Action, Uniformity and Proximity

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1. Introduction

Let X be a Tychonoff space, $\mathcal{H}(X)$ the group of all self-homeomorphisms of X and $e : (f,x) \in \mathcal{H}(X) \times X \to f(x) \in X$ the evaluation function. Of course, there are many different ways to topologise $\mathcal{H}(X)$. For instance, it can be endowed with the subspace topology induced by any of all known function space topologies. Nevertheless, since $\mathcal{H}(X)$ is a group with respect to the usual composition of functions, it seems worthwhile to focus our attention on those topologies which yield continuity of both the group operations, product and inverse function, and, at the same time, yield continuity of the evaluation function. In other words, we will focus on topologies which make $\mathcal{H}(X)$ as a topological group and the evaluation function as a group action of $\mathcal{H}(X)$ on X. Topologies with these two features can be found obviously among uniform topologies. In fact, uniform topologies make as continuous the evaluation function [4]. Furthermore, they make as continuous both product and inverse function at (i,i) and at i respectively, where i is the identity function of X. We will call a *group topology* on $\mathcal{H}(X)$ any topology on $\mathcal{H}(X)$ which makes $\mathcal{H}(X)$ as a topological group. Following [3,4], a topology on $\mathcal{H}(X)$ which makes the evaluation function as a continuous function is called *admissible*. As a matter of fact, *admissible group topologies* on $\mathcal{H}(X)$ are those ones which determine a group action of $\mathcal{H}(X)$ on X.

Being well aware that if X is compact T_2, then the compact-open topology on $\mathcal{H}(X)$, which is also the uniform topology derived from the unique (totally bounded) uniformity on X, is an admissible group topology, [2,6], we will search the admissible group topologies on $\mathcal{H}(X)$ by means of a compact extension procedure. Whenever X is Tychonoff, since any self-homeomorphism of X continuously extends to the Stone-Čech compactification βX of X, then $\mathcal{H}(X)$ embeds as a subgroup in $\mathcal{H}(\beta X)$. Thereby, the relativisation to $\mathcal{H}(X)$ of the compact-open topology of $\mathcal{H}(\beta X)$ is an admissible group topology. Analogously, whenever X is locally compact T_2, $\mathcal{H}(X)$ embeds as a subgroup in $\mathcal{H}(X_\infty)$, where X_∞ is the one-point compactification of X. Thereby, the relativisation of the compact-open topology of $\mathcal{H}(X_\infty)$ to $\mathcal{H}(X)$, called in [3] the g-topology, is an admissible group topology. Accordingly, the previous two significant examples strongly suggest to investigate those uniform topologies on $\mathcal{H}(X)$ derived from totally bounded uniformities on X whose uniform completion is a T_2−compactification of X to which any self-homeomorphism of X continuously extends. We say that a T_2-compactification $\gamma(X)$ of X has *the lifting property* if any self-homeomorphism of X continuously extends to it. Of course, if $\gamma(X)$ is a T_2-compactification of X with the lifting property, then $\mathcal{H}(X)$ embeds as a subgroup in $\mathcal{H}(\gamma(X))$; thereby, the uniform topology derived from the totally bounded uniformity naturally associated with $\gamma(X)$ is an admissible group topology. Furthermore, the compact extension procedure reveals as a powerful method to prove the existence of a least admissible group topology on $\mathcal{H}(X)$. For instance, if X is locally compact T_2, then the g−topology is the least admissible group topology. Besides, if X is the rational numbers space Q, equipped with the Euclidean topology, then the clopen-open topology, which is also the uniform topology derived from the Čech uniformity of Q, is the least admissible group topology of $\mathcal{H}(Q)$, [7,20,21]. Namely, in the acquired results, [3,6,7,8], the least admissible group topology has been constructed as the uniform topology derived from a totally bounded uniformity associated with a T_2−compactification of X with the lifting property.

The issues so far discussed lead us to show the Theorem 4.4: *A uniform topology on $\mathcal{H}(X)$ derived from a totally bounded uniformity on X is a group topology (hence an admissible group topology) if and only if it is derived from a totally bounded uniformity of X associated with a T_2-compactification of X with the lifting property.*

On the other hand, if X is locally compact T_2, then the compact-open topology on $\mathcal{H}(X)$, which is also the topology of uniform convergence on compacta derived from any uniformity on X, is admissible and yields continuity of the product function. Unfortunately in general, the compact-open topology does not provide continuity of the inverse function. But, with the following additional property: (∗) *any point of X has a compact connected neighbourhood*, due to J.J Dijkstra, [11], the compact-open topology becomes a group topology and, as a consequence, the least admissible group topology of $\mathcal{H}(X)$. According to this issue the compact-open topology on $\mathcal{H}(X)$ quotes as the most eligible one if X is a manifold of finite dimension or X is an infinite dimensional manifold modelled on the Hilbert cube.

In looking for topologies of uniform convergence on members of a given family, containing all compact sets, which are admissible group topologies, we focus beyond local compactness. In order to do so, we follow as suggestive example that of bounded sets of an infinite dimensional normed vector space carrying as proximity the metric proximity associated with the norm. We emphasise first that local compactness of X is equivalent to the family of compact sets of X being a *boundedness* of X, [14], which, jointly any EF-proximity of X, gives a *local proximity space*,[16]. As a consequence, we make this particular case to fall within the more general one in which compact sets are substituted with bounded sets in a local proximity space, while the property $(*)$ is replaced by the following one: $(**)$ *for each non empty bounded set* B *there exists a finite number of connected bounded sets* $B_1, \cdots, B_n$ *such that* $B \ll_\delta int(B_1) \cup \cdots \cup int(B_n)$. So doing, we achieve the following issue: Theorem 5.1: *If* $(X, \mathcal{B}, \delta)$ *is a local proximity space with the property* $(**)$ *and any homeomorphism of* X *preserves both boundedness and proximity, then the topology of uniform convergence on bounded sets derived from the unique totally bounded uniformity associated with* δ *is an admissible group topology on* $\mathcal{H}(X)$.

The uniformities so far considered are totally bounded and the concept of totally bounded uniformity can be dually recast as EF-proximity, and then as strong inclusion. As a consequence, it is worthwile to reformulate uniform topologies derived from totally bounded uniformities as proximal set-open topologies. Taking up the common proximity nature of set-open topologies as the compact-open topology, the bounded-open topology and the topology of convergence in proximity, S.A. Naimpally, jointly with the author, introduced as unifying tool the notion of proximal set-open topology, simply replacing the usual inclusion with a strong one, [10]. Let X be a topological space and (Y, δ) an EF-proximity space. Let $C(X, Y)$ denote the set of all continuous functions from X to Y and α a network of X. The *proximal set-open topology relative to* α *and* δ is that having as subbasic open sets the ones of the following form:

$$[C, A]_\delta := \{f \in C(X, Y) : f(C) \ll_\delta A\},$$

where C runs through α, A runs through all open subsets in Y and $\ll_\delta$ is the strong inclusion naturally associated with δ. The proximal set-open topology relative to α and δ is designed by the acronym $PSOT_{\alpha,\delta}$ or, simply, $PSOT_\delta$

when α is the set $CL(X)$ of all non empty closed subsets of X. In [10] it has been proven that if α is a closed and hereditarily closed network of X, then the topology of uniform convergence relative to α on $C(X, Y)$ derived from the unique totally bounded uniformity naturally associated with δ agrees with $PSOT_{\alpha,\delta}$. Consequently, the uniform topology on $C(X, Y)$ derived from the unique totally bounded uniformity compatible with δ agrees with $PSOT_{\delta}$, as already proven in [18].

By endowing $\mathcal{H}(X)$ with $PSOT's$, our two previous results can be reformulated as follows. The former, when α is $CL(X)$, as: *A $PSOT_{\delta}$ is a group topology on $\mathcal{H}(X)$ if and only if it is $PSOT_{\delta'}$ relative to a proximity δ' whose Smirnov compactification has the lifting property.*

After recalling that the concepts of local proximity on a Tychonoff space X and of T_2 local compactification of X are dual,[16], and a T_2 local compactification of X has *the lifting property* if and only if any self-homeomorphism of X continuously extends to it, then the latter result, when α is a boundedness of X which jointly with δ gives a local proximity space, [16], can be recast as : *If $(X, \mathcal{B}, \delta)$ is a local proximity space with the property $(**)$ and the T_2 local compactification of X naturally associated with it has the lifting property, then $PSOT_{\mathcal{B},\delta}$ is an admissible group topology on $\mathcal{H}(X)$.*

2. Preliminaries

In order to give some useful background the definitions, the terminology and the results quoted below are drawn by [5,12,15,17,19,22]. Besides, we expressly remark that the word *uniformity* always means *diagonal or Weil uniformity.*

– *Uniformity, proximity and T_2-compactifications.*

Uniformities, proximities and T_2-compactifications have an intensive reciprocal interaction. EF-proximity and totally bounded uniformity are dual concepts. Any uniformity $\mathcal{U}$ on X naturally determines an EF-proximity on X by setting for $A, B \subseteq X$, $A \not\delta_{\mathcal{U}} B$ if and only if there exists a diagonal neighbourhood $U \in \mathcal{U}$ such that $U[A] \cap B \neq \emptyset$. The class of all uniformities on X determining the same EF-proximity δ on X contains a unique totally bounded uniformity, which is also the least element in the class. In the opposite, by the Smirnov compactification theorem, [19], any EF-proximity δ on X determines, up to homeomorphism, a T_2-compactification $\gamma(X)$ of X, whose unique compatible uniformity in turn induces on X a totally bounded uniformity $\mathcal{U}^*$, whose naturally associated proximity is just the starting δ.

Both proximity and uniformity give rise to exhaustive procedures to generate all T_2-compactifications of a Tychonoff space.

Let (X, δ) be an EF-proximity space, τ_δ the natural underlying topology, $\mathcal{U}^*$ the unique totally bounded uniformity compatible with δ and $\gamma(X)$ the uniform completion of $(X,\mathcal{U}^*)$. Given that $\gamma(X)$ is obviously the Smirnov compactification of (X, δ) up to homeomorphism, it is easily acquired that:

Lemma 2.1. *The following properties are equivalent:*

(a) Any self-homeomorphism of the underlying topological space (X, τ_δ) continuously extends to $\gamma(X)$.

(b) Any self-homeomorphism of X is a proximity function w.r.t. δ.

(c) Any self-homeomorphism of X is a uniformly continuous function w.r.t. $\mathcal{U}^*$.

It is to be reminded that a T_2-compactification $\gamma(X)$ of X has the lifting property if and only if any self-homeomorphism of X continuously extends to it. According with Lemma 2.1 we naturally say that *a proximity has the lifting property* if it satisfies the property (b) in Lemma 2.1 and that *a uniformity has the lifting property* if it satisfies the property (c) in Lemma 2.1.

It is remarkable that, for each positive integer n, any metric uniformity compatible with the space $\mathbb{R}^n$, equipped with the Euclidean topology, for which any homeomorphism is uniformly continuous, or which is equivalent with the lifting property, is totally bounded,[1].

– *Strong inclusion.*

The concept of EF-proximity can be recast as *strong inclusion* or double containment or non-tangential inclusion. For any given EF-proximity δ on a space X the relative dual strong inclusion is the binary relation over the power set $Exp(X)$ of X defined as follows:

$$A \ll_\delta B \text{ iff } A \not\delta X - B.$$

Vice versa, for any given binary relation over $Exp(X)$, $\ll$, which is a strong inclusion the relative dual EF-proximity δ is the binary relation over $Exp(X)$ defined by:

$$A \not\delta B \text{ iff } A \ll_\delta X - B.$$

The relations δ and $\ll_\delta$ are interchangeable.

Furthermore, later on we essentially use the following *betweenness property*. Let δ be an EF-proximity. If $A \ll_\delta B$ then there exists a $\tau_\delta-$closed set C such that $A \ll_\delta \ int(C) \subseteq C \ \ll_\delta B$.

– *Uniform topologies and proximal set-open topologies on* $\mathcal{H}(X)$.

Every uniformity $\mathcal{U}$ compatible with X induces on $\mathcal{H}(X)$ the *uniformity of uniform convergence derived from* $\mathcal{U}$, which admits as basic diagonal neighbourhoods the sets:

$$\hat{U} := \{(f,g) \in \mathcal{H}(X) \times \mathcal{H}(X) : (f(x), g(x)) \in U, \quad \forall x \in X\}$$

as U runs through all diagonal neighbourhoods in $\mathcal{U}$. The uniformity of uniform convergence derived from $\mathcal{U}$ on $\mathcal{H}(X)$ generates in turn the *uniform topology* or the *topology of uniform convergence derived from* $\mathcal{U}$, that we denote as τ_U.

Different uniformities can generate the same uniform topology.

Some already stated basic facts are then summarised further. For a uniformisable space X, every uniform topology on $\mathcal{H}(X)$ is admissible, [4]. Furthermore, every uniform topology on $\mathcal{H}(X)$ provides continuity of the inverse function at i, and continuity of the product at (i,i), where i is the identity function of X.

Let α stand for a family of non empty subsets of X. The *topology of uniform convergence on members of* α *derived from* $\mathcal{U}$, which we denote as $\tau_{\alpha,\,\mathcal{U}}$, is that admitting as subbasic open sets at any $f \in \mathcal{H}(X)$ the following ones:

$$(A, U, f) := \{h \in \mathcal{H}(X) : (f(x), h(x)) \in U, \ \forall x \in A\}$$

where A runs through α and U varies in $\mathcal{U}$.

Since the uniform topologies so far considered are relative to totally bounded uniformities, it is worthwhile to reformulate them as proximal set-open topologies. To unify the concepts of compact-open topology, bounded-open topology and topology of proximity convergence, [18], S.A. Naimpally, jointly with the author, introduced the unifying tool of *proximal set-open topology relative to a network and a proximity*,[10]. This recasting takes up the opportunity of reformulating topologies of uniform convergence on members of a network, when the range space carries a proximity.

A collection α of subsets of a topological space X is said to be a *network* on X provided that for any point x in X and any open subset A of X containing x there is a member C in α such that $x \in C \subseteq A$. A network α is a *closed network* if any element in α is closed and is a *hereditarily closed network* if any closed subset of any element in α is again in α.

Let (Y, δ) be an EF-proximity space and α a network in X, then the *proximal set-open topology relative to* α *and* δ, in short denoted by the acronym $PSOT_{\alpha,\delta}$ or, simply, $PSOT_\delta$ when α is the network $CL(X)$ of all non empty closed subsets of X, is that admitting as subbasic open sets the following ones:

$$[A : W]_\delta := \left\{ f \in Y^X \ : \ f(A) \ll_\delta W \right\}$$

where A runs through α and W is open in Y. Whenever α coincides with the family of all closed subsets of X, the relative $PSOT_\delta$ on $C(X, Y)$ is the topology of proximity convergence relative to δ. We note that a net of functions $\{f_\lambda\}$ *converges in proximity w.r.t.* δ to f if and only if for $A \subseteq X$ and $B \subseteq Y$ then $f(A) \not\delta B$ implies $f_\lambda(A) \not\delta B$ eventually. For continuous functions the topology of proximity convergence relative to an EF-proximity δ agrees with the uniform topology derived from the unique totally bounded uniformity naturally associated with δ,[18]. Besides, when α is the family of all compact subsets of X, for any proximity we get the compact-open topology, which is the prototype within the class of set-open topologies.

Different proximities can induce the same $PSOT$.

The proximal set-open topologies have remarkable properties. Later on, the following Arens-type result,[10], will reveal as very useful.

Theorem 2.1. *Let* α *be a closed, hereditarily closed network in* X *and* δ *an*

EF-proximity on Y. Then $PSOT_{\alpha,\delta}$ is the topology of uniform convergence on members of α derived from the unique totally bounded uniformity compatible with δ.

– *Boundedness plus proximity.*

Blending proximity with boundedness gives local proximity. Local proximities play the same role in the construction of T_2 local compactifications of a Tychonoff space X as that of EF-proximities in the construction of T_2-compactifications of X.

Let X be a Tychonoff space. Any given T_2 local compactification $l(X)$ of X takes up two features of X. Whereas the former one is the separated EF-proximity on X induced by the one-point compactification of $l(X)$, the latter one is the boundedness made by all subsets of X whose closures in $l(X)$ are compact. By joining proximity and boundedness in the unique concept of *local proximity,* S. Leader put this example in abstract, [16].

A non empty collection $\mathcal{B}$ of subsets of a set X is called a *boundedness* in X if and only if:
(a) $A \in \mathcal{B}$ *and* $B \subseteq A$ *implies* $B \in \mathcal{B}$ *and (b)* $A, B \in \mathcal{B}$ *implies* $A \cup B \in \mathcal{B}$.
The elements of $\mathcal{B}$ are called *bounded sets.* It is to be underlined that in [14] S.T. Hu proposed the notion of space with a boundedness as a natural generalisation of that of metric space.

We expressly remark that we look at a local proximity as localisation of an EF-proximity modulo a free regular filter,[16]. A *local proximity space* $(X, \delta, \mathcal{B})$ consists of a set X, together with an EF-proximity δ on X and a boundedness $\mathcal{B}$ in X containing all singletons, which satisfies the following axiom: *If* $A \in \mathcal{B}, C \subseteq X$ *and* $A \ll C$ *then there exists some* $B \in \mathcal{B}$ *such that* $A \ll B \ll C$*, where* $\ll$ *is the strong inclusion of* δ.
A local proximity space $(X, \delta, \mathcal{B})$ is said to be *separated* when δ is separated.

It is remarkable that the boundedness in a local proximity space $(X, \delta, \mathcal{B})$ is also a uniformly Urysohn family w.r.t. the unique totally bounded uniformity naturally associated with δ, [5].

Now we consider some interesting properties of boundedness in proximity setting, which we will use further. In a local proximity space the closure of a bounded set is again bounded. Every compact subset of a local proximity space is bounded. Every local proximity space is also locally bounded. As a matter of fact, proximity spaces are just those ones where the underlying set X is bounded.

Finally, we add the following result: For a Tychonoff space X there exists a bijection between the set of all, up to equivalence, T_2 locally compact dense extensions of X and the set of all separated local proximities on X, [16]. If X is bounded the T_2 local compactification associated with $(X, \delta, \mathcal{B})$ is just the Smirnov compactification relative to δ, while, if X is unbounded, it can be obtained by removing from the Smirnov compactification relative to δ the point determined in that by the free regular filter $\mathcal{F} = \{X \setminus B : \ B \in \mathcal{B}\}$.

3. Proximity and homeomorphism groups

Let (X, δ) be an EF-proximity space. It is easy to show that:

Theorem 3.1. *Let $\mathcal{G}(X)$ be a subgroup of the full group $\mathcal{H}(X)$ of self-homeomorphisms of the underlying topological space X. Assuming that $\mathcal{G}(X)$ is equipped with $PSOT_\delta$, then the evaluation function $e : (f, x) \in \mathcal{G}(X) \times X \to f(x) \in X$ is continuous.*

Proof – As previously remarked, $PSOT_\delta$ is a uniform topology and any uniform convergence implies continuous convergence, which in turn provides continuity of the evaluation function, [4]. □

Furthermore, given that a proximity-isomorphism or δ-isomorphism is a self-homeomorphism of X that preserves proximity in both ways, then we can show that:

Theorem 3.2. *If (X, δ) is an EF-proximity space, then $PSOT_\delta$ is a group topology on the full group of δ-isomorphisms of X.*

Proof – We show first that the product operation is continuous. In order to do so, we assume that $g \circ f \in [C : W]_\delta$, or equivalently $g \circ f(C) \ll_\delta W$, with C closed and W open in X. Since g^{-1} preserves strong inclusion, then

$f(C) \ll_\delta g^{-1}(W)$. Moreover, the betweenness property implies that there exists an open set A in X such that $f(C) \ll_\delta A \subseteq ClA \ll_\delta g^{-1}(W)$. Of course, $[C : A]_\delta$ and $[ClA : W]_\delta$ are $PSOT_\delta$-neighbourhoods of f and g, respectively. Furthermore, if $h \in [C : A]_\delta$ and $k \in [ClA : W]_\delta$, then $k \circ h \in [C : W]_\delta$, so that the first result is acquired. Next, the symmetry property of δ, jointly with the preservation of strong inclusion, yields:

$$f \in [C : W]_\delta \text{ iff } f^{-1} \in [X - W : X - C]_\delta$$

for each δ-isomorphism of (X, δ). Thereby, the inverse function is $PSOT_\delta$-continuous. □

We summarise the previous two results as follows:

Theorem 3.3. *If (X, δ) is an EF-proximity space, then the full group of δ-isomorphisms of X, equipped with $PSOT_\delta$, is a topological group which continuously acts on X by the evaluation function e.*

Besides:

Theorem 3.4. *Whenever X is a T_2 locally compact space, the PSOT associated with the Alexandroff proximity, known as the $g-$topology, is the least admissible group topology on $\mathcal{H}(X)$, [3,6].*

Secondly:

Theorem 3.5. *Whenever X is a T_2, rim-compact and locally connected space, the PSOT associated with the Freudenthal proximity is the least admissible group topology on $\mathcal{H}(X)$, [7].*

Finally:

Theorem 3.6. *Whenever X is the rational numbers space Q, equipped with the Euclidean topology, the PSOT associated with the Čech proximity is the least admissible group topology on $\mathcal{H}(Q)$, [7,8].*

4. Compact extension procedure

Let $\mathcal{U}$ be a collection of subsets of $X \times X$. For any $U \in \mathcal{U}$ and any $h \in \mathcal{H}(X)$ put:

$$U_h := \{(x,y) \in X \times X : (h(x), h(y)) \in U\}.$$

Furthermore, set:

$$\mathcal{S}_{\mathcal{H}} := \{U_h : U \in \mathcal{U},\ h \in \mathcal{H}(X)\}.$$

Theorem 4.1. *Let $\mathcal{U}$ be a uniformity on X. Then the following hold:*

(a) *The family $\mathcal{S}_{\mathcal{H}}$ is a subbase for a uniformity $\mathcal{U}_{\mathcal{H}}$ on X, which is separated whenever $\mathcal{U}$ is so.*

(b) *The uniformity $\mathcal{U}_{\mathcal{H}}$ is totally bounded whenever $\mathcal{U}$ is so.*

(c) *Any self-homeomorphism of X is a uniformly continuous function w.r.t. $\mathcal{U}_{\mathcal{H}}$, or equivalently $\mathcal{U}_{\mathcal{H}}$ has the lifting property.*

(d) *The uniformity $\mathcal{U}_{\mathcal{H}}$ is the least uniformity with the lifting property finer than $\mathcal{U}$.*

Proof – (a) Trivially, for any $U \in \mathcal{U}$ and $h \in \mathcal{H}(X)$, the diagonal $\Delta = \{(x,x) : x \in X\}$ is contained in U_h. Besides, if $U \in \mathcal{U}, V \in \mathcal{U}$ and $V^2 \subseteq U$, then $V_h \circ V_h \subseteq U_h$ for any $h \in \mathcal{H}(X)$. Again trivially, if $U \in \mathcal{U}$ is symmetric, i.e., $U = U^{-1} = \{(x,y) : (y,x) \in U\}$, then it happens that $U_h = U_h^{-1}$ for any $h \in \mathcal{H}(X)$. Finally, from $\mathcal{U} \subseteq \mathcal{U}_{\mathcal{H}}$, it follows :

$$\cap \{U : U \in \mathcal{U}\} = \Delta \ \Rightarrow\ \cap \{U_h : U \in \mathcal{U},\ h \in \mathcal{H}(X)\} = \Delta.$$

(b) By assuming $\mathcal{U}$ totally bounded, it follows that for any diagonal neighbourhood $U \in \mathcal{U}$ there exists a finite number of points $x_1, \cdots, x_n$ in X such that $X = U[x_1] \cup \cdots \cup U[x_n]$. Therefore, if $h \in \mathcal{H}(X)$ and $x_i = h(y_i), i = 1, \cdots, n,$ then $X = U_h[y_1] \cup \cdots \cup U_h[y_n]$. And the above condition is sufficient for $\mathcal{U}_{\mathcal{H}}$ being in turn totally bounded.

(c) It is enough to observe that for any $h, k \in \mathcal{H}(X)$ it happens that if $(x,y) \in V_{h \circ k}$ then $(h(x), h(y)) \in V_k$.

(d) Let $\mathcal{V}$ be a uniformity on X with the lifting property finer than $\mathcal{U}$. In order to verify that $\mathcal{V}$ is finer than $\mathcal{U}_{\mathcal{H}}$ it is enough to show that $\mathcal{V}$ contains the subbase $\mathcal{S}_{\mathcal{H}}$ of $\mathcal{U}_{\mathcal{H}}$. For any diagonal neighbourhood U of $\mathcal{U}$ and any $h \in \mathcal{H}(X)$, since $\mathcal{U} \subseteq \mathcal{V}$ and h is uniformly continuous w.r.t. $\mathcal{V}$, then there exists a diagonal neighbourhood V in $\mathcal{V}$ such that if $(x, y) \in V$ then $(h(x), h(y)) \in U$. But this yields $V \subseteq U_h$, so $U_h \in \mathcal{V}$. And the result follows. □

For every uniformity $\mathcal{U}$ the property (d) in theorem 4.1 motivates us to refer to $\mathcal{U}_{\mathcal{H}}$ as the *minimal $\mathcal{H}(X)$-enlargement* of $\mathcal{U}$. Minimal $\mathcal{H}(X)$-enlargements have interesting properties.

Theorem 4.2. *Let $\mathcal{U}$ be a totally bounded uniformity on X. Then the uniform topology $\tau_{\mathcal{U}_{\mathcal{H}}}$ on $\mathcal{H}(X)$ derived from $\mathcal{U}_{\mathcal{H}}$ is a group topology, hence it is an admissible group topology.*

Proof – The result follows from theorem 3.2. In fact, being $\mathcal{U}_{\mathcal{H}}$ a totally bounded uniformity with the lifting property, then $\tau_{\mathcal{U}_{\mathcal{H}}}$ agrees with $PSOT_\delta$ where δ is the proximity naturally associated with $\mathcal{U}_{\mathcal{H}}$. □

In the case $\mathcal{U}$ is totally bounded the previous result induces us to refer to the uniform topology $\tau_{\mathcal{U}_{\mathcal{H}}}$ as the *fine group topology associated with $\mathcal{U}$.*

Theorem 4.3. *Let $\mathcal{U}$ be a totally bounded uniformity on X. Then the uniform topology on $\mathcal{H}(X)$, $\tau_{\mathcal{U}}$, derived from $\mathcal{U}$ is a group topology if and only if it agrees with the uniform topology $\tau_{\mathcal{U}_{\mathcal{H}}}$ derived from $\mathcal{U}_{\mathcal{H}}$.*

Proof – A net $\{f_\lambda\}$ uniformly converges to the identity function f w.r.t. $\mathcal{U}_{\mathcal{H}}$ if and only if all nets $\{h \circ f_\lambda\}$ uniformly converge to h w.r.t. $\mathcal{U}$, h running through $\mathcal{H}(X)$. Thereby, whenever $\tau_{\mathcal{U}}$ is a group topology, then, if $\{f_\lambda\}$ uniformly converges to the identity function f w.r.t. $\mathcal{U}$, it is obvious that any net $\{h \circ f_\lambda\}$ uniformly converges to h w.r.t. $\mathcal{U}$, h running through $\mathcal{H}(X)$. But this exactly means $\{f_\lambda\}$ uniformly converges to the identity function f w.r.t. $\mathcal{U}_{\mathcal{H}}$. Being both $\tau_{\mathcal{U}_{\mathcal{H}}}$ and $\tau_{\mathcal{U}}$ group topologies the above condition is enough for $\tau_{\mathcal{U}_{\mathcal{H}}} \subseteq \tau_{\mathcal{U}}$, hence $\tau_{\mathcal{U}_{\mathcal{H}}} = \tau_{\mathcal{U}}$. The vice versa is obviously trivial. □

The previuos result can be summarised as follows:

Theorem 4.4. *A uniform topology on $\mathcal{H}(X)$ derived from a totally bounded uniformity on X is a group topology (hence an admissible group topology) if and only if it is derived from a totally bounded uniformity of X associated with a T_2-compactification of X with the lifting property.*

5. Locally compact extension procedure.

We say that a T_2 local compactification has the *lifting property* if and only if any homeomorphism preserves both boundedness and proximity, i.e., any homeomorphic image of a bounded set is bounded and if $B \ll_\delta W$, then $f(B) \ll_\delta f(W)$, where f runs through $\mathcal{H}(X)$, B is bounded and W is open.

It is to be recalled that a local proximity space $(X, \mathcal{B}, \delta)$ verifies the property $(**)$ if and only if *for each non empty bounded set B there exists a finite number of connected bounded sets $B_1, \cdots, B_n$ such that $B \ll_\delta int(B_1) \cup \cdots \cup int(B_n)$.*

Whenever $(X, \mathcal{B}, \delta)$ is a local proximity space, then the subcollection of $\mathcal{B}$ of all closed bounded subsets of X is a closed, hereditarily closed network of X. Accordingly, due to theorem 2.1, $PSOT_{\mathcal{B},\delta}$ is the topology of uniform convergence on members of $\mathcal{B}$ derived from the unique totally bounded uniformity associated with δ. Unfortunately, $PSOT_{\mathcal{B},\delta}$ is not in general an admissible group topology, neither a group topology.

Nevertheless, what stated above is sufficient to draw the following final issue:

Theorem 5.1. *If $(X, \mathcal{B}, \delta)$ is an unbounded local proximity space with the property $(**)$ and any self-homeomorphism of X preserves both boundedness and proximity, then the topology of uniform convergence on bounded sets derived from the unique totally bounded uniformity associated with δ is an admissible group topology on $\mathcal{H}(X)$.*

Proof – Admissibility comes from local boundedness. The proof of continuity of the group operation is similar to the one in theorem 3.2. The inverse function is continuous as well. Let $f \in [F : X \setminus B]_\delta$, with F closed and B bounded, then, $f^{-1}(B)$ is bounded too. So, from $(**)$ it follows that:

$$f^{-1}(B) \ll_\delta int(B_1) \cup \cdots \cup int(B_n),$$

for some connected bounded sets $B_1, \cdots, B_n$. This entails that there exist n points in B, $x_1, \cdots, x_n$, such that $f^{-1}(x_i) \in int(B_i)$, $i = 1, \cdots, n$. As a consequence, $f \in [\{x_i\} : f(int(B_i))]_\delta$. In turn, $C = B_1 \cup \cdots \cup B_n$, being bounded, is distinct from X. For that C can be strongly embedded in a closed bounded set $C^{'}$, again distinct from X. Of course, $f \in [C^{'} \cap F : X \setminus B]_\delta$. We distinguish two cases. In the former one, where X is connected, the boundary of $C^{'}, \partial C^{'}$, is not empty. Consequently, $\partial C^{'} \ll_\delta X \setminus C$. Thus, $f \in [\partial C^{'} : f(X \setminus C)]_\delta$. Clearly, the following subset of $\mathcal{H}(X)$:

$$\mathcal{U} := \cap\Big\{[\{x_i\} : f(\text{int}(B_i))]_\delta : i = 1, \cdots, n\Big\} \cap [F \cap C^{'} : X \setminus B]_\delta \cap [\partial C^{'} : f(X \setminus C)]_\delta$$

is a $PSOT$-neighbourhood of f. We show that if $h \in \mathcal{U}$, then $h \in [F : X \setminus B]_\delta$. Therefore, from $f^{-1}(B) \ll_\delta C$ and from the lifting property, it follows that there exists a diagonal neighbourhood U such that $U[B] \subseteq f(C)$. Also, since $h \in [F \cap C^{'} : X \setminus B]_\delta$, there exists a diagonal neighbourhood V such that $V[h(F \cap C^{'})] \subseteq X \setminus B$. Given a symmetric diagonal neighbourhood $W \subseteq U \cap V$, if $h \notin [F : X \setminus B]_\delta$, then $W[h(F)] \cap B \neq \emptyset$. This implies there exists a point x in B and a point y in F such that $x \in W[h(y)]$. Of course, $h(y) \in W[x] \subseteq f(C)$. Thus, $h(y) \in f(B_i)$ for some $i = 1, \cdots, n$. Yet, $h(x_i)$ also belongs to $f(B_i)$, so $h^{-1}(f(B_i))$ is a connected set which intersects both the interior of $C^{'}$ and the interior of its complement, which implies that it has to intersect also the boundary $\partial C^{'}$ of $C^{'}$. At the same time, $h \in [\partial C^{'} : f(X \setminus C)]_\delta$ implies that $h(\partial C^{'})$ cannot intersect $f(B_i)$, which is an evident contradiction. In the latter case, where X is not connected, the property $(**)$ implies that any bounded set can intersect only a finite number of components, so that we can proceed in every component exactly as before. □

This final result can be recast as the following:

Theorem 5.2. *Whenever $(X, \mathcal{B}, \delta)$ a local proximity space with the property $(**)$ and the T_2 local compactification associated with it has the lifting property, then $PSOT_{\mathcal{B},\delta}$ is an admissible group topology on $\mathcal{H}(X)$.*

References

[1] Alas, O.T. and Di Concilio, A.: Uniformly continuous homeomorphisms, *Topology Appl.* **84** (1998), 33–42.

[2] Arens, R.: A topology for spaces of transformations, *Ann. of Math.* **47** (1946), 480–495.

[3] Arens, R.: Topologies for homeomorphism groups, *Amer. J. Math.* **68** (1946), 593–610.

[4] Arens, R. and Dugundji, J.: Topologies for function spaces, *Pacific J. Math.* **1** (1951), 5–31.

[5] Beer, G.: Topologies on closed and closed convex sets, *Mathematics and Its Applications* **268**, Kluwer Academic Publishers (1993).

[6] Birkhoff, G.: The topology of transformation sets, *Annals of Mathematics* (4) **35** (1934), 861–875.

[7] Di Concilio, A.: Topologizing homeomorphism groups of rim-compact spaces, *Topology Appl.* **153** (2006), 1867–1885.

[8] Di Concilio, A.: Group action on 0-dimensional spaces, *Topology Appl.* **154** (2007), 2050–2055.

[9] Di Concilio, A.: Action on $\mathbb{R} \times \mathbb{Q}$ and fine group topologies, *Topology Appl.* **156** (2009), 956–962.

[10] Di Concilio, A. and Naimpally, S.A.: Proximal set-open topologies, *Boll. Unione Mat. Ital.* (1-B) **8** (2000), 173–191.

[11] Dijkstra, J.J.: On homeomorphism groups and the compact-open topology, *Amer. Math. Monthly* **112** (2005), 910–912.

[12] Engelking, R.: *General Topology*, Polish Scientific Publishers, Warsaw (1977).

[13] Gillman, L. and Jerison, M.: Rings of continuous functions, *Graduate Texts in Mathematics* **43**, Springer-Verlag (1976).

[14] Hu, S.T.: Introduction to general topology, *Holden-Day Series in Mathematics* (1966).

[15] James, I.M.: Introduction to uniform spaces, *London Math. Soc. Lecture Note Series* **144**, Cambridge University Press (1990).

[16] Leader, S.: Local proximity spaces, *Math. Ann.* **169** (1967), 275–281.

[17] Mynard, F. and Pearl, E. editors: Beyond Topology, *Contemporary Mathematics*, Amer. Math. Soc., to appear.

[18] Nachmann, L.J.: On a conjecture of Leader, *Fund. Math.* **65** (1969), 153–155.

[19] Naimpally, S.A. and Warrack, B.D.: *Proximity spaces*, Cambdridge University Press, Cambridge (1970).

[20] Park, W.R.: Convergence structures on homeomorphism groups, *Math. Ann.* **199** (1972), 45–54.

[21] Park, W.R.: A note on the homeomorphism group of the rational numbers, *Proc. Amer. Math. Soc.* **42** (1974), 625–626.

[22] Willard, S.: *General Topology*, Addison-Wesley, Reading (1970).