THE VaR OF THE MATHEMATICAL PROVISION: CRITICAL ISSUES

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ABSTRACT[1]
The paper addresses the original question of calculation and application of the Value at Risk of the mathematical provision in a fair valuation context. The VaR calculation poses both methodological and numerical problems. The first issue concerns the choice of the VaR models and the number of risk factors, while the second one regards the calculation technique. The paper provides for an insight into the determinants of the VaR of the mathematical provision and for a calculation performed by using a simulation approach. As far as the applications are concerned, managerial, regulatory and solvability implementations are explored and discussed.

Keywords: Value at Risk, Life insurance, Reserve, Solvency, Fair Value.

[1] Although the paper is a result of a joint study of the authors, §1 and §2 are due to R. Cocozza, whilst §3 and §4 are due to E. Di Lorenzo, A. Orlando and M. Sibillo.

§1 – INTRODUCTION

At the end of March 2004, the International Accounting Standards Board issued the International Financial Reporting Standard 4 Insurance Contracts. For the first time, it provides guidance on accounting for insurance contracts, and marks the first step in the IASB's project to achieve the convergence of widely varying insurance industry accounting practices around the world. More specifically, the IFRS "permits an insurer to change its accounting policies for insurance contracts only if, as a result, its financial statements present information that is more relevant and no less reliable, or more reliable and no less relevant". Moreover, "it permits the introduction of an accounting policy that involves remeasuring designated insurance liabilities consistently in each period to reflect current market interest rates (and, if the insurer so elects, other current estimates and assumptions)", thus giving rise to a potential reclassification of some or all financial assets as "at fair value through profit or loss". As known, the recognition of a fair value disclosure requirement has to comply with the lack of agreement upon a definition of fair value as well as of any guidance from the Board on how the fair value has to be calculated. According to the majority of commentators, this uncertainty may lead to fair value disclosures that are unreliable and inconsistently measured among insurance entities. Since market valuations do not exist for many items on the insurance balance sheet, it is necessary to rely on entity specific measurement for determining insurance contract and asset fair values. As a consequence, such values may be subject to wide ranges of judgment and to significant abuse. Ultimately, they may provide information that is not at all comparable among companies. The cause for this concern is the risk margin component of the fair value. Risk margins are clearly a part of market values for uncertain assets and liabilities, but with respect to many insurance contracts, their value cannot be reliably calibrated to the market. Hence, a market-based valuation basis for them would produce irrelevant information.

This statement gives rise to a wider trouble. If there is an amendment in the evaluation criteria for the reserve from one year to another – according to current market yields or even to current mortality tables – there is a possible change in the value of the reserve according to the application of a more stringent or, at the opposite, a more flexible criterion. This may turn into a proper fair valuation risk (Cocozza et al., 2005). In the accounting perspective, the introduction of an accounting policy involving remeasuring designated insurance liabilities consistently in each period to reflect current market interest rates (and, if the insurer so elects, other current estimates and assumptions) implies that the fair value of the mathematical provision is properly a *current value* or a *present value*. Consistently, the fair value of the mathematical provision could be properly defined as the net present value of the residual debt towards the policyholders evaluated *at current interest rates and, eventually, at current mortality rates*. In a sense, this is marking to market. However, this is the crucial point. The International Accounting Standards Committee defines the fair value as "the amount for which an asset could be exchanged, or a liability settled, between knowledgeable, willing parties in an arm's-length transaction", while the Financial Accounting Standard Board defines the fair value as "an estimate of an exit price determined by market interactions". Hence, the CAS Fair Value Task Force defined fair value as "the market value, if a sufficiently active market exists, or an estimated market value, otherwise". From the accounting side a fair value is not necessarily an equilibrium price, but *merely* a market price and, in the case of the mathematical provision, an *estimated market price*: therefore, there is an *assortment* of prices and not an unique reference. The variation of such *evaluated price* gives rise to the recalled fair valuation risk and opens the path to a quantification of contingency provisions.

In this perspective, the paper addresses the original question of calculation and application of the Value at Risk of the mathematical provision in a fair valuation context. It starts with a survey of the issues involved into the calculation of the VaR of the mathematical provision (section 2) and goes on with the mathematical formalization (section 3) and a numerical solution (section 4) to the problem.

§2 – THE RATIONALE OF THE VaR

Typically, VaR is defined as "the predicted worst-case loss at a specific confidence level over a certain period of time". This definition is based on the identification of the probability density function for the one-period profit or loss of a portfolio and on the capability of summarizing the distribution with a single statistic, often reported as the maximum loss that can occur within a given confidence interval. When this approach is applied to a balance sheet item and namely to a liability, such as the mathematical provision, the classical definition needs some specifications.

To begin with, worst cases coincide with an increases in the value of the liability, because these rises are the counterparty of expenses or, better, additional costs which may result in either a profit shrinkage or a proper loss. Therefore, the classical portfolio return distribution can be redesigned as a *liability cost distribution*, where critical values lie in the right-hand tail.

Moreover, as for assets the portfolio return distribution is centred on the expected return, so for a liability the cost distribution has to be centred on the *expected cost*. In the case of the mathematical provision the expected cost can be easily linked to the expected value of the reserve at the end of the risk horizon, which is the value of the reserve without any modification of relevant risk factors (Cocozza *et al.*, 2004c; 2005), a part from the time passage. In this perspective the *expected reserve* is the predicted value of the reserve at end of the risk horizon calculated using the informative set available at the beginning of the risk horizon.

Finally, there is the risk horizon. In a balance sheet approach, the risk horizon should coincide with the balance period, that is to say with a year by year evaluation if the balance is annual or a semester evaluation if the balance is six-monthly. In a risk management perspective the risk horizon could be even different from the balance period.

Therefore, the VaR of the mathematical provision can be originally defined as *the predicted worst case additional cost at a specific confidence level over a period of time consistent with the risk analysis*.

As far as the risk factors are concerned, a full VaR estimation should consider both financial and actuarial risk factors. Setting apart the actuarial risk components, a first order approximation of the performance variation due to a change in the evaluation interest rate can be obtained through the cash flow mapping of the reserve. The mathematical provision can be assimilated to a portfolio of zero coupon bonds – whose nominal value is the certain-equivalent of the conditional payment – with maturity equal to each single critical (remaining) instant and with portfolio contribution equal to the ratio of the individual present value of each cash flow to the total present value.

If we concentrate only on the financial risk factor, the expected reserve, in a fair valuation context, is that value of the mathematical provision calculated by means of the forwards rates implied by current spot rates. Therefore, the expected cost of the mathematical provision can be defined as the algebraic summation of the initial provision and the expected reserve. In any case in which the summation of the initial and the effective final provision is higher then the expected cost there is an *accounting loss*. The corresponding potential loss can be effectively measured by means of the reserve VaR, assuming it is possible to produce a significant estimate of the relevant risk factors (Cocozza *et al.* 2005). In this perspective, it is possible to estimate at the beginning of each risk horizon (namely of each year) the maximum additional cost due to a change in the interest rates that is possible to experiment at a specific confidence level.

The estimation of such Value at Risk can have multiple applications both internal and external. From a managerial point of view, this measure can serve as a benchmark for the quantification of the contingency provisions and as a guide in the profit distribution. From an institutional perspective, the measure can be effectively applied for comparison among companies, given that a fair valuation system may reduce the comparability of result across different insurance entities. In this perspective, the VaR as comparability measure could be usefully applied both for accounting standards and for solvency requirement.

§3: THE MATHEMATICAL MODEL

3.a: The quantile reserve and the Value at Risk

The quantification of a liability fair value can be approached introducing the replicating portfolio, that is a portfolio of financial instruments giving origin to a cash flow matching that one underlying the liability itself. Of course, in the case of liability traded in an existing market, the fair value would coincide with the market value itself. The fair valuation of life insurance liabilities, since referring on cash flows depending on the human life duration and so not trading in an existing market, can be considered existing in the economic reality, as stated in Buhlmann 2004 and can be measured, as a consequence, by means of a portfolio of financial instruments.

In the fair valuation order of ideas, the stochastic nature of the cash flows underlying the life policies appears crucial and as therefore the need of using the probability distribution of liabilities according to stochastic projections.

Within the stochastic scenario described for the liability pattern, it is possible to introduce quantitative tools, significant in actuarial practice, such as the *quantile reserve*. Indicating by R(t) the financial position at time t, that is, in this case, the stochastic mathematical provision of a life insurance contract, or a portfolio of contracts, the quantile reserve at confidence level α ($0<\alpha<1$), is expressed by the value $R_\alpha^*(t)$ in the following equation:

$$P\{R(t)> R_\alpha^*(t)\}=\alpha$$

This representation gives rise to a market consistent value of the insurance liability. As a consequence, assessment can be significantly measured by stochastic methods such as *Value-at-Risk*, calculated at the beginning of each year of the life contract.

Let us consider the time interval $[t,t+h]$ and indicate the financial position at its extremes respectively $r(t)$ and $R(t+h)$. The potential periodic loss is defined as:

$$L=r(t)-R(t+h).$$

At confidence level α, the Value-at-Risk $VaR(\alpha)$ is given by the requirement:

$$P\{L> VaR(\alpha)\}=\alpha$$

representing the probable maximum expense, in other words it means that, in $(1-\alpha)100\%$ of the cases, the expense is smaller or equal to $VaR(\alpha)$. In terms of the function F, representing the distribution of L, the following expression holds:

$$VaR(\alpha)=F^{-1}(1-\alpha).$$

3.b: The valuation scheme – the model

Let us introduce two probability spaces (Ω, F',P'), (Ω, F'',P''), where F' and F'' are the σ-algebras containing, respectively, the *financial events* and the *life duration events*.

Assuming the independence of the randomness in mortality on the fluctuations of interest rates, we denote by (Ω, F,P) the probability space generated by the preceding two. F contains the information flow about both mortality and financial history, represented by the filtration $\{F_k\}\subset F$ ($F_k = F'_k \cup F''_k$ with $\{F'_k\}\subset F'$ and $\{F''_k\}\subset F''$).

If N_j is the number of claims (living or dead according to the kind of life contract) at time j within a portfolio of identical policies, we evaluate at time t the stochastic stream of cash-flow $\mathbf{X}^t = N_{t+1}X_{t+1}, N_{t+2}X_{t+2}, \ldots, N_nX_n$, that is the stochastic loss at time t, referring to a portfolio perspective.

As usually assumed in a *fair valuation* framework, the market is frictionless, with continuous trading, no restrictions on borrowing or short-sales, the zero-bond and the stocks are both infinitely divisible.
The fair value of the reserve is given by

$$V_t = E\left[\sum_{j>t} N_j X_j v(t,j)/F_t\right] \tag{1}$$

according to a risk-neutral valuation, where *v(t,j)* is the present value at time t of one monetary unit due at time j and *E* represents the expectation under the risk-neutral probability measure, whose existence derives by well known results, based on the completeness of the market.

In the recent literature several authors remarked that the demographic valuation is not supported by the hypothesis of the completeness of the market, and to this extent appropriate probability measure are introduced, as recalled in (De Felice and Moriconi, 2004).

Really the current valuation (cf. Cocozza *et al.* 2005) can be represented by means of the expectation consistently with the best prediction of the demographic scenario. In this context, a fair valuation procedure involves the latest information on the two main factors bringing risk to the business, properly interest rates and mortality.
In a portfolio perspective and in the case of surviving benefits, if *c* is the number of policies at time 0 we can write (cf. Coppola *et al*. 2005)

$$V_t = E\left[\sum_{j>t} c1_{\{k_{x,t>j}\}} X_j v(t,j)/F_t\right] \tag{2}$$

where the indicator function $\mathbf{1}_{\{Kx,t>j\}}$ takes the value 1 if the curtate future lifetime of the insured, aged *x* at issue, takes values greater than $t + j$ (j =1, 2, . . .) (equivalently if the insured aged $x + t$ survives up to the time $t + j$), 0 otherwise. By virtue of the basic assumptions on the risk sources, we get

$$V_t = \sum_{j>t} cX_j E\left[1_{\{k_{x,t>j}\}}/F_t\right] E\left[v(t,j)/F_t\right] \tag{3}$$

that is the price in *t* of a portfolio of ZCB with maturities in *j* (cf. Coppola *et al*. 2005).

Formula (3) is obtained in a general view according to a *forward* perspective, beginning from an initial position in 0. It is obvious that the model can be recalibrated in a *spot* perspective, according to a year by year valuation. In this order of ideas, formula (3) provides the expected value of the reserve at the end of the year, valued at the beginning of the year itself.

§4: A NUMERICAL SOLUTION

4.a: Interest rates and mortality scenario

The aim of this section is to provide a practical application of the mathematical and accounting tools presented in the previous sections. In particular, in a year by year valuation perspective, we will quantify at the beginning of the year the two critical values $R_{0,95}*(t)$ and $R_{0,99}*(t)$ of the reserve distribution, with the meaning reported in section 3.a.

The example of application we propose is referred to an immediate unitary life annuity contract issued on a life aged 40. In order to compute the quantile reserve $R_\alpha*(t)$ at time *t* at confidence level 95% and 99%, we introduce the stochastic background for the interest rate distribution.
The valuation of the financial instruments composing the ZCB portfolio will be made assuming a term structure of interest rates based on the Cox-Ingersoll-Ross stochastic differential equation:

$$dr_t = -\alpha(r_t - \mu)dt + \sigma\sqrt{r_t}\,dW_t \tag{4}$$

with α and σ positive constants, μ the long term mean and W_t a Wiener process.

The survival probabilities are deduced by the Italian Male Survival Table RG48, which is a projected mortality table taking into account the improvement in the mortality trend, that is the longevity phenomenon.

4.b: The computational procedure

In order to obtain the distribution of the reserve *R(t)* we propose a simulation procedure, which can be schematized by the following two basic steps.

The first one consists in the discretization of the SDE in formula (4). The discretized process we consider can be represented by the sequence $\{r_{\Delta}, r_{2\Delta}, \ldots\ldots, r_{k\Delta}\}$, where k is the number of time steps, Δ is a constant and $k\Delta = T$ is the time horizon. In the perspective of our example, concerning the valuation of the annual reserve, we pose $\Delta = 1$. Moreover, since T represents the residual duration of the contract at time t, it results $T = \omega-(t+x)$., ω and x being the upper limit age and the age at issue of the policyholder respectively. In our example $t=10$, $\omega=110$ and $x=40$.

We discretize the model by means of the Euler scheme, expressed by the following relation:

$$r_k = r_{k-1} + \alpha(\mu - r_{k-1})\Delta + \sigma\sqrt{r_{k-1}\Delta}\cdot\varepsilon_k \qquad k=1,2,..,T \qquad (5)$$

characterized by an easy implementation and a simple interpretation of the results.

The T random values for ε_k we need, providing one simulated path for the stochastic process r_t, are obtained considering as the starting point the value of the annual rate $r_0 = 0.0279$.

The second step consists in the computation of the reserve *R(t)* by means of the simulated path.

4.c: Numerical results

Within the numerical example we consider, we report the results obtained considering N=1000, 10000, 75000 simulations.

In the following table the characteristic values of the simulated distribution of the reserve, obtained by means of the procedure proposed in section 4.b, are reported, corresponding to the number of simulation indicated in the column.

N	1000	10000	75000
Mean	19.51930	19.52049	19.52179
Median	19.51922	19.52006	19.52198
Maximum	19.70779	19.72898	19.78047
Minimum	19.27175	19.29468	19.27176
St.Dev.	0.063360	0.064026	0.063131
Kurtosis	3.156597	2.920521	3.001561
Sweekness	-0.122263	0.001204	-0.007368
Jarque-Bera	3.513159	2.0634449	0.686215
Probability	0.172634	0.267878	0.709562
R*(99%)	19.66380	19.66960	19.66832
R*(95%)	19.61981	19.62704	19.62558

Table 1: **Reserve distribution** results obtained for N=1000, N=10000 and N=75000 simulation paths.

We can easily observe that better results are obtained as N increases and in particular, looking at kurtosis, sweekness and Jarque-Bera test, we can observe that the higher N is, the more *R(t)* approximates a normal distribution, according to the central limit theorem. This can be seen graphically by means of the histograms and the Quantile-Quantile plots shown below for each value of N.

Figure 1: Simulated reserve distribution: histograms and Quantile-Quantile plots N=1000

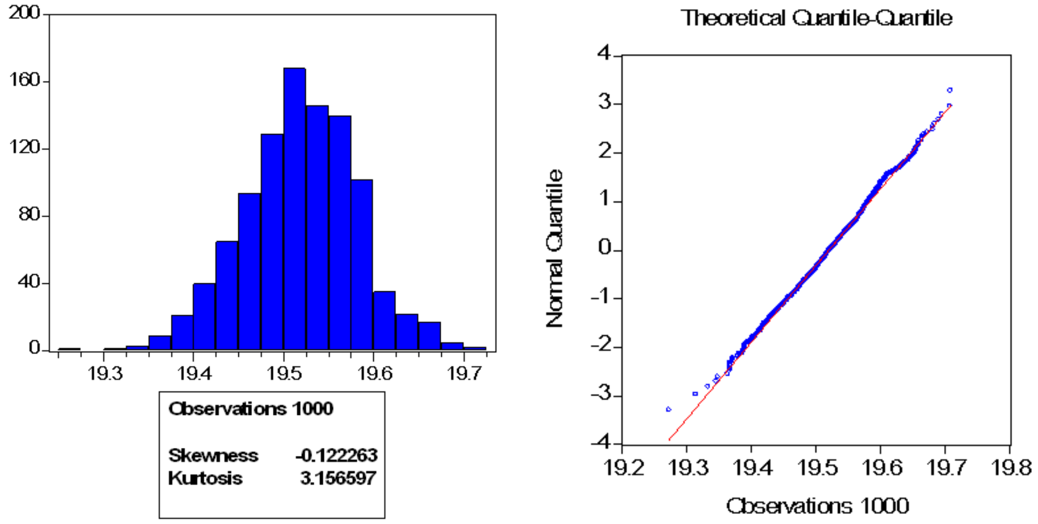


Figure 2: Simulated reserve distribution: histograms and Quantile-Quantile plots N=10000

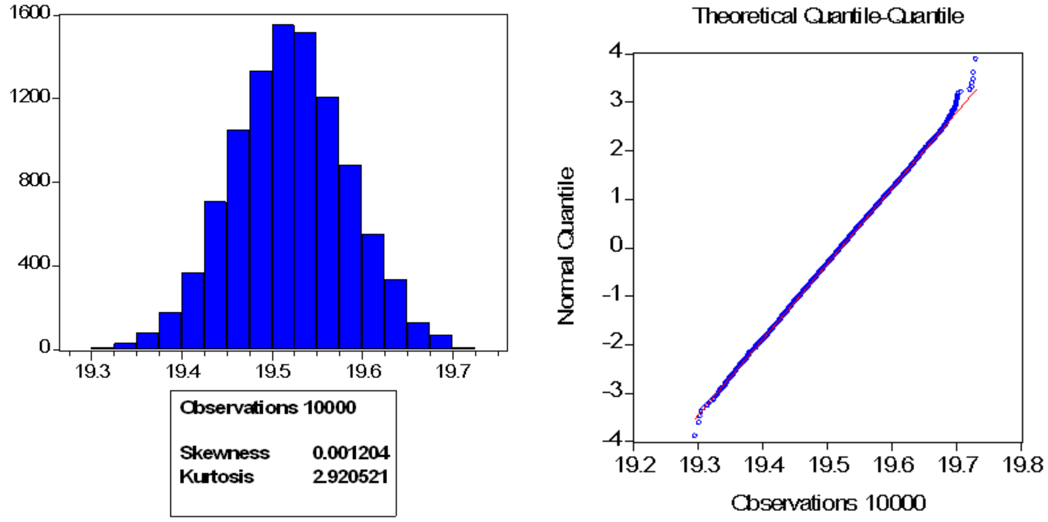
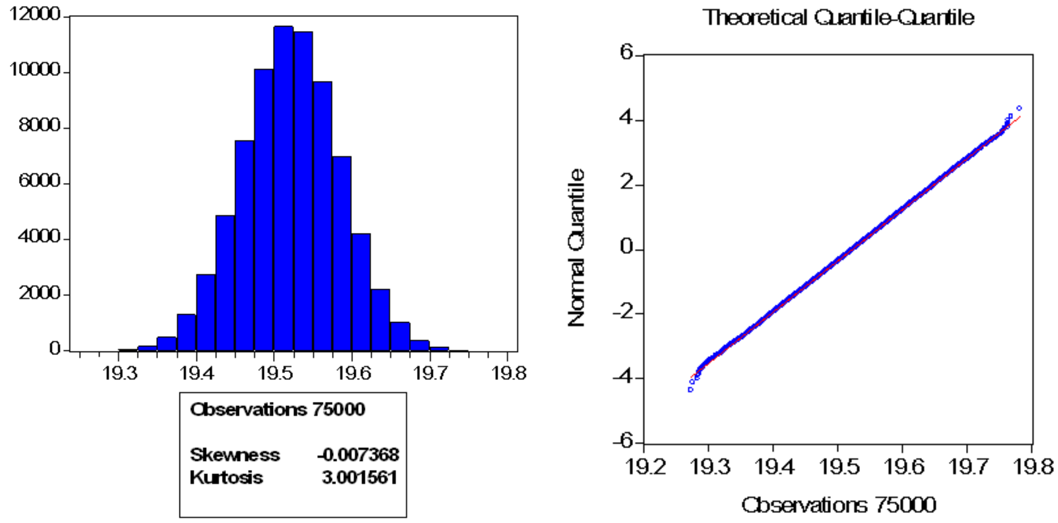


Figure 3: Simulated reserve distribution: histograms and Quantile-Quantile plots N=75000



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