Multistep Collocation Methods for Volterra Integral Equations

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Abstract

We introduce multistep collocation methods for the numerical integration of Volterra Integral Equations, which depend on the numerical solution in a fixed number of previous time steps. We describe the constructive technique, analyze the order of the resulting methods and their linear stability properties. Numerical experiments confirm the theoretical expectations.

Key words: Numerical methods for Volterra Integral Equations, Multistep collocation methods, Multistep Runge–Kutta methods.
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1 Introduction

Volterra Integral Equations (VIEs) are models of evolutionary problems with memory arising in many applications (see [2,3] and related bibliography), and have the form

$$y(t) = g(t) + \int_0^t k(t, \tau, y(\tau))d\tau \quad t \in I := [0, T], \tag{1.1}$$

where $k \in C(D \times \mathbb{R})$, with $D := \{(t, \tau) : 0 \leq \tau \leq t \leq T\}$, and $g \in C(I)$.

In the literature many authors (see [2,3] and references therein contained) have analyzed one–step collocation methods for VIEs. As it is well known, a

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collocation method is based on the idea of approximating the exact solution of a given integral equation with a suitable function belonging to a chosen finite dimensional space, usually a piecewise algebraic polynomial which satisfies the integral equation exactly on a certain subset of the integration interval (called the set of collocation points).

In this paper we propose a new extension to the multistep case, for which a first analysis appeared in [7]. More precisely here we derive a general class of multistep collocation methods, which depend on a fixed number r of previous time steps, with the aim of increasing the order of classical one–step collocation methods without increasing the computational cost. In analogy to the case of Ordinary Differential Equations (ODEs) [8,10–12], we construct multistep collocation methods by adding interpolation conditions in the previous r step points. We prove that while classical one–step collocation methods have uniform order m for any choice of the collocation parameters and local superconvergence order in the mesh points of $2m-2$ (Gauss and Lobatto points) or $2m-1$ (Radau II points) [3], the r–steps m–points collocation methods have uniform order $m+r$, and order of local superconvergence $2m+r-1$.

After recalling in Section 2 the one–step collocation methods for VIEs, in Section 3 we describe the construction of the new multistep collocation methods. In Section 4 we determine the order of convergence and superconvergence of the new methods and in Section 5 we analyze the linear stability, by providing the recurrence relation and the stability matrix. In Section 6 we show numerical experiments which confirm the theoretical expectations.

2 Collocation methods for VIEs

For sake of completeness we recall the basic material on collocation methods for VIEs [2,3]. Let us discretize the interval I by introducing a uniform mesh

$$I_h = \{t_n := nh, n = 0, ..., N,\ \ h \geq 0, Nh = T\}\,.$$

The equation (1.1) can be rewritten, by relating it to this mesh, as

$$y(t) = F_n(t) + \Phi_n(t) \qquad t \in [t_n, t_{n+1}],$$

where $F_n(t) := g(t) + \int_0^{t_n} k(t, \tau, y(\tau))d\tau$ and $\Phi_n(t) := \int_{t_n}^{t} k(t, \tau, y(\tau))d\tau$ represent respectively the *lag term* and the *increment function*. Let us fix m collocation parameters $0 \leq c_1 < ... < c_m \leq 1$ and denote by $t_{n,j} = t_n + c_j h$ the collocation points. The collocation polynomial, restricted to the interval

$[t_n, t_{n+1}]$, is of the form:

$$u_n(t_n + sh) = \sum_{j=1}^{m} L_j(s) U_{n,j} \quad s \in [0,1] \quad n = 0, ..., N-1 \tag{2.1}$$

where $L_j(s)$ is the $j-th$ Lagrange fundamental polynomial with respect to the collocation parameters and $U_{n,j} := u_n(t_{n,j})$. *Exact* collocation methods are obtained by imposing that the collocation polynomial (2.1) exactly satisfies the VIE (1.1) in the collocation points $t_{n,i}$ and by computing $y_{n+1} = u_n(t_{n+1})$:

$$\begin{cases} U_{n,i} = F_{n,i} + \Phi_{n,i} \\ y_{n+1} = \sum\limits_{j=1}^{m} L_j(1) U_{n,j} \end{cases}, \tag{2.2}$$

where

$$F_{n,i} = g(t_{n,i}) + h \sum_{\nu=0}^{n-1} \int_0^1 k(t_{n,i}, t_\nu + sh, u_\nu(t_\nu + sh)) ds \quad i = 1, ..., m \tag{2.3}$$

$$\Phi_{n,i} = h \int_0^{c_i} k(t_{n,i}, t_n + sh, u_n(t_n + sh)) ds \quad i = 1, ..., m. \tag{2.4}$$

Note that the first equation in (2.2) represents a system of m nonlinear equations in the m unknowns $U_{n,i}$. We obtain an approximation $u(t)$ of the solution $y(t)$ of the integral equation (1.1) in $[0, T]$, by considering

$$u(t)|_{(t_n, t_{n+1}]} = u_n(t) \tag{2.5}$$

where $u_n(t)$ given by (2.1). We recall that generally $u(t)$ is not continuous in the mesh points, as

$$u(t) \in S_{m-1}^{(-1)}(I_h), \tag{2.6}$$

where

$$S_\mu^{(d)}(I_h) = \left\{ v \in C^d(I) : v|_{(t_n, t_{n+1}]} \in \Pi_\mu \ (0 \le n \le N-1) \right\}.$$

Here, Π_μ denotes the space of (real) polynomials of degree not exceeding μ.

The classical collocation methods have uniform order m for any choice of the collocation parameters, and can achieve local superconvergence in the mesh points by opportunely choosing the collocation parameters, as stated by the following theorem [3].

Theorem 2.1 *Suppose that the given functions describing the VIE (1.1) satisfy $k \in C^{(m)}(D)$, $g \in C^{(m)}(I)$. Then, for any choice of the collocation parameters $0 \le c_1 < ... < c_m \le 1$, the error $\varepsilon(t) = u(t) - y(t)$ satisfies*

$$\|\varepsilon\|_\infty = O(h^m).$$

Suppose moreover that $k \in C^{(2m-v)}(D)$, $g \in C^{(2m-v)}(I)$, *for some* $v \in \{0,1,2\}$, *then*

- *If the collocation parameters are the Radau II points for* $(0,1]$ *we have, for* $v = 1$,
$$\max_{n=0,\ldots,N} |\varepsilon(t_n)| = O(h^{2m-1})$$
- *If the collocation parameters are* m *Lobatto points for* $[0,1]$ *or* $m-1$ *Gauss-Legendre points for* $[0,1]$ *with* $c_m = 1$, *we have, for* $v = 2$,
$$\max_{n=0,\ldots,N} |\varepsilon(t_n)| = O(h^{2m-2})$$

Discretized collocation methods are determined by using suitable quadrature formulas $\bar{F}_{n,i} \simeq F_{n,i}$ and $\bar{\Phi}_{n,i} \simeq \Phi_{n,i}$ for approximating the lag term (2.3) and the increment function (2.4). Such methods preserve, under suitable hypothesis on the quadrature formulas, the same order of the exact collocation methods [3].

3 Construction of the multistep collocation method

The multistep collocation methods are obtained by introducing in the collocation polynomial the dependence from r previous time steps; namely we seek for a collocation polynomial, whose restriction to the interval $[t_n, t_{n+1}]$ takes the form

$$u_n(t_n + sh) = \sum_{k=0}^{r-1} \varphi_k(s) y_{n-k} + \sum_{j=1}^{m} \psi_j(s) U_{n,j} \quad s \in [0,1] \quad n = r, ..., N-1 \tag{3.1}$$

where again

$$U_{n,j} := u_n(t_{n,j}) \tag{3.2}$$

and $\varphi_k(s)$, $\psi_j(s)$ are polynomials of degree $m + r - 1$ to be determined by imposing the interpolation conditions at the points t_{n-k}, that is $u_n(t_{n-k}) = y_{n-k}$, and by satisfying (3.2). The method is then constructed by imposing the collocation conditions, which will be described in the following. The starting values $y_1, y_2, \ldots, y_r$, needed in (3.1), may be obtained by using a suitable starting procedure, based on a classical one step method.

The interpolation conditions at t_{n-k}, $k = 0, ..., r-1$, together with the condition (3.2), lead to the following linear system:

$$\begin{aligned} &\varphi_l(-k) = \delta_{lk}, \quad \varphi_l(c_j) = 0, \qquad l, k = 0, ..., r-1 \\ &\psi_i(-k) = 0, \qquad \psi_i(c_j) = \delta_{ij}, \qquad i, j = 1, ..., m. \end{aligned} \tag{3.3}$$

The (3.3) represents a linear system of $(r+m)^2$ equations where the $(r+m)^2$ unknowns are the coefficients of the polynomials $\varphi_k(s)$ and $\psi_j(s)$, considering the c_j, $j = 1, ..., m$ as fixed parameters.

Remark 3.1 *Assuming that $c_i \neq c_j$ and $c_1 \neq 0$, then the unique solution of the system (3.3) assumes the form:*

$$\begin{aligned} \varphi_k(s) &= \prod_{i=1}^{m} \tfrac{s-c_i}{-k-c_i} \cdot \prod_{\substack{i=0 \\ i\neq k}}^{r-1} \tfrac{s+i}{-k+i}, \\ \psi_j(s) &= \prod_{i=0}^{r-1} \tfrac{s+i}{c_j+i} \cdot \prod_{\substack{i=1 \\ i\neq j}}^{m} \tfrac{s-c_i}{c_j-c_i}. \end{aligned} \tag{3.4}$$

The expression (3.4) follows immediately by observing that $\{\varphi_k(s), \psi_j(s)\}$ defined by mean of (3.3) represent the fundamental Lagrange polynomials with respect to the nodes $\{-k, c_j | k = 0, ..., r-1, j = 1, ..., m\}$.

The exact multistep collocation method is then obtained by imposing the collocation conditions, i.e. that the collocation polynomial (3.1) exactly satisfies the VIE (1.1) at the collocation points $t_{n,i}$, and by computing $y_{n+1} = u_n(t_{n+1})$:

$$\begin{cases} U_{n,i} = F_{n,i} + \Phi_{n,i} \\ y_{n+1} = \sum_{k=0}^{r-1} \varphi_k(1) y_{n-k} + \sum_{j=1}^{m} \psi_j(1) U_{n,j} \end{cases}, \tag{3.5}$$

where the lag–term $F_{n,i}$ and increment–term $\Phi_{n,i}$ are given by (2.3) and (2.4) respectively. The discretized multistep collocation method is then obtained by using suitable quadrature formulas for the lag–term and the increment–term approximation. The discretized multistep collocation polynomial, denoted by $P_n(t)$, is then of the form

$$P_n(t_n + sh) = \sum_{k=0}^{r-1} \varphi_k(s) y_{n-k} + \sum_{j=1}^{m} \psi_j(s) Y_{n,j} \quad s \in [0,1] \quad n = 0, ..., N-1 \tag{3.6}$$

where the functions $\varphi_k(s)$ and $\psi_j(s)$ are given by (3.4), and $Y_{n,j} := P_n(t_{n,j})$ are determined by the solution of the following nonlinear system

$$\begin{cases} Y_{n,i} = \bar{F}_{n,i} + \bar{\Phi}_{n,i} \\ y_{n+1} = \sum_{k=0}^{r-1} \varphi_k(1) y_{n-k} + \sum_{j=1}^{m} \psi_j(1) Y_{n,j} \end{cases}. \tag{3.7}$$

The lag–term and increment–term approximations

$$\bar{F}_{n,i} = g(t_{n,i}) + h \sum_{\nu=0}^{n-1} \sum_{l=0}^{\mu_1} b_l k(t_{n,i}, t_\nu + \xi_l h, P_\nu(t_\nu + \xi_l h)) \quad i = 1, ..., m \tag{3.8}$$

$$\bar{\Phi}_{n,i} = h\sum_{l=0}^{\mu_0} w_{il}k(t_{n,i}, t_n + d_{il}h, P_n(t_n + d_{il}h)) \quad i = 1, ..., m \tag{3.9}$$

are obtained by using quadrature formulas of the form

$$(\xi_l, b_l)_{l=1}^{\mu_1}, \quad (d_{il}, w_{il})_{l=1}^{\mu_0}, \ i = 1, ..., m, \tag{3.10}$$

where the quadrature nodes ξ_l and d_{il} satisfy $0 \leq \xi_1 < ... < \xi_{\mu_1} \leq 1$ and $0 \leq d_{i1} < ... < d_{i\mu_0} \leq 1$, μ_0 and μ_1 are positive integers and w_{il}, b_l are suitable weights.

Remark 3.2 *The discretized multistep collocation method (3.6)-(3.7) provides a continuous approximation $P(t)$ of the solution $y(t)$ of the integral equation (1.1) in $[0, T]$, by considering*

$$P(t)|_{(t_n, t_{n+1}]} = P_n(t) \tag{3.11}$$

where $P_n(t)$ is given by (3.6). We note that usually the polynomial constructed in the collocation methods for VIEs doesn't interpolate the numerical solution in the previous step points, resulting a discontinuous approximation of the solution (2.6). In this new extension, the collocation polynomial is instead a continuous approximation to the solution, i.e. $u(t) \in S_{m-1}^{(0)}(I_h)$, as a result of the application of our technique.

Remark 3.3 *The discretized multistep collocation method (3.6)-(3.7) can be regarded as a multistep Runge–Kutta method for VIEs:*

$$\begin{cases} Y_{n,i} = \bar{F}_n(t_{n,i}) + h\sum\limits_{l=1}^{\mu_0} w_{il}k\left(t_n + e_{il}h, \ t_n + d_{il}h, \ \sum\limits_{k=0}^{r-1}\gamma_{ilk}y_{n-k} + \sum\limits_{j=1}^{m}\beta_{ilj}Y_{n,j}\right) \\ y_{n+1} = \sum\limits_{k=0}^{r-1}\theta_k y_{n-k} + \sum\limits_{j=1}^{m}\lambda_j Y_{n,j} \end{cases}, \tag{3.12}$$

where

$$\bar{F}_n(t) = g(t) + h\sum_{\nu=0}^{n-1}\sum_{l=1}^{\mu_1} b_l k\left(t, \ t_\nu + \xi_l h, \ \sum_{k=0}^{r-1}\delta_{lk}y_{\nu-k} + \sum_{j=1}^{m}\eta_{lj}Y_{\nu,j}\right) \tag{3.13}$$

and

$$\begin{aligned} &e_{il} = c_i, \ \gamma_{ilk} = \varphi_k(d_{il}), \ \beta_{ilj} = \psi_j(d_{il}), \\ &\theta_k = \varphi_k(1), \ \lambda_j = \psi_j(1), \\ &\delta_{lk} = \varphi_k(\xi_l), \ \eta_{lj} = \psi_j(\xi_l). \end{aligned}$$

Many multistep Runge–Kutta methods for ODEs have already appeared in the literature also in the context of General Linear Methods for ODEs (see [9] and the related bibliography). At the best of our knowledge (3.12)-(3.13) represent the first version of multistep Runge-Kutta methods for VIEs.

The reason of interest of the constructed methods lies in the fact that they increase the order of convergence of collocation methods without increasing the computational cost, except for the cost due to the starting procedure. As a matter of fact, in advancing from t_n to t_{n+1}, we make use of the approximations y_{n-k}, $k = 0, ..., r-1$, which have already been evaluated at the previous steps. This permits to increase the order, as we will show in the next section, by maintaining in (3.5) or (3.7) the same dimension m of the nonlinear system (2.2).

4 Determination of the order

In this section we will analyze the order of convergence and superconvergence of the exact multistep collocation method (3.1)-(3.5) and of the discretized multistep collocation method (3.6)-(3.7).

Lemma 4.1 *Let us consider the linear VIE*

$$y(t) = g(t) + \int_0^t k(t,\tau)y(\tau)d\tau \quad t \in I, \tag{4.1}$$

with $y \in \mathbb{R}$, $k \in C^{(m+r)}(D)$, $g \in C^{(m+r)}(I)$. Then, for any choice of distinct collocation abscissas $0 < c_1 < ... < c_m \leq 1$, the exact solution $y(t)$ of the VIE (4.1) satisfies

$$y(t_n + sh) = \sum_{k=0}^{r-1} \varphi_k(s)y(t_{n-k}) + \sum_{j=1}^{m} \psi_j(s)y(t_{n,j}) + h^{m+r}R_{m,r,n}(s), \quad s \in [0,1], \tag{4.2}$$

where the functions $\varphi_k(s)$ and $\psi_j(s)$ are given by (3.4), and

$$R_{m,r,n}(s) = \int_{-r+1}^{1} K_{m,r}(s,\nu)y^{(m+r)}(t_n + \nu h)d\nu$$

$$K_{m,r}(s,\nu) = \frac{1}{(m+r+1)!}\left\{(s-\nu)_+^{m+r-1} - \sum_{k=0}^{r-1}\varphi_k(s)(-k-\nu)_+^{m+r-1} - \sum_{j=1}^{m}\psi_j(s)(c_j-\nu)_+^{m+r-1}\right\}.$$

Proof. The hypothesis $k \in C^{(m+r)}(D)$, $g \in C^{(m+r)}(I)$ assure that $y \in C^{(m+r)}(I)$. By Peano theorems for interpolation [2], it follows that the thesis is true for $s \in [-r+1, 1]$. ■

In the following theorem we prove that the r-step m-point exact collocation method (3.1)-(3.5) has uniform convergence order of at least $p = m + r$, for any choice of distinct collocation abscissas $0 < c_1 < ... < c_m \leq 1$..

Theorem 4.2 *Let $\varepsilon(t) = y(t) - u(t)$ be the error of the exact collocation method (3.1)-(3.5) and $p = m + r$. Suppose that*

i. *the given functions describing the VIE (1.1) satisfy $k \in C^{(p)}(D \times \mathbb{R})$, $g \in C^{(p)}(I)$.*
ii. *the starting error is $\|\varepsilon\|_{\infty,[0,t_r]} = O(h^p)$.*
iii. *$\rho(\mathbf{A}) < 1$, where*

$$\mathbf{A} = \left[\begin{array}{c|c} \mathbf{0}_{r-1,1} & \mathbf{I}_{r-1} \\ \hline \varphi_{r-1}(1) & \varphi_{r-2}(1), ..., \varphi_0(1) \end{array}\right] \tag{4.3}$$

and ρ denotes the spectral radius.

Then

$$\|\varepsilon\|_\infty = O(h^{m+r}).$$

Proof. We will carry out the proof in the case of a linear VIE (4.1). The proof can be straightforwardly extended to the case of a nonlinear VIE (1.1) by using the mean value theorem (see [2,3,6] for more details).

By Lemma 4.1 we have

$$y(t_n+sh) = \sum_{k=0}^{r-1} \varphi_k(s) y(t_{n-k}) + \sum_{j=1}^{m} \psi_j(s) y(t_{n,j}) + h^p R_{m,r,n}(s) \quad s \in [0,1]. \tag{4.4}$$

It follows, by subtracting (4.4) and (3.1), that the exact collocation error $\varepsilon(t)$ has the local representation

$$\varepsilon(t_n + sh) = \sum_{k=0}^{r-1} \varphi_k(s) \varepsilon_{n-k} + \sum_{j=1}^{m} \psi_j(s) \varepsilon_{n,j} + h^p R_{m,r,n}(s) \quad n \geq r \tag{4.5}$$

with $\varepsilon_{n-k} = \varepsilon(t_{n-k})$, $\varepsilon_{n,j} = \varepsilon(t_{n,j})$. The first equation in (3.5), together with (2.3)-(2.4) with linear kernel $k(t,\tau,y) = k(t,\tau)y$, and (3.1)-(3.2), lead to

$$u_n(t_{n,i}) = g(t_{n,i}) + h \sum_{l=0}^{n-1} \int_0^1 k(t_{n,i}, t_l+sh) u_l(t_l+sh) ds + h \int_0^{c_i} k(t_{n,i}, t_n+sh) u_n(t_n+sh) ds. \tag{4.6}$$

By evaluating (4.1) for $t = t_{n,i}$, by splitting the integral over $[0, t_{n,i}]$ in the sum of the integrals in the subintervals $[t_l, t_{l+1}]$, $l = 0, \ldots, n-1$, $[t_n, t_{n,i}]$ and by considering the change of variable $\tau = t_l + sh$, $l = 0, \ldots, n$ in each of them, we obtain

$$y(t_{n,i}) = g(t_{n,i}) + h \sum_{l=0}^{n-1} \int_0^1 k(t_{n,i}, t_l+sh) y(t_l+sh) ds + h \int_0^{c_i} k(t_{n,i}, t_n+sh) y(t_n+sh) ds. \tag{4.7}$$

By subtracting (4.6) from (4.7), we get

$$\varepsilon_{n,i} = h\sum_{l=0}^{n-1}\int_0^t k(t_{n,i}, t_l + sh)\varepsilon(t_l + sh)ds + h\int_0^{c_i} k(t_{n,i}, t_n + sh)\varepsilon(t_n + sh)ds. \tag{4.8}$$

By the hypothesis on the starting error it follows that

$$\varepsilon(t_l + sh) = h^p q_l(s) \quad l = 0, ..., r-1 \quad s \in [0,1] \tag{4.9}$$

with $\|q_l\|_\infty \leq C_1$ independent of h. By substituting the expressions (4.5) and (4.9) in the equation (4.8) we obtain

$$\epsilon_n^{(2)} - h\tilde{\mathbf{B}}_n\epsilon_n^{(2)} = h\sum_{l=r}^{n-1}\tilde{\mathbf{B}}_n^{(l)}\epsilon_l^{(2)} + h\sum_{l=r}^{n}\bar{\mathbf{B}}_n^{(l)}\epsilon_l^{(1)} + h^{p+1}\sum_{l=0}^{n}\bar{\rho}_n^{(l)} \quad n \geq r \tag{4.10}$$

where $\epsilon_l^{(1)} \in \mathbb{R}^r$, $\epsilon_l^{(2)}, \bar{\rho}_n^{(l)} \in \mathbb{R}^m$, $\bar{\mathbf{B}}_n^{(l)} \in \mathbb{R}^{m\times r}$, $\tilde{\mathbf{B}}_n, \tilde{\mathbf{B}}_n^{(l)} \in \mathbb{R}^{m\times m}$ are defined as

$$\epsilon_l^{(1)} = [\varepsilon_{l-r+1}, ..., \varepsilon_l]^T \quad \epsilon_l^{(2)} = [\varepsilon_{l,1}, ..., \varepsilon_{l,m}]^T$$

$$\left(\bar{\mathbf{B}}_n^{(l)}\right)_{ik} = \begin{cases} \int_0^1 k(t_{n,i}, t_l + sh)\varphi_k(s)ds, & l = r, r+1, ..., n-1, \\ \int_0^{c_i} k(t_{n,i}, t_n + sh)\varphi_k(s)ds, & l = n, \end{cases}$$

$$\left(\tilde{\mathbf{B}}_n\right)_{ij} = \int_0^{c_i} k(t_{n,i}, t_n + sh)\psi_j(s)ds, \quad \left(\tilde{\mathbf{B}}_n^{(l)}\right)_{ij} = \int_0^1 k(t_{n,i}, t_l + sh)\psi_j(s)ds$$

$$\left(\bar{\rho}_n^{(l)}\right)_i = \begin{cases} \int_0^1 k(t_{n,i}, t_l + sh)q_l(s)ds, & l = 0, ..., r-1, \\ \int_0^1 k(t_{n,i}, t_l + sh)R_{m,r,l}(s)ds, & l = r, ..., n-1, \\ \int_0^{c_i} k(t_{n,i}, t_n + sh)R_{m,r,n}(s)ds, & l = n. \end{cases}$$

Expression (4.5) with $l-1$ in place of n and $s = 1$ leads to

$$\epsilon_l^{(1)} = \mathbf{A}\epsilon_{l-1}^{(1)} + \mathbf{S}\epsilon_{l-1}^{(2)} + h^p\tilde{\rho}_{m,r,l-1}, \quad l \geq r \tag{4.11}$$

where $\mathbf{A}$ is given by (4.3),

$$\mathbf{S} = \begin{bmatrix} \mathbf{0}_{r-1,m} \\ \psi^T(1) \end{bmatrix}, \quad \tilde{\rho}_{m,r,j} = \begin{bmatrix} \mathbf{0}_{r-1,1} \\ R_{m,r,j}(1) \end{bmatrix},$$

$$\psi(1) = [\psi_0(1), ..., \psi_m(1)]^T.$$

The solution of the difference equation (4.11) is

$$\epsilon_l^{(1)} = \mathbf{A}^{l-r+1}\epsilon_{r-1}^{(1)} + \sum_{j=r-1}^{l-1}\mathbf{A}^{l-j-1}\left(\mathbf{S}\epsilon_j^{(2)} + h^p\tilde{\rho}_{m,r,j}\right), \tag{4.12}$$

which, inserted in (4.10), leads to

$$
\begin{aligned}
(I - h\tilde{\mathbf{B}}_n)\epsilon_n^{(2)} = h\sum_{l=r}^{n-1}\tilde{\mathbf{B}}_n^{(l)}\epsilon_l^{(2)} + h\sum_{j=r}^{n-1}\left(\sum_{l=j+1}^{n}\bar{\mathbf{B}}_n^{(l)}\mathbf{A}^{l-j-1}\mathbf{S}\right)\epsilon_j^{(2)} + \\
+ h\left(\sum_{l=r}^{n}\bar{\mathbf{B}}_n^{(l)}\mathbf{A}^{l-r+1}\right)\epsilon_{r-1}^{(1)} + h\left(\sum_{l=r}^{n}\bar{\mathbf{B}}_n^{(l)}\mathbf{A}^{l-r}\mathbf{S}\right)\epsilon_{r-1}^{(2)} \qquad (4.13)\\
+ h^{p+1}\sum_{j=r-1}^{n-1}\left(\sum_{l=j+1}^{n}\bar{\mathbf{B}}_n^{(l)}\mathbf{A}^{l-j-1}\right)\tilde{\rho}_{m,r,j} + h^{p+1}\sum_{l=0}^{n}\bar{\rho}_n^{(l)} \quad n \geq r
\end{aligned}
$$

We have from (4.9) that

$$
\left\|\epsilon_{r-1}^{(1)}\right\|_1 \leq rC_1h^p, \quad \left\|\epsilon_{r-1}^{(2)}\right\|_1 \leq mC_1h^p.
$$

By setting

$$
\begin{array}{ll}
\left\|\bar{\mathbf{B}}_n^{(l)}\right\|_1 \leq D_1, & \left\|\tilde{\mathbf{B}}_n^{(l)}\right\|_1 \leq D_2,\\
\left\|\left(I - h\tilde{\mathbf{B}}_n\right)^{-1}\right\|_1 \leq D_0, & \|\mathbf{S}\|_1 \leq D_3,\\
M_{m,r} = \left\|y^{(m+r)}\right\|_\infty & K_{m,r} := \max\limits_{s\in[0,1]}\int_{-r+1}^{1}|K_{m,r}(s,\nu)|\,d\nu\\
\bar{K} = \max\limits_{t\in I}\int_0^t |k(t,\tau)|\,d\tau & \alpha_{m,r} = m\bar{K}K_{m,r}M_{m,r}\\
\beta_{m,r} = K_{m,r}M_{m,r}, & \gamma = m\bar{K}C_1,
\end{array}
$$

where D_i, $i = 0, 1, 2, 3$ are constants, we obtain

$$
\left\|\bar{\rho}_n^{(l)}\right\|_1 \leq \begin{cases} \gamma & l = 0, ..., r-1 \\ \alpha_{m,r} & l = r, ..., n \end{cases}, \quad \|\tilde{\rho}_{m,r,j}\|_1 \leq \beta_{m,r}.
$$

Moreover, since $\rho(\mathbf{A}) < 1$, there exists a constant D_4 such that

$$
\sum_{l=0}^{k}\left\|\mathbf{A}^l\right\|_1 \leq D_4,
$$

independently of $k \in \mathbb{N}$.

Then, from (4.13),

$$
\left\|\epsilon_n^{(2)}\right\|_1 \leq h\gamma_1\sum_{l=r}^{n-1}\left\|\epsilon_l^{(2)}\right\|_1 + \gamma_2h^p,
$$

with $\gamma_1 = D_0D_2 + D_0D_1D_3D_4$, $\gamma_2 = D_0D_1D_4C_1(r + mD_3)h + r\gamma h + T(\alpha_{m,r} + D_0D_1D_4\beta_{m,r})$.The above generalised discrete Gronwall inequality leads to the estimate

$$
\left\|\epsilon_n^{(2)}\right\|_1 \leq B_2h^p,
$$

with $B_2 = \gamma_2 e^{T\gamma_1}$, and then, from (4.12),

$$\left\|\epsilon_n^{(1)}\right\|_1 \leq B_1 h^p,$$

with $B_1 = rC_1D_4 + D_4D_3B_2 + D_4\beta_{m,r}$.

Using the local error representation (4.5) the two above inequalities yield

$$|\varepsilon(t_n + sh)| \leq \Lambda_{m,r}(\left\|\epsilon_n^{(1)}\right\|_1 + \left\|\epsilon_n^{(2)}\right\|_1) + h^p K_{m,r}M_{m,r} \leq C_2 h^p,$$

with $C_2 = \Lambda_{m,r}(B_2 + B_1) + K_{m,r}M_{m,r}$, uniformly for $s \in [0,1]$ and $n \geq r$, where

$$\Lambda_{m,r} := \max\left\{\|\varphi_k\|_\infty, \|\psi_j\|_\infty,\ k = 0, ..., r-1,\ j = 1, ..., m\right\}.$$

This, together to the expression (4.9) for the starting error, is equivalent to the estimate

$$\|\varepsilon\|_\infty \leq Ch^p, \tag{4.14}$$

with $C = \max\{C_1, C_2\}$. ■

Remark 4.3 *We will show in Section 5 (see Remark 5.3) that the hypothesis iii. of Theorem 4.2 is equivalent to the zero-stability of the multistep collocation method.*

The following theorem provides a condition on the collocation parameters in order to obtain local superconvergence in the mesh points.

Theorem 4.4 *Let us suppose that*

- *the hypothesis of the Theorem 4.2 hold with* $p = 2m + r - 1$.
- *the collocation parameters* $c_1, ..., c_m$ *are the solution of the system*

$$\begin{cases} c_m = 1 \\ \frac{1}{i+1} - \sum\limits_{k=0}^{r-1} \beta_k(-k)^i - \sum\limits_{j=1}^{m} \gamma_j (c_j)^i = 0 \quad i = m+r, ..., 2m+r-2 \end{cases} \tag{4.15}$$

with

$$\beta_k = \int_0^1 \varphi_k(s)ds, \quad \gamma_j = \int_0^1 \psi_j(s)ds \tag{4.16}$$

then

$$\max_{n=0,...,N} |\varepsilon(t_n)| = O(h^{2m+r-1}).$$

Proof. We will carry out the proof in the case of a linear VIE (4.1). The proof can be immediately generalised to the case of nonlinear VIEs by employing

the Taylor's theorem with quadratic remainder term (see [2] pag. 115). Since $u(t)$ satisfies the integral equation at the collocation points, we have

$$u(t) = g(t) + \int_0^t k(t,\tau)u(\tau)d\tau - \delta(t) \quad t \in I,$$

with $\delta(t_{n,i}) = 0$. By subtracting (4.1) we obtain

$$\varepsilon(t) = \delta(t) + \int_0^t k(t,\tau)\varepsilon(\tau)d\tau \quad t \in I,$$

whose solution is

$$\varepsilon(t) = \delta(t) + \int_0^t R(t,\tau)\delta(\tau)d\tau \quad t \in I,$$

where $R(t,\tau)$ denotes the resolvent kernel for $k(t,\tau)$. It follows that the error in a mesh point t_n assumes the form

$$\varepsilon(t_n) = \delta(t_n) + h\sum_{\nu=0}^{n-1}\int_0^1 R(t_n, t_\nu + sh)\delta(t_\nu + sh)ds. \tag{4.17}$$

Let us consider the quadrature formula

$$\int_0^1 f(s)ds \approx \sum_{k=0}^{r-1}\beta_k f(-k) + \sum_{j=1}^{m}\gamma_j f(c_j) \tag{4.18}$$

for the computation of the integrals in (4.17), where β_k and γ_j are given by (4.16). By denoting with $E_{n,\nu}$ the corresponding error terms we obtain:

$$\varepsilon(t_n) = \delta(t_n) + h\sum_{\nu=0}^{n-1}\left[\sum_{k=0}^{r-1}\beta_k R(t_n, t_{\nu-k})\delta(t_{\nu-k}) + \sum_{j=1}^{m}\gamma_j R(t_n, t_{\nu,j})\delta(t_{\nu,j})\right] + h\sum_{\nu=0}^{n-1}E_{n,\nu}.$$

The hypothesis $c_m = 1$ assures that $t_{\nu-k}$ are collocation points for each ν. Since the defect function vanishes in the collocation points, we have $\delta(t_{\nu-k}) = 0$, $\delta(t_{\nu,j}) = 0$, hence

$$\max_{n=0,\ldots,N}|\varepsilon(t_n)| = \max_{n=0,\ldots,N}\left|h\sum_{\nu=0}^{n-1}E_{n,\nu}\right| \leq T\max_{0\leq\nu\leq n-1\leq N-1}|E_{n,\nu}|.$$

Thus the order of $\varepsilon(t_n)$ coincides with that of the error term $E_{n,\nu}$ associated with the quadrature formula (4.18). Such quadrature formula has degree of precision of at least $m + r - 1$ by construction of the polynomials φ_k and ψ_j. The condition (4.15) on the collocation parameters $c_1, ..., c_m$ assures that the quadrature formula (4.18) has infact degree of precision of at least $p = 2m + r - 2$, i.e. $|E_{n,\nu}| = O(h^{2m+r-1})$ (by Peano Theorem 8.8.4 in [2]). Then the thesis immediately follows. ■

The following theorem shows that, as in the exact case, the r-step m-point discretized collocation method (3.6)-(3.7) has convergence order of at least $p = m+r$, for any choice of distinct collocation abscissas $0 < c_1 < ... < c_m \leq 1$.

Theorem 4.5 *Let $e(t) := y(t) - P(t)$ be the error of the discretized collocation method (3.6)-(3.7) and let $p = m + r$. Suppose that*

i. *the given functions describing the VIE (1.1) satisfy $k \in C^{(p)}(D)$, $g \in C^{(p)}(I)$;*
ii. *the lag–term and increment–term quadrature formulas (3.10) are of order respectively at least $p + 1$ and p;*
iii. *the starting error is $\|e\|_{\infty,[0,t_r]} = O(h^p)$.*
iv. *$\rho(\mathbf{A}) < 1$, where A is given by (4.3).*

Then

$$\|e\|_\infty = O(h^{m+r}).$$

Proof. The thesis follows from Theorem 4.2 by observing that $\|e\|_\infty \leq \|\varepsilon\|_\infty + \|u - P\|_\infty$ and by the hypothesis on the order of the lag-term and increment-term quadrature formulas. ■

We obtain an analogous result concerning the local superconvergence:

Theorem 4.6 *Let us suppose that*

- *the hypothesis of the Theorem 4.5 hold with $p = 2m + r - 1$.*
- *the collocation parameters $c_1, ..., c_m$ are the solution of the system (4.15).*

Then

$$\max_{n=0,...,N} |e(t_n)| = O(h^{2m+r-1}).$$

Proof. The thesis follows from Theorem 4.4 by observing that $|e(t_n)| \leq |\varepsilon(t_n)| + |u(t_n) - P(t_n)|$ and by the hypothesis on the order of the lag-term and increment-term quadrature formulas. ■

Remark 4.7 *In order to obtain superconvergence, in (3.8)-(3.9)-(3.10) we can choose $\mu_1 = m + \left\lceil \frac{r-1}{2} \right\rceil$, $\mu_0 = m - 1 + \left\lceil \frac{r}{2} \right\rceil$, the abscissas ξ_l as μ_1 Gauss-Legendre nodes in $[0, 1]$ and the abscissas d_{il} as μ_0 Gauss-Legendre nodes in the interval $[0, c_i]$.*

Table 1 gives a comparison, in terms of computational cost, between classical superconvergent one–step gaussian collocation methods and our new superconvergent multistep collocation methods based on Gauss-Legendre quadrature formulas for lag–term and increment–function approximation. The values reported in the table hold when r is odd, and analogous results hold when it is even. The table shows (see columns 2 and 4) that, if we consider the same

number m of collocation points, and hence the same dimension of the nonlinear system, then the r–step m–point collocation methods have higher order, i.e. order $2m + r - 1$ instead of $2m - 2$, which is the order of the classical one–step gaussian method. Analogously (compare columns 3 and 4 in Table 1), to obtain classical collocation methods having the same order $2m + r - 1$ of r–step m–point collocation methods, we have to use more collocation parameters and hence increase to $m + \frac{r-1}{2} + 1$ the dimension of the nonlinear system.

Table 1
Comparison with classical gaussian methods

	Classic one–step		Multistep
Number of collocation points	m	$m + \frac{r-1}{2} + 1$	m
dimension of nonlinear system	m	$m + \frac{r-1}{2} + 1$	m
number of lag-term quadrature nodes	m-1	$m + \frac{r-1}{2}$	$m + \frac{r-1}{2}$
order	$2m - 2$	$2m + r - 1$	$2m + r - 1$

Example 4.8 *Let us consider the case $m = 1$, $r = 3$ and denote by c the collocation abscissa. The polynomials φ_k, $k = 0, 1, 2$ and ψ_1, obtained by (3.4), have the following expression:*

$$\begin{aligned} \varphi_0(s) = \tfrac{(c-s)(2+3s+s^2)}{2c}, \quad \varphi_1(s) = -\tfrac{s(c-s)(2+s)}{1+c}, \\ \varphi_2(s) = \tfrac{s(c-s)(1+s)}{2(2+c)}, \quad \psi_1(s) = \tfrac{s(2+3s+s^2)}{c(2+3c+c^2)}. \end{aligned} \tag{4.19}$$

By choosing in (3.10) the $2-$points Gauss-Legendre quadrature formulas in $[0, 1]$ and in $[0, c_i]$, we obtain a one-stage method of order 4 for any choice of c. In this case the order of superconvergence is 4 and coincides with the order of uniform convergence. In the following section we will show how to choose the collocation abscissa c in order to heighten the linear stability properties of the method.

Example 4.9 *Let us consider the case $m = 2$, $r = 3$. The polynomials φ_k, $k = 0, 1, 2$ and ψ_j, $j = 1, 2$ obtained by (3.4), have the following expression:*

$$\begin{aligned} \varphi_0(s) = \tfrac{(c_1-s)(c_2-s)(2+3s+s^2)}{2c_1c_2}, \quad \varphi_1(s) = -\tfrac{s(c_1-s)(c_2-s)(2+s)}{(1+c_1)(1+c_2)}, \\ \varphi_2(s) = \tfrac{s(c_1-s)(c_2-s)(1+s)}{2(2+c_1)(2+c_2)}, \\ \psi_1(s) = \tfrac{s(c_2-s)(2+3s+s^2)}{c_1(2+3c_1+c_1^2)(c_2-c_1)}, \quad \psi_2(s) = \tfrac{s(c_1-s)(2+3s+s^2)}{c_2(2+3c_2+c_2^2)(c_1-c_2)}. \end{aligned} \tag{4.20}$$

and we obtain a two-stage method of order 5 *for any choice of* $\{c_1, c_2\}$*. The system of equations (4.15) reduces to:*

$$\begin{cases} c_2 = 1 \\ 76 - 97c_2 - 97c_1 + 135c_1c_2 = 0 \end{cases},$$

with solution $\{c_1, c_2\} = \left\{\frac{21}{38}, 1\right\}$*. The resulting method is of mesh order* $p = 2m + r - 1 = 6$*, if we consider in (3.10) the* $3-$*points Gauss-Legendre quadrature formulas in* $[0, 1]$ *and in* $[0, c_i]$*.*

5 Linear stability analysis

In this section we will study the stability properties of the multistep collocation method with respect to the basic test equation

$$y(t) = 1 + \lambda \int_0^t y(\tau)d\tau \quad t \in [0, T], \quad Re(\lambda) \leq 0, \tag{5.1}$$

usually employed in literature for the stability analysis of numerical methods for VIEs (see for example [3,1] and their references). Since the exact solution $y(t)$ of (5.1) tends to zero when t goes to $+\infty$, it is natural to require that the numerical solution has the same behaviour. We recall (see [1]) that a numerical method is said to be stable for given $z := h\lambda \in C$ if the numerical solution y_n, obtained by applying the method to the test equation (5.1) with fixed stepsize h, tends to zero when $n \to +\infty$. The region of absolute stability of the method is the set of all values $z \in C$ for which the above condition is satisfied. Furthermore the method is said A-stable if its region of absolute stability includes the negative complex half plane.

Let us define

$$\Omega_{ik} = \int_0^{c_i} \varphi_k(s)ds, \quad \rho_{ij} = \int_0^{c_i} \psi_j(s)ds$$
$$\beta_k = \int_0^1 \varphi_k(s)ds, \quad \gamma_j = \int_0^1 \psi_j(s)ds,$$

and introduce the vectors

$$\mathbf{U}_n = [U_{n,1}, ..., U_{n,m}]^T, \quad \mathbf{y}_n^{(r)} = [y_n, ..., y_{n-r+1}]^T,$$

$$\beta = [\beta_0, ..., \beta_{r-1}]^T \quad \gamma = [\gamma_1, ..., \gamma_m]^T, \quad \mathbf{u} = [1, ..., 1]^T \in \mathbb{R}^m,$$
$$\psi(1) = [\psi_1(1), ..., \psi_m(1)]^T, \quad \varphi(1) = [\varphi_0(1), ..., \varphi_{r-1}(1)]^T,$$

and the matrices

$$\mathbf{\Omega} = (\Omega_{ik}) \in \mathbb{R}^{m\times r}, \quad \rho = (\rho_{ij}) \in \mathbb{R}^{m\times m}, \quad \mathbf{E} = \left[\begin{array}{c} -\psi(1)^T \\ \hline \mathbf{0}_{r,m} \end{array}\right],$$

$$\mathbf{F} = \left[\begin{array}{c|c} \mathbf{0}_{1,r} & 0 \\ \hline \mathbf{I}_r & \mathbf{0}_{r,1} \end{array}\right], \quad \mathbf{G} = \left[\begin{array}{c|c} 1 & -\varphi(1)^T \\ \hline \mathbf{0}_{r,1} & \mathbf{I}_r \end{array}\right].$$

Theorem 5.1 *The exact multistep collocation method (3.1)-(3.5), applied to the test equation (5.1), leads to the following recurrence relation*

$$\begin{bmatrix} y_{n+1} \\ \mathbf{y}_n^{(r)} \\ \mathbf{U}_n \end{bmatrix} = \mathbf{R}(z) \begin{bmatrix} y_n \\ \mathbf{y}_{n-1}^{(r)} \\ \mathbf{U}_{n-1} \end{bmatrix} \tag{5.2}$$

where the stability matrix is given by

$$\mathbf{R}(z) = [\mathbf{Q}(z)]^{-1}\, \mathbf{M}(z) \tag{5.3}$$

with

$$\mathbf{Q}(z) = \left[\begin{array}{c|c|c} \mathbf{0}_{m,1} & -z\mathbf{\Omega} & \mathbf{I}_m - z\ \rho \\ \hline \multicolumn{2}{c|}{\mathbf{G}} & \mathbf{E} \end{array}\right]$$

$$\mathbf{M}(z) = \left[\begin{array}{c|c|c} \mathbf{0}_{m,1} & z(\mathbf{u}\beta^T - \mathbf{\Omega}) & z(\mathbf{u}\gamma^T - \rho) \\ \hline \multicolumn{2}{c|}{\mathbf{F}} & \mathbf{0}_{r+1,m} \end{array}\right].$$

Proof. By applying the method (3.1)-(3.5) to the test equation (5.1) we obtain

$$y_{n+1} = \varphi(1)^T \mathbf{y}_n^{(r)} + \psi(1)^T \mathbf{U}_n \tag{5.4}$$

where

$$\mathbf{U}_n = \bar{\mathbf{F}}_n + z\left(\mathbf{\Omega}\mathbf{y}_n^{(r)} + \rho\mathbf{U}_n\right), \tag{5.5}$$

$$\bar{\mathbf{F}}_n = \mathbf{u} + z\sum_{\nu=0}^{n-1}(\mathbf{b}^T\mathbf{y}_\nu^{(r)} + \gamma^T\mathbf{U}_\nu)\mathbf{u}. \tag{5.6}$$

From the expression (5.6) we derive

$$\bar{\mathbf{F}}_n - \bar{\mathbf{F}}_{n-1} = z(\mathbf{b}^T\mathbf{y}_{n-1}^{(r)} + \gamma^T\mathbf{U}_{n-1})\mathbf{u}. \tag{5.7}$$

The computation of the difference $\mathbf{U}_n - \mathbf{U}_{n-1}$ by substituting the expression (5.5) for both terms $\mathbf{U}_n$ and $\mathbf{U}_{n-1}$, and by using (5.7), leads to

$$(\mathbf{I}_m - z\rho)\,\mathbf{U}_n - z\mathbf{\Omega}\mathbf{y}_n^{(r)} = z(\mathbf{u}\beta^T - \mathbf{\Omega})\mathbf{y}_{n-1}^{(r)} + z(\mathbf{u}\gamma^T - \rho)\mathbf{U}_{n-1}.$$

From (5.4) it follows that

$$\mathbf{G}\begin{bmatrix} y_{n+1} \\ \mathbf{y}_n^{(r)} \end{bmatrix} + \mathbf{E}\mathbf{U}_n = \mathbf{F}\begin{bmatrix} y_n \\ \mathbf{y}_{n-1}^{(r)} \end{bmatrix}.$$

Thus we obtain

$$\mathbf{Q}(z)\begin{bmatrix} y_{n+1} \\ \mathbf{y}_n^{(r)} \\ \mathbf{U}_n \end{bmatrix} = \mathbf{M}(z)\begin{bmatrix} y_n \\ \mathbf{y}_{n-1}^{(r)} \\ \mathbf{U}_{n-1} \end{bmatrix}$$

and the thesis follows. ■

Let us define

$$\tilde{\Omega}_{ik} = \sum_{l=1}^{\mu_0} \omega_{il}\varphi_k(d_{il}), \quad \tilde{\rho}_{ij} = \sum_{l=1}^{\mu_0} \omega_{il}\psi_j(d_{il}),$$
$$\tilde{\beta}_k = \sum_{l=1}^{\mu_1} b_l\varphi_k(\xi_l), \quad \tilde{\gamma}_j = \sum_{l=1}^{\mu_1} b_l\psi_j(\xi_l),$$

and introduce the vectors

$$\mathbf{Y}_n = [Y_{n,1}, ..., Y_{n,m}]^T$$
$$\tilde{\beta} = [\tilde{\beta}_0, ..., \tilde{\beta}_{r-1}]^T \quad \tilde{\gamma} = [\tilde{\gamma}_1, ..., \tilde{\gamma}_m]^T,$$

and the matrices

$$\tilde{\mathbf{\Omega}} = (\tilde{\Omega}_{ik}) \in \mathbb{R}^{m\times r}, \quad \tilde{\rho} = (\tilde{\rho}_{ij}) \in \mathbb{R}^{m\times m}.$$

Theorem 5.2 *The discretized multistep collocation method (3.6)-(3.7), applied to the test equation (5.1), leads to the following recurrence relation*

$$\begin{bmatrix} y_{n+1} \\ \mathbf{y}_n^{(r)} \\ \mathbf{Y}_n \end{bmatrix} = \mathbf{R}(z)\begin{bmatrix} y_n \\ \mathbf{y}_{n-1}^{(r)} \\ \mathbf{Y}_{n-1} \end{bmatrix} \tag{5.8}$$

where the stability matrix is given by

$$\mathbf{R}(z) = \left[\tilde{\mathbf{Q}}(z)\right]^{-1}\tilde{\mathbf{M}}(z) \tag{5.9}$$

with

$$\tilde{\mathbf{Q}}(z) = \left[\begin{array}{c|c|c} \mathbf{0}_{m,1} & -z\tilde{\mathbf{\Omega}} & \mathbf{I}_m - z\ \tilde{\rho} \\ \hline \multicolumn{2}{c|}{\mathbf{G}} & \mathbf{E} \end{array}\right]$$

$$\tilde{\mathbf{M}}(z) = \left[\begin{array}{c|c|c} \mathbf{0}_{m,1} & z(\mathbf{u}\tilde{\beta}^T - \tilde{\mathbf{\Omega}}) & z(\mathbf{u}\tilde{\gamma}^T - \tilde{\rho}) \\ \hline \multicolumn{2}{c|}{\mathbf{F}} & \mathbf{0}_{r+1,m} \end{array}\right].$$

Proof. The proof is analogous to that of Theorem 5.1. ■

Remark 5.3 *By analyzing the stability behaviour of the exact or discretized multistep collocation method when $z = h\lambda = 0$, from (5.3) or (5.9) we get*

$$\mathbf{R}(0) = \left[\begin{array}{cc} \varphi^T(1) & \mathbf{0}_{1,m+1} \\ \mathbf{I}_r & \mathbf{0}_{r,m+1} \\ \mathbf{0}_{m,r} & \mathbf{0}_{m,m+1} \end{array}\right],$$

with $\varphi(1) = [\varphi_0(1), ..., \varphi_{r-1}(1)]^T$, for both exact and discretized collocation methods. An easy computation shows that

$$\rho\left(\mathbf{R}(0)\right) = \rho\left(\mathbf{A}\right),$$

where the matrix $\mathbf{A}$ is defined by (4.3). Thus the zero-stability is equivalent to the condition iii. of Theorem 4.2.

If the eigenvalues of the stability matrix $\mathbf{R}(z)$ are within the unit circle, then the multistep collocation method is stable. The region of absolute stability of the method is thus the set $\mathcal{S} := \{z \in C : |eig(\mathbf{R}(z))| < 1\}$. . The method is A–stable if $\mathcal{S} \supseteq \{z \in C : Re(z) < 0\}$.

In Figure 1 we report the stability region of the method with $m = 1$ and $r = 3$ described in Example 4.8, with collocation parameter $c = 1$. By drawing the stability regions for different values of the abscissa c, we observed that the stability region is larger when we increase c, thus the choice $c = 1$ guarantees the largest stability region. In Figures 2-3 we report the stability regions of the method with $m = 2$ and $r = 3$ described in Example 4.9, with collocation parameters $\{c_1 = \frac{21}{38}, c_2 = 1\}$ and $\{c_1 = 0.7, c_2 = 1\}$ respectively. In particular we note that the stability region of the superconvergent method in Figure 2 is bounded, while the stability region in Figure 3 is unbounded. The parameters used in Figure 3 have been found through an extensive symbolic-numerical search.

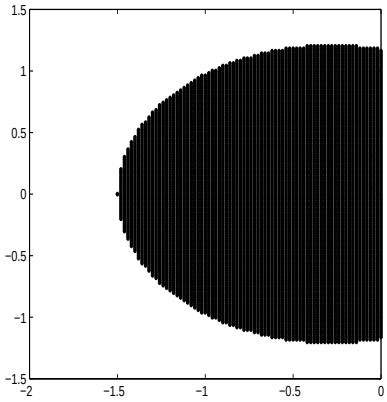

Fig. 1. Stability region with $m = 1$, $r = 3$, $c = 1$.

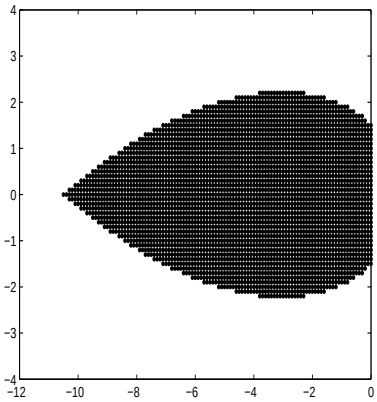

Fig. 2. Stability region with $m = 2$, $r = 3$, $c_1 = \frac{21}{38}$, $c_2 = 1$ (superconvergence).

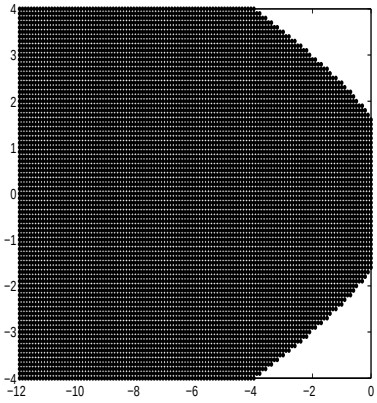

Fig. 3. Stability region with $m = 2$, $r = 3$, $c_1 = 0.7$, $c_2 = 1$.

6 Numerical Experiments

In this section numerical experiments will be carried out in order to validate the order proved in Section 4 and to compare the performances of the new

methods with respect to the classical collocation methods. The methods have been implemented in MATLAB and we report the numerical results on the following three test problems:

- the linear VIE

$$y(t) = e^t + \int_0^t 2\cos(t-\tau)y(\tau)d\tau \quad t \in [0,10], \tag{6.1}$$

  with exact solution $y(t) = e^t(1+t^2)$;
- the nonlinear VIE

$$y(t) = 1 + \sin^2(t) - \int_0^t 3\sin(t-\tau)y^2(\tau)d\tau, \quad t \in [0,10], \tag{6.2}$$

  with exact solution $y(t) = cos(t)$;
- the nonlinear VIE

$$y(t) = 2 - \cos(t) - \int_0^t \sin(ty(\tau)-\tau)d\tau, \quad t \in [0,5], \tag{6.3}$$

  with exact solution $y(t) \equiv 1$.

We consider the multistep collocation methods with $m = 2, r = 3$, given by Example 4.9 with the following choices of the collocation abscissas: $\{c_1 = 0.7, c_2 = 1\}$ and $\left\{c_1 = \frac{21}{38}, c_2 = 1\right\}$. The methods have respectively order $p = 5$ and $p = 6$ (superconvergence). The starting values have been obtained from the known exact solutions.

The accuracy is defined by the number of correct significant digits cd at the end point (the maximal absolute end point error is written as 10^{-cd}). The order of the method is estimated with the formula $p(h) = \frac{cd(h)-cd(2h)}{log_{10}2}$ for a fixed h.

The results listed in Tables 2-3 clearly show that our methods produce the expected order ($N = \frac{T}{h}$ represents the number of mesh points in the integration interval).

Table 2
Estimated order - Problem (6.1)

N		32	64	128	256	512	1024
$c_1 = 0.7$	cd	3, 73	5, 20	6, 69	8, 19	9, 69	11, 19
$c_2 = 1$	$p(10/N)$		4, 90	4, 95	4, 97	4, 99	4, 99
$c_1 = \frac{21}{38}$	cd	4, 62	6, 36	8, 14	9, 93	11, 73	13, 56
$c_2 = 1$	$p(10/N)$		5, 80	5, 90	5, 95	5, 97	6, 09

Table 3
Estimated order - $c_1 = \frac{21}{38}$, $c_2 = 1$

N		16	32	64	128	256
Problem (6.2)	cd	3,53	5,48	7,36	9,18	11,00
	$p(10/N)$		6,49	6,22	6,07	6,02
Problem (6.3)	cd	7,54	9,34	11,15	12,96	14,78
	$p(5/N)$		5,99	6,00	6,02	6,04

Now we compare, in terms of computational cost, the performances of the new multistep collocation methods with respect to the classical one–step gaussian collocation methods. Namely we consider the methods:

- NewSuperconv2: superconvergent multistep collocation method of Example 4.9 with $m = 2$, $r = 3$ and $\left\{c_1 = \frac{21}{38}, c_2 = 1\right\}$, having order $p = 6$;
- ClassicSuperconv4: superconvergent classical gaussian one-step collocation method with the same order $p = 6$, needing $m = 4$ collocation parameters;
- ClassicSuperconv2: superconvergent classical gaussian one-step collocation method with the same number $m = 2$ of collocation parameters, having order $p = 2$.

We report in Figures 4-6 the number of kernel evaluations with respect to the correct significant digits for the three test problems. As we expected from Table 1, the figures show that the improvement in efficiency of multistep collocation method is higher when problem is strongly nonlinear, since in this case the reduction of the dimension of the nonlinear system considerably reduces the computational cost.

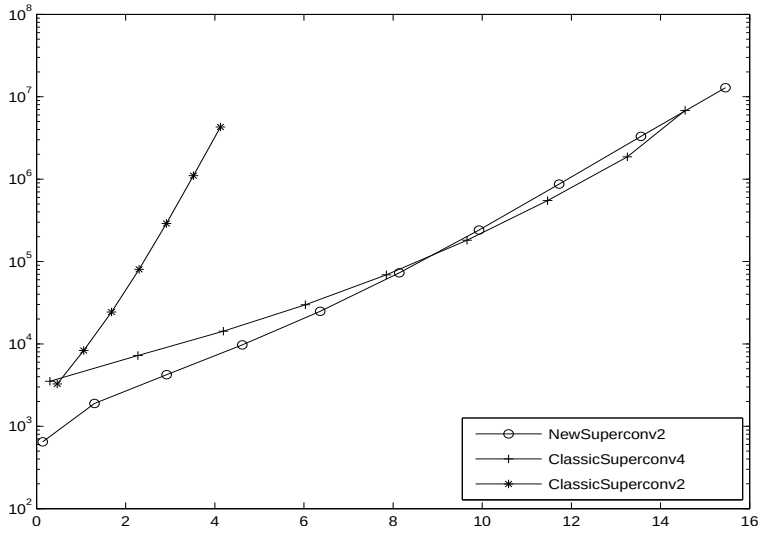

Fig. 4. Number of kernel evaluations for (6.1)

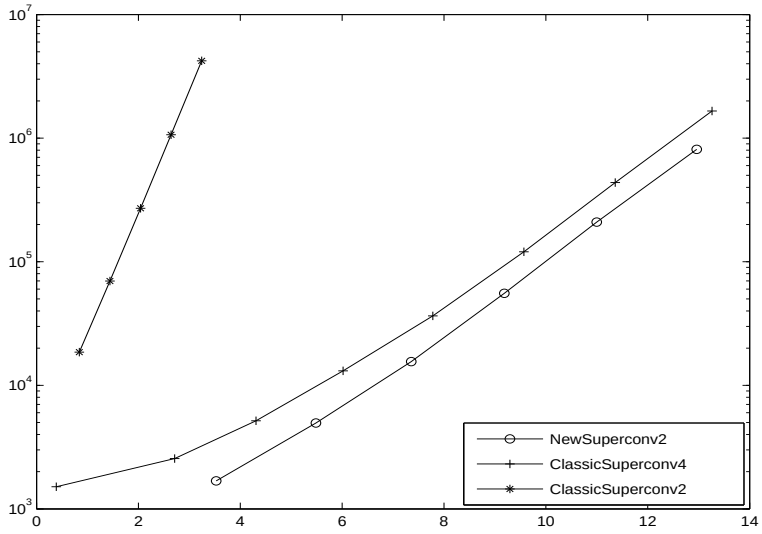

Fig. 5. Number of kernel evaluations for (6.2)

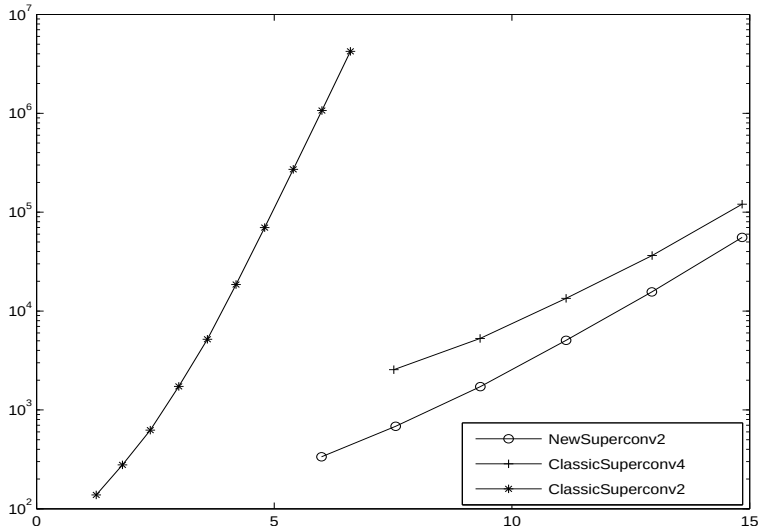

Fig. 6. Number of kernel evaluations for (6.3)

7 Conclusions

In this paper we extend in a multistep manner the collocation technique for the numerical solution of VIEs, up to now considered only in the context of one–step methods. The introduction of previous step points allows to heighten the order of the resulting methods. We didn't find A–stable methods within this class, but wide stability regions exist. We think that the knowledge of the collocation polynomial, which provides a continuous approximation of uniform order of the solution, will allow a cheap variable stepsize implementation. Indeed, when the stepsize changes, the new approximation values can be computed by simply evaluating the collocation polynomial, without running into problems of order reduction, as a consequence of the uniform order.

In [6] we introduce a modification in the technique, thus obtaining two-step *almost* collocation methods, also for systems of VIEs, by relaxing some of the collocation conditions and by introducing some previous stage values, in order to further increase the order and to have free parameters in the method, to be used to get A-stability.

The good behaviour in terms of accuracy and efficiency makes these methods interesting for a further development, starting from the analysis of multistep Runge-Kutta methods of the form (3.12)-(3-13), apart from the original idea of collocation, in order to further decrease the computational cost of the non-linear system in the unknowns $Y_{n,j}$ at each time step.

When we need to solve an integral equation on a long time interval, the lag–term computation can be very expensive, and despite of the reduction of the nonlinear system's dimension, our multistep collocation methods may still be not sufficiently efficient. In this case, for special kernels (i.e. Hammerstein convolution kernels) we can think about employing our multistep collocation methods together with techniques for fast lag–term computation, already used in the case of one-step collocation and Runge–Kutta methods for VIEs [5,4,13].

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