ x \in I_+$$

and

$$\varepsilon_- \leq K + \left| \frac{2^{1-3n/2}}{9} \left| f'\left(x_-\right)\right| \psi_k^n \left(x_- + \delta\right) \right| + \left| \frac{6}{\pi^2} f''\left(x_-\right) \left[\varphi\left(\delta\right) - 1\right] \right| , x \in I_-$$

from where, (4.6) follows.

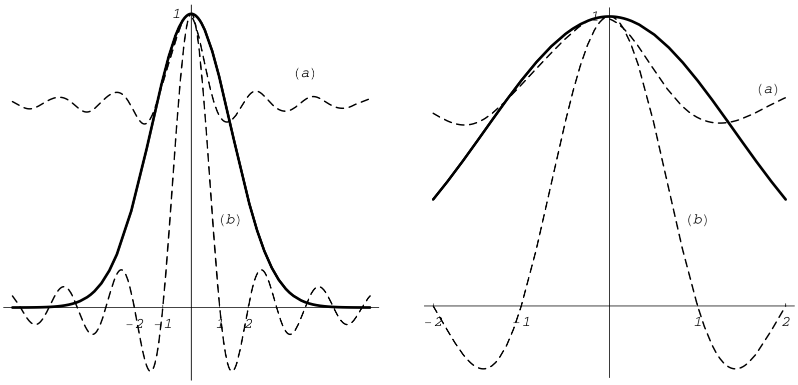

FIGURE 5. *The function* $e^{-x^2/4}$ *and its approximation with* $n = 0, k = -1$ *with: (a)* (4.4) *and (b)* (4.7).

$\square$

If we compare (4.4) with the wavelet reconstruction (2.7) where $N = 0, K = 0$

$$f(x) \cong \alpha_0 \, \varphi_0^0(x) + \beta_0^0 \psi_0^0(x) \tag{15}$$

we notice (Fig. 5) that there are some advantages in using (4.4). In fact, up to the second order approximation, (4.4) is better than (2.7) moreover (4.4) can be used for a more general class of functions C^2 while (2.7) is restricted only to the $L_2(\mathbb{R})$ -functions. In fact, only in this case the integrals (2.6) are finite and the wavelet coefficients can be computed.

5. **Conclusions**

It has been shown that any C^2-function can be easily approximated by the basic scaling and Shannon function. The approximation error is computed and a comparison with the wavelet reconstruction of $L_2(\mathbb{R})$-functions is also given. In the comparison it is shown that the proposed approximation is more efficient.

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