High performance parallel numerical methods for Volterra equations with weakly singular kernels

Giovanni Capobianco

Dipartimento S.T.A.T., Università degli Studi del Molise, Contrada Fonte Lappone, I-86170 Pesche (IS), Italy

Dajana Conte

Dipartimento di Matematica e Informatica, Università di Salerno, Via Ponte Don Melillo, I-84084 Fisciano (SA), Italy

Ida Del Prete

Dipartimento di Matematica e Applicazioni, R. Caccioppoli, Università degli Studi Napoli, Federico II, Via Cintia, Monte S. Angelo, I-80126 Napoli, Italy

Abstract

Non stationary discrete time waveform relaxation methods for Abel systems of Volterra integral equations using fractional linear multistep formulae are introduced. Fully parallel discrete waveform relaxation methods having an optimal convergence rate are constructed. A significant expression of the error is proved, which permits to estimate the number of iterates needed to satisfy a prescribed tolerance and allows to identify the problems where the optimal methods offer the best performances. The numerical experiments confirm the theoretical expectations.

Key words: Abel equations, parallel methods, waveform relaxation methods
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Email addresses: giovanni.capobianco@unimol.it (Giovanni Capobianco), dajconte@unisa.it (Dajana Conte), ida.delprete@unina.it (Ida Del Prete).

1 Introduction

The aim of this work is to develop high performance numerical methods for large systems of Volterra integral equations (VIEs) with weakly singular kernel (Abel type) of the form

$$y(t) = f(t) + \int_0^t \frac{k(t,s,y(s))}{(t-s)^\alpha} ds, \quad t \in [0,T], \quad 0 < \alpha < 1, \quad (1.1)$$
$$y, f, k \in R^d, \quad d >> 1.$$

Such equations arise from many applications such as reaction–diffusion problems in small cells [15] or from the semidiscretization in space of Volterra–Fredholm integral equations with weakly singular kernel and of partial Abel integral or integro–differential equations. The last kind of equations occur as mathematical model in linear quasi–static viscoelasticity problems (see [25], [26] and their lists of references) and in the description of anomalous diffusion processes and wave propagation in viscoelastic materials [14] [16] [19] [20] [22] [23] [24]. In many of the cited examples the spatial semidiscretization leads to VIEs of the form (1.1) with linear convolution kernel [14], [25], [26]. Volterra–Fredholm integral equations with singular kernels occur for example in the modelling of the coding mechanism in the transmission of nervous signals among neurons [17].

Numerical methods for (1.1) require a very high computational cost, because both of the hereditary nature of the problem and of the presence of the singularity. Of course, the computational cost grows with the dimension of the problem and it can become prohibitive for large systems of Abel VIEs.

A parallel approach can be a possible solution in order to solve (1.1) into a reasonable time frame. The iterative waveform relaxation methods (WR methods) [3] [5] [7] [10] [11] [12] [13] are especially suitable for large systems of VIEs since they allow a parallelism "across the system". The idea of WR methods is to construct a sequence of functions $\{y^{(\nu)}(t)\}_{\nu \in N}$, said *"waveforms"*, which converges to the solution of (1.1). The resulting system, at each iteration, can be decoupled into independent subsystems that can be solved in parallel. Since the parallel WR methods are, gener-

ally, slowly convergent it is convenient to use nonstationary WR methods (NSWR methods) (see [8], [9]).

In this paper we construct fast convergent discrete time NSWR methods for (1.1) using fractional linear multistep formulae. Section 2 contains the detailed construction of the methods. In Section 3 we perform the convergence analysis (with respect to the iterations) for nonlinear and linear systems, providing conditions on the stepsize that ensure the convergence of the methods. In Section 4 we construct the family of Richardson fully parallel discrete time NSWR methods. We give the expression of the parameters of the optimal methods with respect to the convergence rate. These methods will be called "fast methods". Moreover, a significant error expression is proved, that allows to estimate a priori the number of iterations to be performed in order to achieve the required precision and to characterize the class of problems on which the proposed methods have the best performances. In Section 5 we report numerical experiments that confirm the theoretical results and show that the convergence improvement with respect to the optimal stationary methods varies from the 33% to the 75%. We briefly discuss a strategy to develop a dynamical integration window and the optimal reordering of the parameters for the construction of an efficient parallel algorithm based on the fast discrete time Richardson NSWR methods.

2 Discrete time NSWR methods

It is known that the most suitable numerical methods for solving VIEs of Abel type are the fractional linear multistep methods [21] and the collocation methods [1] [2] [4].

Here we treat the fractional linear multistep methods; analogous results hold for a family of methods generalizing the one point collocation methods [2].

Let us discretize the integration range through the mesh points:

$$t_n = nh, \qquad n = 0, 1, \ldots, \mathrm{N}, \quad h = \frac{T}{\mathrm{N}} \quad .$$

In [21] the class of fractional linear multistep methods for VIEs of Abel type was constructed starting from a linear multistep method $\omega = (\rho, \sigma)$, ρ and σ being the first and second characteristic polynomial of the method respectively. For the equation (1.1) a method of such class is given by

$$\begin{aligned} y_n = f(t_n) + h^{1-\alpha} \sum_{j=0}^{s} w_{n,j}(\alpha) k(t_n, t_j, y_j) + \\ + h^{1-\alpha} \sum_{j=0}^{n} \omega_{n-j}(\alpha) k(t_n, t_j, y_j), \quad n = 0, 1, \ldots, \mathrm{N} \quad , \\ y_n \in R^d, \end{aligned} \tag{2.1}$$

where s is the number of starting points, $\{w_{n,j}\}$ and $\{\omega_{n-j}\}$ represent respectively the starting and the convolution weights, which both depend on α. The starting values $y_1, \ldots, y_s$ are determined by some special starting procedure and then the approximations y_n, for $n = s+1, \ldots, \mathrm{N}$, are recursively determined.

Let us consider a NSWR method of the form

$$\begin{aligned} y^{(\nu)}(t) = f(t) + \int_0^t \frac{\mathcal{G}_\nu(t, s, y^{(\nu-1)}(s), y^{(\nu)}(s))}{(t-s)^\alpha} ds, \\ t \in [0, T], \quad \nu = 1, 2 \ldots \quad , \\ y^{(0)}(t) := f(t), \end{aligned} \tag{2.2}$$

where $\{\mathcal{G}_\nu(t, s, u, v)\}_{\nu \in \mathrm{N}}$ are such that

$$\mathcal{G}_\nu(t, s, u, u) = k(t, s, u) \quad \forall \nu. \tag{2.3}$$

By applying (2.1) to the system (2.2) we obtain the following

discrete time NSWR method

$$
\begin{aligned}
y_n^{(\nu)} = f(t_n) + h^{1-\alpha} \sum_{j=0}^{s} w_{n,j}(\alpha) \mathcal{G}_\nu(t_n, t_j, y_j^{(\nu-1)}, y_j^{(\nu)}) + \quad (2.4) \\
+ h^{1-\alpha} \sum_{j=0}^{n} \omega_{n-j}(\alpha) \mathcal{G}_\nu(t_n, t_j, y_j^{(\nu-1)}, y_j^{(\nu)}) \quad , \\
n = s+1, \ldots, \mathrm{N} \quad , \qquad \nu = 1, 2, \ldots \quad ,
\end{aligned}
$$

where $y_n^{(\nu)}$ represents an approximation $y^{(\nu)}(t_n)$. In (2.4) $y_0 = f(t_0)$ and we do not iterate on the starting values $y_1, \ldots, y_s$ which are determined by the starting procedure of the fractional linear multistep method (2.1) applied to equation (2.2).

Parallel methods are obtained by choosing the functions $\mathcal{G}_\nu$ such that, at each iteration, the system (2.4) is decoupled into independent subsystems.

The formulation (2.4) of the method is the most immediate, but not the most convenient. As a matter of fact, the convergence clearly slows down when n grows. In order to overcome this problem, we divide the integration range into subintervals, said "windows", and then we apply the method "window after window". Suppose that the interval $[t_s, T]$ is subdivided in γ windows of length bh, with b a positive integer, namely

$$
[0, T] = [0, t_s] \cup \bigcup_{r=0}^{\gamma-1} (t_{s+rb}, t_{s+(r+1)b}].
$$

The application of the fractional linear discrete time NSWR method (2.4) to the window $(t_{s+rb}, t_{s+(r+1)b}]$ leads to:

$$
\begin{aligned}
y_n^{(\nu)} = \mathcal{Q}_n + h^{1-\alpha} \sum_{j=s+rb+1}^{n} \omega_{n-j}(\alpha) \mathcal{G}_\nu(t_n, t_j, y_j^{(\nu-1)}, y_j^{(\nu)}) \\
\mathcal{Q}_n = f(t_n) + h^{1-\alpha} \sum_{j=0}^{s} w_{n,j}(\alpha) k(t_n, t_j, y_j) + \quad (2.5) \\
+ h^{1-\alpha} \sum_{j=s+1}^{s+rb} \omega_{n-j}(\alpha) k(t_n, t_j, y_j^{(m_j)}), \qquad \nu = 1, 2, \ldots , \\
n = s + rb + 1, \ldots, s + (r+1)b, \qquad r = 0, ..., \gamma - 1.
\end{aligned}
$$

As in (2.4) $y_0 = f(t_0)$ and the starting values $y_1, \ldots, y_s$ are determined by the starting procedure of the fractional linear multistep method (2.1) applied to equation (2.2).

In the points t_n belonging to the r-th window the iteration process is continued until a required accuracy on $\left\{y_n^{(\nu)}\right\}_{n=s+rb+1}^{s+(r+1)b}$ is obtained. The number of iterations m_n performed in the mesh point t_n depends on n, and is constant in the same window, i.e.:

$$m_n = \bar{m}_r, \qquad n = s + rb + 1, \ldots, s + (r+1)b.$$

The final solution $y_n^{(m_n)} \approx y_n$, of course, inherits all the properties (order, convergence, stability) of the solution given by the method (2.1).

When the length of the window coincides with the stepsize h, the corresponding method is said time–point (or time–step) NSWR method and assumes the form

$$\begin{aligned}
y_n^{(\nu)} &= \mathcal{Q}_n + h^{1-\alpha}\omega_0(\alpha)\mathcal{G}_\nu(t_n, t_n, y_n^{(\nu-1)}, y_n^{(\nu)}) \quad , \qquad (2.6)\\
&\qquad\qquad n = s+1, \ldots, \mathrm{N} \quad ,\\
&\qquad\qquad \nu = 1, \ldots, m_n \quad ,\\
y_n^{(0)} &:= \mathcal{Q}_n \quad ,\\
\mathcal{Q}_n &= f(t_n) + h^{1-\alpha}\sum_{j=0}^{s} w_{n,j}(\alpha)k(t_n, t_j, y_j) +\\
&\quad + h^{1-\alpha}\sum_{j=0}^{n-1}\omega_{n-j}(\alpha)k(t_n, t_j, y_j^{(m_j)}) \quad .
\end{aligned}$$

Remark 2.1 *It is easy to verify that the method (2.6) can be equivalently obtained applying the fractional linear multistep method to the system (1.1) and then applying the waveform relaxation scheme to the resulting nonlinear system.*

3 Convergence analysis

Let us assume that the functions $\mathcal{G}_\nu$ satisfy the following Lipschitz type condition:

$$\|\mathcal{G}_\nu(t,s,u,v) - \mathcal{G}_\nu(t,s,u',v')\| \le \ell_1^{(\nu)}\|u-u'\| + \ell_2^{(\nu)}\|v-v'\|, \quad (3.1)$$
$$\forall \nu, \qquad \forall u,u',v,v' \in R^d, \qquad t,s \in [0,T], \quad \ell_1^{(\nu)}, \ell_2^{(\nu)} \ge 0.$$

The following theorem establishes the convergence of the NSWR method (2.5) to the fractional linear method (2.1).

Theorem 3.1 *If the functions $\mathcal{G}_\nu$ satisfy (3.1) with $\ell_1^{(\nu)}$ and $\ell_2^{(\nu)}$ uniformly bounded with respect to ν, then there exists $h_0 > 0$ such that, for $h < h_0$,*

$$\lim_{\nu\to\infty} y_n^{(\nu)} = y_n,$$

where $y_n^{(\nu)}$ and y_n are computed respectively through the methods (2.5) and (2.1). The convergence is ensured in the norm for which (3.1) holds.

Proof:
Let us put $e_n^{(\nu)} =: y_n - y_n^{(\nu)}$ and suppose that the point t_n belongs to the r-th window. Then, by setting $m = s + rb$, there exists an index $1 \le i \le b$ such that the point t_n is of the form $t_n = t_{m+i}$. Subtracting (2.5) from (2.1) and exploiting (2.3) and (3.1) we obtain

$$\|e_{m+i}^{(\nu)}\| \le h^{1-\alpha}\sum_{j=1}^{i} |\omega_{i-j}(\alpha)| \left(\ell_1^{(\nu)}\|e_{m+j}^{(\nu-1)}\| + \ell_2^{(\nu)}\|e_{m+j}^{(\nu)}\|\right).$$

By setting $E_m^{(\nu)} = \left(\|e_{m+1}^{(\nu)}\|, ..., \|e_{m+b}^{(\nu)}\|\right)^T$, $C = \sum_{j=1}^{b} |\omega_{i-j}(\alpha)|$, it follows that

$$\|E_m^{(\nu)}\|_\infty \le h^{1-\alpha} C \left(\ell_1^{(\nu)}\|E_m^{(\nu-1)}\|_\infty + \ell_2^{(\nu)}\|E_m^{(\nu)}\|_\infty\right).$$

Therefore,

$$\left(1 - h^{1-\alpha} C \ell_2^{(\nu)}\right) \|E_m^{(\nu)}\|_\infty \leq h^{1-\alpha} C \ell_1^{(\nu)} \|E_m^{(\nu-1)}\|_\infty.$$

By choosing h such that

$$h < \frac{1}{\left(C \ell_2^{(\nu)}\right)^{\frac{1}{1-\alpha}}} \quad ,$$

we are able to write

$$\|E_m^{(\nu)}\|_\infty \leq \frac{h^{1-\alpha} C \ell_1^{(\nu)}}{1 - h^{1-\alpha} C \ell_2^{(\nu)}} \|E_m^{(\nu-1)}\|_\infty.$$

The convergence is ensured if

$$\frac{h^{1-\alpha} C \ell_1^{(\nu)}}{1 - h^{1-\alpha} C \ell_2^{(\nu)}} < 1 \quad \forall \nu. \tag{3.2}$$

By setting $h_0 = \frac{1}{(C(L_1+L_2))^{\frac{1}{1-\alpha}}}$ and $L_i = \sup_\nu \ell_i^{(\nu)}, i = 1, 2$, the condition (3.2) is satisfied when $h < h_0$.

The convergence theorem of the fractional linear multistep methods given in [21] ensures the following result.

Corollary 3.2 *If the linear multistep method $\omega(\rho, \sigma)$ is implicit, stable and consistent, and all zeros of $\sigma(\zeta)$ are inside the unit disk, under the hypotheses of the Theorem 3.1, the method (2.5) converges to the exact solution of (1.1).*

Proof:
Let us denote as ${\epsilon_n}^\nu := y(t_n) - y_n^{(\nu)}$ the error of the NSWR method after ν iterations and $\delta_n := y(t_n) - y_n$ the error of the method (2.1). The proof immediately follows by observing that ${\epsilon_n}^\nu = {e_n}^\nu + \delta_n$. ∎

3.1 Linear VIEs

For the particular case of time–point NSWR methods (2.6) applied to linear systems with kernel of the form

$$k(t,s,y(s)) = k(t,s)y(s) \tag{3.3}$$

we are able to provide further sufficient conditions for the convergence of the method.

Let us choose the functions $\mathcal{G}_\nu$ preserving the linearity, that is:

$$\mathcal{G}_\nu(t,s,u,v) = M_\nu(t,s)v + N_\nu(t,s)u. \tag{3.4}$$

Then, the condition (2.3) becomes

$$M_\nu(t,s) + N_\nu(t,s) = k(t,s) \tag{3.5}$$

and the time-point NSWR (2.6) is

$$\begin{aligned}
y_n^{(\nu)} &= \mathcal{Q}_n + h^{1-\alpha}\omega_0(\alpha)\left[M_\nu(t_n,t_n)y_n^{(\nu)} + N_\nu(t_n,t_n)y_n^{(\nu-1)}\right], \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (3.6)\\
&\qquad\qquad\qquad\qquad n = s+1,\ldots,\mathrm{N} \quad , \\
&\qquad\qquad\qquad\qquad \nu = 1,\ldots,m_n \quad , \\
y_n^{(0)} &:= \mathcal{Q}_n \quad , \\
\mathcal{Q}_n &= f(t_n) + h^{1-\alpha}\sum_{j=0}^{s} w_{n,j}(\alpha)k(t_n,t_j)y_j + \\
&\quad + h^{1-\alpha}\sum_{j=0}^{n-1}\omega_{n-j}(\alpha)k(t_n,t_j)y_j^{(m_j)} \quad .
\end{aligned}$$

As regards the convergence of this method let us assume that $\|N_\nu(t_n,t_n)\|$ and $\| \left[I - h^{1-\alpha}\omega_0(\alpha)M_\nu\left(t_n,t_n\right)\right]^{-1} \|$ are uniformly bounded with respect to ν, the following theorem can be proved:

Theorem 3.3 *The method (3.6), when $\nu \to \infty$, converges to the*

fractional linear multistep method (2.1) if

$$h < \frac{1}{[\omega_0(\alpha)]^{\frac{1}{1-\alpha}}}.$$

Proof: The thesis immediately follows by observing that for the method (3.6) the error $e_n^{(\nu)} := y_n - y_n^{(\nu)}$ satisfies:

$$e_n^{(\nu)} = \left[h^{1-\alpha}\omega_0(\alpha)\right]^{\nu} \left\{ \prod_{j=1}^{\nu} \left[[I - h^{1-\alpha}\omega_0(\alpha) M_j(t_n, t_n)]^{-1} N_j(t_n, t_n) \right] \right\} e_n^{(0)}.$$

■

4 Fully parallel methods for linear VIEs

In this section we construct fast convergent and fully parallel time–point NSWR methods for linear VIEs.

A fully parallel method can be obtained by choosing, in (3.5), $M_\nu(t, s)$ as a diagonal matrix, in this way the system (3.6), that has order d, is decoupled into d independent equations, that can be solved in parallel.

In particular, let us choose:

$$M_\nu(t_n, t_n) = \mu_\nu^{(n)} \mathbf{I_d}. \tag{4.1}$$

where $\mathbf{I}_d$ is the d-by-d identity matrix, then the corresponding method is said the Richardson method. If $\mu_\nu^{(n)} = 0$, we obtain the functional iteration method.

Let us define, for every n, the following polynomial of degree ν

$$P_\nu^{(n)}(t) := \prod_{k=1}^{\nu} \left(t - \mu_k^{(n)}\right) \tag{4.2}$$

Theorem 4.1 *The error of the non stationary Richardson time–point relaxation method is given by*

$$e_n^{(\nu)} = \frac{P_\nu^{(n)}\left(k(t_n,t_n)\right)}{P_\nu^{(n)}\left(\frac{1}{h^{1-\alpha}\omega_0(\alpha)}\right)} e_n^{(0)} \quad . \tag{4.3}$$

Proof:
Using (4.1) in (3.6), for the error $e_n^{(\nu)}$ it holds:

$$\begin{aligned} e_n^{(\nu)} &= h^{1-\alpha}\omega_0(\alpha)\left[\mathbf{I}_d - h^{1-\alpha}\omega_0(\alpha)\mu_\nu^{(n)}\mathbf{I}_d\right]^{-1}\left[k(t_n,t_n) - \mu_\nu^{(n)}\mathbf{I}_d\right] e_n^{(\nu-1)} \\ &= \frac{h^{1-\alpha}\omega_0(\alpha)}{1 - h^{1-\alpha}\omega_0(\alpha)\mu_\nu^{(n)}}\left[k(t_n,t_n) - \mu_\nu^{(n)}\mathbf{I}_d\right] e_n^{(\nu-1)} \\ &= \frac{\left[k(t_n,t_n) - \mu_\nu^{(n)}\mathbf{I}_d\right]}{\frac{1}{h^{1-\alpha}\omega_0(\alpha)} - \mu_\nu^{(n)}} e_n^{(\nu-1)} \end{aligned}$$

By replying on $e_n^{(\nu-1)}$ we have

$$e_n^{(\nu)} = \frac{\left[k(t_n,t_n) - \mu_\nu^{(n)}\mathbf{I}_d\right]}{\frac{1}{h^{1-\alpha}\omega_0(\alpha)} - \mu_\nu^{(n)}} \frac{\left[k(t_n,t_n) - \mu_{\nu-1}^{(n)} I\right]}{\frac{1}{h^{1-\alpha}\omega_0(\alpha)} - \mu_{\nu-1}^{(n)}} e_n^{(\nu-2)}$$

and so the thesis follows by recursion. ∎

The Theorem 4.1 will be used to construct methods with the optimal convergence rate.

If $k(t_n,t_n)$ has real eigenvalues

$$\lambda_1^{(n)} \le \lambda_2^{(n)} \le \ldots \le \lambda_d^{(n)}$$

then the following theorem holds.

Theorem 4.2 *The non stationary Richardson time–point relaxation NSWR method with $\mu_i^{(n)} = \lambda_i^{(n)}$, $i = 1,\ldots,d$ gives the exact solution in d iterates.*

Proof:

In this case $P_d^{(n)}(t)$ is the characteristic polynomial of $k(t_n, t_n)$. Therefore $P_d^{(n)}\left(k(t_n, t_n)\right) = 0$ and $e_n^{(d)} = 0$ ∎

This theorem looses its interest when the dimension of the system is very large. In this case it is more convenient (sometimes necessary) to accept an approximate solution (up to a prescribed precision) by performing a number of iterations smaller than d. The parameters of the method are then chosen in order to minimize the error (4.3). More precisely, fixed the number m_n of iterations to perform at the point t_n, the optimal parameters $\mu_\nu^{(n)}$ are then determined in order to minimize $\left\| P_{m_n}^{(n)}(k(t_n, t_n) \right\|$. The corresponding method will be said "fast".

The optimal parameters are determined according to the following theorem

Theorem 4.3 *The fast non stationary Richardson time–point method is obtained by choosing:*

$$\mu_i^{(n)} = \frac{\lambda_d^{(n)} - \lambda_1^{(n)}}{2} \cos\left[\frac{(2i-1)\pi}{2m_n}\right] + \frac{\lambda_d^{(n)} + \lambda_1^{(n)}}{2} \quad , \qquad i = 1, \ldots, m_n \quad , \tag{4.4}$$

and the error satisfies

$$\left\| e_n^{(m_n)} \right\| \leq \frac{1}{\left| T_{m_n}\left(\frac{\lambda_d^{(n)} + \lambda_1^{(n)} - \frac{2}{h^{1-\alpha}\omega_0(\alpha)}}{\lambda_d^{(n)} - \lambda_1^{(n)}} \right) \right|} \| e_n^{(0)} \| \quad . \tag{4.5}$$

where $T_{m_n}(x)$ is the Chebyshev polynomial of degree m_n.

Proof:
The proof follows by the known minimax properties of the Chebyshev polynomials, observing that the parameters $\mu_i^{(n)}$ are the zeros of the Chebyshev polynomial shifted in the range $\left[\lambda_1^{(n)}, \lambda_d^{(n)}\right]$ ∎

Remark 4.1 *The error estimate derived in the Theorem 4.3 can be used also to compute a priori how many iterations should be performed in order to achieve a given tolerance.*

Remark 4.2 *Recalling that the Chebyshev polynomial grows quickly outside the range* $[-1,1]$*, the values* $T_{m_n}\left(\frac{\lambda_d^{(n)}+\lambda_1^{(n)}-\frac{2}{h^{1-\alpha}\omega_0(\alpha)}}{\lambda_d^{(n)}-\lambda_1^{(n)}}\right)$ *grow when* $\lambda_1^{(n)}$ *increases and* $\lambda_d^{(n)}-\lambda_1^{(n)}$ *decreases. Thus, from (4.5) we can observe that these methods are especially suitable for systems with large eigenvalues and small spectrum size.*

Corollary 4.1 *If the kernel (3.3) is of convolution type, i.e.* $k(t,s) = k(t-s)$ *then the parameters* $\mu_i^{(n)}$ *are independent of* n *and they are the zeros of the Chebyshev polynomial shifted in the range* $[\lambda_1, \lambda_d]$ *where* λ_1 *and* λ_d *are respectively the minimum and maximum eigenvalue of* $k(0)$*.*

5 Numerical experiments

In the previous sections we introduced the non stationary discrete time WR methods in order to have both fully parallel and fast convergent methods. In this section we analyze the performances of the fully parallel Richardson NSWR method compared to the corresponding stationary method, in order to verify the real improvement in the convergence rate. We recall that the optimal stationary WR (OSWR) method is obtained by setting in (4.1)

$$\mu_\nu^{(n)} = \mu = \frac{\lambda_1 + \lambda_d}{2}$$

according to the results contained in [9].

5.1 The methods

In our experiments we have considered the Richardson time–point NSWR method using fractional linear multistep methods based on backward differentiation formulae of order $p = 2$ and

$p = 3$ for $\alpha = \frac{1}{2}$, denoted by $(\text{BDF2})^{-\frac{1}{2}}$ and $(\text{BDF3})^{-\frac{1}{2}}$. The application of these methods to the equation (1.1) with kernel (3.3) leads to

$$\begin{aligned}
y_n^0 &= F_n \\
y_n^{\nu+1} &= \sqrt{h}\mu_\nu \tilde{\omega}_0 y_n^{\nu+1} + F_n^\nu, \qquad n = 1, \ldots, \mathrm{N}\,, \qquad \nu = 0, 1 \ldots\,, \\
F_n^\nu &= \sqrt{h}\tilde{\omega}_0 \left(\mu_\nu I - k(t_n, t_n)\right) y_n^\nu + F_n\,, \\
F_n &= \sqrt{h} \sum_{j=s+1}^{n-1} \tilde{\omega}_{n-j} k(t_n, t_j) y_j^{m_j} + \sqrt{h} \sum_{j=0}^{s} \tilde{w}_{n,j} k(t_n, t_j) y_j + f(t_n)
\end{aligned}$$

with

$$\begin{aligned}
\tilde{\omega}_n &:= \tilde{\omega}_n\left(\frac{1}{2}\right) = \Gamma\left(\frac{1}{2}\right)\omega_n, \quad n = 0, \ldots, N - s - 1\,, \\
\tilde{w}_{n,j} &:= \tilde{w}_{n,j}\left(\frac{1}{2}\right) = \Gamma\left(\frac{1}{2}\right) w_{n,j}, \quad n = 1, \ldots, N\,, \quad j = 0, \ldots, s\,,
\end{aligned}$$

where ω_n and $w_{n,j}$ are the convolution and the starting quadrature weights of the backward differentiation formulae respectively (see [18] for more details).

5.2 The test examples

The numerical experiments have been carried out on the linear constant coefficient convolution problem

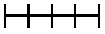
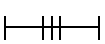
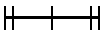
$$y(t) = f(t) + \int_0^t \frac{A \cdot y(s)}{(t-s)^{\frac{1}{2}}} ds \quad , \qquad t \in [0, 1], \qquad A \in R^{20\times 20}. \tag{5.1}$$

Here $f(t) = t$ except for Table 5 where $f(t) = \sqrt{t}$, the stepsize is $h = 10^{-3}$ and in the point t_n the iterates stop when

$$\frac{\left\| y_n^{(\nu)} - y_n^{(\nu-1)} \right\|_\infty}{\left\| y_n^{(\nu)} \right\|_\infty} < 10^{-5}. \tag{5.2}$$

We can deduce from Remark 4.2 that the convergence properties of the NSWR methods depend on the eigenvalues of the matrix A

rather than its entries. For this reason we constructed the matrices A such that they differ one from another for the interval $[a, b]$, on the real axis, containing the spectrum of A and for the position of the eigenvalues in $[a, b]$. We considered three possible distributions of the eigenvalues inside $[a, b]$:

-  linearly spaced between a and b,
-  close to the middle $\frac{a+b}{2}$,
-  close to the edges a and b.

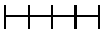
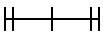
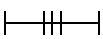
To construct A we have considered, in the linearly spaced case, the diagonally matrix whose eigenvalues are, for $i = 0, \ldots, 19$, $\lambda_i = b + i \cdot l$, $l = -\frac{b-a}{19}$; in the close to the edges case, the eigenvalues are, for $i = 0, \ldots, 9$, $\lambda_i = b + i^2 \cdot l$ and $\lambda_{i+10} = a - i^2 \cdot l$ with $l = -0.4\frac{b-a}{11^2}$; in the close to the middle case, the eigenvalues are $\lambda_0 = b$, $\lambda_{19} = a$ and, for $i = 0, \ldots, 8$, $\lambda_{i+1} = b - .49(b-a) - i^2 \cdot l$ and $\lambda_{i+10} = a + .49(b-a) + i^2 \cdot l$, $l = -.2(b-a)/7^2$.

5.3 The numerical results

The Tables from 1 to 3 show the results obtained with the method $(\text{BDF2})^{-\frac{1}{2}}$. Here, we report, in the first column, the interval $[a, b]$ and in the second column the position of the eigenvalues in $[a, b]$. In the last two columns, we report the number of the iterations, n_{fNS}, needed by the fast NSWR Richardson method to satisfy (5.2) and the number of iterations, n_{oST}, required by the OSWR Richardson one.

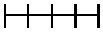
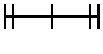
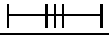
The Tables 4 and 5 show the results obtained with the method $(\text{BDF3})^{-\frac{1}{2}}$.

Tab. 1

Eigenv. interval	*Eigenv. position*	n_{fNS}	n_{oST}
$[-2,-1]$		3	5
		3	5
		3	5

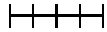
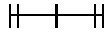
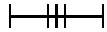
In this test, where the spectrum of the eigenvalues of A is contained in a small and close to the origin interval $[a,b]$, the fast NSWR method is always better than the OSWR one. We can note that the position of the eigenvalues in $[a,b]$ does not influence the rate of convergence.

Tab. 2

Eigenv. interval	*Eigenv. position*	n_{fNS}	n_{oST}
$[-100,-1]$		20	43
		18	72
		18	33

In this test, where the spectrum is large, the fast NSWR method is much better than the OSWR one: it goes from a good improvement when the eigenvalues are close to the middle of $[a,b]$, or linearly spaced (the rate of convergence is almost twice), to an excellent one when they are close to the edges (in this case the improvement is almost three times as much). The relative position of the eigenvalues in $[a,b]$ influences only the convergence of the OSWR.

Tab. 3

Eigenv. interval	*Eigenv. position*	n_{fNS}	n_{oST}
$[-100, -90]$		3	5
		3	5
		4	6

Both methods have good performances when the spectrum is quite small but not close to the origin. However, the fast NSWR method has an average improvement of the 35–40%. The relative position of the eigenvalues has just a small influence on both methods.

Tab. 4

Eigenv. interval	n_{fNS}	n_{oST}
$[-10, -1]$	5	6
$[-100, -90]$	3	3
$[-100, -10]$	10	17
$[-1000, -100]$	13	25
$[-10000, -100]$	13	25

Tab. 5

Eigenv. interval	n_{fNS}	n_{oST}
$[-10, -1]$	7	9
$[-100, -90]$	3	5
$[-100, -10]$	13	25
$[-1000, -100]$	17	34
$[-10000, -100]$	18	39

The tests reported in Tables 4–5 differ only from the forcing function $f(t)$. From these tables we can observe that the fast NSWR method is always better than the OSWR one and we can observe that, if the number of iterations of the fast NSWR method grows, then the number of iterations of the OSWR method grows in an equal or greater amount (in $[-10, -1]$ we pass from 5 iterations of the non stationary versus 6 of the stationary one to 7 versus 9; in $[-10000, -100]$ from 13 versus 25 to 18 versus 39).

Moreover, we can observe that the NSWR method shows the best performances, in the absolute sense, when the spectrum is quite small but not close to the origin (when the eigenvalues of A are in $[-100, -90]$ only 3 iterates are needed for the required accuracy). On the contrary, the relative (i.e. non stationary versus stationary) best performances occur if the spectrum is quite large (when the eigenvalues are in $[-10000, -100]$ and $[-1000, -100]$ the gain is of about the 50%).

Remark 5.1 *The efficient implementation of the fast NSWR Richardson method into a parallel algorithm requires some expedients as for example:*

- *the construction of a dynamical integration window in order to allow a balance between the convergence rate and the communication exchange among the processors,*
- *the optimal ordering of the parameters of the Richardson method in order to effectively reduce the error in the first iterates.*

The detailed description of the development of a parallel code based on the fast NSWR Richardson method is contained in [6].

5.4 Conclusions

We have done numerical experiments in order to evaluate the performances of the fast time–point NSWR methods versus the OSWR ones. The results of these tests and the theoretical results of Theorem 4.3 permit us to affirm that the fast NSWR methods

have always the best performances. These are, in absolute sense, if the spectrum size of the matrix A is quite small and, with an equal size of the spectrum, if the spectrum is not close to the origin. On the contrary, even if the number of iterations is quite high, we have the best improvements of the convergence rate respect to the optimal stationary method, when A has a large spectrum and the eigenvalues are close to its edges.

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