MODELLING THE HYPERBOLOID ELASTIC SHELL

Carlo CATTANI[1], Paul E. STERIAN[2]

În articol se abordează ecuațiile de stare fără momente ale unei pânze elastice hiperboloidale care este obținută ca o suprafață de revoluție. Sunt cercetate condițiile de compatibilitate care sunt utilizate ulterior pentru obținerea unei clase de soluții analitice. Modelele pe calculator prezentate ale problemelor corespunzătoare cu condiții inițiale şi de frontieră date evidențiază unele efecte neliniare cum ar fi curbarea suprafeței de undă care se propagă.

This paper deals with the momentless state equations of a hyperboloid elastic shell which is obtained as a surface of revolution. The compatibility conditions are investigated and used to derive a class of analytical solutions. A computer model of given initial-boundary problems show some nonlinear effects such as the bending of the wave propagation surface.

Keywords: hyperboloid elastic shell, momentless state equations

1. Introduction

The theory of shells has been deeply investigated in the last 50 years in many different fields from theoretical to applicative ones: Engineering applications, radio towers, biological, bio-mechanics etc. [1-10].

The main directions refer to the geometrical structure of the shell: of revolution (cylinder, ellipse, spherical, hyperboloid), with small defects etc. Some attention was paid to cylindrical shells (see e.g. [11, 17] and the references there) while hyperboloid shells were very little investigated.

Hyperboloid shells have some different behaviour respect to other revolution surface shells, but the most important is that they are more stable under infinitesimal variation of the metrics like e.g. light wind load. Therefore they have some applications in cooling towers (for nuclear plants).

In the following we will consider the fundamental elastic equations on the hyperboloid shell and by using the compatibility conditions we will define a class of solutions [12-16]. The solutions will be investigated by adding some initial-boundary conditions and some nonlinear effects of the wave propagation will be shown by using computer modelling.

[1]Prof., diFarma, University of Salerno, Via Ponte Don Melillo, 84084 Fisciano (SA) Italy, ccattani@unisa.it.
[2] Prof., Physics Department, University POLITEHNICA of Bucharest, Romania, e-mail: sterian@physics.pub.ro., Academy of Romanian Scientists, www.aosromania.ro

2. Momentless state equations of a hyperboloid shell

In the following the fundamental equations of an elastic shell having the form of a hyperboloid revolution surface are considered. The shell is assumed to be unbounded (with respect to the symmetry axis) and the equations are referred to a local frame made by the normal (to surface) vector n and two tangent vectors t_1, t_2 and a local system of coordinates (Fig. 1): $0 \le \tau < 2\pi$, $-\pi \le \theta < \pi$.

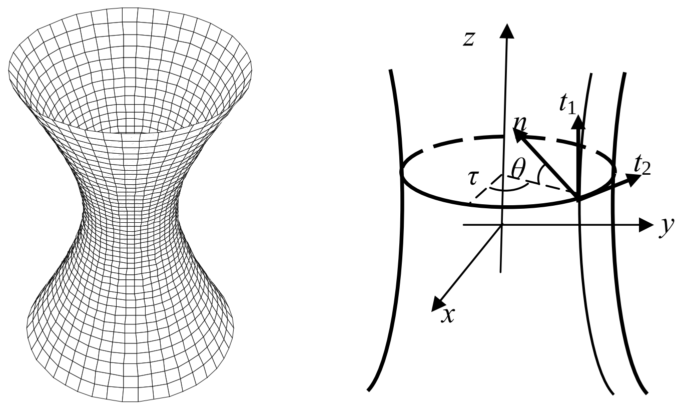

Fig. 1. Hyperboloid (left) with the local frame and coordinates on its surface.

The equations of equilibrium of the hyperboloid shell in the momentless theory of shells are [16-18]

$$\begin{cases} \dfrac{1}{R_1}\dfrac{\partial T_1}{\partial\theta} + \dfrac{\mathrm{ctg}\,\theta}{R_2}(T_1 - T_2) + \dfrac{1}{R_2\sin\theta}\dfrac{\partial S}{\partial\tau} + q_1 & = & 0 \\ \dfrac{1}{R_1}\dfrac{\partial S}{\partial\theta} + 2\dfrac{\mathrm{ctg}\,\theta}{R_2}S + \dfrac{1}{R_2\sin\theta}\dfrac{\partial T_2}{\partial\tau} + q_2 & = & 0 \\ \dfrac{T_1}{R_1} + \dfrac{T_2}{R_2} & = & q_n \end{cases} \tag{2.1}$$

where R_1, R_2 (both negative) are the main Gaussian curvatures, T_1, T_2, S are the tangent and normal components of stress, q_1, q_2, q_n are superficial loadings on the tangent and normal directions respectively.

By neglecting T_2 and defining

$$U = T_1 R_2 \sin^2\theta \quad , \quad V = S R_2^2 \sin^2\theta \tag{2.2}$$

we obtain the equivalent system

$$\begin{cases} \dfrac{R_2{}^2 \sin\theta}{R_1} \dfrac{\partial U}{\partial \theta} + \dfrac{\partial V}{\partial \tau} + q_1 = (q_n \cos\theta - q_1 \sin\theta) R_2^3 \sin^2\theta \\ \dfrac{\partial V}{\partial \theta} + \dfrac{R_2}{\sin\theta} \dfrac{\partial U}{\partial \tau} = -(q_2 \sin\theta + \dfrac{\partial q_n}{\partial \tau}) R_1 R_2^2 \sin\theta \end{cases} \tag{2.3}$$

Or by assuming

$$q_n = Const. \tag{2.4}$$

we have

$$\begin{cases} \dfrac{R_2{}^2 \sin\theta}{R_1} \dfrac{\partial U}{\partial \theta} + \dfrac{\partial V}{\partial \tau} + q_1 = (q_n \cos\theta - q_1 \sin\theta) R_2^3 \sin^2\theta \\ \dfrac{\partial V}{\partial \theta} + \dfrac{R_2}{\sin\theta} \dfrac{\partial U}{\partial \tau} = -q_2 R_1 R_2^2 \sin^2\theta \end{cases} \tag{2.5}$$

The solution of this system in the special case that the unknown functions are assumed to be periodic could be found by using the Galerkin method and Fourier expansions.

In the more general case when the initial conditions are localized or not periodic the solution could be easily obtain by the Galerkin method and the wavelet series expansions [18].

3. Compatibility conditions and analytical solutions

From (2.3) it is

$$\begin{cases} \dfrac{\partial V}{\partial \tau} = -\dfrac{R_2{}^2}{R_1} \sin\theta \dfrac{\partial U}{\partial \theta} + \left(q_n \cos\theta - q_1 \sin\theta\right) R_2^3 \sin^2\theta - q_1 \\ \dfrac{\partial V}{\partial \theta} = -\dfrac{R_2}{\sin\theta} \dfrac{\partial U}{\partial \tau} - \left(q_2 \sin\theta + \dfrac{\partial q_n}{\partial \tau}\right) R_1 R_2^2 \sin\theta \end{cases}$$

and by further derivatives,

$$\begin{cases} \dfrac{\partial^2 V}{\partial \theta\, \partial \tau} = -\dfrac{R_2{}^2}{R_1} \cos\theta \dfrac{\partial U}{\partial \theta} - \dfrac{R_2{}^2}{R_1} \sin\theta \dfrac{\partial^2 U}{\partial \theta^2} + \left(q_n \cos\theta - q_1 \sin\theta\right) R_2^3 \sin 2\theta \\ \qquad\qquad - \left(q_n \sin\theta + q_1 \cos\theta\right) R_2^3 \sin^2\theta \\ \dfrac{\partial^2 V}{\partial \tau \partial \theta} = -\dfrac{R_2}{\sin\theta} \dfrac{\partial^2 U}{\partial \tau^2} - \dfrac{\partial^2 q_n}{\partial \tau^2} R_1 R_2^2 \sin\theta \end{cases}$$

so that for the integrability conditions on $V(\theta)$ there follows:

$$\frac{R_2{}^2}{R_1}\cos\theta\frac{\partial U}{\partial\theta}+\frac{R_2{}^2}{R_1}\sin\theta\frac{\partial^2 U}{\partial\theta^2}-\left(q_n\cos\theta-q_1\sin\theta\right)R_2^3\sin 2\theta$$
$$+\left(q_n\sin\theta+q_1\cos\theta\right)R_2^3\sin^2\theta$$
$$=\frac{R_2}{\sin\theta}\frac{\partial^2 U}{\partial\tau^2}+\frac{\partial^2 q_n}{\partial\tau^2}R_1R_2^2\sin\theta$$

or, under the hypothesis (2.4),

$$\frac{R_2{}^2}{R_1}\cos\theta\frac{\partial U}{\partial\theta}+\frac{R_2{}^2}{R_1}\sin\theta\frac{\partial^2 U}{\partial\theta^2}-\frac{R_2}{\sin\theta}\frac{\partial^2 U}{\partial\tau^2}=$$
$$\left(q_n\cos\theta-q_1\sin\theta\right)R_2^3\sin 2\theta-\left(q_n\sin\theta+q_1\cos\theta\right)R_2^3\sin^2\theta \tag{3.1}$$

Let us solve first the homogeneous problem:

$$\frac{R_2{}^2}{R_1}\cos\theta\frac{\partial U}{\partial\theta}+\frac{R_2{}^2}{R_1}\sin\theta\frac{\partial^2 U}{\partial\theta^2}-\frac{R_2}{\sin\theta}\frac{\partial^2 U}{\partial\tau^2}=0\ . \tag{3.2}$$

By assuming

$$U_0(\theta,\tau)=X(\theta)Y(\tau) \tag{3.3}$$

the general solution is

$$\begin{cases} X(\theta) & = & C_1\log\left|\tan\dfrac{\theta}{2}\right|+C_2 \\ Y(\tau) & = & A_1\,\tau+A_2 \end{cases} \tag{3.4}$$

so that Eq. (3.3) gives

$$U_0(\theta,\tau)=\left(A_1\,\tau+A_2\right)\left(C_1\log\left|\tan\frac{\theta}{2}\right|+C_2\right) \tag{3.5}$$

and finally, by considering the inhomogeneous equation, we have as integral of (3.1):

$$U(\theta,\tau)=\left(A_1\,\tau+A_2\right)\left(C_1\log\left|\tan\frac{\theta}{2}\right|+C_2\right)+B_1\tau+B_2$$
$$+\frac{R_1R_2}{12}\left(-3\,q_n\cos 2\theta+2\,q_1\sin 2\theta-4\,q_1\,\theta\right) \tag{3.6}$$

Under the same hypothesis (2.4), from the second equation (2.3) and the above, we get:

$$V(\theta,\tau) = -\frac{1}{2} q_2 R_1 R_2 \,\theta + \frac{1}{4} q_2 R_1 R_2^2 \,\sin 2\theta \;+ D_3 R_2 \log\left|\cot\frac{\theta}{2}\right| - \frac{1}{2} D_2 R_2 \log^2\left|\tan\frac{\theta}{2}\right|$$
$$-\frac{R_2^{\,2}}{2R_1} D_2 \;\tau^2 \;-\left[\frac{R_2^{\,3}}{3} q_1 \sin^3\theta \;+ q_1 + \frac{R_2^{\,2}}{R_1} D_4\right]\tau \;+ D_1 \qquad (3.7)$$

Eqs. (3.6), (3.7) give the analytic solution of system (2.5) which fulfils the compatibility conditions (3.1).

4. Computer Modelling

Let us consider the solutions (3.3), (3.5) by removing the singularities

$$U(\theta,\tau) = \left(A_1\,\tau + A_2\right) + \frac{R_1 R_2}{12}\left(-\;3\,q_n \cos 2\theta + 2\,q_1 \;\; \sin 2\theta - 4\,q_1\,\theta\right)$$

$$V(\theta,\tau) = -\frac{1}{2} q_2 R_1 R_2 \,\theta + \frac{1}{4} q_2 R_1 R_2^2 \;\sin 2\theta$$
$$-\frac{R_2^{\,2}}{2R_1} D_2 \;\tau^2 \;-\left[\frac{R_2^{\,3}}{3} q_1 \sin^3\theta \;+ q_1 + \frac{R_2^{\,2}}{R_1} D_4\right]\tau \;+ D_1 \qquad (4.1)$$

under the boundary conditions:

$$U(0,0) = V(0,2\pi) = 0, \quad U(0,2\pi) = V(0,0) = 1,$$

and the following values of the curvatures and parameters

$$R_1 = -\;3.9\;\;,\; R_2 = -5.2,\;\; q_1 = 0.01,\;\; q_2 = q_n = 7\;\;.$$

The initial conditions imply the following values of the integration constants:

$$A_1 = \frac{1}{2\pi}\;\;,\; A_2 = \frac{R_1 R_2}{4} q_n \;\;, D_1 = D_2 = 1 \;\;,\;\; D_4 = -\frac{R_1 + 2\pi R_1 q_1}{2\pi R_2^2}$$

so that we can model the surfaces of solutions.
From Figs. 2, 3 it can be seen the solution for the functions $U(\theta,\tau)$, $V(\theta,\tau)$. The solution $V(\theta,\tau)$ shows the nonlinear effect of surface bending due to the presence of the nonlinear terms τ^2 [19-20].

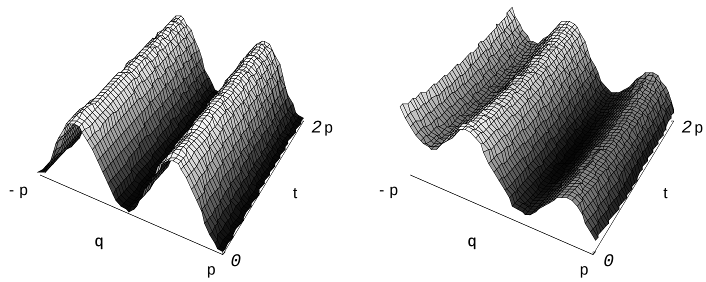

Fig. 2. Solution for $U(\theta,\tau)$ (left) and $V(\theta,\tau)$.

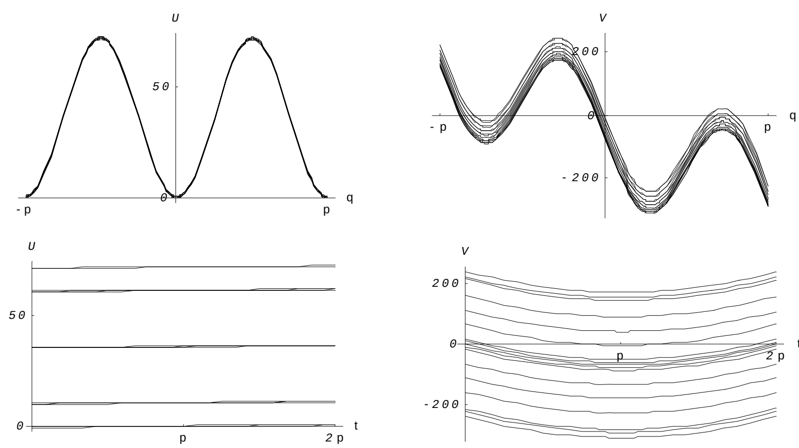

Fig. 3. Sections for the solution $U(\theta,\tau)$ (left column) and $V(\theta,\tau)$.

As a second example let us take the axisymmetric tangent loading [19]

$$q_1 = q_n = \frac{\sin \pi\theta}{\pi\theta}, \quad q_2 = 0 \, ,$$

so that Eq. (4.1) become

$$U(\theta,\tau) = (A_1\,\tau + A_2) + \frac{R_1 R_2}{12}\left(-\,3\,\cos 2\,\theta + 2\sin 2\,\theta - 4\,\theta\right)\frac{\sin \pi\,\theta}{\pi\theta}$$

$$V(\theta,\tau) = -\frac{{R_2}^2}{2R_1} D_2\,\tau^2 - \left[\frac{{R_2}^3}{3}\frac{\sin \pi\,\theta}{\pi\theta}\sin^3\theta + \frac{\sin \pi\,\theta}{\pi\theta} + \frac{{R_2}^2}{R_1} D_4\right]\tau + D_1$$

This loading is a localized function of θ, e.g. with max value in the origin and slow decay to 0, as it usually happens when on the surface there is a localized pressure in correspondence to a given value of θ.

Together with this kind of loading we assume also that the same value of the boundary conditions of the previous example hold thus obtaining the surfaces of Figs. 4, 5.

As already shown in [19-21], the surface $V(\theta,\tau)$ shows a strong nonlinear effect of resonance with increasing amplitude which is combined with the surface bending.

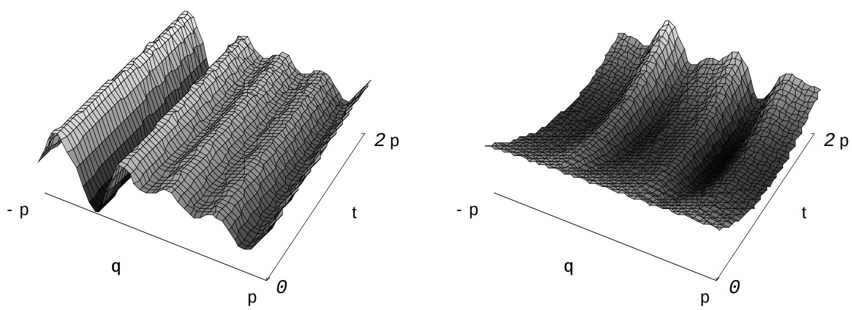

Fig. 4. Solution for $U(\theta,\tau)$ (left column) and $V(\theta,\tau)$ in presence of localized load

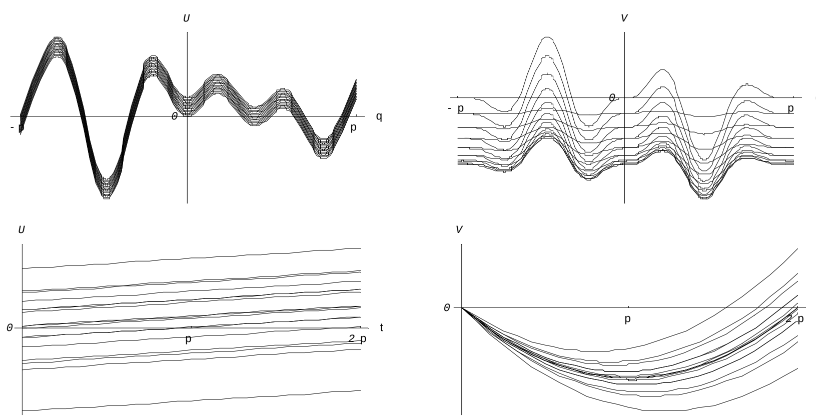

Fig. 5. Sections for the solution $U(\theta,\tau)$ (left column) and $V(\theta,\tau)$ in presence of a localized load.

5. Conclusions

In this paper we have derived a class of functions which are solution of the elastic hyperboloid shella. We have also shown by using a computer modelling the existence of some nonlinear effects as the bending of the solution as well as its increasing amplitude.

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