Coexistence of unlimited bipartite and genuine multipartite entanglement: Promiscuous quantum correlations arising from discrete to continuous-variable systems

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Quantum mechanics imposes "monogamy" constraints on the sharing of entanglement. We show that, despite these limitations, entanglement can be fully "promiscuous," i.e., simultaneously present in unlimited two-body and many-body forms in states living in an infinite-dimensional Hilbert space. Monogamy just bounds the divergence rate of the various entanglement contributions. This is demonstrated in simple families of N-mode ($N \geqslant 4$) Gaussian states of light fields or atomic ensembles, which therefore enable infinitely more freedom in the distribution of information, as opposed to systems of individual qubits. Such a finding is of importance for the quantification, understanding, and potential exploitation of shared quantum correlations in continuous variable systems. We discuss how promiscuity gradually arises when considering simple families of discrete variable states, with increasing Hilbert space dimension towards the continuous variable limit. Such models are somehow analogous to Gaussian states with asymptotically diverging, but finite, squeezing. In this respect, we find that non-Gaussian states (which in general are more entangled than Gaussian states) exhibit also the interesting feature that their entanglement is more shareable: in the non-Gaussian multipartite arena, unlimited promiscuity can be already achieved among three entangled parties, while this is impossible for Gaussian, even infinitely squeezed states.

I. INTRODUCTION

Entanglement is a core concept in quantum mechanics [1], and a key resource for quantum communication and information processing [2]. Unlike classical correlations, entanglement cannot be freely distributed [3]. In a multipartite compound system of two-level quantum objects (qubits), if two subsystems are maximally entangled, they cannot share any residual form of quantum correlations with the other remaining parties [4,5]. Analogous *monogamy* relations have been recently established for entanglement between canonical conjugate variables of continuous-variable (CV) systems [6,7], like harmonic oscillators, light modes, and atomic ensembles, endowed with an infinite-dimensional Hilbert space. In the general case of a state distributed among N parties (each owning a single qubit, or a single mode, respectively), the monogamy constraint on bipartite entanglement takes the form of the Coffman-Kundu-Wootters inequality [4],

$$E_{S_i|(S_1\cdots S_{i-1}S_{i+1}\cdots S_N)} \geqslant \sum_{j \neq i}^{N} E_{S_i|S_j}, \qquad (1)$$

where the global system is multipartitioned in subsystems S_k ($k=1,\ldots,N$), each owned by a respective party, and E is a proper measure of bipartite entanglement. The left-hand side of inequality (1) quantifies the bipartite entanglement between a probe subsystem S_i and the remaining subsystems taken as a whole. The right-hand side quantifies the total

bipartite entanglement between S_i and each one of the other subsystems $S_{j \neq i}$ in the respective reduced states. The nonnegative difference between these two entanglements, minimized over all choices of the probe subsystem, is referred to as the *residual multipartite entanglement*. It quantifies the purely quantum correlations that are not encoded in pairwise form, so it includes all manifestations of genuine K-partite entanglement, involving K subsystems at a time, with $2 < K \leqslant N$. In the simplest nontrivial instance of $N=3$, the residual entanglement has the (physical and mathematical) meaning of the genuine tripartite entanglement shared by the three subsystems [4,6,8]. In the general case $N > 3$, a natural way to discriminate between all those entanglement contributions has been very recently advanced in Ref. [9], and proven successful when addressing entanglement shared by harmonic systems under complete permutation invariance. The study of entanglement sharing and monogamy constraints thus offers a natural framework to interpret and quantify entanglement in multipartite quantum systems [3].

From an operational perspective, qubits are the main logical units for standard realizations of quantum information (QI) protocols [2]. Also CV Gaussian entangled resources have been proven useful for all known implementations of QI processing [10], including quantum computation [11], sometimes outperforming more traditional qubit-based approaches as in the case of teleportation [12]. It is therefore important to understand if special features of entanglement appear in states of infinite-dimensional Hilbert spaces, which are unparalleled in the corresponding states of qubits. Such

findings may lead to new ways of manipulating QI in the CV setting [13], and may in general contribute to a deeper and more complete understanding of entanglement in complex systems.

In this paper, we address this motivation on a clear-cut physical ground, aiming in particular to show whether the unboundedness of the mean energy characterizing CV states enables a qualitatively richer structure for distributed quantum correlations. We prove that multimode Gaussian states exist, that can possess simultaneously arbitrarily large pairwise bipartite entanglement between some pairs of modes and arbitrarily large genuine multipartite entanglement among all modes. In particular, we focus on a four-mode family of Gaussian states which are producible with standard optical means, the achievable amount of entanglement being technologically limited only by the attainable degree of squeezing. These states asymptotically reach the form of two perfectly entangled Einstein-Podolsky-Rosen (EPR) pairs that can, moreover, be arbitrarily intercorrelated quantumly, as well as share an asymptotically diverging genuine four-partite entanglement among all modes. We show that this *promiscuity* is fully compatible with the monogamy inequality (1) for CV entanglement as established in Refs. [6,7], and even with the stronger monogamy constraints on distributed bipartite and multipartite quantum correlations recently put forward in Ref. [9]. This original feature of entanglement sheds new light on the actual extent to which the basic laws of quantum mechanics curtail the distribution of information; on a more applicative ground, it may serve as a prelude to implementations of QI processing in an infinite-dimensional scenario that *cannot* be achieved with single qubit resources. While promiscuity might seem an unexpected property of quantum correlations, we will show that it indeed arises naturally and gradually with increasing dimension of the Hilbert space associated to the entanglement-sharing parties. In the limit of continuous-variable systems, non-Gaussian states will be shown to be in general sensibly more powerful than Gaussian states from the point of view of (promiscuous) entanglement sharing.

II. ENTANGLEMENT SHARING AMONG GAUSSIAN MODES

Entanglement in CV systems is encoded in the form of EPR correlations [14]. Let us consider the motional degrees of freedom of two particles, or the quadratures of a two-mode radiation field, where mode $k=i,j$ is described by the ladder operators $\hat{a}_k, \hat{a}_k^\dagger$ satisfying the bosonic commutation relation $[\hat{a}_k, \hat{a}_k^\dagger]=1$. An arbitrarily increasing degree of entanglement [15] (and of mean energy) can be encoded in a two-mode squeezed state $|\psi^{sq}\rangle_{i,j}=U_{i,j}(r)(|0\rangle_i \otimes |0\rangle_j)$ with increasing squeezing factor $r \in \mathbb{R}$, where the (phase-free) two-mode squeezing operator is [16] $U_{i,j}(r)=\exp\left[\frac{r}{2}(\hat{a}_i^\dagger \hat{a}_j^\dagger - \hat{a}_i \hat{a}_j)\right]$, and $|0\rangle_k$ denotes the vacuum state in the Fock space of mode k. In the limit of infinite squeezing ($r \to \infty$), the state approaches the ideal (unnormalizable, infinite-energy and unphysical) EPR state [14], simultaneous eigenstate of total momentum and relative position of the two subsystems, which thus share infinite entanglement. The entanglement of
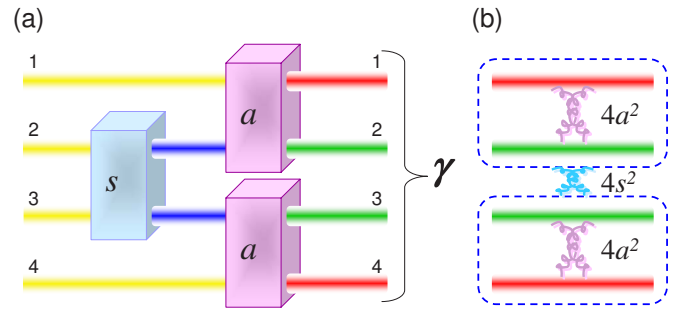

FIG. 1. (Color online) (a) Preparation of the four-mode Gaussian states $\boldsymbol{\gamma}$ of Eq. (2) via two-mode squeezers (boxes) applied on vacuum modes (yellow beams). (b) Structure of bipartite entanglement.

squeezed states and, more generally, *Gaussian states*, has been intensively studied in recent times [17]. Gaussian states of N modes are completely described in phase space by the $2N \times 2N$ real symmetric covariance matrix (CM) σ of the second moments $\frac{1}{2}\langle \hat{X}_i \hat{X}_j + \hat{X}_j \hat{X}_i \rangle$ of the canonical bosonic operators $\hat{X}_k \equiv \hat{q}_k = \hat{a}_k + \hat{a}_k^\dagger$, $\hat{X}_{N+k} \equiv \hat{p}_k = \hat{a}_k - \hat{a}_k^\dagger (k=1\cdots N)$. In this representation the unitary two-mode squeezing operator amounts to a symplectic matrix $S_{i,j}(r)=\begin{pmatrix} c_r & s_r \\ s_r & c_r \end{pmatrix} \oplus \begin{pmatrix} c_r & -s_r \\ -s_r & c_r \end{pmatrix}$ (where $c_r = \cosh r$, $s_r = \sinh r$, which acts by congruence on the CM, $\boldsymbol{\sigma} \mapsto S\boldsymbol{\sigma}S^T$.

Entanglement sharing has been addressed for Gaussian states in Refs. [6,7,9], by adapting and extending the original Coffman-Kundu-Wootters analysis [4] (see also [18] where a different analysis is performed, which, nonetheless, has implications for the sharing of correlations in multimode Gaussian states). In the most basic multipartite CV setting, namely that of three-mode Gaussian states, a partial "promiscuity" of entanglement can be achieved. Permutation-invariant states exist which are the Gaussian simultaneous analogs of Greenberger-Horne-Zeilinger (GHZ) and W states of qubits [8,19]. They possess unlimited tripartite entanglement (with increasing squeezing) and nonzero, accordingly increasing bipartite entanglement which, nevertheless, stays finite even for infinite squeezing [6]. We will now show that in CV systems with more than three modes, entanglement can be distributed in an *infinitely* promiscuous way. To illustrate the existence of such a phenomenon, we consider the simplest nontrivial instance of a family of four-mode Gaussian states, endowed with a partial symmetry under mode exchange.

III. FOUR-MODE STATES: STRUCTURAL AND ENTANGLEMENT PROPERTIES

We start with an uncorrelated state of four modes, each one initially in the vacuum of the respective Fock space, whose corresponding CM is the identity. We apply a two-mode squeezing transformation with squeezing s to modes 2 and 3, then two further two-mode squeezing transformations (redistributing the initial pairwise entanglement among all modes) with squeezing a to the pairs of modes 1,2 and 3,4 [see Fig. 1(a)]. For any value of the parameters s and a, the

output is a pure four-mode Gaussian state with CM $\boldsymbol{\gamma}$,

$$\boldsymbol{\gamma} = S_{3,4}(a)S_{1,2}(a)S_{2,3}(s)S_{2,3}^T(s)S_{1,2}^T(a)S_{3,4}^T(a). \quad (2)$$

A state of this form is invariant under the double exchange of modes $1 \leftrightarrow 4$ and $2 \leftrightarrow 3$, as $S_{i,j} = S_{j,i}$ and operations on disjoint pairs of modes commute.

In a pure four-mode Gaussian state and in its reductions, bipartite entanglement is equivalent to negativity of the partially transposed CM, obtained by reversing time in the subspace of any chosen single subsystem [20,21]. This inseparability criterion is readily verified for the family of states in Eq. (2) yielding that, for all nonzero values of the squeezings s and a, $\boldsymbol{\gamma}$ is entangled with respect to any global bipartition of the modes. The state is thus said to be *fully inseparable* [2], i.e., it contains genuine four-partite entanglement. Following previous studies on CV entanglement sharing we use the "contangle" τ [6] to quantify bipartite entanglement, an entanglement monotone under Gaussian local operations and classical communication. It is defined for pure states as the squared logarithmic negativity [22] (which quantifies how much the partially transposed state fails to be positive) and extended to mixed states via Gaussian convex roof [23,24], i.e., as the minimum of the average pure-state entanglement over all decompositions of the mixed state in ensembles of pure Gaussian states. If $\boldsymbol{\sigma}_{i|j}$ is the CM of a (generally mixed) bipartite Gaussian state where subsystem i comprises one mode only, then the contangle τ can be computed as

$$\tau(\boldsymbol{\sigma}_{i|j}) \equiv \tau(\boldsymbol{\sigma}_{i|j}^{opt}) = g[m_{i|j}^2], \quad g[x] = \text{arcsinh}^2[\sqrt{x-1}]. \quad (3)$$

Here $\boldsymbol{\sigma}_{i|j}^{opt}$ corresponds to a pure Gaussian state, and $m_{i|j} \equiv m(\boldsymbol{\sigma}_{i|j}^{opt}) = \sqrt{\text{Det}\,\boldsymbol{\sigma}_i^{opt}} = \sqrt{\text{Det}\,\boldsymbol{\sigma}_j^{opt}}$, with $\boldsymbol{\sigma}_{i(j)}^{opt}$ being the reduced CM of subsystem $i(j)$ obtained by tracing over the degrees of freedom of subsystem $j(i)$. The CM $\boldsymbol{\sigma}_{i|j}^{opt}$ denotes the pure bipartite Gaussian state which minimizes $m(\boldsymbol{\sigma}_{i|j}^p)$ among all pure-state CMs $\boldsymbol{\sigma}_{i|j}^p$ such that $\boldsymbol{\sigma}_{i|j}^p \leq \boldsymbol{\sigma}_{i|j}$. If $\boldsymbol{\sigma}_{i|j}$ is a pure state, then $\boldsymbol{\sigma}_{i|j}^{opt} = \boldsymbol{\sigma}_{i|j}$, while for a mixed Gaussian state Eq. (3) is mathematically equivalent to constructing the Gaussian convex roof. For a separable state $m(\boldsymbol{\sigma}_{i|j}^{opt}) = 1$. The contangle τ is completely equivalent to the Gaussian entanglement of formation [23], which quantifies the cost of creating a given mixed Gaussian state out of an ensemble of pure, entangled Gaussian states.

A. Structure of bipartite entanglement

In the four-mode state with CM $\boldsymbol{\gamma}$, we can compute the bipartite contangle in closed form [24] for all pairwise reduced (mixed) states of two modes i and j, described by a CM $\boldsymbol{\gamma}_{i|j}$. By applying the partial transpose criterion [20], we find that the two-mode states indexed by the partitions $1|3$, $2|4$, and $1|4$, are separable. For the remaining two-mode

states we find $m_{1|2} = m_{3|4} = \cosh 2a$, while $m_{2|3}$ is equal to

$$\frac{-1 + 2\cosh^2(2a)\cosh^2 s + 3\cosh(2s) - 4\sinh^2 a \sinh(2s)}{4[\cosh^2 a + e^{2s}\sinh^2 a]}$$

if

$$a < \text{arcsinh}[\sqrt{\tanh s}],$$

and to 1 (implying separability) otherwise. Accordingly, one can compute the pure-state entanglements between one probe mode and the remaining three modes. One finds $m_{1|(234)} = m_{4|(123)} = \cosh^2 a + \cosh(2s)\sinh^2 a$ and $m_{2|(134)} = m_{3|(124)} = \sinh^2 a + \cosh(2s)\cosh^2 a$.

Concerning the structure of bipartite entanglement, the contangle in the mixed two-mode states $\boldsymbol{\gamma}_{1|2}$ and $\boldsymbol{\gamma}_{3|4}$ is $4a^2$, irrespective of the value of s. This quantity is exactly equal to the degree of entanglement in a pure two-mode squeezed state $S_{i,j}(a)S_{i,j}^T(a)$ of modes i and j generated with the same squeezing a. In fact, the two-mode mixed state $\boldsymbol{\gamma}_{1|2}$ (and, equivalently, $\boldsymbol{\gamma}_{3|4}$) serves as a proper resource for CV teleportation [12], realizing a perfect transfer (unit fidelity [25]) in the limit of infinite squeezing a. The four-mode state $\boldsymbol{\gamma}$ reproduces thus (in the regime of very high a) the entanglement content of two EPR-like pairs (1,2 and 3,4). Remarkably, there is an additional, independent entanglement *between* the two pairs, given by $\tau(\boldsymbol{\gamma}_{(12)|(34)}) = 4s^2$ [the original entanglement in the two-mode squeezed state $S_{2,3}(s)S_{2,3}^T(s)$ after the first construction step, see Fig. 1(a)], which can be itself increased arbitrarily with increasing s. This peculiar distribution of bipartite entanglement [see Fig. 1(b)] is a first interesting signature of an unmatched freedom of entanglement sharing in multimode Gaussian states as opposed for instance to states of the same number of qubits, where a similar situation is *impossible*. Specifically, if in a pure state of four qubits the first two approach unit entanglement and the same holds for the last two, the only compatible global state reduces necessarily to a product state of the two singlets: no interpair entanglement is allowed by the monogamy constraint [4,5].

B. Residual entanglement

We can now move to a closer analysis of entanglement distribution and genuine multipartite quantum correlations. A first step is to verify whether the monogamy inequality (1) is satisfied on the four-mode state $\boldsymbol{\gamma}$ distributed among the four parties (each one owning a single mode) [26]. In fact, the problem reduces to proving that $\min\{g[m_{1|(234)}^2] - g[m_{1|2}^2], g[m_{2|(134)}^2] - g[m_{1|2}^2] - g[m_{2|3}^2]\}$ is nonnegative. The first quantity always achieves the minimum yielding

$$\tau^{res}(\boldsymbol{\gamma}) \equiv \tau(\boldsymbol{\gamma}_{1|(234)}) - \tau(\boldsymbol{\gamma}_{1|2})$$

$$= \text{arcsinh}^2\{\sqrt{[\cosh^2 a + \cosh(2s)\sinh^2 a]^2 - 1}\} - 4a^2. \quad (4)$$

Since $\cosh(2s) > 1$ for $s > 0$, it follows that $\tau^{res} > 0$. The entanglement in the global Gaussian state is therefore correctly distributed according to the monogamy law, in such a way
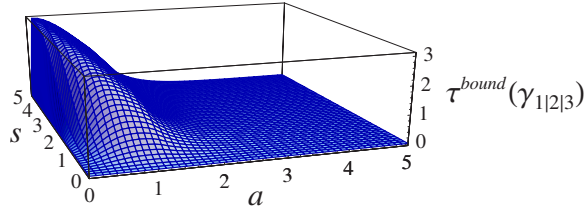
FIG. 2. (Color online) Upper bound $\tau^{bound}(\gamma_{1|2|3})$, Eq. (5), on the tripartite entanglement between modes 1, 2, and 3 (and equivalently 2, 3, and 4) of the four-mode Gaussian state defined by Eq. (2), plotted as a function of the squeezing parameters s and a. Focusing on the tripartition $1|2|3$, the bipartite contangle $\tau(\gamma_{i|(jk)})$ (with i,j,k a permutation of 1,2,3) is bounded from above by the corresponding bipartite contangle $\tau(\sigma^p_{i|(jk)})$ in any pure, three-mode Gaussian state with CM $\sigma^p_{i|(jk)} \leq \gamma_{i|(jk)}$. The state $\sigma^p = S_{1,2}(a)S_{2,3}(t)S^T_{2,3}(t)S^T_{1,2}(a)$ of the three modes 1, 2 and 3, with $t = \frac{1}{2}\text{arccosh}\left[\frac{1+\text{sech}^2 a \tanh^2 s}{1-\text{sech}^2 a \tanh^2 s}\right]$, satisfies this condition and from Eq. (3) we have $\tau(\gamma_{i|(jk)}) \leq g[(m^{bound}_{i|(jk)})^2]$. This leads to Eq. (5), where the quantity $g[(m^{bound}_{2|(13)})^2] - g[m^2_{1|2}] - g[m^2_{2|3}]$ with $m^{bound}_{2|(13)} = \sinh^2 a + m^{bound}_{3|(12)}\cosh^2 a$ is not included in the minimization, being always larger than the other terms.

that the residual contangle τ^{res} quantifies the multipartite entanglement not stored in couplewise form. Those quantum correlations can be either tripartite involving three of the four modes, and/or genuinely four-partite among all of them. Concerning the tripartite entanglement, we first observe that in the tripartitions $1|2|4$ and $1|3|4$ the tripartite entanglement is zero, as mode 4 is not entangled with the block of modes 1,2, and mode 1 is not entangled with the block of modes 3,4.

The only tripartite entanglement present, if any, is equal in content (due to the symmetry of the state γ) for the tripartitions $1|2|3$ and $2|3|4$, and can be quantified by the residual contangle (a Gaussian entanglement monotone [6]) determined by the corresponding three-mode monogamy inequality (1). One can show (see caption of Fig. 2) that such genuine tripartite entanglement $\tau^{tri}(\gamma_{1|2|3})$ is bounded from above by the quantity

$$\tau^{bound}(\gamma_{1|2|3}) \equiv \min\{g[(m^{bound}_{1|(23)})^2] - g[m^2_{1|2}], g[(m^{bound}_{3|(12)})^2]$$
$$- g[m^2_{2|3}]\}, \tag{5}$$

with $m^{bound}_{3|(12)} = \frac{1+\text{sech}^2 a \tanh^2 s}{1-\text{sech}^2 a \tanh^2 s}$, $m^{bound}_{1|(23)} = \cosh^2 a + m^{bound}_{3|(12)}\sinh^2 a$. The upper bound $\tau^{bound}(\gamma_{1|2|3})$ is always non-negative (as a consequence of monogamy [6]), is decreasing with increasing squeezing a, and vanishes in the limit $a \to \infty$, as shown in Fig. 2. Therefore in the regime of increasingly high a, eventually approaching infinity, any form of tripartite entanglement among any three modes in the state γ is negligible (exactly vanishing in the limit). As a crucial consequence, in that regime the residual entanglement $\tau^{res}(\gamma)$ determined by Eq. (4) is all stored in four-mode quantum correlations and quantifies the *genuine* four-partite entanglement.

We finally observe that $\tau^{res}(\gamma)$ is an increasing function of a for any value of s (see Fig. 3), and it *diverges* in the limit $a \to \infty$. This proves that the class of pure four-mode Gaussian
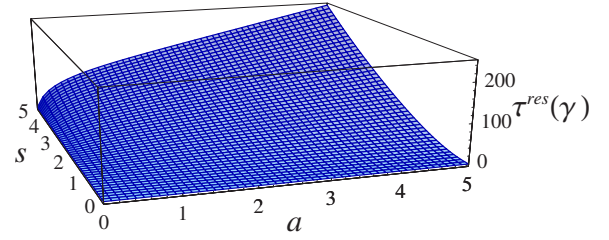

FIG. 3. (Color online) Residual multipartite entanglement $\tau^{res}(\gamma)$ [Eq. (4)], which in the regime of large squeezing a is completely distributed in the form of genuine four-partite quantum correlations.

states with CM γ given by Eq. (2) exhibits genuine four-partite entanglement which grows unboundedly with increasing squeezing a and, simultaneously, possesses pairwise bipartite entanglement in the mixed two-mode reduced states of modes 1,2 and 3,4, that increases unboundedly as well with increasing a [15]. Moreover, as previously shown, the two pairs can themselves be arbitrarily entangled with each other with increasing squeezing s. Notice that usual monogamy inequalities do not typically constrain "hybrid" entanglement distributions such as 2×2 vs 1×1 or 1×3, even though it is reasonable to expect that some limitations exist also when entanglement is shared over such partitions.

C. Strong monogamy and genuine four-partite entanglement

It would be desirable to have a clear distinction between tripartite and four-partite quantum correlations also in the range in which the former are not negligible, i.e., for small a. A simple idea comes from Ref. [9], which adapted to our setting states the following. The minimum residual entanglement Eq. (4) between one probe mode and the remaining three, is equal to the total three-mode entanglement involving the probe mode and any two of the others, plus the genuine four-partite entanglement shared among all modes. Such a construction, which provides an additional decomposition for the difference between the left-hand side and the right-hand side of Eq. (1), leads in general to the postulate that a *stronger* monogamy constraint acts on entanglement shared by N parties, which imposes a trade-off on both the bipartite and all forms of genuine multipartite entanglement on the same ground. Existence of such a constraint, which directly generalizes the Coffman-Kundu-Wootters ($N=3$) case [4], has been indeed proven for arbitrary Gaussian states endowed with permutation invariance (i.e., for the generalized N-mode Gaussian GHZ/W states and their mixed reductions), and a *bona fide* quantification of the genuine N-party entanglement (computed again by means of the contangle) has been obtained on such states [9].

We will now show that strong monogamy also holds in the partially symmetric four-mode Gaussian states of Eq. (2), which is an important indication of the generality of the approach presented in [9]. Indeed, strong monogamy immediately follows from Eqs. (4),(5), as $\tau^{res}(\gamma) - \tau^{tri}(\gamma_{1|2|3}) \geq \tau^{res}(\gamma) \geq \tau^{bound}(\gamma_{1|2|3}) \equiv \tau^{Abound}(\gamma_{1|2|3|4}) \geq 0$. The quantity

$\tau^{Abound}(\boldsymbol{\gamma}_{1|2|3|4})$ is a *lower* bound to the genuine four-partite entanglement shared by the four modes in the state $\boldsymbol{\gamma}$, for any a and s. Its functional dependence on these squeezing parameters is very similar to that of $\tau^{res}(\boldsymbol{\gamma})$ (one needs only to subtract the surface in Fig. 2, which extends over a very narrow scale, from the surface in Fig. 3). $\tau^{res}(\boldsymbol{\gamma})$ represents actually an upper bound for the four-partite entanglement, which gets asymptotically tight for $a \gg 1$: in this limit, upper and lower bounds coincide and as already remarked, the residual entanglement *is* all in four-partite form. This brief analysis is very relevant on one hand for the purposes of Ref. [9] as it embodies the first proof of strong monogamy beyond strong symmetry requirements; while on the other hand, in the scope of the present paper, it enables us to regard the considerations of the previous subsection as mathematically valid over all the parameter space, and not only in the limit of high squeezing a.

IV. DISCUSSION: WHERE DOES PROMISCUITY COME FROM?

A. Some operational considerations

From a practical point of view, two-mode squeezing transformations are basic tools in the domain of quantum optics [16] (they occur, e.g., in parametric down-conversions), and the amount of producible squeezing in experiments is constantly improving [27]. Only technological, no *a priori* limitations need to be overcome to increase a and/or s to the point of engineering excellent approximations to the demonstrated promiscuous entanglement structure in multimode states of light and atoms (see also [28]). To make an explicit example, already with realistic squeezing degrees like $s = 1$ and $a = 1.5$ (corresponding to ~ 3 and 10 dB, respectively), one has a bipartite entanglement of $\tau(\boldsymbol{\gamma}_{1|2}) = \tau(\boldsymbol{\gamma}_{3|4}) = 9$ ebits (corresponding to a Gaussian entanglement of formation [23] of ~ 3.3 ebits), coexisting with a residual multipartite entanglement of $\tau^{res}(\boldsymbol{\gamma}) \simeq 5.5$ ebits, of which the tripartite portion is at most $\tau^{bound}(\boldsymbol{\gamma}_{1|2|3}) \simeq 0.45$ ebits. This means that one can simultaneously extract at least three qubit singlets from each pair of modes $\{1,2\}$ and $\{3,4\}$, and more than a single copy of genuinely four-qubit entangled states (like cluster states). Albeit with imperfect efficiency, this entanglement transfer can be realized by means of Jaynes-Cummings interactions [29], representing a key step for a reliable physical interface between fields and qubits in a distributed QI processing network.

B. Emergence of promiscuity from discrete to continuous variable systems, and beyond Gaussian states

By constructing a simple and feasible example we have shown that, when the quantum correlations arise among degrees of freedom spanning an infinite-dimensional space of states (characterized by unbounded mean energy), an accordingly infinite freedom is tolerated for QI to be doled out. The construction presented here can be straightforwardly extended to investigate the increasingly richer structure of entanglement sharing in N-mode (N even) Gaussian states via additional pairs of two-mode squeezing operations which

further redistribute entanglement. Notice once more that the promiscuous entanglement distribution happens with no violation of the *a priori* monogamy constraints that retain their general validity in quantum mechanics. The motivation is the following. In the CV scenario, naively speaking, if one writes from Eq. (1) that the entanglement between one mode and the rest is equal to the sum of bipartite entanglements plus the residual multipartite entanglement, one can in principle approach the limiting "fully promiscuous" case "$\infty = \infty + \infty$." The only restriction imposed by monogamy is to bound the divergence rates of the individual entanglement contributions as the squeezing parameters are increased. Within the restricted Hilbert space of four or more qubits, instead, an analogous entanglement structure between the single qubits is strictly forbidden (one can either have "$1 = 1 + 0$" or "$1 = 0 + 1$").

Our results open original perspectives for the understanding and characterization of entanglement in multiparticle systems. Gaussian states with finite squeezing (finite mean energy) are somehow analogous to discrete systems with an effective dimension related to the squeezing degree. As the promiscuous entanglement sharing arises in Gaussian states by asymptotically increasing the squeezing to infinity, it is natural to expect that dimension-dependent families of states will exhibit an entanglement structure gradually more promiscuous with increasing Hilbert space dimension towards the CV limit. Let us further clarify this point. In the bipartite, pure-state instance, a two-mode squeezed Gaussian state $|\psi_{sq}(r)\rangle$ with squeezing $r \equiv (\log d)/2$ has exactly the same entanglement $E = \log d$ (quantified by the logarithmic negativity [22]) as a maximally entangled state of two *qudits*, $|\Phi_d\rangle = d^{-1/2} \sum_{i=1}^{d} |ii\rangle \in \mathbb{C}^d \otimes \mathbb{C}^d$. Thus for any finite squeezing, Gaussian entanglement is not, in a sense, taking advantage of the full infinite-dimensional Hilbert space, but only of a restricted, finite section of it. Only for $r \to \infty$, and accordingly $d \to \infty$, the states $|\psi_{sq}(r \to \infty)\rangle$ and $|\Phi_{d \to \infty}\rangle$ tend towards the (unphysical) EPR state, and the entanglement and the mean energy asymptotically diverge [15].

With such an analogy in mind, let us now construct the simplest conceivable family of tripartite states, in which we will witness promiscuity of distributed entanglement arising with increasing dimensionality of the Hilbert space. We define the pure permutationally invariant state $|\Psi_{2N}\rangle \in \mathbb{C}^{6N}$ of a system of $3N$ qubits (where $N \geq 2$ is an even integer, and we label the state with the subscript "$2N$" for future convenience) as the tensor product of $N/2$ copies of the three-qubit GHZ state [19], and of $N/2$ copies of the three-qubit W state [8]. In formula,

$$|\Psi_{2N}\rangle = |\psi_{\text{GHZ}}\rangle^{\otimes(N/2)} \otimes |\psi_W\rangle^{\otimes(N/2)}, \qquad (6)$$

where $|\psi_{\text{GHZ}}\rangle = (|000\rangle + |111\rangle)/\sqrt{2}$ and $|\psi_W\rangle = (|001\rangle + |010\rangle + |100\rangle)/\sqrt{3}$. By grouping our $3N$ qubits into three parties A, B, and C, each formed by N qubits (each qubit, in turn, taken from a single copy of either the GHZ or the W state), the state $|\Psi_d\rangle \in \mathbb{C}^d \otimes \mathbb{C}^d \otimes \mathbb{C}^d$ of Eq. (6) becomes the tripartite state of the three *qudits ABC* with $d \equiv 2N$. A graphical depiction of such a construction is provided in Fig. 4. It is straightforward now to evaluate the entanglement properties of $|\Psi_d\rangle$. Using the tangle as an entanglement measure [30]
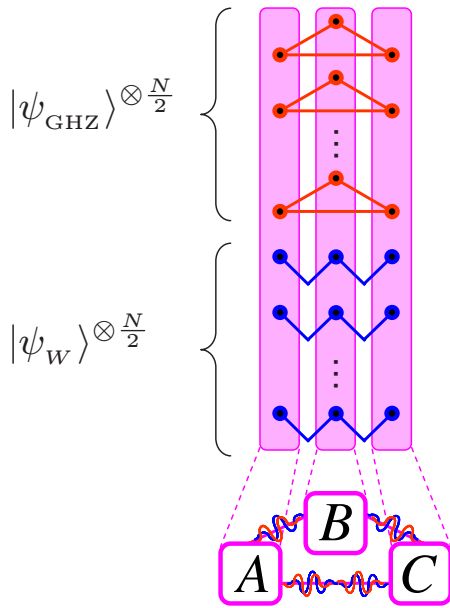
FIG. 4. (Color online) A tripartite pure state of three *qudits* ($d=2N$ with N even) A, B, and C, is constructed as the tensor product of $N/2$ copies of a three-qubit GHZ state, and $N/2$ copies of a three-qubit W state. In the limit $d \to \infty$, ABC constitute a tripartite CV system whose (non-Gaussian) state exhibits full promiscuity, i.e., unlimited genuine tripartite entanglement and unlimited bipartite entanglement between any two parties.

we find that the three *qudits ABC* share a genuine tripartite residual entanglement $\mathcal{T}^{A|B|C} = d/4$, as the GHZ state has unit three-tangle, and the W state has zero three-tangle [4,8]. Similarly, the bipartite entanglement between any two *qudits* is $\mathcal{T}_d^{A|B} = \mathcal{T}_d^{A|C} = \mathcal{T}_d^{B|C} = d/9$, as the two-qubit reductions of W states have a tangle of $4/9$, while the two-qubit reductions of GHZ states are separable. Monogamy is clearly satisfied (the tangle between one party and the other two is $\mathcal{T}^{A|(BC)} = 17d/36$), and one can see that both—tripartite and every couplewise—forms of entanglement are linearly increasing with the dimension d, and are so monotonically increasing functions of each other. Bipartite and genuine multipartite entanglement can grow unboundedly (i.e., diverging for $d \to \infty$), and are mutually enhanced: this is exactly what the concept of promiscuity means. Only for a three-qubit system there is mutual exclusion between maximum tripartite entanglement (the GHZ case) and maximum two-party reduced entanglement (the W case). As soon as *qudits* with $d > 2$ are concerned, an increasingly more freedom is let for entanglement distribution compatibly with monogamy. An alternative demonstration of the full promiscuity of entanglement in the state of Eq. (6) is provided in the Appendix by employing the "squashed entanglement" [31] as bipartite entanglement monotone.

From this simple example we learn several things. First, that promiscuity gradually arises with increasing dimension d of the Hilbert space, mediating between the prohibitive qubit scenario, and the libertine CV one. Second, that the analogy between finite-squeezing CV states and finite-dimensional systems can be extended to the multipartite scenario. Third, and perhaps more interestingly, we see that

Gaussian states are too structurally simple as compared to general non-Gaussian states of CV systems, and are hence sometimes unable to achieve the full power of entanglement distribution in infinite-dimensional systems even in the limit of a diverging mean energy per mode. What we mean is quite straight. Taking the limit $d \to \infty$ in the above *qudit* picture, the state $|\Psi_{d \to \infty}\rangle$ exhibits asymptotically diverging entanglement both in genuine tripartite form and in couplewise form between any two parties of the CV systems A, B, and C, i.e., an *unlimited*, full promiscuous structure. We know instead that three-mode Gaussian states can never achieve such a distribution of entanglement [6] (the reduced two-mode entanglement stays finite even for infinite squeezing), and in this paper we had to resort to four-mode states with broken symmetry to demonstrate unlimited promiscuity in the Gaussian scenario.

The state $|\Psi_{d \to \infty}\rangle$ is indeed a highly non-Gaussian state: a recently introduced quantifier of non-Gaussianity [32], which measures the normalized Hilbert-Schmidt distance δ between a non-Gaussian state and a reference Gaussian state with the same first and second moments, yields [33]

$$\delta(|\Psi_d\rangle) = \frac{1}{2} + 2^{-(3d/4)-1}3^{-d/4} - 2^{d/2}3^{-3d/2}7^{d/4}. \qquad (7)$$

On a scale ranging from 0 to 1, $\delta(|\Psi_d\rangle) \geq 0.48$ for any $d \geq 4$ (recall that $d=4$ corresponds to just one copy of the GHZ state and one copy of the W state), and increases with d converging to $1/2$ for $d \to \infty$. As a term of comparison, $1/2$ is the asymptotic non-Gaussianity of the nth single-mode Fock state $|n\rangle$ for $n \gg 1$ [32]. We have thus given the first evidence that monogamy of CV entanglement holds beyond the Gaussian case (although the state we considered is highly nongeneric, see the discussion in [34]), and that, crucially, non-Gaussian states can achieve a strictly more promiscuous structure in the sharing of quantum correlations than Gaussian states. More promiscuity means more and stronger forms of entanglement encoded in different multipartitions and simultaneously accessible for quantum information purposes.

One can readily provide generalizations of the above example to convince oneself that for an arbitrary number of *qudits* in the limit $d \to \infty$, a fully promiscuous structure of entanglement is available (under and beyond permutation invariance) for the corresponding multipartite non-Gaussian CV states. Permutation-invariant Gaussian states (the so-called Gaussian GHZ/W states [6]), instead, are always promiscuous for any N, but never *fully* promiscuous, in the sense that the reduced bipartite and/or K-partite entanglements ($K < N$) in the various partitions always stay finite (and increasingly low with higher N) even in the limit of infinite squeezing [9]. And we stress that this is not a consequence of restricting to a single mode per party: even considering a big MN-mode CV system, partitioned into N subsystems each comprising M modes, the corresponding N-partite Gaussian entanglement (under permutation invariance) is exactly *scale invariant*, i.e., it is equal to the entanglement among N single modes (i.e., to the case $M=1$) [9], hence can never be fully promiscuous. Of course, while Gaussian states cannot reach the shareability of non-

Gaussian states, mimicking the former via the latter is always possible. If one wishes to reproduce the N-mode permutation-invariant Gaussian instance in terms of *qudits* with varying d, it is sufficient, e.g., for $N=3$ to modify the construction of Fig. 4 by unbalancing (as a function of d) the fraction of GHZ vs W copies, and keeping the latter number finite while allowing the former to diverge with d. In fact, as just shown, this Gaussian-like setting represents clearly not the best entanglement sharing structure that states in infinite-dimensional Hilbert spaces may achieve. It is worth pointing out, however, that in the current experimental practice, should a certain degree of promiscuity be required, it would be surely easier to produce and manipulate Gaussian states of four modes, with a quite high squeezing, than tensor products of many three-qubit states.

C. Concluding remarks and outlook

Inspired by the above discussion on the origin of entanglement promiscuity, a more extended investigation into the huge moat of *qudits* appears as the next step to pursue, in order to aim at developing the complete picture of entanglement sharing in many-body systems [3]. Here, we have initiated this program by establishing a sharp discrepancy between the two *extrema* in the ladder of Hilbert space dimensions: namely, entanglement of CV systems in the limit of infinite mean energy has been proven infinitely more shareable than that of individual qubits. We have also given indication that such a transition between these two diametrically opposite cases is not sharp, but smeared on the whole scale of Hilbert space dimensionalities $2 < d < \infty$, as promiscuity gradually and smoothly arises with increasing d.

Once a more comprehensive understanding will be available of the distributed entanglement structure in high-dimensional and CV systems (also beyond Gaussian states), the interesting task of devising new protocols to translate such potential into full-power QI processing implementations can be addressed as well. In this respect, it is worth remarking again that non-Gaussian states, which by virtue of the extremality theorem [35] are systematically more entangled than Gaussian ones at fixed second moments in a bipartite scenario, appear also to have a richer structure when entanglement sharing is addressed. In the tripartite case, for instance, we have constructed an example of a fully promiscuous non-Gaussian state, while promiscuity of three-mode Gaussian entanglement is always partial. This provides an additional motivation for further characterizing non-Gaussian entanglement also in the multipartite scenario, together with its possible exploitation, e.g., to perform highly efficient teleportation networks [36] or to solve the Byzantine agreement problem [37].

APPENDIX: NON-GAUSSIAN (FULL) VS GAUSSIAN (PARTIAL) PROMISCUITY OF TRIPARTITE ENTANGLEMENT SHARING VIA THE SQUASHED ENTANGLEMENT

Here we provide an alternative comparison (and contrast) between the structure of entanglement sharing in the permutationally invariant pure CV state obtained by taking the limit $d \to \infty$ of the three-*qudit* state in Eq. (6), and the permutationally invariant pure three-mode Gaussian states (Gaussian GHZ/W states) introduced in Ref. [6]. Even though more abstract, this analysis is perhaps more conclusive as the same entanglement measure is employed for both states. The *squashed entanglement* E_{sq} is an interesting information-theoretic entanglement monotone which is, among the other properties, additive on tensor product states [31]. Moreover, it has been proven to fulfill monogamy, in the sense of Eq. (1), for arbitrary N-partite systems, regardless of the associated Hilbert space dimensions [38]. Unfortunately, this measure is uncomputable for almost all entangled states, even though it is known to be upper bounded by the entanglement cost, and lower bounded by the distillable entanglement (hence equal to the von Neumann entropy of entanglement on pure states). Let us bound the different entanglement contributions in the state of Eq. (6) in terms of squashed entanglement. The 1×2 entanglement between one *qudit*, say A, and the other two, is

$$E_{sq}^{A|(BC)}(|\Psi_d\rangle) = (d/4)[E_{sq}^{A|(BC)}(|\psi_{GHZ}\rangle) + E_{sq}^{A|(BC)}(|\psi_W\rangle)]$$
$$\approx 0.48d \tag{A1}$$

(in this case the squashed entanglement reduces on each copy to the computable von Neumann entropy of one qubit). Concerning two-party entanglement, the GHZ states have all separable two-qubit reductions; on the other hand, the bipartite squashed entanglement in the reduced two-qubit states of each W copy amounts to some quantity, say ω, which is strictly positive, as every entangled two-qubit state has a nonzero distillable entanglement (positive partial transposition criterion [39]). The total bipartite squashed entanglement between any two *qudits* in the state Eq. (6) is thus

$$E_{sq}^{A|B}(|\Psi_d\rangle) = \omega d/4, \quad \omega > 0. \tag{A2}$$

From the general monogamy of the squashed entanglement, we know that the residual three-partite entanglement is well defined and positive,

$$E_{sq}^{A|B|C}(|\Psi_d\rangle) = E_{sq}^{A|(BC)}(|\Psi_d\rangle) - 2E_{sq}^{A|B}(|\Psi_d\rangle).$$

Exploiting in particular the fact that squashed entanglement is monogamous on each W copy, $E_{sq}^{A|(BC)}(|\psi_W\rangle) \geqslant 2\omega$, we immediately obtain a lower bound on the three-*qudit* residual entanglement,

$$E_{sq}^{A|B|C}(|\Psi_d\rangle) \geqslant d/4, \tag{A3}$$

where we have used the fact that for each GHZ copy $E_{sq}^{A|B|C}(|\psi_{GHZ}\rangle) = 1$. Looking at Eqs. (A1)–(A3), we observe once more that all forms of (squashed) entanglement are mutually increasing functions of each other on the state of Eq. (6), and diverging in the limit $d \to \infty$. This definitely proves

that the corresponding asymptotic (non-Gaussian) CV state, is definitely fully promiscuous, compatibly with the monogamy of entanglement sharing.

On the other hand, let us repeat the same analysis for the Gaussian counterparts, represented by pure permutation-invariant three-mode "GHZ/W" Gaussian states [6] with asymptotically diverging squeezing (and mean energy) in any single mode. The 1×2 squashed entanglement, equal to the von Neumann entropy of one mode, diverges with the squeezing (for $r \gg 0$ it goes $\sim 2r$). The reduced two-mode squashed entanglement, instead, is smaller than the corre-

sponding entanglement of formation (which equates the entanglement cost as its additivity is proven in this special case [23]). The latter increases with the squeezing, converging to the finite value ≈ 0.278 in the limit $r \to \infty$. This unambiguously proves that permutation-invariant tripartite Gaussian states exhibit only a partial promiscuity, as the genuine tripartite entanglement (obtained by difference) diverges for infinite squeezing, while the reduced bipartite entanglement stays constant. It is also known that no three-mode Gaussian states can be more promiscuous than these GHZ/W states [6].

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