Math. Struct. in Comp. Science (2006), *vol.* 16, *pp.* 1–23.
doi:10.1017/S0960129506005111 Printed in the United Kingdom

Quasi-metric spaces and point-free geometry

ANNA DI CONCILIO and GIANGIACOMO GERLA

Dipartimento di Matematica e Informatica, Via Ponte don Melillo, 84084 Fisciano (Salerno), Italy

Received 1 June 2005; revised 24 November 2005

An approach to point-free geometry based on the notion of a quasi-metric is proposed in which the primitives are the regions and a non-symmetric distance between regions. The intended models are the bounded regular closed subsets of a metric space together with the Hausdorff excess measure.

1. Introduction

The interest in Computer Science in point-free geometry has increased recently in connection with the question of a suitable formalisation of naive spatial knowledge. The motivation for this new field of research lies in a dissatisfaction, from a computational point of view, with the complexity of Euclidean geometry based on the notion of a point. The possibility of considering a geometry in which the notion of a point is not assumed as a primitive was first examined by A. N. Whitehead in *An Inquiry Concerning the Principles of Natural Knowledge*, *The Concept of Nature* and *Process and Reality*. In particular, in this last book the primitives are the *regions* and the *connection relation*, that is, the relation between two regions that either overlap or have at least a common boundary point. Such a point-free approach to geometry was formalised and investigated by several authors (see, for example, Clarke (1981) and Gerla (1994)). Namely, one considers structures (Re, C) where the elements in Re are called *regions* and C is a binary relation in Re called a *connection relation*. The *inclusion relation* $\leqslant$ is defined by setting $x \leqslant y$ if and only if $C(x) \subseteq C(y)$ where, as usual, for any region z, we set $C(z) = \{z' \in Re : zCz'\}$. Later, Gerla (1990) proposed the notion of a *pointless pseudo-metric space* $(Re, \leqslant, m, D)$ in which the inclusion, the distance $m : Re \times Re \to R^+$ and the diameter $D\colon Re \to R^+$ are all assumed as primitives. A 'canonical' model is obtained by setting: Re equal to the class of bounded regular open subsets of a metric space (M, δ); $\leqslant$ equal to the set theoretical inclusion; and by defining m and D by setting

$$m(x, y) = \inf\{\delta(P, Q) : P \in x, Q \in y\}$$
$$D(x) = \sup\{\delta(P, P') : P, P' \in x\}$$

for any pair x, y of subsets of M. Such a class of structures was previously defined in Weihrauch and Schreiber (1981) in the framework of computability theory.

In this note we sketch a new approach to point-free geometry where the unique primitive is the notion of a *quasi-metric*, that is, a distance-like measure lacking a symmetry property (see, for example, Di Concilio (1971), Reilly (1992), Seda (1997) and Smyth (1987)). Namely, we examine a particular class of quasi-metrics, the *quasi-metric spaces of regions*.

The intended model is the *excess measure* e_δ defined by setting, for any pair x and y of non-empty closed bounded subsets of a metric space (M,δ),

$$e_\delta(x, y) = \sup\{\delta(P, y) : P \in x\},$$

where, in turn,

$$\delta(P, y) = \inf\{\delta(P, Q) : Q \in y\}.$$

Such a measure is well-known in the literature since the Hausdorff distance d_H is defined by setting $d_H(x, y) = \max\{e_\delta(x, y), e_\delta(y, x)\}$. An advantage of such an approach with respect to the work cited above is that we are not forced to assume the inclusion relation and the diameter as primitives. Indeed, these notions can be defined in a very simple way from the quasi-metric. Obviously, the main step in our theory is the definition of a *point* and of a *distance between points* in order to associate any quasi-metric space of regions (Re, d) with a point-based metric space.

Note that this paper in its present form does not address the computational dimension of point-free geometry, which, on the basis of the recent literature, lies in generalised metric spaces. However, it appears to be possible to reformulate the notions and the results we present here in constructive terms.

Finally, we wish to thank the referees for their fruitful suggestions and comments.

2. Preliminaries

In the following, R denotes the set of real numbers and $R^+ = \{x \in R : x \geqslant 0\}$.

Definition 2.1. A *quasi-metric space* is a structure (Re, d) such that Re is a non-empty set and $d : Re \times Re \to R^+$ is a mapping such that, for any $x, y, z \in Re$:

d1: $d(x, x) = 0$;
d2: $d(x, y) = 0$ and $d(y, x) = 0 \Rightarrow x = y$;
d3: $d(x, y) \leqslant d(x, z) + d(z, y)$.

Then, the metric spaces are the quasi-metric spaces satisfying the symmetric property

d0: $d(x, y) = d(y, x)$.

The proof of the following proposition is trivial.

Proposition 2.2. Let (Re, d) be a quasi-metric space and define the mapping $d_H : Re \times Re \to R^+$ by setting

$$d_H(x, y) = d(x, y) \vee d(y, x).$$

Then (Re, d_H) is a metric space.

We call (Re, d_H) the *symmetrisation* of (Re, d). The quasi-metric spaces are related to the partial orders as follows.

Proposition 2.3. Let (Re, d) be a quasi-metric space. Then the relation $\leqslant$ defined by setting

$$x \leqslant y \quad \Leftrightarrow \quad d(x, y) = 0$$

for every $x, y \in Re$ is a partial order. Conversely, let $\leqslant$ be any partial order in a set Re and define the mapping $d : Re \times Re \to R^+$ by setting

$$d(x, y) = \begin{cases} 0 & \text{if } x \leqslant y \\ 1 & \text{otherwise.} \end{cases}$$

Then (Re, d) is a quasi-metric space whose associated partial order is $\leqslant$.

Since our goal is to give a basis for point-free geometry, we call the elements of Re *regions* and the relation $\leqslant$ defined in Proposition 2.3 an *inclusion relation*. Also, we define the diameter of a region as follows.

Definition 2.4. Given a quasi-metric space (Re, d), we define the *diameter* of a region $x \in Re$ to be the number

$$d(x) = \sup\{d(x_1, x_2) : x_1 \leqslant x,\ x_2 \leqslant x\}. \tag{2.1}$$

We say that x is *bounded* if $d(x) \neq \infty$.

Observe that the notion of diameter is assumed as a primitive by several authors (see for example Pultr (1984; 1988) and Banaschewski and Pultr (1998)). Obviously, $d(x) = 0$ if and only if x is an atom. When (Re, d) is a metric space, the associated partial order $\leqslant$ coincides with the identity relation, and thus all diameters are equal to zero and all regions are atoms. When the quasi-metric space is defined by a partial order as in Proposition 2.3, we have that $d(x) = 0$ if x is an atom and $d(x) = 1$ otherwise.

Proposition 2.5. Any quasi-metric $d : Re \times Re \to R^+$ is order-preserving with respect to the first variable and order-reversing with respect to the second variable. Also, the diameter $d : Re \to R^+$ is order-preserving and, for any region x,

$$d(x) = \sup\{d(x, x') : x' \leqslant x\}. \tag{2.2}$$

Proof. Assume that $x' \leqslant x$. Then $d(x', y) \leqslant d(x', x) + d(x, y) = d(x, y)$. Assume that $y' \leqslant y$. Then $d(x, y) \leqslant d(x, y') + d(y', y) = d(x, y')$. The proof of the remaining part of the proposition is trivial. □

Definition 2.6. Given two quasi-metric spaces (Re, d) and (Re', d') and a mapping $h : Re \to Re'$, we say that h is *non-expansive* if $d'(h(x), h(y)) \leqslant d(x, y)$. We say that h is an *isometry* if $d(x, y) = d'(h(x), h(y))$.

We conclude this section by noticing that the class of quasi-metric spaces defines a category in a natural way.

Proposition 2.7. The class of quasi-metric spaces defines a category QMS provided we assume as morphisms the non-expansive mappings. Let ORD be the category whose objects are the ordered sets and the morphisms are the order preserving maps. Then Proposition 2.3 defines a functor from QMS to ORD and a functor from ORD to QMS.

Proof. Let (Re, d) and (Re', d') be two quasi-metric spaces and $(Re, \leqslant), (Re', \leqslant')$ be the associated partial orders. Then, if $h : Re \to Re'$ is non-expansive, we have that $x \leqslant y$

entails $h(x) \leqslant' h(y)$, so h is a morphism from $(Re, \leqslant)$ to $(Re', \leqslant')$. Consequently, the map associating any quasi-metric space with the related partial order and any non-expansive map h with the same map h is a functor from QMS to ORD. Likewise, the map associating any partial order with the quasi-metric defined in Proposition 2.3 and any order-preserving map h with h is a functor from ORD to QMS. □

3. The notion of point

In this section we will propose a suitable definition of a point and the distance between points in order to associate any quasi-metric space with a metric space in a natural way. To this end, recall that a *pseudo-metric space* is a structure (M, d) satisfying d0, d1 and d3, and that any pseudo-metric space (M, d) is associated with a metric space (M', d'), which we call *the quotient* of (M, d). Namely, we define an equivalence relation $\equiv$ in M by setting $x \equiv y$ if and only if $d(x, y) = 0$, and we set M' equal to the quotient of M modulo $\equiv$. Moreover, we define the distance between two classes $[x]$ and $[y]$ by setting $d'([x], [y]) = d(x, y)$.

Definition 3.1. A sequence $\langle p_n \rangle_{n \in N}$ of regions of a quasi-metric space (Re, d) is called a *point-representing* if:

a. $\lim_{n \to \infty} d(p_n) = 0$

b. $\forall \varepsilon > 0\ \exists m : h \geqslant m, k \geqslant m \Rightarrow d(p_h, p_k) < \varepsilon$.

We use Pr to denote the class of point-representing sequences. When (Re, d) is a metric space, the notion of a point-representing sequence coincides with the usual notion of a Cauchy sequence. There are quasi-metric spaces in which no point-representing sequence exists. So, we add the following axiom:

d4: A point-representing sequence exists.

Proposition 3.2. For any $\langle p_n \rangle_{n \in N}$ and $\langle q_n \rangle_{n \in N}$ in Pr, the sequence

$$\langle d(p_n, q_n) \rangle_{n \in N}$$

is convergent.

Proof. We have to prove that

$$\forall \varepsilon > 0\ \exists m (h \geqslant m \text{ and } k \geqslant m \Rightarrow |d(p_h, q_h) - d(p_k, q_k)| < \varepsilon).$$

Indeed, since

$$d(p_h, q_h) \leqslant d(p_h, p_k) + d(p_k, q_k) + d(q_k, q_h),$$

we have

$$d(p_h, q_h) - d(p_k, q_k) \leqslant d(p_h, p_k) + d(q_k, q_h).$$

Similarly,

$$d(p_k, q_k) - d(p_h, q_h) \leqslant d(p_k, p_h) + d(q_h, q_k).$$

Consequently,

$$|d(p_h, q_h) - d(p_k, q_k)| \leqslant d_H(p_k, p_h) + d_H(q_k, q_h).$$

Given $\varepsilon > 0$, let m be such that for any $h \geqslant m$ and $k \geqslant m, d_H(p_k, p_h) < \varepsilon/2, d(q_k, q_h) < \varepsilon/2$. Then we have that $|d(p_h, q_h) - d(p_k, q_k)| < \varepsilon$ for any $h \geqslant m, k \geqslant m$, and this completes the proof. □

In accordance with this proposition, we define in Pr the map $d_c : Pr \times Pr \to R^+$ by setting

$$d_c(\langle p_n\rangle_{n\in N}, \langle q_n\rangle_{n\in N}) = \lim_{n\to\infty} d(p_n, q_n) \tag{3.1}$$

for any $\langle p_n\rangle_{n\in N}$ and $\langle q_n\rangle_{n\in N}$ in Pr.

Proposition 3.3. The structure (Pr,d_c) satisfies d1 and d3.

Proof. Axiom d1 is immediate. To prove d3, observe that if $\langle p_n\rangle_{n\in N}, \langle q_n\rangle_{n\in N}$ and $\langle r_n\rangle_{n\in N}$ are elements in Pr, then

$$\begin{aligned} d_c(\langle p_n\rangle_{n\in N}, \langle q_n\rangle_{n\in N}) &= \lim_{n\to\infty} d(p_n, q_n) \leqslant \lim_{n\to\infty}(d(p_n, r_n) + d(r_n, q_n)) \\ &= \lim_{n\to\infty} d(p_n, r_n) + \lim_{n\to\infty} d(r_n, q_n) \\ &= d_c(\langle p_n\rangle_{n\in N}, \langle r_n\rangle_{n\in N}) + d_c(\langle r_n\rangle_{n\in N}, \langle q_n\rangle_{n\in N}). \end{aligned}$$ □

It is easy to prove that d_c is not symmetric in general (see Proposition 5.6). To obtain this property we have to add a further axiom to quasi-metric spaces. As an example, we propose the following one:

d5: $|d(x, y) - d(y, x)| \leqslant d(x) + d(y)$.

This axiom is in accordance with the idea that *'small'* regions are approximations of ideal points. In fact, it says that in the class of *'small'* regions the mapping d is approximately symmetric and therefore that the class of *'small'* regions can be regarded (approximately) as a metric space. Observe that all the results in this paper remain valid if in d5 we substitute the maximum $\mathrm{Max}\{d(x), d(y)\}$ for the sum $d(x) + d(y)$.

Definition 3.4. We call any structure (Re,d) satisfying d1–d5 a *quasi-metric space of regions.*

Trivially, the set of atoms of a quasi-metric space of regions is a metric space, and the metric spaces coincide with the quasi-metric spaces of regions in which all the regions have diameter zero. Observe also that while any subset Re' of a quasi-metric space (Re, d) defines a quasi-metric space, when (Re, d) satisfies d5 it is possible that (Re', d) does not satisfy d5. This is because the notion of the diameter in (Re, d) is different from the notion of the diameter in (Re', d).

Proposition 3.5. The structure (Pr, d_c) associated with a quasi-metric space of regions is a pseudo-metric space.

Proof. To prove the symmetric property, observe that, since $|d(p_n, q_n) - d(q_n, p_n)| \leqslant d(p_n) + d(q_n)$, it is $\lim_{n\to\infty} d(p_n, q_n) = \lim_{n\to\infty} d(q_n, p_n)$. □

This proposition enables us to propose the following definition.

Definition 3.6. We call the quotient $(\bar{M}, \bar{\delta})$ of the pseudo-metric space (Pr, d_c) the *metric space associated with* (Re, d). We call any element in $\bar{M}$ a *point*.

Thus, the metric space $(\bar{M}, \bar{\delta})$ associated with a metric space of regions (Re, d) is defined by:

— considering the class Pr of point-representing sequences
— setting $\bar{M}$ equal to the quotient of Pr modulo the equivalence $\equiv$ defined by

$$\langle p_n \rangle_{n\in N} \equiv \langle q_n \rangle_{n\in N} \Leftrightarrow \lim_{n\to\infty} d(p_n, q_n) = 0$$

— defining $\bar{\delta} : \bar{M} \times \bar{M} \to R^+$ by the equation

$$\bar{\delta}(P, Q) = \lim_{n\to\infty} d(p_n, q_n)$$

where $P = [\langle p_n \rangle_{n\in N}]$ and $Q = [\langle q_n \rangle_{n\in N}]$ are points in $\bar{M}$.

Observe that if (Re, d) is a metric space, the associated metric space $(\bar{M}, \bar{\delta})$ is the completion of (Re, d). Indeed, since all diameters are equal to zero, the proposed notion of a point-representing sequence coincides with the usual notion of a Cauchy sequence.

Proposition 3.7. Let (Re, d) be a quasi-metric space of regions and (Re, d_H) be its symmetrisation. Then the associated metric space $(\bar{M}, \bar{\delta})$ is a subspace of the metric completion of (Re, d_H).

Proof. By definition, Pr is the class of Cauchy sequences of (Re, d_H) whose d-diameters are vanishing. Also, for any $\langle p_n \rangle_{n\in N}, \langle q_n \rangle_{n\in N}$ in Pr,

$$d_H(\langle p_n \rangle_{n\in N}, \langle q_n \rangle_{n\in N}) = d_c(\langle p_n \rangle_{n\in N}, \langle q_n \rangle_{n\in N}) \vee d_c(\langle q_n \rangle_{n\in N}, \langle p_n \rangle_{n\in N}).$$

Since d_c is symmetric,

$$d_H(\langle p_n \rangle_{n\in N}, \langle q_n \rangle_{n\in N}) = d_c(\langle p_n \rangle_{n\in N}, \langle q_n \rangle_{n\in N}). \qquad \square$$

The quasi-metric space associated with an ordered set with an atom is a quasi-metric space of regions. The associated metric space is isometric with the metric space of the atoms equipped with the discrete distance.

4. Distance between points and regions, and completeness

The following proposition is useful for defining the notion of the distance between a point and a region.

Proposition 4.1. Let $\langle p_n \rangle_{n\in N}$ be an element in Pr and x be a region. Then, both the sequences $\langle d(x, p_n) \rangle_{n\in N}$ and $\langle d(p_n, x) \rangle_{n\in N}$ are convergent. Moreover,

$$\langle p_n \rangle_{n\in N} \equiv \langle p'_n \rangle_{n\in N} \quad \Rightarrow \lim_{n\to\infty} d(x, p_n) = \lim_{n\to\infty} d(x, p'_n)$$

and

$$\langle p_n\rangle_{n\in N} \equiv \langle p'_n\rangle_{n\in N} \;\Rightarrow\; \lim_{n\to\infty} d(p_n, x) = \lim_{n\to\infty} d(p'_n, x).$$

Proof. By hypothesis, for any $\varepsilon > 0$ there is a natural number m such that $d(p_k, p_h) \leqslant \varepsilon$ for any $k \geqslant m$ and $h \geqslant m$. Also, since $d(x, p_h) \leqslant d(x, p_k) + d(p_k, p_h)$, we have that $d(x, p_h) - d(x, p_k) \leqslant d(p_k, p_h) \leqslant \varepsilon$. Since $d(x, p_k) \leqslant d(x, p_h) + d(p_h, p_k)$, it is $d(x, p_k) - d(x, p_h) \leqslant d(p_h, p_k) \leqslant \varepsilon$. Then $|d(x, p_h) - d(x, p_k)| \leqslant \varepsilon$ for any $k \geqslant m$ and $h \geqslant m$, and this proves that $\langle d(x, p_n)\rangle_{n\in N}$ is a Cauchy sequence in the space of real numbers and therefore convergent. Likewise, since $d(p_h, x) \leqslant d(p_h, p_k) + d(p_k, x)$, we have that $d(p_h, x) - d(p_k, x) \leqslant d(p_h, p_k) \leqslant \varepsilon$ and, since $d(p_k, x) \leqslant d(p_k, p_h) + d(p_h, x)$, we have that $d(p_k, x) - d(p_h, x) \leqslant d(p_k, p_h) \leqslant \varepsilon$. Then $|d(p_h, x) - d(p_k, x)| \leqslant \varepsilon$ for any $k \geqslant m$ and $h \geqslant m$, and this proves that $\langle d(p_n, x)\rangle_{n\in N}$ is a Cauchy sequence, and thus convergent.

Assume that $\langle p_n\rangle_{n\in N}$ and $\langle p'_n\rangle_{n\in N}$ are two equivalent point-representing sequences. Then, from $d(x, p_n) \leqslant d(x, p'_n) + d(p'_n, p_n)$ it follows that

$$d(x, p_n) - d(x, p'_n) \leqslant d(p'_n, p_n) \leqslant d(p'_n, p_n) + d(p_n) + d(p'_n).$$

Since $d(x, p'_n) \leqslant d(x, p_n) + d(p_n, p'_n)$, we have

$$d(x, p'_n) - d(x, p_n) \leqslant d(p_n, p'_n) \leqslant d(p'_n, p_n) + d(p_n) + d(p'_n).$$

So

$$|d(x, p'_n) - d(x, p_n)| \leqslant d(p'_n, p_n) + d(p_n) + d(p'_n),$$

and this proves that $lim_{n\to\infty} d(x, p_n) = \lim_{n\to\infty} d(x, p'_n)$.

The second implication is proved by similar reasoning. □

Proposition 4.1 enables us to give the following definitions.

Definition 4.2. Let x be a region and $P = [\langle p_n\rangle_{n\in N}]$ be a point. Then we set

$$\underline{d}(P, x) = \lim_{n\to\infty} d(p_n, x) \text{ and } \underline{d}(x, P) = \lim_{n\to\infty} d(x, p_n).$$

Trivially, $\underline{d}(P, x)$ is order-reversing with respect to the second variable and $\underline{d}(x, P)$ is order-preserving with respect to the first variable. Also, in general, $\underline{d}(P, x) \neq \underline{d}(x, P)$.

Proposition 4.3. Let x, x' be two regions and P, P' two points. Then the following inequalities hold true,

$$\bar{\delta}(P, P') \leqslant \underline{d}(P, x) + \underline{d}(x, P') \quad (1)$$

$$d(x, x') \leqslant \underline{d}(x, P) + \underline{d}(P, x') \quad (2)$$

$$\underline{d}(P, x) \leqslant \bar{\delta}(P, P') + \underline{d}(P', x) \quad (3)$$

$$\underline{d}(P, x) \leqslant \underline{d}(P, x') + d(x', x) \quad (4)$$

$$\underline{d}(x, P) \leqslant \underline{d}(x, P') + \bar{\delta}(P', P) \quad (5)$$

$$\underline{d}(x, P) \leqslant d(x, x') + \underline{d}(x', P) \quad (6)$$

$$\underline{d}(P, x) \leqslant \underline{d}(x, P) + d(x) \quad (7)$$

$$\underline{d}(x, P) \leqslant \underline{d}(P, x) + d(x) \quad (8)$$

$$|\underline{d}(P, x) - \underline{d}(x, P)| \leqslant d(x). \quad (9)$$

Proof. To prove 1, observe that

$$\bar{\delta}(P, P') = \lim_{n\to\infty} d(p_n, p'_n) \leqslant \lim_{n\to\infty} d(p_n, x) + \lim_{n\to\infty} d(x, p'_n) = \underline{d}(P, x) + \underline{d}(x, P').$$

To prove 2, observe that $d(x, x') \leqslant d(x, p_n) + d(p_n, x')$ and therefore that

$$d(x, x') \leqslant \lim_{n\to\infty} d(x, p_n) + \lim_{n\to\infty} d(p_n, x) \leqslant \underline{d}(x, P) + \underline{d}(P, x').$$

We can prove 3–6 using similar reasoning. To prove 7, observe that since $d(p_n, x) \leqslant d(x, p_n) + d(x) + d(p_n)$ and $\lim_{n\to\infty} d(p_n) = 0$, we have

$$\underline{d}(P, x) = \lim_{n\to\infty} d(p_n, x) \leqslant \lim_{n\to\infty} d(x, p_n) + d(x) = \underline{d}(x, P) + d(x).$$

In a similar way we can prove 8. Finally, 9 is an immediate consequence of 7 and 8. □

Theorem 4.4. Let (Re, d) be a quasi-metric space of regions. Then the associated metric space $(\bar{M}, \bar{\delta})$ is complete.

Proof. To prove that $(\bar{M}, \bar{\delta})$ is complete, observe that if $P = [\langle p_n \rangle_{n\in N}]$ is an element of $\bar{M}$, then for any $\varepsilon > 0$ there is a region s such that $d(s) \leqslant \varepsilon, \underline{d}(P, s) < \varepsilon$ and $\underline{d}(s, P) \leqslant \varepsilon$. In fact, let $m \in N$ be such that $d(p_h) \leqslant \varepsilon$ and $d(p_h, p_k) \leqslant \varepsilon$ for any $h \geqslant m$ and $k \geqslant m$. Then, in particular, $d(p_m) \leqslant \varepsilon, d(p_m, p_n) \leqslant \varepsilon$ and $d(p_n, p_m) \leqslant \varepsilon$ for any $n \geqslant m$ and, therefore, by setting $s = p_m$, we get that $d(s) \leqslant \varepsilon$ and that $\underline{d}(s, P) = \lim_{n\to\infty} d(p_m, p_n) \leqslant \varepsilon$ and $\underline{d}(P, s) = \lim_{n\to\infty} d(p_n, p_m) \leqslant \varepsilon$. Let $\langle P_n \rangle_{n\in N}$ be a Cauchy sequence of elements of the metric space $(\bar{M}, \bar{\delta})$, and, for any $n \in N$, let s_n be a region such that $d(s_n) \leqslant 1/n, \underline{d}(s_n, P_n) \leqslant 1/n$ and $\underline{d}(P_n, s_n) \leqslant 1/n$. Then,

$$d(s_h, s_k) \leqslant \underline{d}(s_h, P_h) + \bar{\delta}(P_h, P_k) + \underline{d}(P_k, s_k) \leqslant 1/h + \bar{\delta}(P_h, P_k) + 1/k,$$

and thus $\langle s_n \rangle_{n\in N}$ is a sequence representing a point $P \in \bar{M}$. Also, since

$$\bar{\delta}(P, P_n) \leqslant \underline{d}(P, s_n) + \underline{d}(s_n, P_n) \leqslant \underline{d}(P, s_n) + 1/n$$

and $\lim_{n\to\infty} \underline{d}(P, s_n) = 0$, we have that $P = \lim_{n\to\infty} P_n$. □

5. Canonical examples: the Hausdorff excess spaces

An interesting class of quasi-metric spaces is related to the Hausdorff distance. Indeed, assume that (M, δ) is a metric space. Then, given $P \in M$ and x a non-empty subset of M, we define $\delta(P, x)$ by setting

$$\delta(P, x) = \inf\{\delta(P, Q) : Q \in x\}. \tag{5.1}$$

If x, y are non-empty subsets of M, we set

$$m(x, y) = \inf\{\delta(P, Q) : P \in x, Q \in y\} \tag{5.2}$$

or, equivalently,

$$m(x, y) = \inf\{\delta(P, y) : P \in x\}. \tag{5.3}$$

Also, we define the *excess function* e_δ by setting, for any x and y in $P(M) - \{\varnothing\}$,

$$e_\delta(x, y) = \sup\{\delta(P, y) : P \in x\}. \tag{5.4}$$

Obviously, it is possible that $e_\delta(x,y)=\infty$. However, if we confine ourselves to the class $B(M)$ of all closed, bounded, non-empty subsets of M, then $e_\delta(x,y)$ is always finite. Both the maps m and e_δ extend the distance δ; indeed, for any $P,Q\in M$,

$$e_\delta(\{P\},\{Q\}) = m(\{P\},\{Q\}) = \delta(P,Q).$$

We define the diameter $D(x)$ of an element x in $B(M)$ by setting

$$D(x)=\sup\{\delta(P,P') : P\in x, P'\in x\}. \tag{5.5}$$

Observe that, given any $x\in P(M)-\{\varnothing\}$ and using $cl(x)$ to denote the closure of x, we have

$$cl(x)=\{P\in M : \delta(P,x)=0\}. \tag{5.6}$$

Then, it is immediate to prove that, for any $x,y\in P(M)-\{\varnothing\}$,

$$e_\delta(x,y)=e_\delta(cl(x),cl(y)), \tag{5.7}$$

$$m(x,y)=m(cl(x),cl(y)) \tag{5.8}$$

and

$$D(x)=D(cl(x)). \tag{5.9}$$

Proposition 5.1. Let P and Q be elements in M and x,y be elements in $B(M)$. Then

$$\delta(P,x)\leqslant\delta(P,Q)+\delta(Q,x). \tag{5.10}$$

$$m(x,y)\leqslant e_\delta(x,y)\leqslant m(x,y)+D(x). \tag{5.11}$$

$$|e_\delta(x,y)-e_\delta(y,x)|\leqslant\max\{D(x),D(y)\}. \tag{5.12}$$

Proof. To prove (5.10), observe that,

$$\begin{aligned}\delta(P,x) &= \inf\{\delta(P,P') : P'\in x\}\leqslant\inf\{\delta(P,Q)+\delta(Q,P') : P'\in x\}\\ &= \delta(P,Q)+\inf\{\delta(Q,P') : P'\in x\}=\delta(P,Q)+\delta(Q,x).\end{aligned}$$

To prove (5.11) observe that the inequality $m(x,y)\leqslant e_\delta(x,y)$ is trivial. Also, for any $P,P'\in x$,

$$\delta(P,y)\leqslant\delta(P,P')+\delta(P',y)\leqslant D(x)+\delta(P',y)$$

and therefore

$$\delta(P,y)\leqslant D(x)+\inf\{\delta(P',y) : P'\in x\}=D(x)+m(x,y).$$

Finally, to prove (5.12), observe that $m(x,y)=m(y,x)\leqslant e_\delta(y,x)$, and therefore, by (5.11), that $e_\delta(x,y)\leqslant m(x,y)+D(x)\leqslant e_\delta(y,x)+D(x)$. This proves that

$$e_\delta(x,y)-e_\delta(y,x)\leqslant D(x)\leqslant\max\{D(x),D(y)\}.$$

Since in the same manner we can prove that $e_\delta(y,x)-e_\delta(x,y)\leqslant\max\{D(x),D(y)\}$, (5.12) follows. □

Theorem 5.2. Let (M,δ) be a metric space and $e_\delta : B(M)\times B(M)\to R^+$ be the related excess function. Then $(B(M),e_\delta)$ is a quasi-metric space of regions whose associated partial order is the set theoretical inclusion and whose diameter is the diameter function D defined by (5.5).

Proof. To prove the triangle inequality, observe that,

$$\delta(P,y) \leqslant \delta(P,Q) + \delta(Q,y) \leqslant \delta(P,Q) + \sup_{Q' \in z} \delta(Q',y) = \delta(P,Q) + e_\delta(z,y)$$

whenever Q belongs to z. Therefore,

$$\delta(P,y) \leqslant \inf_{Q \in z} \delta(P,Q) + e_\delta(z,y) = \delta(P,z) + e_\delta(z,y).$$

Consequently,

$$\begin{aligned} e_\delta(x,y) &= \sup\{\delta(P,y) : P \in x\} \leqslant \sup\{\delta(P,z) + e_\delta(z,y) : P \in x\} \\ &= \sup\{\delta(P,z) : P \in x\} + e_\delta(z,y) = e_\delta(x,z) + e_\delta(z,y). \end{aligned}$$

Let x, y be elements in $B(M)$. Then, since y is a closed set,

$$e_\delta(x,y) = 0 \Leftrightarrow \delta(P,y) = 0 \text{ for any } P \in x \Leftrightarrow \subseteq y.$$

This proves both d1 and d2 and that the partial order associated with $(B(M), e_\delta)$ is the inclusion. To prove that $e_\delta(x) = D(x)$, observe that, since $e_\delta(x,x') \leqslant e_\delta(x,\{P\})$ for any $P \in x'$,

$$\begin{aligned} e_\delta(x) &= \sup\{e_\delta(x,x') : x' \subseteq x, x' \in B(M)\} \\ &\leqslant \sup\{e_\delta(x,\{P'\}) : P' \in x'\} = \sup_{P \in x} \sup_{P' \in x} e_\delta(\{P\},\{P'\}) = D(x). \end{aligned}$$

Also,

$$\begin{aligned} e_\delta(x) &= \sup\{e_\delta(x_1,x_2) : x_1 \subseteq x, x_2 \subseteq x, x_1 \in B(M), x_2 \in B(M)\} \\ &\geqslant \sup\{e_\delta(\{P_1\},\{P_2\}) : P_1 \in x, P_2 \in x\} \\ &= \sup\{\delta(P_1,P_2) : P_1 \in x, P_2 \in x\} = D(x). \end{aligned}$$

By (5.12) we can conclude that $(B(M), e_\delta)$ is a quasi-metric space of regions. □

Observe that the symmetrisation of $(B(M), e_\delta)$ is the well-known Hausdorff distance.

Definition 5.3. Let (M, δ) be a metric space. Then we call the space $(B(M), e_\delta)$ a *full Hausdorff excess space* and any subspace of $(B(M), e_\delta)$ a *Hausdorff excess space*.

Vitolo (1995) proved that any quasi-metric space is isometric to a Hausdorff excess space (see also Gerla (2004)). As an immediate consequence, we obtain the following extension theorem.

Theorem 5.4. Any quasi-metric space can be extended into a quasi-metric space of regions.

Theorem 5.5. Let (M, δ) be a metric space, and $(\bar{M}, \bar{\delta})$ be the metric space associated with $(B(M), e_\delta)$. Also, define the map $h : M \to \bar{M}$ by setting $h(P) = [\langle p_n \rangle_{n \in N}]$, for any $P \in M$, where $p_n = \{P\}$ for any $n \in N$. Then h is an isometry such that $h(M)$ is dense in $\bar{M}$. Consequently, $(\bar{M}, \bar{\delta})$ is the metric completion of (M, δ) and, when (M, δ) is complete, $(\bar{M}, \bar{\delta})$ coincides with (M, δ).

Proof. It is evident that h is an isometry. To prove that $h(M)$ is dense in $\bar{M}$, let $P = [\langle p_n \rangle_{n\in N}]$ be any element in $\bar{M}$. Moreover, for any $n \in N$, let P_n be an element in p_n. We claim that $\lim_{n\to\infty} h(P_n) = P$, that is, that

$$\lim_{n\to\infty} \bar{\delta}(h(P_n), P) = \lim_{n\to\infty}(\lim_{m\to\infty} e_\delta(\{P_n\}, p_m)) = 0.$$

Indeed,

$$e_\delta(\{P_n\}, p_m) \leqslant e_\delta(\{P_n\}, p_n) + e_\delta(p_n, p_m) = e_\delta(p_n, p_m).$$

Since $\langle p_n \rangle_{n\in N}$ is a point-representing sequence, given any $\varepsilon > 0$, there exists an integer h such that $e_\delta(p_n, p_m) \leqslant \varepsilon$ for any $n \geqslant h$ and $m \geqslant h$. Consequently,

$$\bar{\delta}(h(P_n), P) = \lim_{m\to\infty} e_\delta(\{P_n\}, p_m) \leqslant \lim_{m\to\infty} e_\delta(p_n, p_m) \leqslant \varepsilon$$

for any $n \geqslant h$. Thus, $\lim_{n\to\infty} \bar{\delta}(h(P_n), P) = 0$ and this proves that $h(M)$ is dense in $\bar{M}$. Since by Theorem 4.5 the space $(\bar{M}, \bar{\delta})$ is complete, we can conclude that $(\bar{M}, \bar{\delta}\ )$ is the metric completion of (M, δ). □

In accordance with this theorem, in the following we identify any point P in M with the point $h(P)$ in $\bar{M}$. Then we consider $\bar{\delta}$ as an extension of δ, and the excess $e_{\bar{\delta}}$ in $(\bar{M}, \bar{\delta})$ as an extension of the excess e_δ in (M, δ). Finally, observe that if $x \in B(M)$, it is possible that $h(x)$ is not closed in the space $(\bar{M}, \bar{\delta})$ and thus it is possible that $h(x)$ is not an element of $B(\bar{M})$.

A suitable modification of the excess function shows the independence of d5.

Proposition 5.6. Let (M, δ) be a metric space and set, for any $x \in B(M)$ and $y \in B(M)$,

$$d_\delta(x, y) = e_\delta(x, y) + |e_\delta(x) - e_\delta(y)|. \tag{5.13}$$

Then $(B(M), d_\delta)$ is a quasi-metric space such that the map $d_c : Pr \times Pr \to R^+$ defined by (3.1) is not symmetric. Therefore $(B(M), d_\delta)$ does not satisfy d5.

Proof. Trivially, $d_\delta(x, x) = 0$. To prove the triangle inequality, observe that

$$\begin{aligned} d_\delta(x, y) &= e_\delta(x, y) + |e_\delta(x) - e_\delta(y)| \\ &\leqslant e_\delta(x, z) + e_\delta(z, y) + |e_\delta(x) - e_\delta(z)| + |e_\delta(z) - e_\delta(y)| = d_\delta(x, z) + d_\delta(z, y). \end{aligned}$$

Also, if $\leqslant$ is the partial order defined by d_δ,

$$x \leqslant y \Leftrightarrow x \subseteq y \text{ and } e_\delta(x) = e_\delta(y). \tag{5.14}$$

This shows that both d1 and d2 hold, and thus that $(B(M), d_\delta)$ is a quasi-metric space.

Let P, Q and R be points such that $\delta(P, Q) < \delta(P, R)$, let $\langle p_n \rangle_{n\in N}$ be the sequence constantly equal to P and $\langle q_n \rangle_{n\in N}$ be the sequence constantly equal to $\{P, Q\}$. Then, since both $\{P\}$ and $\{P, Q\}$ are atoms, these sequences belong to Pr. On the other hand,

$$d_c(\langle p_n \rangle_{n\in N}, \langle q_n \rangle_{n\in N}) = d(P, \{Q, R\}) = \delta(P, Q) + \delta(Q, R)$$

while

$$d_c(\langle q_n \rangle_{n\in N}, \langle p_n \rangle_{n\in N}) = d(\{Q, R\}, P) = \delta(P, R) + \delta(Q, R)$$

and, therefore,

$$d_c(\langle p_n\rangle_{n\in N}, \langle q_n\rangle_{n\in N}) \neq d_c(\langle q_n\rangle_{n\in N}, \langle p_n\rangle_{n\in N}). \qquad \square$$

6. The set of points of a region

We relate points and regions by the following definition.

Definition 6.1. Let P be a point and r be a region. Then we say that P is *a point of* r provided that $\underline{d}(P, r) = 0$. We use $Pt(r)$ to denote the set of points of r.

Proposition 6.2. For any region r, $Pt(r)$ *is a closed subset of* $(\bar{M}, \bar{\delta})$.

Proof. Let $(P_n)_{n\in N}$ be a sequence of elements in $Pt(r)$ and assume that such a sequence is convergent to a point P. Then, since by 3 of Proposition 4.3 it is $\underline{d}(P, r) \leqslant \bar{\delta}(P, P_n) + \underline{d}(P_n, r) = \bar{\delta}(P, P_n)$, we have $\underline{d}(P, r) \leqslant \lim_{n\to\infty} \bar{\delta}(P, P_n) = 0$, and thus $P \in Pt(r)$. $\square$

Proposition 6.3. Let $P = [\langle p_n\rangle_{n\in N}]$ be a point. Then

$$\lim_{n\to\infty} \underline{d}(P, p_n) = \lim_{n\to\infty} \underline{d}(p_n, P) = 0. \tag{6.1}$$

Let $(P_n)_{n\in N}$ be a sequence of points such that $P_n \in Pt(p_n)$. Then

$$\lim_{n\to\infty} P_n = P. \tag{6.2}$$

Proof. Given $\varepsilon > 0$, there is a natural number m such that $d(p_h, p_k) \leqslant \varepsilon$ for any $h \geqslant m$ and $k \geqslant m$. This entails that $\underline{d}(P, p_k) = \lim_{h\to\infty} d(p_h, p_k) \leqslant \varepsilon$ for any $k \geqslant m$ and therefore that $\lim_{k\to\infty} \underline{d}(P, p_k) = 0$. In a similar way one proves that $\lim_{h\to\infty} \underline{d}(p_h, P) = 0$. Moreover, since $\bar{\delta}(P_n, P) \leqslant \underline{d}(P_n, p_n) + \underline{d}(p_n, P) = \underline{d}(p_n, P)$,

$$\lim_{n\to\infty} \bar{\delta}(P_n, P) \leqslant \lim_{n\to\infty} \underline{d}(p_n, P) = 0.$$

Note that while $\underline{d}(P, y)$ is defined as the limit $\lim_{n\to\infty} d(p_n, y)$, we have that $\bar{\delta}(P, Pt(y))$ is the value $\inf\{\bar{\delta}(P, P') : P' \in Pt(y)\}$. As we will show in the following sections, there are examples of quasi-metrics in which $\underline{d}(P, y) \neq \bar{\delta}(P, Pt(y))$. The following proposition says that in any case $\underline{d}(P, y) \leqslant \bar{\delta}(P, Pt(y))$. $\square$

Proposition 6.4. Let P be a point and y be a region. Then

$$\underline{d}(P, y) \leqslant \bar{\delta}(P, Pt(y)). \tag{6.3}$$

Proof. If $Pt(y) = \varnothing$, then $\bar{\delta}(P, Pt(y)) = \infty$ and (6.3) is trivial. Otherwise, since $\underline{d}(P, y) \leqslant \bar{\delta}(P, P') + \underline{d}(P', y)$, we have that $\underline{d}(P, y) \leqslant \bar{\delta}(P, P')$ for any $P' \in Pt(y)$ and therefore that

$$\underline{d}(P, y) \leqslant \inf\{\bar{\delta}(P, P') : P' \in Pt(y)\} = \bar{\delta}(P, Pt(y)).$$

Such a proposition suggests we consider quasi-metric spaces such that:

d6:$\underline{d}(P, y) = \bar{\delta}(P, Pt(y))$.

This axiom claims the existence of sufficiently many points, in some sense. In particular, d6 entails that any region y contains a point. Otherwise, $\bar{\delta}(P, Pt(y)) = \infty$ while $\underline{d}(P, y)$ is always finite. $\square$

Theorem 6.5. Let (M, δ) be a metric space. Then $(B(M), e_\delta)$ satisfies d6. Moreover, if $(\bar{M}, \bar{\delta})$ is the metric space associated with $(B(M), e_\delta)$, then, for every $y \in B(M), Pt(y)$ is the closure of y in $(\bar{M}, \bar{\delta})$. In particular, if (M, δ) is complete, since $(\bar{M}, \bar{\delta})$ coincides with (M, δ), we have $Pt(y) = y$.

Proof. Let $P = [\langle p_n \rangle_{n\in N}] \in \bar{M}$ be a point and $y \in B(M)$ be a region. Then, by Proposition 6.4, it is sufficient to prove that

$$\lim_{n\to\infty} e_\delta(p_n, y) \geqslant \inf\{\bar{\delta}(P, P') : P' \in \bar{M} \text{ and } P' \in Pt(y)\}.$$

Let $(P_n)_{n\in N}$ be a sequence of elements in M such that $P_n \in p_n$. Then, by Proposition 6.3, $\lim_{n\to\infty} P_n = P$ and, therefore, since the function $f : \bar{M} \to R$ defined by setting $f(P) = \bar{\delta}(P, y)$ is continuous, and $e_\delta(p_n, y) \geqslant \delta(P_n, y)$,

$$\begin{aligned}\lim_{n\to\infty} e_\delta(p_n, y) &\geqslant \lim_{n\to\infty} \delta(P_n, y) = \bar{\delta}(\lim_{n\to\infty} P_n, y) = \bar{\delta}(P, y)\\ &= \inf\{\bar{\delta}(P, Q) : Q \in \bar{M} \text{ and } Q \in y\} \geqslant \inf\{\bar{\delta}(P, P') : P' \in Pt(y)\}.\end{aligned}$$

To prove the second part of the theorem, we use $cl(y)$ to denote the closure of y in the space $(\bar{M}, \bar{\delta})$. Then, since $Pt(y)$ is a closed set containing y, we have $Pt(y) \supseteq cl(y)$. Moreover, let $P = [\langle p_n \rangle_{n\in N}]$ be a point and, for any $n \in N$, let P_n be an element of p_n. Then

$$\begin{aligned}P \in Pt(y) &\Leftrightarrow \lim_{n\to\infty} e_\delta(p_n, y) = 0 \Rightarrow \lim_{n\to\infty} \delta(P_n, y) = 0 \Leftrightarrow \bar{\delta}(\lim_{n\to\infty} P_n, y) = 0\\ &\Leftrightarrow \bar{\delta}(P, y) = 0 \Leftrightarrow P \in cl(y).\end{aligned}$$

This proves that $Pt(y) \subseteq cl(y)$ and therefore that $Pt(y) = cl(y)$. □

In order to prove the independence of d6, we propose an example inspired by the notion of fuzzy subset of a metric space. Namely, we confine ourselves to the three-valued fuzzy subsets, which we represent as a pair (x, y) of subsets such that $x \subseteq y$. The set x is interpreted as the set of elements whose membership degree is 1, and y as the set of elements whose membership degree is greater than or equal to 0.5. Accordingly, any classical subset x is identified with a pair (x, x). This enables us to prove the following proposition where, given two real numbers x and y, $x \oplus y$ denotes the average $(x + y)/2$.

Proposition 6.6. Let (M, δ) be a metric space, set $Re = \{(x_1,x_2) \in B(M) \times B(M) : x_1 \subseteq x_2\}$ and define $\underline{e}_\delta$ by setting

$$\underline{e}_\delta((x_1, x_2), (y_1, y_2)) = e_\delta(x_1, y_1) \oplus e(x_2, y_2),$$

for every (x_1, x_2) and (y_1, y_2) in Re. Then $(Re, \underline{e}_\delta)$ is a quasi-metric space of regions. Such a space is a proper extension of $(B(M), e_\delta)$ whose associated metric space coincides with the one of $(B(M), e_\delta)$ and in which d6 fails.

Proof. Since

$$\begin{aligned}(x_1, x_2) \leqslant (y_1, y_2) &\Leftrightarrow \underline{e}_\delta((x_1, x_2), (y_1, y_2)) = e_\delta(x_1, y_1) \oplus e_\delta(x_2, y_2) = 0\\ &\Leftrightarrow e_\delta(x_1, y_1) = 0 \text{ and } e_\delta(x_2, y_2) = 0 \Leftrightarrow x_1 \subseteq y_1 \text{ and } x_2 \subseteq y_2,\end{aligned}$$

we have that $\underline{e}_\delta$ satisfies d1 and d2. To prove d3, observe that, given $(x_1, x_2), (y_1, y_2)$ and (z_1, z_2) in Re,

$$\begin{aligned} \underline{e}_\delta((x_1,x_2),(y_1,y_2)) &= e_\delta(x_1,y_1) \oplus e_\delta(x_2,y_2) \qquad (*) \\ &\leqslant (e_\delta(x_1,z_1) + e_\delta(z_1,y_1)) \oplus (e_\delta(x_2,z_2) + e_\delta(z_2,y_2)) \\ &= (e_\delta(x_1,z_1) \oplus e_\delta(x_2,z_2)) + (e_\delta(z_1,y_1) \oplus e_\delta(z_2,y_2)) \\ &= \underline{e}_\delta((x_1,x_2),(z_1,z_2)) + \underline{e}_\delta((z_1,z_2),(y_1,y_2)). \end{aligned}$$

Axiom d4 is immediate. To prove d5, first observe that if we use $D((x, y))$ to denote the diameter of (x, y) in the space $(Re, \underline{e}_\delta)$, we have

$$\begin{aligned} D((x,y)) &= \sup\{\underline{e}_\delta((x,y),(x',y')) : x' \subseteq x \text{ and } y' \subseteq y\} \\ &= \sup\{e_\delta((x,x') \oplus e_\delta(y,y')) : x' \subseteq x \text{ and } y' \subseteq y\} \\ &= \sup\{e_\delta(x,x') : x' \subseteq x\} \oplus \sup\{e_\delta(y,y') : y' \subseteq y\} = D(x) \oplus D(y). \end{aligned}$$

Consequently, by $(*)$,

$$\begin{aligned} &|\underline{e}_\delta((x_1,x_2),(y_1,y_2)) - \underline{e}_\delta((y_1,y_2),(x_1,x_2))| \\ &\quad = |e_\delta(x_1,y_1) \oplus e_\delta(x_2,y_2) - e_\delta(y_1,x_1) \oplus e_\delta(y_2,x_2))| \\ &\quad = |(e_\delta(x_1,y_1) - e_\delta(y_1,x_1)) \oplus (e_\delta(x_2,y_2) - e_\delta(y_2,x_2))| \\ &\quad \leqslant |e_\delta(x_1,y_1) - e_\delta(y_1,x_1)| + |e_\delta(x_2,y_2) - e_\delta(y_2,x_2)| \\ &\quad \leqslant D(x_1) \oplus D(y_1) + D(x_2) \oplus D(y_2) \\ &\quad = D((x_1,y_1)) + D((x_2,y_2)). \end{aligned}$$

Trivially, the map $i : B(M) \to Re$ defined by setting $i(x) = (x, x)$ is an isometry from $(B(M), e_\delta)$ into $(Re, \underline{e}_\delta)$. It is obvious that such an isometry is not surjective. Let $\bar{M}$ be the set of points in $(B(M), e_\delta)$ and $\underline{M}$ be the set of points in $(Re, \underline{e}_\delta)$. Then we can define the map $\underline{i} : \bar{M} \to \underline{M}$ by setting $\underline{i}(P) = [\langle i(p_n)\rangle_{n\in N}]$ in $\underline{M}$ for any point $P = [\langle p_n\rangle_{n\in N}]$ in $\bar{M}$. Trivially, this map is an isometry. To prove that $\underline{i}$ is surjective, given any point $[\langle (p_n, q_n)\rangle_{n\in N}]$ in $\underline{M}$, we observe that $\langle p_n\rangle_{n\in N}$ is a point-representing sequence. Indeed,

$$\lim_{n\to\infty} D(p_n) \leqslant \lim_{n\to\infty} 2 \cdot D((p_n, q_n)) = 0.$$

Moreover, given any $\varepsilon > 0$, let m be such that for any $h \geqslant m$ and $k \geqslant m$,

$$\underline{e}_\delta((p_h, q_h), (p_k, q_k)) \leqslant \varepsilon/2.$$

Then, since $e_\delta(p_h, p_k) \leqslant 2 \cdot \underline{e}_\delta((p_h, q_h), (p_k, q_k))$, we have also that $e_\delta(p_h, p_k) \leqslant \varepsilon$ for any $h \geqslant m$ and $k \geqslant m$. Moreover, since $\lim_{n\to\infty} \underline{e}_\delta((p_n, p_n), (p_n, q_n)) = 0$, we have that $\underline{i}([\langle (p_n)\rangle_{n\in N}]) = [\langle (p_n, p_n)\rangle_{n\in N}] = [\langle (p_n, q_n)\rangle_{n\in N}]$, and this proves that i is surjective.

To prove that d6 is not satisfied, it is enough to consider two distinct points P and Q, the point $\underline{P}$ defined by the sequence constantly equal to $(\{P\}, \{P\})$ and the region $(\{P\}, \{P, Q\})$. □

7. Abstract excess spaces

We are interested in the spaces of regions (Re, d) for which the mapping $Pt : Re \to B(\bar{M})$ is an isometry, that is, one for which

$$d(x, y) = e_{\bar{\delta}}(Pt(x), Pt(y)).$$

The following proposition shows what happens in the general case.

Proposition 7.1. Let P be a point and x and y be regions. Then

$$d(x, y) \geqslant \sup\{\underline{d}(P, y) : P \in Pt(x)\}. \tag{7.1}$$

Consequently, if d6 is satisfied,

$$d(x, y) \geqslant e_{\bar{\delta}}(Pt(x), Pt(y)) \tag{7.2}$$

and

$$d(x) \geqslant e_{\bar{\delta}}(Pt(x)). \tag{7.3}$$

Proof. To prove (7.1), observe that $\underline{d}(P, y) \leqslant \underline{d}(P, x) + d(x, y) = d(x, y)$ for any $P \in Pt(x)$. (7.2) is trivial. Also, for every $P, Q \in Pt(x)$,

$$\begin{aligned} \bar{\delta}(P, Q) &= \lim_{n\to\infty} d(p_n, q_n) \leqslant \lim_{n\to\infty} d(p_n, x) + d(x, q_n) \\ &\leqslant \lim_{n\to\infty} (d(p_n, x) + d(q_n, x) + d(q_n) + d(x)) \\ &= \lim_{n\to\infty} d(p_n, x) + \lim_{n\to\infty\text{g}} (q_n, x) + \lim_{n\to\infty} d(q_n) + d(x) = d(x) \end{aligned}$$

and therefore $e_{\bar{\delta}}(Pt(x)) = \sup\{\bar{\delta}(P, Q) : P \in Pt(x), Q \in Pt(x)\} \leqslant d(x)$. □

This proposition suggests the following definition.

Definition 7.2. An *abstract excess space* is a quasi-metric space of bounded regions (Re, d) satisfying d6 and such that, for any point P and x, $y \in Re$:

d7: $d(x, y) = \sup\{\underline{d}(P, y) : P \in Pt(x)\}$

Proposition 7.3. Every full Hausdorff excess space is an abstract excess space.

Proof. Let (M, δ) be a metric space and x, y be regions in $B(M)$. Then, using cl to denote the closure operator in $(\bar{M}, \bar{\delta})$,

$$\begin{aligned} e_\delta(x, y) &= e_{\bar{\delta}}(cl(x), cl(y)) \\ &= e_{\bar{\delta}}(Pt(x), Pt(y)) \\ &= \sup\nolimits_{P\in Pt(x)} \bar{\delta}(P, Pt(y)) \\ &= \sup\nolimits_{P\in Pt(x)} \underline{d}(P, y). \end{aligned}$$ □

It is simple to prove the following representation theorem for abstract excess spaces.

Theorem 7.4. Let (Re, d) be an abstract excess space and $(\bar{M}, \bar{\delta})$ be the associated metric space. Then:

(i) $d(x, y) = e_{\bar{\delta}}(Pt(x), Pt(y))$,
(ii) $d(x) = e_{\bar{\delta}}(Pt(x))$

and thus $Pt : Re \to B(\bar{M})$ is an isometry from (Re, d) into $(B(\bar{M}), e_{\bar{\delta}})$ preserving diameters. Consequently, every abstract excess space is isometric to a Hausdorff excess space.

Proof. Since $d(x, y) = \sup_{P \in Pt(x)} \underline{d}(P, y)$ and $\underline{d}(P, y) = \bar{\delta}(P, Pt(y)) = \inf\{\bar{\delta}(P, Q) : Q \in Pt(y)\}$, we have that

$$d(x, y) = \sup_{P \in Pt(x)} \inf_{Q \in Pt(y)} \bar{\delta}(P, Q) = e_{\bar{\delta}}(Pt(x), Pt(y)).$$

To prove (ii), observe that

$$d(x) = \sup\{d(x, x') : x' \leqslant x\} = \sup\{e_{\bar{\delta}}(Pt(x), Pt(x')) : x' \leqslant x\} \leqslant e_{\bar{\delta}}(Pt(x)).$$

So, by (7.3), $d(x) = e_{\bar{\delta}}(Pt(x))$.

□

The following proposition shows that d7 is independent of the remaining axioms. We use $x \div y$ to denote the value $x - y$ if $x \geqslant y$ and 0 otherwise.

Proposition 7.5. Let (M, δ) be a metric space and set, for any $x, y \in B(M)$,

$$d(x, y) = e_\delta(x, y) + e_\delta(x) \div e_\delta(y). \tag{7.4}$$

Then $(B(M), d)$ is a quasi-metric space of regions such that the partial order is the inclusion relation, $d(x) = 2 \cdot e_\delta(x)$, and the associated metric space coincides with the metric space $(\bar{M}, \bar{\delta})$ of (M, δ). Moreover, though d6 is satisfied, d7 does not hold.

Proof. Trivially,

$$d(x, y) = 0 \Leftrightarrow e_\delta(x, y) = 0 \text{ and } e_\delta(x) \leqslant e_\delta(y) \Leftrightarrow x \subseteq y.$$

This proves both d1 and d2. To prove d3, observe that

$$\begin{aligned} d(x, y) &= e_\delta(x, y) + e_\delta(x) \div e_\delta(y) \\ &\leqslant e_\delta(x, z) + e_\delta(z, y) + e_\delta(x) \div e_\delta(z) + e_\delta(z) \div e_\delta(y) = d(x, z) + d(z, y). \end{aligned}$$

Also,

$$\begin{aligned} d(x) &= \sup\{d(x, x') : x' \subseteq x\} \\ &= \sup\{e_\delta(x, x') + e_\delta(x) \div e_\delta(x') : x' \subseteq x\} \\ &= \sup\{e_\delta(x, x') + e_\delta(x) - e_\delta(x') : x' \subseteq x\} \\ &= e_\delta(x) + \sup\{e_\delta(x, x') - e_\delta(x') : x' \subseteq x\} \\ &= e_\delta(x) + \sup\{e_\delta(x, \{p\}) - e_\delta(\{p\}) : p \in x\} \\ &= e_\delta(x) + \sup\{e_\delta(x, \{p\}) : p \in x\} = 2 \cdot e_\delta(x). \end{aligned}$$

Axiom d4 is trivial. To prove d5, observe that e_δ satisfies d5, and thus

$$\begin{aligned} |d(x, y) - d(y, x)| &= |e_\delta(x, y) - e_\delta(y, x) + e_\delta(x) \div e_\delta(y) - e_\delta(y) \div e_\delta(x)| \\ &\leqslant |e_\delta(x, y) - e_\delta(y, x)| + |e_\delta(x) - e_\delta(y)| \\ &\leqslant |e_\delta(x, y) - e_\delta(y, x)| + \max\{e_\delta(x), e_\delta(y)\} \\ &\leqslant e_\delta(x) + e_\delta(y) + e_\delta(x) + e_\delta(y) = d(x) + d(y). \end{aligned}$$

Let $\langle p_n \rangle_{n\in N}$ be a point-representing sequence in the space $(B(M), e_\delta)$. Then $\lim_{n\to\infty} d(p_n) = \lim_{n\to\infty} 2\cdot e_\delta(p_n) = 0$. Moreover, given $\varepsilon > 0$, let m be such that $e_\delta(p_h, p_k) < \varepsilon/3, e_\delta(p_h) < \varepsilon/3$ and $e_\delta(p_k) < \varepsilon/3$ for any $h \geqslant m$ and $k \geqslant m$. Then

$$d(p_h, p_k) = e_\delta(p_h, p_k) + e_\delta(p_h) \div e_\delta(p_k) \leqslant e_\delta(p_h, p_k) + e_\delta(p_h) + e_\delta(p_k) \leqslant \varepsilon$$

for any $h \geqslant m$ and $k \geqslant m$. This proves that $\langle p_n \rangle_{n\in N}$ is a point-representing sequence in the space $(B(M), d)$. Conversely, since $e_\delta \leqslant d$, any point-representing sequence in the space $(B(M), d)$ is a point-representing sequence in the space $(B(M), d)$.

Let $\langle p_n \rangle_{n\in N}$ and $\langle q_n \rangle_{n\in N}$ be two point-representing sequences. Then, since $\lim_{n\to\infty}(e_\delta(p_n) \div e_\delta(q_n)) = 0$,

$$\begin{aligned} \lim_{n\to\infty} d(p_n, q_n) &= \lim_{n\to\infty} (e_\delta(p_n, q_n) + e_\delta(p_n) \div e_\delta(q_n)) \\ &= \lim_{n\to\infty} e_\delta(p_n, q_n) + \lim_{n\to\infty} (e_\delta(p_n) \div e_\delta(q_n)) = \lim_{n\to\infty} e_\delta(p_n, q_n). \end{aligned}$$

This means that the metric space associated with $(B(M), d)$ coincides with the metric space associated with $(B(M), e_\delta)$, and thus with $(\bar{M}, \bar{\delta})$, the metric completion of (M, δ).

To prove d6, let $P = [\langle p_n \rangle_{n\in N}]$ be an element in $\bar{M}$ and $y \in B(M)$. Then, using $e_{\bar{\delta}}$ to denote the Hausdorff excess induced by $\bar{\delta}$,

$$\bar{\delta}(P, Pt(y)) = e_{\bar{\delta}}(\{P\}, Pt(y)) = \lim_{n\to\infty} e_\delta(p_n, y) = \lim_{n\to\infty} d(p_n, y) = \underline{d}(P, y).$$

Finally, to prove that d7 does not hold, let x and y be two regions such that $e_\delta(x) > e_\delta(y)$. Then

$$\begin{aligned} d(x, y) &= e_\delta(x, y) + e_\delta(x) \div e_\delta(y) > e_\delta(x, y) = e_{\bar{\delta}}(Pt(x), Pt(y)) \\ &= \sup\{\bar{\delta}(P, Pt(y)) : P \in Pt(x)\} \\ &\geqslant \sup\{\underline{d}(P, y) : P \in Pt(x)\}. \end{aligned}$$

□

8. Atom-free spaces

So far no axiom excludes the existence of regions that are atoms. This enables us to obtain a theory extending the theory of metric spaces. Alternatively, in accordance with Whitehead's program, we can decide to confine ourselves to atom-free spaces.

Definition 8.1. We define an *atom-free quasi-metric space of regions* to be any quasi-metric space of regions satisfying:

d8: No atom exists in *Re*.

It is obvious that the space $(B(M), e_\delta)$ is not atom-free, so in order to define a notion of canonical models for the atom-free space theory, we have to look for a more reasonable definition of regions in a metric space. As an example, we will just consider the regular subsets in $B(M)$.

Definition 8.2. Let (M, δ) be a metric space and use $cl : P(M) \to P(M)$ and $\text{int} : P(M) \to P(M)$ to denote the closure and interior operators, respectively. Also, define $reg : P(M) \to P(M)$ by setting $reg(x) = cl(\text{int}(x))$. Then we say any fixed point of reg is a *regularly closed set*, or *regular set* for short.

It is easy to prove that in the class of the closed subsets of (M, δ), the operator *reg* satisfies the following properties:

(i) $reg(\emptyset) = \emptyset$
(ii) $x \subseteq y \Rightarrow reg(x) \subseteq reg(y)$
(iii) $reg(x) \subseteq x$
(iv) $reg(reg(x)) = reg(x)$.

Also, the class of regular sets is a Boolean algebra. We use $Re(M)$ to denote the class of regular elements in $B(M)$. Equation (iv) entails that $Re(M) = \{reg(x) : x \in B(M)\} - \{\emptyset\}$. An interesting class of elements in $Re(M)$ is defined by setting, for any $P \in M$ and $n \in N$,

$$B_n(P) = cl(\{P' \in M : \delta(P', P) < 1/n\}). \tag{8.1}$$

Theorem 8.3. Let (M, δ) be a metric space. Then $(Re(M), e_\delta)$ is a quasi-metric space of regions whose diameter coincides with the diameter D defined in (5.5) and whose order is the set-theoretical inclusion.

Proof. We use $e_\delta(x)$ to denote the diameter of an element $x \in Re(M)$ in the space $(Re(M), e_\delta)$. Then, by Theorem 5.2,

$$\begin{aligned} e_\delta(x) &= \sup\{e_\delta(x, x") : x" \subseteq x, x" \in Re(M)\} \\ &\leqslant \sup\{e_\delta(x, x") : x" \subseteq x, x" \in B(M)\} = D(x). \end{aligned}$$

Also, observe that for any $x \in Re(M)$ and $x' \in B(M)$, we have $x' \subseteq x$ if and only if $reg(x') \subseteq x$. Then,

$$\begin{aligned} D(x) &= \sup\{e_\delta(x, x') : x' \subseteq x, x' \in B(M)\} \\ &\leqslant \sup\{e_\delta(x, reg(x')) : x' \subseteq x, x' \in B(M)\} \\ &= \sup\{e_\delta(x, reg(x')) : reg(x') \subseteq x, x' \in B(M)\} \\ &= \sup\{e_\delta(x, x") : x" \subseteq x, x" \in Re(M)\} \\ &= e_\delta(x). \end{aligned}$$

Since the diameter in $(Re(M), e_\delta)$ coincides with the diameter in $(B(M), e_\delta)$, we also have that d4 is satisfied. □

In the following we call the space $(Re(M), e_\delta)$ a *small Hausdorff excess space* .

Theorem 8.4. Let (M, δ) be a metric space and use $(\bar{M}, \bar{\delta})$ to denote the metric space associated with $(Re(M), e_\delta)$. Also, use $k : M \to \bar{M}$ to denote the map defined by setting for any $P \in M$,

$$k(P) = [\langle B_n(P)\rangle_{n \in N}]. \tag{8.2}$$

Then k is an isometry such that $k(M)$ is dense in $\bar{M}$. Consequently, $(\bar{M}, \bar{\delta})$ is the completion of (M, δ) and therefore $(\bar{M}, \bar{\delta})$ is isometric with the metric space associated with $(B(M), e_\delta)$.

Proof. Observe that, given $P \in M, \langle B_n(P)\rangle_{n \in N}$ is a point-representing sequence of elements in $Re(M)$. To prove that k is an isometry, let P and Q be two elements in M, and observe that, for any $P' \in B_n(P)$ and $Q' \in B_n(Q)$,

$$\delta(P, Q) \leqslant \delta(P, P') + \delta(P', Q') + \delta(Q', Q) \leqslant 2/n + \delta(P', Q'),$$

and thus

$$\delta(P,Q) \leqslant 2/n + \delta(P', B_n(Q)) \leqslant 2/n + e_\delta(B_n(P), B_n(Q)).$$

As a consequence,

$$\delta(P,Q) \leqslant \lim_{n\to\infty} e_\delta(B_n(P), B_n(Q)) = \underline{\delta}(k(P), k(Q)).$$

Similarly, since

$$\delta(P',Q') \leqslant \delta(P',P) + \delta(P,Q) + \delta(Q,Q') \leqslant 2/n + \delta(P,Q),$$

we have $e_\delta(B_n(P), B_n(Q)) \leqslant 2/n + \delta(P,Q)$ and thus

$$\delta(k(P), k(Q)) = \lim_{n\to\infty} e_\delta(B_n(P), B_n(Q)) \leqslant \delta(P,Q).$$

Then, $\delta(P,Q) = \underline{\delta}(k(P), k(Q))$, which proves that $h : M \to \bar{M}$ is an isometry. To prove that $k(M)$ is dense in $(\bar{M}, \bar{\delta})$, let $P = [\langle p_n \rangle_{n\in N}]$ be any element in $\bar{M}$. Moreover, for any $n \in N$, let $Q_n \in M$ be an element of the set p_n. We claim that $\lim_{n\to\infty} k(Q_n) = P$, that is, that

$$\lim_{n\to\infty} \underline{\delta}(k(Q_n), P) = \lim_{n\to\infty}(\lim_{m\to\infty} e_\delta(B_m(Q_n), p_m)) = 0.$$

Indeed, if we use m to denote the minimum distance defined by (5.2),

$$\begin{aligned} e_\delta(B_m(Q_n), p_m) &\leqslant e_\delta(B_m(Q_n), p_n) + e_\delta(p_n, p_m) \\ &\leqslant m(B_m(Q_n), p_n) + D(B_m(Q_n)) + e_\delta(p_n, p_m) \\ &= D(B_m(Q_n)) + e_\delta(p_n, p_m) \leqslant 2/m + e_\delta(p_n, p_m). \end{aligned}$$

On the other hand, since $\langle p_n \rangle_{n\in N}$ is a point-representing sequence, given any $\varepsilon > 0$, an integer h exists such that $e_\delta(p_n, p_m) \leqslant \varepsilon$ for any $n \geqslant h$ and $m \geqslant h$. Consequently,

$$\underline{\delta}(k(Q_n), P) = \lim_{m\to\infty} e_\delta(B_m(Q_n), p_m) \leqslant \lim_{m\to\infty} e_\delta(p_n, p_m) \leqslant \varepsilon$$

for any $n \geqslant h$. Thus, $\lim_{n\to\infty} \underline{\delta}(k(Q_n), P) = 0$, which proves that $k(M)$ is dense in $(\bar{M}, \bar{\delta})$. Since, by Theorem 4.5, the space $(\bar{M}, \bar{\delta})$ is complete, we can conclude that $(\bar{M}, \bar{\delta})$ is the completion of (M, δ). □

Theorem 8.5. Let (M, δ) be a metric space. Then $(Re(M), e_\delta)$ is an abstract excess space. If (M, δ) has no isolated point, then $(Re(M), e_\delta)$ is atom-free.

Proof. Since the points in $(Re(M), e_\delta)$ coincide with the points in $(B(M), e_\delta)$, it is evident that $(Re(M), e_\delta)$ satisfies d6 and d7. To prove the second part of the theorem, we prove that an element x in $Re(M)$ is an atom iff there is an isolated point $P \in M$ such that $x = \{P\}$. Indeed, if P is an isolated point, it is evident that $\{P\}$ is a bounded regular subset and therefore an atom in $Re(M)$. Conversely, let x be an atom in $Re(M)$ and let P be an element of int(x). We claim that P is an isolated point such that $x = \{P\}$. Indeed, if we assume that $x \neq \{P\}$, then a point $Q \in x$ exists such that $Q \neq P$. Accordingly, there exists $n \in N$ such that $B_n(P) \subseteq x$ and $Q \notin B_n(P)$. Then $B_n(P)$ is a proper sub-region of x, which contradicts the hypothesis that x is an atom. Since $x = \{P\}$, and x is regular, we have also that P is an isolated point. □

For the case in which there are isolated points in (M, δ), we can again define an atom-free space by the notion of a formal ball. Indeed, for any quasi-metric space (Re, δ), we define a *closed formal ball with center p and radious r*, to be every pair (p, r), where $p \in Re$ and r is a positive real number. We define in the class $Ball(Re)$ of closed formal balls in Re the function

$$d((p, \lambda), (q, \mu)) = \max\{\delta(p, q) + \lambda - \mu, 0\}.$$

It is routine to prove that $(Ball(Re), d)$ is a quasi-metric space. Also, if $\leqslant$ is the order associated with d, then

$$(p, \lambda) \leqslant (q, \mu) \Leftrightarrow d((p, \lambda), (q, \mu)) = 0 \Leftrightarrow \delta(p, q) + \lambda \leqslant \mu.$$

Moreover, $d((p, \lambda)) = 2 \cdot \lambda$. Also, while d4 is satisfied, since it is

$$\begin{aligned} |d((p, \lambda), (q, \mu)) - d((q, \lambda), (p, \mu))| &\leqslant |\delta(p, q) - \delta(q, p)| + 2 \cdot \lambda + 2 \cdot \mu \\ &= |\delta(p, q) - \delta(q, p)| + d((p, \lambda)) + d((q, \mu)), \end{aligned}$$

when (Re, δ) is a metric space, d5 is satisfied. It is also evident that such a space has no atom. In our opinion it would be interesting to compare these ideas with the completion of a generalised metric space via formal open balls, which was proposed in Vickers (2005).

9. Defining the points by nested sequences of regions

In the literature on point-free geometry the notion of a point is usually defined by referring to the class of nested sequences of regions (see, for example, Gerla (1990) and Whitehead (1929)). We can proceed in the same way in our theory of quasi-metric spaces of regions.

Definition 9.1. Given a quasi-metric space (Re, d), we define *nested-representing sequences* as any order-reversing sequence $\langle p_n \rangle_{n \in N}$ of regions with vanishing diameters, that is, such that

$$\lim_{n \to \infty} d(p_n) = 0.$$

We use Nr to denote the class of nested-representing sequences. Obviously, any nested-representing sequence is a point-representing sequence in accordance with Definition 3.1. To prove that Nr is non-empty, we have to consider an axiom analogous to Axiom d4:

d4′: Any region x contains a region x' such that $d(x') \leqslant d(x)/2$.

Trivially, d4′ entails that any region contains a nested-representing sequence.

Definition 9.2. Let (Re, d) be a quasi-metric space of regions satisfying d4′. Then the *nested metric space associated with* (Re, d) is the metric space (M', δ') where

$$M' = \{[\langle p_n \rangle_{n \in N}] \in \bar{M} : \langle p_n \rangle_{n \in N} \in Nr\}$$

and δ' is the restriction of $\bar{\delta}$ to M'.

In general, the space (M', δ') is different from $(\bar{M}, \bar{\delta})$. For example, if (Re, d) is a metric space, then, while $(\bar{M}, \bar{\delta})$ is the completion of (Re, d), (M', δ') coincides with (Re, d). Indeed, in such a case the only point-representing sequences are the sequences constantly equal to an element of Re. This observation is in accordance with the following theorem.

Theorem 9.3. Let (Re, d) be a quasi-metric space of regions satisfying d4′, and $(\bar{M}, \bar{\delta})$ and (M', δ') be the associated metric space and nested metric space, respectively. Then $(\bar{M}, \bar{\delta})$ is the metric completion of (M', δ').

Proof. To prove that (M', δ') is dense in $(\bar{M}, \bar{\delta})$, let $P = [\langle p_n \rangle_{n \in N}]$ be any element in $\bar{M}$. Then we can consider for any $n \in N$ a point P_n in M' such that $P_n \in Pt(p_n)$. Then, since by (6.1) $\lim_{n\to\infty} \underline{d}(p_n, P) = 0$ and

$$\bar{\delta}(P_n, P) \leqslant \underline{d}(P_n, p_n) + \underline{d}(p_n, P) = \underline{d}(p_n, P),$$

we have that $lim_{n\to\infty}\bar{\delta}(P_n, P) = 0$. Thus every element of $\bar{M}$ is a limit of a sequence of elements of M', and, therefore, by the completeness of $(\bar{M}, \bar{\delta})$, the space $(\bar{M}, \bar{\delta}\ )$ is the metric completion of (M', δ'). □

In accordance with Theorem 4.4, the metric space associated with a quasi-metric space of regions is complete. The question arises as to whether the associated nested metric space satisfies some completeness property.

Definition 9.4. Let (M, δ) be a metric space. Then we say that (M, δ) is *weakly complete* if any nested sequence of non-empty regularly closed subsets with vanishing diameters has a non-empty intersection. We say that a metric space (M', δ') is a *weak* completion of (M, δ) if (M', δ') is weakly complete and (M, δ) is dense in (M', δ').

Theorem 9.5. Let (M, δ) be a metric space. Then the nested metric space (M', δ') associated with $(Re(M), e_\delta)$ is a weak completion of (M, δ).

Proof. By mimicking Theorem 8.4 we have that (M, δ) is isometric to a dense subspace of (M', δ'). Also, observe that any regularly closed subset x' of M' is the closure in (M', δ') of some $x \in Re(M)$. Let $\langle x'_n \rangle_{n \in N}$ be any nested sequence of elements in $R_e(M')$ with vanishing diameters and let $x_n \in Re(M)$ be such that its closure in (M', δ') is x'_n. Then $\langle x_n \rangle_{n \in N}$ is a nested representing sequence and therefore it determines a point P in M' that belongs to the closure x'_n of x_n in (M', δ'). □

The definition of a point by the nested-representing sequences refers only to the inclusion relation between regions and to the diameter of a region. So, the question arises as to whether a possible approach to point-free geometry can be based on these two notions as primitives. A reasonable proposal should be as follows. We start from a structure $(R, \leqslant, D)$ where $\leqslant$ is a partial order and $D : R \to R^+$ a map. In this structure we define the notion of a nested-representing sequence as in Definition 9.1. In the set Pr of nested-representing sequences we can set

$$d(\langle x_n \rangle_{n \in N}, \langle y_n \rangle_{n \in N}\}) = \inf\{D(x) : xOx_n \text{ and } xOy_n \text{ for any } n \in N\}$$

where O is the overlapping relation defined by setting xOy provided that a region z exists contained in both x and y. By imposing suitable properties on the diameter D, it should be possible to prove that (Nr, d) is a pseudo-metric space and therefore to define a metric space as in Section 4. With regard to this idea, observe that the partial order and the diameter induced by a quasi-metric do not exhaust the information carried by the quasi-metric. Namely the following proposition holds true.

Proposition 9.6. Let R be the set of real numbers. Then there are two canonical quasi-metric spaces in R^2 that are not isometric but define the same diameter and the same inclusion relation.

Proof. Let (R^2, d) be the Euclidean metric space and let δ be the taxi-metric, that is, set

$$\delta((x_1, y_1), (x_2, y_2)) = |x_1 - x_2| + |y_1 - y_2|.$$

Then the balls in such a space are the squares whose sides have the direction of the diagonals. It can be shown easily that (R^2, δ) is topologically equivalent to (R^2, d). Nevertheless, these spaces are not isometric. Indeed, in (R^2, δ) the four points (-1,0), (1,0), (0,-1), (0,1) define a square whose diagonals are equal to the sides and in (R^2, d) such a point configuration cannot exist. Consider the Hausdorff excesses defined in these spaces by the class of taxi-balls. It is evident that they are not isometric. Also, given a closed taxi-ball of radius ε, both its Euclidean-diameter and taxi-diameter are equal to $2 \cdot \varepsilon$. Moreover, in both cases the partial order associated with the Hausdorff excess is the usual inclusion relation. □

10. Open questions and future work

L. M. Blumenthal proved in Blumenthal (1970) that, given an integer $n \in N$, it is possible to add to the theory of metric spaces MS a suitable set of axioms ES to obtain a theory $T = MS \cup ES$ for the Euclidean n-dimensional metric space, that is, a theory whose models coincide with the metric space of the Euclidean space whose dimension is n. Obviously, the axioms in T refer to the points and the distance between points as primitives. Now, assume as primitives the regions and a distance between regions. Then it is an open question as to whether a system of axioms ES can be added to the axioms d1–d8 to obtain a theory T whose models are the atom-free quasi-metric spaces of regions whose associated metric space (M, δ) is a Euclidean metric space. Such a theory should be a point-free approach to Euclidean geometry in accordance with Whitehead's ideas.

Furthermore, it should be interesting to study the category whose objects are the quasi-metric spaces of regions and whose morphisms are the non-expansive maps.

Finally, in accordance with the recent literature, it is important to explore the computability dimension of the proposed notions.

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