Harmonic wavelet solution of Poisson's problem

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Abstract. The multiscale (wavelet) decomposition of the solution is proposed for the analysis of the Poisson problem. The approximate solution is computed with respect to a finite dimensional wavelet space [4, 5, 7, 8, 16, 15] by using the Galerkin method. A fundamental role is played by the connection coefficients [2, 7, 11, 9, 14, 17, 18], expressed by some hypergeometric series. The solution of the Poisson problem is compared with the approach based on Daubechies wavelets [18].

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1 Introduction

The wavelet solution of Poisson problem was obtained in [17, 18] by using the Daubechies wavelets [13] The main problem was to fulfill the boundary conditions of the Poisson problem and was solved by using compactly supported wavelets with some constraints on the boundaries. The solution was obtained by the Galerkin method and the computation of the connection coefficients [9, 10, 11, 12, 14]. This coefficients consist in the projection of the derivatives of the wavelet basis into a suitable wavelet space. Latto proposed [14] to take into account the refinement equations for their computation. More in general, there exists many different families of wavelets and it is quite difficult to choose, for a given problem, the most suitable family. There follows that even for the simplest Poisson's problem the Daubechies wavelets implies some cumbersome computations. In the following the same problem will be solved by using the harmonic wavelets [4, 15, 16]. With this choice the solution is the trivial one.

Wavelets are ∞^2 functions $\psi_k^n(x)$ which depend on two parameters: n and k, the scale and localization (translation) parameter respectively. These functions fulfill the fundamental axioms of multiresolution analysis [13] so that by a suitable choice of the scale and translation parameter one is able to easily and quickly approximate any function (even tabular) with decay to infinity. One of the major drawback of the wavelet theory is the arbitrariness of their choice and the existence of many families of wavelets which satify the multiresolution axioms. Among the many families of wavelets, harmonic wavelets [4, 5, 7, 8, 16, 15] are the most expedient tool for studying

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processes which are localized in Fourier domain, as it happens, e.g. in dispersive wave propagation [2, 3, 6].

Comparing with Daubechies wavelets, the harmonic wavelets are analytically defined, infinitely differentiable, and band limited [16, 2, 3] thus enabling us to easily study, their differentiable properties. The corresponding connection coefficients (also called refinable integrals [11, 9]) can be explicitly (and finitely) computed at any order [4], while in the available literature were given only for a few order of derivatives [15], or like in the case of the Daubechies wavelets, by some approximate formulas [14, 18, 17].

By using the Frobenius method and the connection coefficients the Poisson problem can be transformed into an infinite dimensional algebraic system, which can be solved by fixing a finite scale of approximation.

2 Harmonic wavelet space structure

The harmonic scaling function [16] and wavelets are the complex function (see e.g. [2])

$$\boxed{\left\{\begin{array}{lcl}\varphi^n_k(x) & \equiv & 2^{n/2}\dfrac{e^{2\pi i\,(2^n x-k)}-1}{2\pi i(2^n x-k)}\\ \\ \psi^n_k(x) & \equiv & 2^{n/2}\dfrac{e^{4\pi i(2^n x-k)}-e^{2\pi i(2^n x-k)}}{2\pi i(2^n x-k)}\end{array}\right.}\ , \tag{2.1}$$

with $n, k \in \mathbb{Z}$. The dilated and translated instances of the corresponding Fourier transform of (2.1), (see e.g. [4]) are

$$\boxed{\left\{\begin{array}{lcl}\widehat{\varphi}^n_k(\omega) & = & \dfrac{2^{-n/2}}{2\pi}e^{-i\omega k/2^n}\chi(2\pi+\omega/2^n)\\ \\ \widehat{\psi}^n_k(\omega) & = & \dfrac{2^{-n/2}}{2\pi}e^{-i\omega k/2^n}\chi(\omega/2^n)\end{array}\right.}\ , \tag{2.2}$$

where $\chi(\omega)$ is the characteristic function

$$\chi(\omega) \stackrel{\text{def}}{=} \left\{\begin{array}{lcl}1 & , & 2\pi \leq \omega \leq 4\pi\ ,\\ 0 & , & \text{elsewhere}\ .\end{array}\right. \tag{2.3}$$

Both the scaling and wavelet functions are very well localized functions in the frequency domain, despite the slow decay in the space variable.

From the definition of the inner (or scalar or dot) product, of two functions $f(x), g(x)$, and taking into account the Parseval equality

$$\langle f, g\rangle \equiv \int_{-\infty}^{\infty} f(x)\,\overline{g(x)}\mathrm{d}x = 2\pi\int_{-\infty}^{\infty}\widehat{f}(\omega)\,\overline{\widehat{g}(\omega)}\mathrm{d}\omega = 2\pi\left\langle\widehat{f},\widehat{g}\right\rangle\quad , \tag{2.4}$$

it can be shown that

(2.5)
$$\begin{cases} \langle \varphi_k^n(x), \varphi_h^m(x)\rangle = \delta^{nm}\delta_{kh} \quad , \quad \langle \overline{\varphi}_k^n(x), \overline{\varphi}_h^m(x)\rangle = \delta^{nm}\delta_{kh} \quad , \quad \langle \varphi_k^n(x), \overline{\varphi}_h^m(x)\rangle = 0 \ , \\ \langle \psi_k^n(x), \psi_h^m(x)\rangle = \delta^{nm}\delta_{hk} \ , \left\langle \overline{\psi}_k^n(x), \overline{\psi}_h^m(x)\right\rangle = \delta^{nm}\delta_{kh} \quad , \quad \left\langle \psi_k^n(x), \overline{\psi}_h^m(x)\right\rangle = 0 \ , \\ \left\langle \varphi_k^n(x), \overline{\psi}_h^m(x)\right\rangle = 0 \quad , \quad \langle \overline{\varphi}_k^n(x), \psi_h^m(x)\rangle = 0 \quad . \end{cases}$$

where δ^{nm} (δ_{hk}) is the Kronecker symbol.

3 Wavelet reconstruction of functions

Let us consider the class of (real or complex) functions $f(x)$, such that the following integrals

$$\begin{cases} \alpha_k & = & \langle f(x), \varphi_k^0(x)\rangle = \displaystyle\int_{-\infty}^{\infty} f(x)\overline{\varphi}_k^0(x)\mathrm{d}x \\ \alpha_k^* & = & \langle f(x), \overline{\varphi}_k^0(x)\rangle = \displaystyle\int_{-\infty}^{\infty} f(x)\varphi_k^0(x)\mathrm{d}x \\ \beta_k^n & = & \langle f(x), \psi_k^n(x)\rangle = \displaystyle\int_{-\infty}^{\infty} f(x)\overline{\psi}_k^n(x)\mathrm{d}x \\ \beta^{*}{}_k^n & = & \langle f(x), \overline{\psi}_k^n(x)\rangle = \displaystyle\int_{-\infty}^{\infty} f(x)\psi_k^n(x)\mathrm{d}x \end{cases} \tag{3.1}$$

exist and are finite.

From the orthogonality conditions (2.5), the function $f(x)$ can be reconstructed in terms of harmonic wavelets as (see e.g. [16])

(3.2)
$$f(x) = \left[\sum_{k=-\infty}^{\infty} \alpha_k \varphi_k^0(x) + \sum_{n=0}^{\infty}\sum_{k=-\infty}^{\infty} \beta_k^n \psi_k^n(x)\right] + \left[\sum_{k=-\infty}^{\infty} \alpha_k^* \overline{\varphi}_k^0(x) + \sum_{n=0}^{\infty}\sum_{k=-\infty}^{\infty} \beta^{*}{}_k^n \overline{\psi}_k^n(x)\right]$$

which involve the basis and (for a complex function) its conjugate basis. For a real function $(f(x) = \overline{f}(x))$ it is $\alpha_k^*(x) = \alpha_k(x)$, $\beta^{*}{}_k^n(x) = \beta_k^n(x)$.

The approximation of (3.2) up to the scale $N \leq \infty$ and to a finite translation $M \leq \infty$ is

$$\begin{aligned} f(x) \cong \Pi^{N,M} f(x) &= \left[\sum_{k=0}^{M} \alpha_k \varphi_k^0(x) + \sum_{n=0}^{N}\sum_{k=-M}^{M} \beta_k^n \psi_k^n(x)\right] \\ &+ \left[\textstyle\sum_{k=0}^{M} \alpha_k^* \overline{\varphi}_k^0(x) + \displaystyle\sum_{n=0}^{N}\sum_{k=-M}^{M} \beta^{*}{}_k^n \overline{\psi}_k^n(x)\right] \ . \end{aligned} \tag{3.3}$$

According to (2.2),(2.4), in the Fourier domain, it is

$$
(3.4)\qquad \left\{
\begin{array}{lcl}
\alpha_k & = & 2\pi\langle\widehat{f(x)},\widehat{\varphi^0_k(x)}\rangle = \displaystyle\int_{-\infty}^{\infty}\widehat{f}(\omega)\overline{\widehat{\varphi}^0_k(\omega)}\mathrm{d}\omega = \int_0^{2\pi}\widehat{f}(\omega)e^{i\omega k}\mathrm{d}\omega \\
\alpha^*_k & = & 2\pi\langle\widehat{f(x)},\widehat{\overline{\varphi}^0_k(x)}\rangle = \ldots = \displaystyle\int_0^{2\pi}\widehat{f}(\omega)e^{-i\omega k}\mathrm{d}\omega \\
\beta^n_k & = & 2\pi\langle\widehat{f(x)},\widehat{\psi^n_k(x)}\rangle = \ldots = 2^{-n/2}\displaystyle\int_{2^{n+1}\pi}^{2^{n+2}\pi}\widehat{f}(\omega)e^{i\omega k/2^n}\mathrm{d}\omega \\
\beta^{*n}_{\ k} & = & \langle\widehat{f(x)},\widehat{\overline{\psi}^n_k(x)}\rangle = \ldots = 2^{-n/2}\displaystyle\int_{2^{n+1}\pi}^{2^{n+2}\pi}\widehat{f}(\omega)e^{-i\omega k/2^n}\mathrm{d}\omega\ ,
\end{array}
\right.
$$

being $\overline{\widehat{f(x)}} = \overline{\widehat{f}(-\omega)}$.

Since wavelets are localized, they can capture with few terms the main features of functions defined in a short range interval. It should be noticed, however, that, for a non trivial function $f(x) \neq 0$ the corresponding wavelet coefficients (3.4), in general, vanish when either

$$
\widehat{f}(\omega) = 0\ ,\ \forall k \qquad \text{or} \qquad \widehat{f}(\omega) = Cnst.\ ,\ k \neq 0\ .
$$

In particular, it can be seen that the wavelet coefficients (3.4) trivially vanish when

$$
(3.5)\qquad \left\{
\begin{array}{lcll}
f(x) & = & \sin(2k\pi x) & ,\quad k \in \mathbb{Z} \\
f(x) & = & \cos(2k\pi x) & ,\quad k \in \mathbb{Z}\quad (k \neq 0)
\end{array}
\right.
$$

For instance from $(3.1)_1$, for $\cos(2k\pi x)$ it is

$$
\begin{aligned}
\alpha_k &= \int_{-\infty}^{\infty}\cos(2k\pi x)\overline{\varphi}^0_k(x)\mathrm{d}x = \frac{1}{2}\int_{-\infty}^{\infty}\left(e^{-2ih\pi x} + e^{2ih\pi x}\right)\overline{\varphi}^0_k(x)\mathrm{d}x \\
&= \frac{1}{2}\left[\int_{-\infty}^{\infty}e^{-2ih\pi x}\overline{\varphi}^0_k(x)\mathrm{d}x + \int_{-\infty}^{\infty}e^{2ih\pi x}\overline{\varphi}^0_k(x)\mathrm{d}x\right]
\end{aligned}
$$

from where by the change of variable $2\pi x = \xi$ there follows

$$
\alpha_k = \frac{1}{2}\left[\widehat{\overline{\varphi}^0_k(x)} + \widehat{\varphi}^0_k(x)\right]_{x=2\pi h}\ .
$$

According to (2.2) and to

$$
(3.6)\qquad e^{\pi i n} = \left\{
\begin{array}{lll}
1, & n = 2k\ , & k \in \mathbb{Z} \\
-1, & n = 2k+1\ , & k \in \mathbb{Z}
\end{array}
\right.
$$

it is

$$
\widehat{\varphi}^0_k(2\pi h) = \frac{1}{2}e^{-i2\pi hk}\chi(2\pi + 2\pi h) = \frac{1}{2}\chi(2\pi + 2\pi h)
$$

and, because of (2.3)

$$
\chi(2\pi + 2\pi h) = 1 \quad ,\quad 0 < h < 1
$$

so that

$$\widehat{\varphi}^0_k(2\pi h) = 0 \quad , \quad \forall h \neq 0 \quad .$$

There follows that $\alpha_h = 0$, as well as the remaining wavelet coefficients of $\cos(2k\pi x)$ (with $k \in \mathbb{Z}$ and $k \neq 0$). Analogously, it can be shown that all wavelet coefficients of $\sin(2k\pi x)$ ($\forall k \in \mathbb{Z}$) are zero.

As a consequence, a given function $f(x)$, for which the coefficients (3.1) are defined, admits the same wavelet coefficients of

$$f(x) + \sum_{h=0}^{\infty} [A_h \sin(2h\pi x) + B_h \cos(2h\pi x)] - B_0 \ , \tag{3.7}$$

or (by a simple tranformation) in terms of complex exponentials,

$$f(x) - C_0 + \sum_{h=-\infty}^{\infty} C_h e^{2ih\pi x} \ , \tag{3.8}$$

so that the wavelet coefficients of $f(x)$ are defined unless an additional trigonometric series (the coefficients A_h , B_h , C_h being constant) as in (3.7).

4 Differential structure and connection coefficients

Equation (2.4) describes the basic structure of the functional space defined on the basis functions (2.1). The investigation of the differential properties of the basis leads us to the computation of their derivatives. Moreover, in the application of the Frobenius method, it is assumed that a certain unknown functions (with its derivatives) can be expressed in terms of a basis (and its derivatives). For this reason, as a first step, we need the computation of the derivatives of the wavelet basis (see e.g. [4, 5, 6, 11, 12, 14, 15, 17]), through the connection coefficients [4, 5, 9, 10, 14, 17, 18].

The differential properties of wavelets are based on the knowledge of the following inner products:

$$\left\{ \begin{array}{l} \lambda^{(\ell)}{}_{kh} \equiv \left\langle \dfrac{d^\ell}{dx^\ell}\varphi^0_k(x), \varphi^0_h(x) \right\rangle \quad , \quad \Lambda^{(\ell)}{}^m_{kh} \equiv \left\langle \dfrac{d^\ell}{dx^\ell}\varphi^0_k(x), \psi^m_h(x) \right\rangle \\ \\ \gamma^{(\ell)}{}^{nm}_{kh} \equiv \left\langle \dfrac{d^\ell}{dx^\ell}\psi^n_k(x), \psi^m_h(x) \right\rangle \quad , \quad \zeta^{(\ell)}{}^n_{kh} \equiv \left\langle \dfrac{d^\ell}{dx^\ell}\psi^n_k(x), \varphi^0_h(x) \right\rangle \ . \end{array} \right. \tag{4.1}$$

and the corresponding inner products with conjugate functions.

$$\left\{ \begin{array}{l} \overline{\lambda}^{(\ell)}{}_{kh} \equiv \left\langle \dfrac{d^\ell}{dx^\ell}\overline{\varphi}^0_k(x), \overline{\varphi}^0_h(x) \right\rangle \quad , \quad \overline{\Lambda}^{(\ell)}{}^m_{kh} \equiv \left\langle \dfrac{d^\ell}{dx^\ell}\overline{\varphi}^0_k(x), \overline{\psi}^m_h(x) \right\rangle \\ \\ \overline{\gamma}^{(\ell)}{}^{nm}_{kh} \equiv \left\langle \dfrac{d^\ell}{dx^\ell}\overline{\psi}^n_k(x), \overline{\psi}^m_h(x) \right\rangle \quad , \quad \overline{\zeta}^{(\ell)}{}^n_{kh} \equiv \left\langle \dfrac{d^\ell}{dx^\ell}\overline{\psi}^n_k(x), \overline{\varphi}^0_h(x) \right\rangle \ . \end{array} \right. \tag{4.2}$$

The coefficients (4.1) can be easily computed in the Fourier domain(see for a proof [4]) (for the first and second order connection coefficients of *periodic* harmonic wavelets see also [2, 3, 15]), while the mixed connection coefficients $(4.1)_{2,4}$ are trivially zero:

$$\Lambda^{(\ell)}{}_{kh}^{m} = 0 \quad , \quad \zeta^{(\ell)}{}_{kh}^{n} = 0 \quad , \quad \overline{\Lambda}^{(\ell)}{}_{kh}^{m} = 0 \quad , \quad \overline{\zeta}^{(\ell)}{}_{kh}^{n} = 0 \; . \tag{4.3}$$

These coefficients enable us to characterize any order derivative of the basis. In fact, according to (4.1) it is

$$\frac{\mathrm{d}^{\ell}\varphi_k^0(x)}{\mathrm{d}x^{\ell}} = \sum_{m=0}^{\infty}\sum_{h=-\infty}^{\infty} \lambda^{(\ell)}{}_{kh}^{m}\, \varphi_h^m(x) \; . \tag{4.4}$$

A good approximation is obtained by a finite value of M

$$\boxed{\frac{\mathrm{d}^{\ell}\varphi_k^0(x)}{\mathrm{d}x^{\ell}} \cong \sum_{h=0}^{M} \lambda^{(\ell)}{}_{kh}\varphi_h^0(x)} \tag{4.5}$$

Analogously we have,

$$\frac{\mathrm{d}^{\ell}\psi_k^n(x)}{\mathrm{d}x^{\ell}} = \sum_{m=0}^{\infty}\sum_{k,h=-\infty}^{\infty} \gamma^{(\ell)}{}_{kh}^{nm}\, \psi_h^m(x) \tag{4.6}$$

and a good approximation, which depends only on the dilation N and translational parameter M, is

$$\boxed{\frac{\mathrm{d}^{\ell}\psi_k^n(x)}{\mathrm{d}x^{\ell}} \cong \sum_{m=0}^{N}\sum_{h=-M}^{M} \gamma^{(\ell)}{}_{kh}^{nm}\, \psi_h^m(x)} \tag{4.7}$$

with $N \leq n$. Of course, since the harmonic wavelets are oscillating functions the approximation improves by increasing the translational parameters, however the approximation can be considered sufficiently good for a very low value of M.

For the corresponding conjugate functions we have

$$\frac{\mathrm{d}^{\ell}\overline{\varphi}_k^0(x)}{\mathrm{d}x^{\ell}} = \sum_{m=0}^{\infty}\sum_{h=-\infty}^{\infty} -\lambda^{(\ell)}{}_{kh}^{m}\, \overline{\varphi}_h^m(x) \quad , \quad \frac{\mathrm{d}^{\ell}\overline{\varphi}_k^0(x)}{\mathrm{d}x^{\ell}} \cong \sum_{h=0}^{M} \lambda^{(\ell)}{}_{kh}\overline{\varphi}_h^0(x) \tag{4.8}$$

and

(4.9)

$$\frac{\mathrm{d}^{\ell}\overline{\psi}_k^n(x)}{\mathrm{d}x^{\ell}} = \sum_{m=0}^{\infty}\sum_{k,h=-\infty}^{\infty} -\gamma^{(\ell)}{}_{kh}^{nm}\, \overline{\psi}_h^m(x) \quad , \quad \frac{\mathrm{d}^{\ell}\overline{\psi}_k^n(x)}{\mathrm{d}x^{\ell}} \cong \sum_{m=0}^{N}\sum_{h=-M}^{M} \overline{\gamma}^{(\ell)}{}_{kh}^{nm}\, \psi_h^m(x) \; .$$

Thanks to Eqs. (4.4),(4.6), and to the orthonormality conditions (2.5), the remaining mixed coefficients are trivially null.

5 Wavelet solution of the Poisson problem

Let us consider the Poisson problem

$$
\left\{
\begin{array}{l}
\dfrac{\mathrm{d}^2 u}{\mathrm{d}x^2} + \mu^2 u = f(x) \\[2ex]
u(0) = A \quad , \quad u(1) = B \;.
\end{array}
\right.
\tag{5.1}
$$

By using the Frobenius method, the solution is searched in the form of harmonic wavelet series up to a finite scale of approximation (3.3), so that the projection into a finite dimensional wavelet space is:

(5.2)

$$
\left\{
\begin{array}{lcl}
\left\langle \Pi^{N,M} \dfrac{\mathrm{d}^2}{\partial x^2} u(x), \varphi^0_k \right\rangle + \left\langle \Pi^{N,M} u(x), \varphi^0_k \right\rangle & = & \left\langle \Pi^{N,M} f(x), \varphi^0_k \right\rangle \;, \\[2ex]
\left\langle \Pi^{N,M} \dfrac{\mathrm{d}^2}{\partial x^2} u(x), \overline{\varphi}^0_k \right\rangle + \left\langle \Pi^{N,M} u(x), \overline{\varphi}^0_k \right\rangle & = & \left\langle \Pi^{N,M} f(x), \overline{\varphi}^0_k \right\rangle \;, \\[2ex]
\left\langle \Pi^{N,M} \dfrac{\mathrm{d}^2}{\partial x^2} u(x), \psi^n_k(x) \right\rangle + \left\langle \Pi^{N,M} u(x), \psi^n_k(x) \right\rangle & = & \left\langle \Pi^{N,M} u(x), \psi^n_k(x) \right\rangle \;, \\[2ex]
\left\langle \Pi^{N,M} \dfrac{\mathrm{d}^2}{\partial x^2} u(x), \overline{\psi}^n_k(x) \right\rangle + \left\langle \Pi^{N,M} u(x), \overline{\psi}^n_k(x) \right\rangle & = & \left\langle \Pi^{N,M} u(x), \overline{\psi}^n_k(x) \right\rangle \;, \\[2ex]
 & & k = 0, 1, \dots, M \;; n = 0, \dots, N \;,
\end{array}
\right.
$$

where we assume that $u(x)$ is in the form:

$$
\begin{array}{rcl}
u(x) \cong \Pi^{N,M} u(x) & = & \left[\displaystyle\sum_{k=0}^{M} \alpha_k \varphi^0_k(x) + \sum_{n=0}^{N} \sum_{k=-M}^{M} \beta^n_k \psi^n_k(x) \right] \\[2ex]
 & + & \left[\displaystyle\sum_{k=0}^{M} \alpha^*_k \overline{\varphi}^0_k(x) + \sum_{n=0}^{N} \sum_{k=-M}^{M} \beta^{*}{}^n_k \overline{\psi}^n_k(x) \right] \\[2ex]
 & - & B_0 + \displaystyle\sum_{k=0}^{\infty} A_k \sin 2k\pi x + B_k \cos 2k\pi x \;.
\end{array}
\tag{5.3}
$$

The second derivative, on account of (4.5),(4.7),(4.1),(4.2),(4.3), is

(5.4)

$$
\begin{array}{rcl}
\dfrac{\mathrm{d}^2 u}{\mathrm{d}x^2} & \cong & \left[\displaystyle\sum_{k=0}^{M} \alpha_k \sum_{h=0}^{M} \lambda^{(\ell)}{}_{kh} \varphi^0_h(x) + \sum_{n=0}^{N} \sum_{k=-M}^{M} \beta^n_k \sum_{m=0}^{N} \sum_{h=-M}^{M} \gamma^{(\ell)}{}^{nm}_{kh} \psi^m_h(x) \right] \\[2ex]
 & + & \left[\sum_{k=0}^{M} \alpha^*_k \sum_{h=0}^{M} \lambda^{(\ell)}{}_{hk} \overline{\varphi}^0_h(x) + \displaystyle\sum_{n=0}^{N} \sum_{k=-M}^{M} \beta^{*}{}^n_k \sum_{m=0}^{N} \sum_{h=-M}^{M} \gamma^{(\ell)}{}^{nm}_{hk} \overline{\psi}^m_h(x) \right] \\[2ex]
 & - & \displaystyle\sum_{k=0}^{\infty} 4k^2\pi^2 \left(A_k \sin 2k\pi x + B_k \cos 2k\pi x \right) \;.
\end{array}
$$

On the given function $f(x)$ it is assumed that it can be represented in the form (5.3), that is

(5.5)
$$
\begin{aligned}
f(x) \quad &\cong \quad \left[\sum_{k=0}^{M} a_k \varphi_k^0(x) + \sum_{n=0}^{N}\sum_{k=-M}^{M} b_k^n \psi_k^n(x)\right] + \left[\sum_{k=0}^{M} a_k^* \overline{\varphi}_k^0(x) + \sum_{n=0}^{N}\sum_{k=-M}^{M} b^{*n}_{\ k} \overline{\psi}_k^n(x)\right] \\
&- \quad d_0 + \sum_{k=0}^{\infty} c_k \sin 2k\pi x + d_k \cos 2k\pi x \ .
\end{aligned}
$$

By putting (5.3),(5.4),(5.5) into (5.1) and by the scalar product with the basis functions φ_i^0 ,$\overline{\varphi}_i^0$, $\psi_i^r(x)$, $\overline{\psi}_i^r(x)$ we obtain the algebraic system:

$$
(5.6) \quad \left\{
\begin{array}{rcll}
\displaystyle\sum_{k=0}^{M} \alpha_k \lambda^{(2)}{}_{ki} + \mu^2 \alpha_i &=& a_i & , \quad (i = 0, \dots, M) \\
\displaystyle\sum_{k=0}^{M} \alpha_k^* \lambda^{(2)}{}_{ik} + \mu^2 \alpha_i^* &=& a_i^* & , \quad (i = 0, \dots, M) \\
\displaystyle\sum_{n=0}^{N}\sum_{k=-M}^{M} \beta_k^n \gamma^{(2)}{}_{ki}^{nr} + \mu^2 \beta_i^r &=& b_i^r & , \quad (r = 0, \dots, N; i = -M, \dots, M) \\
\displaystyle\sum_{n=0}^{N}\sum_{k=-M}^{M} \beta^{*n}_{\ k} \gamma^{(2)}{}_{ki}^{nr} + \mu^2 \beta^{*r}_{\ i} &=& b_i^r & , \quad (r = 0, \dots, N; i = -M, \dots, M) \\
-4k^2\pi^2 A_k + \mu^2 A_k &=& c_k & , \quad (k = 0, \dots, \infty) \\
-4k^2\pi^2 B_k + \mu^2 B_k &=& d_k & , \quad (k = 1, \dots, \infty) \ .
\end{array}
\right.
$$

Once the wavelet coefficients are know the more general wavelet solution $U(x)$ of (5.1) it is defined unless an arbitrary function $F(x)$ to be specified on account of the boundary conditions:

$$
(5.7) \qquad U(x) = \Pi^{N,M} u(x) + F(x)
$$

where $F(x)$ is an eigenfunction such that

$$
\frac{\mathrm{d}^2 F}{\mathrm{d}x^2} + \mu^2 F(x) = 0 \ .
$$

The boundary conditions $(5.1)_{2,3}$ give two additional equations. Taking into account the definitions (2.1) it is

$$
(5.8) \quad \left\{
\begin{array}{l}
\varphi_k^0(0) = \overline{\varphi}_k^0(0) = \left\{ \begin{array}{l} 1 \ , k = 0 \\ 0 \ , k \neq 0 \end{array} \right. \quad , \quad \varphi_k^0(1) = \overline{\varphi}_k^0(1) = \left\{ \begin{array}{l} 1 \ , k = 1 \\ 0 \ , k \neq 1 \end{array} \right. \\[2ex]
\psi_k^n(0) = \overline{\psi}_k^n(1) = \left\{ \begin{array}{l} 2^{n/2} \ , k = 0 \\ 0 \ , k \neq 0 \end{array} \right. \quad , \quad \psi_k^n(1) = \overline{\psi}_k^n(1) = 0 \ , \forall k \ , \forall n
\end{array}
\right.
$$

so that, from (5.3),(5.7),$(5.1)_{2,3}$ the two additional equations reduce to

$$
(5.9) \qquad \begin{cases} \alpha_0 + \sum_{n=0}^{N} 2^{n/2}\beta_0^n + \alpha_0^* + \sum_{n=0}^{N} 2^{n/2}\beta^{*n}_0 + \sum_{k=1}^{\infty} B_k + F(0) & = & A \\ \alpha_1 + \alpha_1^* + \sum_{k=1}^{\infty} B_k + F(1) & = & B \end{cases}
$$

Test problem 1. Let us consider the following

$$
(5.10) \qquad \begin{cases} \dfrac{\mathrm{d}^2 u}{\mathrm{d}x^2} = f(x) \\ \\ u(0) = 1 \quad , \quad u(1) = 2 \end{cases}
$$

which is obtained by (5.1) when

$$
\mu = 0 \quad , \quad f(x) = -4\pi^2 \sin 2\pi x
$$

and as initial conditions

$$
A = 1 \quad , \quad B = 2.
$$

This problem was considered in [18] to show the application of this method, which was based in that paper on Daubechies wavelets. Due to the choice of the basis (Daubechies wavelets) the computation was obtained only after the computation of the connection coefficients. Indeed this problem in the harmonic basis is a trivial problem (and trivially can be solved). In fact, the given function $f(x)$ is not localized in space, and therefore it should expected that the solution $U(x)$ too will not be localized. Thus making unuseless the wavelet approach. In fact, system (5.6) gives

$$
\alpha_k = 0 \; , \alpha_k^* = 0 \; , \beta_k^n = 0 \; , \beta^{*n}_k = 0 \; , A_1 = 1 \; , (k = 1) \; , B_k = 0
$$

so that, according to (5.7),(5.3)

$$
U(x) = \sin 2\pi x + F(x) \; .
$$

The eigenfunction is

$$
F(x) = \nu x + \rho
$$

and equations (5.9), give

$$
\rho = 1 \; , \; \nu = 1
$$

there follows,

$$
U(x) = \sin 2\pi x + x + 1 \; .
$$

Test problem 2, As a second example let us consider

$$
(5.11) \qquad \begin{cases} \dfrac{\mathrm{d}^2 u}{\mathrm{d}x^2} + \mu^2 u = 0 \\ u(0) = 0 \quad , \quad u(1) = 1 \; , \end{cases}
$$

which is obtained from (5.1) with,

$$\mu = D > 0 \quad , \quad f(x) = 0$$

and initial conditions

$$A = 0 \quad , \quad B = 1 \quad .$$

System (5.6) gives

$$\alpha_k = 0 \ , \alpha^*_k = 0 \ , \beta^n_k = 0 \ , \beta^{*n}_k = 0 \ , B_k = 0 \ , \ k = \mu/(2\pi)$$

so that the solution, according to (5.3),(5.6), is

$$u(x) = A \sin \mu x \ .$$

The constant A is determined by imposing the boundary conditions

$$0 = F(0) \ , \qquad 1 = A \sin \mu + F(1) \ .$$

An eigenfunction is $F(x) = 0$ so that $F(1) = 0$ thus we have

$$u(x) = \frac{\sin \mu x}{\sin \mu} \quad .$$

In any case, the wavelet solution is a trivial solution which doesn't show the presence of the localized functions. Even in this problem 2, the solution is a trivial case for the harmonic wavelets.

Conclusions

It has been given a method for the wavelet solution of ordinary differential equations. In particular it has been shown that by using the harmonic wavelets the Poisson problem can be trivially solved.

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