Applied Mathematics and Computation 215 (2010) 4164–4171

Contents lists available at ScienceDirect

Applied Mathematics and Computation

journal homepage: www.elsevier.com/locate/amc

Harmonic wavelet method towards solution of the Fredholm type integral equations of the second kind ☆

C. Cattani [a], A. Kudreyko [b,*]

[a] diFarma, University of Salerno, Via Ponte Don Melillo, 84084 Fisciano (SA), Italy
[b] Department of Mathematics and Computer Science, University of Salerno, Via Ponte Don Melillo, 84084 Fisciano (SA), Italy

ARTICLE INFO

Keywords:
Harmonic wavelets
Haar wavelets
Legendre wavelets
Fredholm integral equation
Periodic multiresolution analysis
Collocation method

ABSTRACT

Periodic harmonic wavelets (PHW) were applied as basis functions in solution of the Fredholm integral equations of the second kind. Two equations were solved in order to find out advantages and disadvantages of such choice of the basis functions. It is proved that PHW satisfy the properties of the multiresolution analysis.

1. Introduction

Solution of integral equations is one of the main goals in various areas of applied science and engineering. The Fredholm integral equation of the second kind is given as [1]

$$f(x) = \int_a^b K(x,t)f(t)dt + g(x), \tag{1}$$

where a and b are finite or infinite real numbers, $K(x,t)$ and $g(x)$ are the known functions for $K(x,t) \in L^2([a;b] \times [a;b])$, $g(x) \in L^2[a;b]$.

It is well-known that integral equations are usually difficult to solve analytically and exact solutions are very scarce. Therefore, integral equations have been a subject of great interest of many researchers. The computational approach of solution of integral equations is an essential branch of the scientific inquiry. Indeed, in order to resolve integral equations, there were developed many methods: such as collocation, Galerkin method, expansion method, product-integration method, Sinc-collocation method, Taylor's series method, etc.

There were many different orthonormal basis functions, such as the Fourier functions, the Chebyshev polynomials and wavelet functions, had been developed and employed with collocation and the Galerkin methods to approximate solution of partial differential equations as well as integral equations. All these functions have their advantages and disadvantages in its application. Among these basis functions, wavelet basis is the most attractive for researchers due to its good approximation properties and quick rate of convergence of the wavelet series [2,3].

In the present work, we employ PHW as basis functions [4,5] in the collocation method towards approximate solution of the Fredholm type integral equations and make the comparison of our results with other methods and analytical solution on two examples. Harmonic wavelets and its periodization are presented in Section 2. In Section 3, it is proved that PHW fulfill properties of the multiresolution analysis. The method of solution is presented in Section 4. Section 5 is devoted to the

☆ This article is a collaborative effort.
* Corresponding author.
E-mail addresses: ccattani@unisa.it (C. Cattani), akudreyko@unisa.it (A. Kudreyko).

0096-3003/$ - see front matter © 2009 Elsevier Inc. All rights reserved.
doi:10.1016/j.amc.2009.12.037

methods of the Haar and Legendre wavelets, which we employed to compare our results. The application of PHW as basis functions towards the given integral equations is presented in Section 6.

2. Harmonic wavelets

Let the Fourier transform of a given function $\psi(x)$ be defined as follows:

$$\hat{\psi}(\omega) = \begin{cases} 1/2\pi & \text{for } 2\pi \leqslant \omega < 4\pi, \\ 0 & \text{elsewhere}. \end{cases} \tag{2}$$

Then by calculating the inverse Fourier transform of $\hat{\psi}(\omega)$, we obtain [6],

$$\psi(x) = \frac{e^{4\pi ix} - e^{2\pi ix}}{2\pi ix}, \tag{3}$$

which is the fundamental function of the Littlewood–Paley theory [2]. It can be shown that this function is the starting point for the construction of the multiresolution analysis of harmonic wavelets [5].

By changing the argument in (3) from x to $2^j x - k$, where j and k are the scaling and the translation parameters $(j, k \in \mathbb{Z})$, the shape of the wavelet is not changed but its horizontal scale is compressed by the factor 2^j, and its position is translated by k units at the new scale (which is $k/2^j$ units at the original scale). The value of j determines the "level" of the wavelet. At level j it occupies bandwidth from $2\pi\, 2^j$ to $4\pi\, 2^j$ which is j octaves higher up the frequency scale.

The Fourier transform of the scaling function of Eq. (3) is [5,6]

$$\hat{\varphi}(\omega) = \begin{cases} 1/2\pi & \text{for } 0 \leqslant \omega < 2\pi, \\ 0 & \text{elsewhere}, \end{cases} \tag{4}$$

or in the space variable,

$$\varphi(x) = \frac{e^{2\pi ix} - 1}{2\pi ix},$$

which is orthogonal to its own unit translates and the mother wavelet $\psi(x)$ [5].

In general, harmonic wavelets can be referred to a physical family of wavelets because they were proposed for the analysis of physical problems [7,8]. In spite of their slow decay in the space variable $(\simeq x^{-1})$ they have a perfect localization in the frequency domain [5,6].

In their simplest form, orthogonal harmonic wavelets provide a complete set of complex exponential functions whose spectrum is confined to adjacent (non-overlapping) bands of frequency. Their real part is an even function which is identical to the Shannon wavelet [9]. Their imaginary part is akin but an odd function. Their equal spacing along the time axis is twice that of the corresponding set of the so-called Shannon function. Harmonic wavelets have been found to be particularly suitable for vibration and acoustic analysis because their harmonic structure is similar to naturally occurring signal structures and therefore they correlate well with experimental signals [8]. They can also be computed by a numerically efficient algorithm based on the fast Fourier transform (FFT) [5,6,8].

In our paper we propose the PHW as basis functions for the approximate solution of the Fredholm integral equations defined on a finite interval $[a; b]$. Therefore, in the following we will consider a periodic expansion of harmonic wavelets. Periodization is a standard technique in the Fourier analysis. Periodic scaling functions can be constructed by the standard procedure [2,5]

$$\varphi^{per}(x) = \sum_{k=-\infty}^{\infty} \varphi(x-k), \quad k \in \mathbb{Z}. \tag{5}$$

Notice, that periodization is not defined for every function on $\mathbb{R}$, but obviously well defined, when $\varphi \in L^1(\mathbb{R}, \mathbb{C})$. By substituting the Fourier transform of $\varphi(x-k)$ into (5), we obtain

$$\varphi^{per}(x) = \sum_{k=-\infty}^{\infty} \int_{-\infty}^{\infty} \hat{\varphi}(\omega) e^{-i\omega k} e^{-i\omega x} d\omega$$

and from the equality [5]

$$\sum_{k=-\infty}^{\infty} e^{i(\omega_1 - \omega_2)k} = 2\pi \sum_{m=-\infty}^{\infty} \delta(\omega_2 - \omega_1 - 2\pi m)$$

it follows that

$$\sum_{k=-\infty}^{\infty} e^{-i\omega k} = 2\pi \sum_{m=-\infty}^{\infty} \delta(\omega - 2\pi m). \tag{6}$$

Since $\hat{\varphi}(\omega)$ equals zero everywhere except the band $0 \leqslant \omega < 2\pi$, the only the term with $m = 0$ needs to be retained in (6), therefore

$$\varphi^{per}(x) = \int_{-\infty}^{\infty} \hat{\varphi}(\omega) e^{i\omega x} 2\pi i \delta(\omega) d\omega,$$

which with (4) gives

$$\varphi^{per}(x) = 1.$$

The same argument applies when $\varphi(x-k)$ in (5) is replaced by its complex conjugate.

For the general wavelet function $\psi(2^j x - k)$, its periodic (circular) equivalent is

$$\psi^{per}(2^j x - k) = \sum_{m=-\infty}^{\infty} \psi(2^j(x-m) - k) = \sum_{m=-\infty}^{\infty} 2^{-j} \int_{-\infty}^{\infty} e^{-i\omega k/2^j} e^{-i\omega m} \hat{\psi}(\omega/2^j) e^{i\omega x} d\omega, \tag{7}$$

where $j = 0, \ldots, N$ and $k = 0, \ldots, 2^j - 1$. The summation over m is given by (6). Since $\hat{\psi}(\omega/2^j) = 1/2\pi$ for $2\pi 2^j \leqslant \omega < 2\pi 2^{j+1}$ and zero elsewhere. The only nonzero values of m that have to be considered in (6) are $m = 2^j$ to $2^{j+1} - 1$. The substitution of Eq. (6) into (7) yields us the following function:

$$\psi_{j,k}^{per}(2^j x - k) = 2^{-j} \sum_{m=2^j}^{2^{j+1}-1} e^{2\pi i m \left(x - \frac{k}{2^j}\right)}. \tag{8}$$

Thus, we have received a function of periodic harmonic wavelets, which are defined on the unit interval $[0;1]$ as shown in Fig. 1 for several values of translation and dilation parameters, and extended by a unit periodization from $-\infty$ to ∞.

3. Multiresolution analysis of harmonic wavelets

The idea of the decomposition of a function into a sum of approximate and detailed terms by using orthogonal and biorthogonal wavelets is realized in multiresolution analysis [2,3].

According to the stated above and under the assumption that

$$\psi_{j,k}^{per}(x) = \sum_{r \in \mathbb{Z}} \psi_{j,k}(x+r), \quad \varphi_{j,k}^{per}(x) = \sum_{r \in \mathbb{Z}} \varphi_{j,k}(x+r) \tag{9}$$

are bounded functions. Let $V_j,\ j \in \mathbb{Z}$ be a sequence of subspaces of functions in $L^2(\mathbb{R}, \mathbb{C})$. Then the finite set of spaces $V_j,\ j \in \mathbb{Z}$ with periodic harmonic wavelets as basis functions is called a multiresolution analysis (MRA) if the subsequent conditions hold:

(i) *Nested.* $V_j \subset V_{j+1}$;
(ii) for $j = 0, 1, 2, \ldots$ the system $\varphi_{j,k}^{per}$ with $k = 0, 1, \ldots, 2^j - 1$ is an orthonormal basis in V_j;
(iii) for $j = 0, 1, 2, \ldots$ the system $\{1, \psi_{s,k}^{per}\}$ for $s = 0, 1, \ldots, j-1$ and $k = 0, 1, \ldots, 2^{s-1}$ is an orthonormal basis in V_j;
(iv) $\bigcup_{j=0}^{\infty} V_j$ is the dense in $L^2[0;1]$, so the system $\psi_{j,k}^{per}$ is a complete orthonormal system in $L^2[0;1]$.

Then we can formulate the following theorem.

Theorem 1. *Periodic harmonic wavelets fulfill axioms* (i)–(iv) *of the multiresolution analysis.*

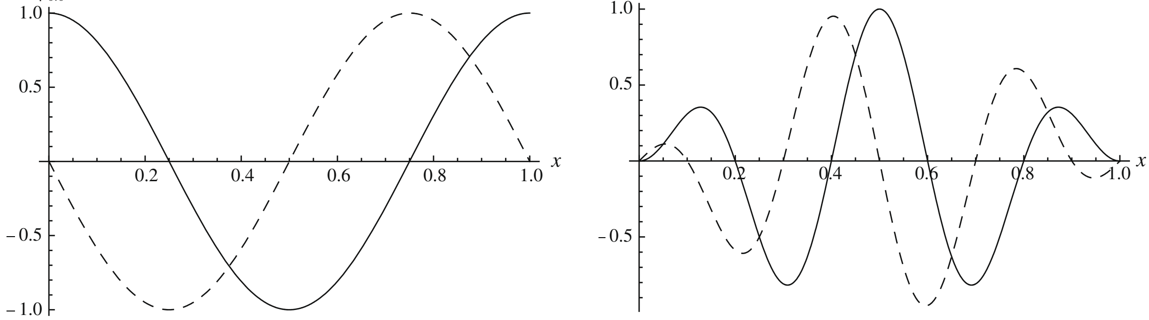

Fig. 1. Real (solid line) and imaginary (dashed line) parts of periodic harmonic wavelets for $\psi_0^0(x)$ and $\psi_1^1(x)$.

Proof. From now on, we assume that $\varphi, \psi \in L^1(\mathbb{R}, \mathbb{C})$. Notice that for $s < j$ we have $\varphi_{s,k} = \sum_{r\in\mathbb{Z}} \alpha_r \varphi_{j,r}$ and $\psi_{s,k} = \sum_{r\in\mathbb{Z}} \beta_r \varphi_{j,r}$. From (9) we infer that

$$\sum_{r\in\mathbb{Z}} |\alpha_r| = \sum_{r\in\mathbb{Z}} |\langle \varphi_{s,k}, \varphi_{j,r}\rangle| \leqslant 2^{s/2+j/2} \sum_{r\in\mathbb{Z}} \int_{-\infty}^{\infty} |\varphi(2^s x - k)| * |\varphi(2^j x - r)| dx = 2^{s/2+j/2} \int_{-\infty}^{\infty} |\varphi(2^s x - k)| * \varphi^{per}(2^j x) dx$$
$$\leqslant C\, 2^{s/2+j/2} \int_{-\infty}^{\infty} |\varphi(2^s x - k)| dx < \infty$$

and the same argument gives $\sum_{r\in\mathbb{Z}} |\beta_r| < \infty$. This implies that $\varphi^{per}_{s,k} \in V_j$ and $\psi^{per}_{s,k} \in V_j$. This gives the proof of (i). Thus, we have to check the orthogonality in (ii)–(iv). For $j \geqslant 0$ we have

$$\int_0^1 \psi^{per}_{j,k}(t) \bar{\psi}^{per}_{j',k'}(t) dt = \sum_{r,s\in\mathbb{Z}} \int_0^1 \psi_{j,k}(t+s) \bar{\psi}_{j',k'}(t+r) dt = \sum_{s\in\mathbb{Z}} \int_{-\infty}^{\infty} \psi_{j,k}(t+s) \bar{\psi}_{j',k'}(t) dt = \sum_{s\in\mathbb{Z}} \int_{-\infty}^{\infty} \psi_{j,k-2^j s}(t) \bar{\psi}_{j',k'}(t) dt. \quad (10)$$

This shows that $\psi^{per}_{j,k}$ and $\psi^{per}_{j',k'}$ are orthogonal unless $j = j'$ and $k = k - 2^j s$ for $s \in \mathbb{Z}$. Therefore, the system, which appears in item (iv) of the properties of the MRA is orthonormal. If we repeat calculations (10) with $j = j'$ and φ instead of ψ, we obtain that the item (ii) holds true. Thus, $\dim V_j = 2^j$.

In order to show the proof of (iv), let us consider the orthogonal projection $\mathscr{P}_j$ from $L^2([0;1), \mathbb{C})$ onto V_j. From (ii) we infer that

$$\mathscr{P}_j f = \sum_{k=0}^{2^j-1} \langle f, \varphi^{per}_{j,k}\rangle \varphi^{per}_{j,k}. \quad (11)$$

Let us fix an exponential $e^{2\pi i r t}$ and compute the rth Fourier coefficient of $\mathscr{P}_j(e^{2\pi i r t})$. From (11) and the consequences of the Parseval's identity it follows that [3]

$$\mathscr{P}_j(e^{2\pi i r t}) = \sum_{k=0}^{2^j-1} \langle e^{2\pi i r t}, \varphi^{per}_{j,k}\rangle \langle \varphi^{per}_{j,k}, e^{2\pi i r t}\rangle = \sum_{k=0}^{2^j-1} |\langle \varphi^{per}_{j,k}, e^{2\pi i r t}\rangle|^2 = \sum_{k=0}^{2^j-1} |\hat{\varphi}^{per}_{j,k}(r)|^2 = 2\pi |\hat{\varphi}^{per}_{j,k}(r)|^2. \quad (12)$$

We may conclude [3] that $\mathscr{P}_j(e^{2\pi i r t}) \to 1$ as $j \to \infty$. Since $\|\mathscr{P}_j\| = 1$ (because $\mathscr{P}_j$ is an orthogonal projection) and $\{e^{2\pi i s}\}_{s\in\mathbb{Z}}$ is an orthonormal system in $L^2[0;1)$, we infer that $\mathscr{P}_j(e^{2\pi i s})$ tends in $L^2[0;1)$ to $e^{2\pi i r t}$ as $j \to \infty$. This implies that for every trigonometric polynomial $f : \mathscr{P}_j(f) \to f$ in $L^2([0;1), \mathbb{C})$. Since (the Weirerstrass theorem) trigonometric polynomials are dense in $L^2([0;1), \mathbb{C})$, we conclude that $\bigcup_{j=0}^{\infty} V_j$ is the dense in $L^2([0;1), \mathbb{C})$. □

4. Wavelet collocation method towards solution of integral equations

According to the previous section, PHW form a MRA and as it was shown in [12], a function $f(x)$, which is defined on $[0;1]$ or a periodic function with a unit period can be represented as the sum of a convergent series

$$f(x) = a_0 + \sum_{j=0}^{\infty} \sum_{k=-\infty}^{\infty} \{a_{j,k}\psi_{j,k}(x) + \tilde{a}_{j,k}\bar{\psi}_{j,k}(x)\},$$

where $a_0,\ a_{j,k}$ and $\tilde{a}^k_j \in \mathbb{C}$. In particular, the projection onto the given subspace for $j = 0, \ldots, N$ and $k = 0, \ldots, 2^j - 1$, is [5]

$$\mathscr{P}_j f(x) \simeq a_0 + \sum_{j=0}^{N} \sum_{k=0}^{2^j-1} \{a_{j,k}\psi_{j,k}(x) + \tilde{a}_{j,k}\bar{\psi}_{j,k}(x)\}. \quad (13)$$

The substitution of (13) into (1) for the lowest level of approximation, i.e. $N = 0$ and $k = 0$ yields

$$a_0 + a_{0,0}\psi_{0,0}(x) + \tilde{a}_{0,0}\bar{\psi}_{0,0}(x) = \int_a^b K(x,t)\{a_0 + a_{0,0}\psi_{0,0}(t) + \tilde{a}_{0,0}\bar{\psi}_{0,0}(t)\} dt + g(x). \quad (14)$$

Let us solve Eq. (14) by using the collocation method. The collocation points are defined as follows: $x_i = \frac{i}{M}$, where M is the number of unknown wavelet coefficients (and also collocation points) $a_{j,k}$ including the scaling coefficient a_0. In the case of $N = 0$, we need to choose two collocation points $x_1, x_2 \in [a,b]$ and recall the equality $\tilde{a}_{0,0} = \bar{a}_{0,0}$ [6]

$$\begin{cases} a_0 + a_{0,0}\psi_{0,0}(x_1) + \tilde{a}_{0,0}\bar{\psi}_{0,0}(x_1) = \int_a^b K(x_1,t)\{a_0 + a_{0,0}\psi_{0,0}(t) + \tilde{a}_{0,0}\bar{\psi}_{0,0}(t)\} dt + g(x); \\ a_0 + a_{0,0}\psi_{0,0}(x_2) + \tilde{a}_{0,0}\bar{\psi}_{0,0}(x_2) = \int_a^b K(x_2,t)\{a_0 + a_{0,0}\psi_{0,0}(t) + \tilde{a}_{0,0}\bar{\psi}_{0,0}(t)\} dt + g(x); \\ \tilde{a}_{0,0} = \bar{a}_{0,0}. \end{cases} \quad (15)$$

Thus, for the lowest level of approximation we have reduced integral equation (1) to a system of linear equations. We skip the detailed analysis of this system, and only notice that it could be solved for example by the Cramer method.

To illustrate the efficiency of the application of PHW as basis functions in collocation method, we found solution of two equations of type (1).

5. The Haar and Legendre wavelet methods

In order to demonstrate the efficiency of the proposed algorithm, we solved two examples and compared its solution with the Haar wavelet method [11] and the method of the Legendre polynomials [10]. But first let us show a brief idea of each of these methods.

5.1. The Haar wavelet method

The Haar wavelet family is defined for $t \in [0;1]$ as follows:

$$h_i = \begin{cases} 1, & \text{for } t \in \left[\frac{k}{m};\frac{k+0.5}{m}\right); \\ 0, & \text{elsewhere}; \\ -1, & \text{for } t \in \left[\frac{k+0.5}{m};\frac{k+1}{m}\right), \end{cases} \tag{16}$$

where the integers $m = 2^j,\ j = 0,1,\dots,N$ indicate the level of the wavelet and $k = 0,1,\dots,m-1$ is the translation parameter. The solution of the unknown function $f(t)$ in (1) is searched in the form

$$f(t) = \sum_{i=1}^{2M} a_i h_i(t), \tag{17}$$

where a_i are the unknown wavelet coefficients and $M = 2^N$ is the number of grid points. The substitution of (17) into Eq. (1) yields us the following expression:

$$\sum_{i=1}^{2M} a_i h_i(x) = \sum_{i=1}^{2M} a_i G_i(x) + g(x),$$

where $G_i(x) = \int_0^1 K(x,t)h_i(t)dt$.

The application of collocation method towards the obtained expression gives us a system of linear equations with the unknown coefficients a_i

$$\sum_{i=1}^{2M} a_i[h_i(x_l) - G_i(x_l)] = g(x_l), \quad l = 1,2,\dots,2M.$$

Thus, the Haar wavelet method reduces the Fredholm integral equation towards a system of linear equations, where the unknown function $f(t)$ represents a numerical superposition of the product of wavelet coefficients a_i and $h_i(t)$.

5.2. Legendre wavelet method

Compactly supported wavelets derived from the Legendre polynomials are termed spherical harmonic or Legendre wavelets. These wavelets $\psi_m^n(x) = \psi(k,\hat{n},m,x)$ have four arguments $k = 2,3,\dots,\ \hat{n} = 2n-1,\ n = 1,2,3,\dots,2^{k-1}$, m is the order of the polynomial, which is defined on $[0;1)$ as follows:

$$\psi_m^n(x) = \begin{cases} 2^{k/2}\left(m+\frac{1}{2}\right)^{1/2} L_m(2^k x - \hat{n}), & \frac{\hat{n}-1}{2^k} \leqslant x \leqslant \frac{\hat{n}+1}{2^k}; \\ 0, & \text{elsewhere}, \end{cases} \tag{18}$$

where $L_m(x)$ are the Legendre polynomials of order m.

The Legendre wavelets serve as basis functions in [10] and a function $f(x) \in L^2[0;1]$ can be expanded as

$$f(x) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} c_m^n \psi_m^n(x).$$

According to [10], integral equation (1) can be reduced to a system of nonlinear equations.

6. Examples

6.1. Example 1

Consider the Fredholm type nonlinear integral equation of the second kind

$$f(x) = x\int_0^1 t\sqrt{f(t)}\,dt + 2 - \frac{x}{3}(2\sqrt{2}-1) - x^2, \tag{19}$$

which has the analytical solution $f(x) = 2 - x^2$.

The projection of the solution on the space with harmonic wavelets as basis functions at the lowest level $N = 0$ with collocation points $x_1 = 0.25$ and $x_2 = 0.5$ give us the values of the unknown wavelet coefficients, i.e. $a_0 = 5/3$ and $a_0^0 = 1/\pi$. The corresponding function of such projection (13) is

$$\mathscr{P}_0 f(x) = \frac{5}{3} + \frac{1}{\pi}\left(\sin 2\pi x - \frac{1}{\pi}\cos 2\pi x\right).$$

The corresponding plot of such projection is presented in Fig. 2 (left). We can see that the modulus of the approximation error (20) is quite high and varies up to $\simeq 0.41$.

On the scale $N = 1$, the projection of the solution on the first level of approximation looks as shown in Fig. 2 (right). The modulus of the absolute error still remains high $\simeq 0.37$, but on the qualitative level we see that the plot of the approximating function better describes the unknown function on the interval $(0; 1)$. The results of numerical computations for $N = 6$ are presented in Table 1 and compared with the results, obtained by the Haar wavelet method.

To estimate the exactness of the achieved results, we considered the maximum of the modulus (at a certain level of approximation) of the difference between the obtained $f(x)$ and exact values $f_{ex}(x)$ of the unknown function

$$\varepsilon = \max_{0.1 \leqslant x \leqslant 0.9} (|f(x) - f_{ex}(x)|). \tag{20}$$

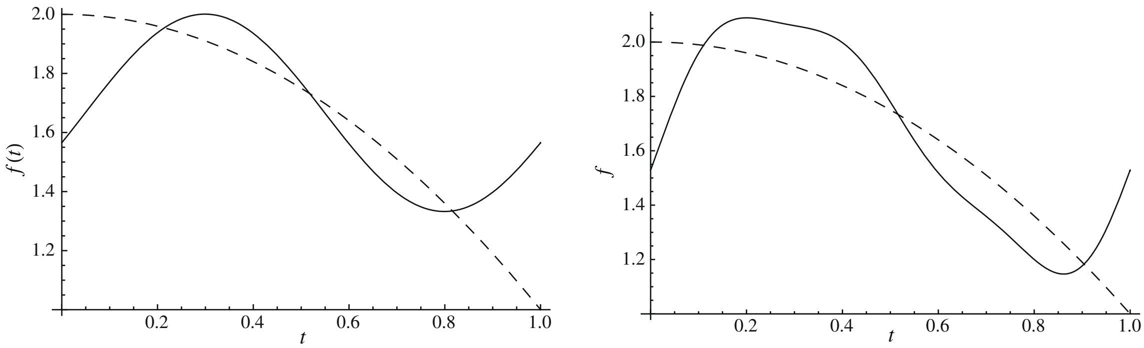

Fig. 2. Exact solution (dashed line) and approximate periodic wavelet solution (solid line) at scales $N = 0$ (left) and $N = 1$ (right).

Table 1
Comparison of error estimation in the case of PHW and the Haar wavelets as basis functions.

N	M	Harmonic wavelets, ε	Haar wavelets, ε
0	2	0.22	–
1	4	0.18	0.0033
2	8	0.094	2.7×10^{-3}
3	16	0.008	0.0011
6	128	8×10^{-7}	3.1×10^{-5}

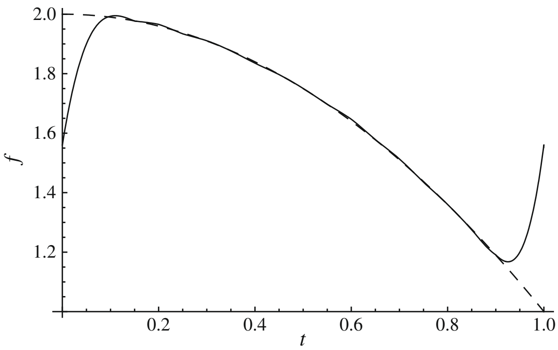

Fig. 3. Exact solution (dashed line) and approximate periodic wavelet solution (solid line) at scale $N = 3$.

Note, that we cut out 1/10 from the boundary intervals. The explanation of such operation will be given later. The numerical results are compared with the wavelet solution obtained by the Haar wavelet method [11].

One can see that for the low values of N, periodic harmonic wavelets give quite poor approximation of $f(t)$. The error for greater N's is getting lower. Fig. 3 represents approximation of the function $f(t)$ when $N = 3$.

6.2. Example 2

As for the second example we will consider the following integral equation:

$$f(x) = e^x - \frac{(1 + 2e^3 x)}{9} + \int_0^1 xt[f(t)]^3 dt \qquad (21)$$

with the analytical solution $f(x) = e^x$ and compare our results with the method, where the Legendre polynomials were used as basis functions [10]. Similar computations that were made in the previous subsection give us the following results, which are presented in Fig. 4 and Table 2.

We can see that for the low levels of approximation by PHW, the absolute value of the error is also quite high. In the paper [10], where the Legendre wavelets were used, the absolute error is about 10^{-4}. The result of such approach for $N = 3$ is presented in Fig. 5.

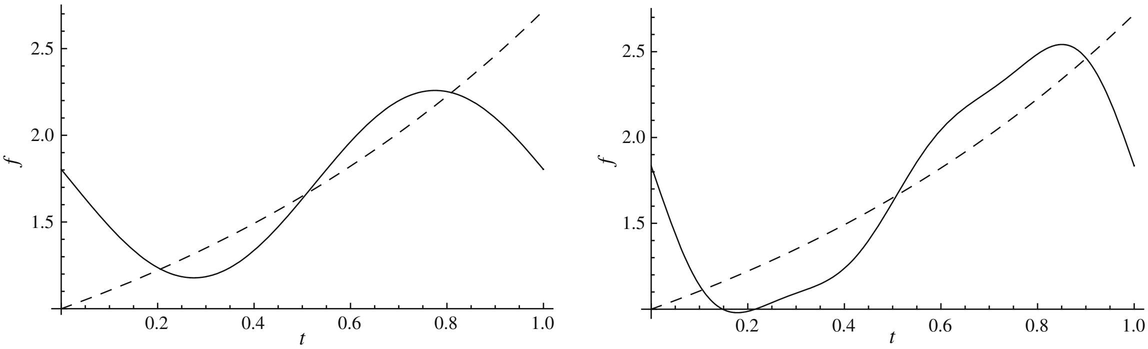

Fig. 4. Exact solution (dashed line) and approximate periodic wavelet solution (solid line) at scales $N = 0$ (left) and $N = 1$ (right).

Table 2
Error estimation in the case of PHW and the Haar wavelets as basis functions.

N	M	Harmonic wavelets, ε
0	2	0.3682
1	4	0.2725
2	8	0.103
6	128	5.3×10^{-6}

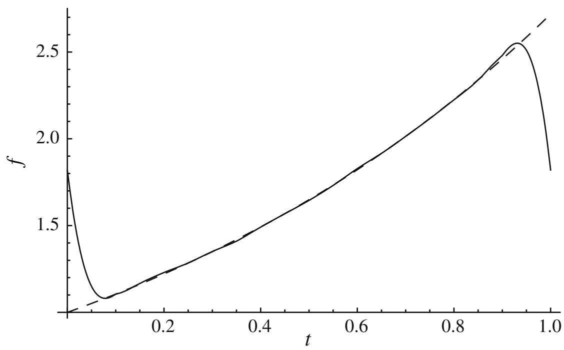

Fig. 5. Exact solution (dashed line) and approximate periodic wavelet solution (solid line) at scale $N = 3$.

7. Conclusion

In this paper, we have studied a method for solving the Fredholm type integral equations by using PHW as basis functions in the collocation method. The problem was reduced to a system of linear equations. In order to demonstrate the applicability of this method, two examples were solved and its results were compared with the Haar wavelet method and the method based on the Legendre polynomials. As a result of such comparison, we see two main disadvantages. The first, and the most visible, is that PHW do not approximate the unknown function nearby bounds. PHW approximate a function so, that this function on the unit interval represents one period of the resulting approximation. In the case if $f(t)$ represents a periodic function, PHW approximate without such jumps [4]. The second disadvantage is that the absolute error at low levels of projection of solution on the finite space with the corresponding basis functions is quite high. In order to get a good approximation, which is comparable with (*for example*) the Haar wavelet method, we need to go on the higher scales.

The advantages of application of PHW are:

- The resulting function of every approximation is defined analytically, which differs it from the Haar approximation, which is numerical.
- The higher orders of scaling parameter j give a good analytical approximation.

In conclusion, we would like to emphasize that our approach does not claim to be a universal, but extends our knowledge on the research of integral equations.

Unfortunately, at the moment wavelet-analysis does not give any clear answer on the question "Why we have chosen this, but not another wavelet?" [13]. Of course, there exist many traditional methods of solution of differential and integral equations. It is known that every of such methods has its own disadvantages, which can be equilibrated by using other methods. In our opinion, wavelet method is one of such methods, which is getting more popular in many applications of engineering and applied mathematics.

Acknowledgement

The work of Aleksey Kudreyko is supported by the Istituto Nazionale di Alta Matematica Francesco Severi (INdAM) under the scholarship U 2008/000564, 21/11/2008.

References

[1] F. Smithies, Integral Equations, Cambridge University Press, 2009. 182p.
[2] I. Daubechies, Ten Lectures on Wavelets, SIAM, Philadelphia, 1992. 357p.
[3] P. Wojtaszczyk, A Mathematical Introduction to Wavelets, London Mathematical Society Student Texts, 1997. 261p.
[4] S.V. Muniandy, I.M. Moroz, Galerkin modeling of the Burgers equation using harmonic wavelets, Phys. Lett. A 235 (1997) 352–356.
[5] D.E. Newland, Harmonic wavelet analysis, Royal Soc. 443 (1993) 203–225.
[6] D.E. Newland, An Introduction to Random Vibrations. Spectral and Wavelet Analysis, Dover Publications, London, 1993. 477p.
[7] C. Cattani, J.J. Rushchitsky, Wavelet and Wave Analysis as Applied to Materials with Micro- or Nanostructure, Series on Advances in Mathematics for Applied Sciences, vol. 74, World Scientific, Singapore, 2007. 476p.
[8] D.E. Newland, Harmonic wavelets in vibrations and acoustics, Royal Soc. 357 (1999) 2607–2625.
[9] C. Cattani, Connection coefficients of Shannon wavelets, Math. Model. Anal. 11 (2) (2006) 1–16.
[10] Y. Mahmoudi, Wavelet Galerkin method for numerical solution of nonlinear integral equation, Appl. Math. Comput. 167 (2005) 1119–1129.
[11] U. Lepik, E. Tamme, Solution of nonlinear Fredholm integral equations via the Haar wavelet method, Proc. Estonian Acad. Sci. Phys. Math. 56 (1) (2007) 17–27.
[12] T. Morita, Expansion in Harmonic Wavelets of a Periodic Function, Interdiscipl. Inform. Sci. 3 (1) (1997) 5–12.
[13] D.E. Newland, Practical signal analysis: do wavelets make any difference? in: Proceedings of 1997 Design Engineering Technical Conferences, September 14–17, 1997, Sacramento, California.