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Limit theorems for number of diffusion processes, which did not absorb by boundaries*

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Abstract: We have random number of independent diffusion processes with absorption on boundaries in some region at initial time $t = 0$. The initial numbers and positions of processes in region is defined by the Poisson random measure. It is required to estimate the number of the unabsorbed processes for the fixed time $\tau > 0$. The Poisson random measure depends on τ and $\tau \to \infty$.

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MSC (2000): 60J60

Let us consider the set of independent random diffusion processes $\xi_k(t)$, $k = \overline{1, N}$, $t \geq 0$, $\xi_k(0) = \gamma_k$, $\gamma_k \in Q \subset R^d$.

Let the domain $Q \subset R^d$ be open connected region and it is limited by the smooth surface ∂Q. All processes $\xi_k(t)$ are diffusion processes with absorption on the boundary ∂Q. These processes are solutions of the following stochastic differential equations in Q

$$d\xi(t) = a(t, \xi(t))dt + \sum_{i=1}^{d} b_i(t, \xi(t))dw_i^{(k)}(t) \tag{1}$$

$$\xi(t) \in R^d\ ;\ x \in R^d \quad b_i(t, x), \quad a(t, x) : R_+ \times R^d \to R^d.$$

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with an initial condition: $\xi(0) = \gamma_k \in D$.

Here $W^{(k)}(t) = (w_i^{(k)}(t), \quad 1 \leq i \leq d), \quad 1 \leq k \leq N$ are independent in totality of d-dimensional Wiener processes.

Thus, these processes have the identical diffusion matrices and shift vectors, but they have different initial states.

Let Q be bounded and the boundary ∂Q is the Lyapunov surface $C^{(1,\lambda)}$.

The initial number and positions of processes is defined by the random Poisson measure $\mu(\cdot, \tau)$ in Q:

$$P(\mu(A,\tau) = k) = \frac{m^k(A,\tau)}{k!} e^{-m(A,\tau)},$$

where $m(\cdot, \tau)$ is finitely additive positive measure on Q for fixed τ.

We are going to investigate distribution of the number of the processes $\xi_k(t)$, which is in Q for all moments of time $t \leq \tau$.

This problem was offered in [1] as the mathematical model of practice problem.

We will do the following suppositions with respect to functions $a(t,x), b_i(t,x), i = \overline{1,d}$. They are sufficient for existence of unique solutions of equations (1) [2, p. 470].

There is such L that for functions $b_{ij}(t,x), a_i(t,x)$ the following inequalities take place

$$|a(s,x) - a(s,y)| + \sum_k^d |b_k(s,x) - b_k(s,y)| \leq L|x-y|$$

$$|a(s,x)|^2 + \sum_k^d |b_k(s,x)|^2 \leq L^2(1+|x|^2) \tag{2}$$

for all $x, y \in R^d$. Here $|y| = \left(\sum_{i=1}^d y_i^2\right)^{\frac{1}{2}}$.

We will define matrix $\sigma = B^T B, \quad B = (b_{ij}(t,x)), \quad \sigma = (\sigma_{ij}(t,x)), 1 \leq i,j \leq d$.

We will consider the following parabolic boundary problem

$$\frac{\partial u(t,x)}{\partial t} = \frac{1}{2}\sum_{j,i=1}^d \sigma_{ij}(\tau - t, x)\frac{\partial^2 u(t,x)}{\partial x_i \partial x_j} + \sum_{i=1}^d a_i(\tau - t, x)\frac{\partial u(t,x)}{\partial x_i} \quad 0 \leq t \leq \tau \tag{3}$$

$$u(0,x) = 1,\ x \in Q; \quad u(t,x) = 0,\ x \in \partial Q, \quad t \in [0,\tau]$$

In addition to (2) we will assume that the coefficients of operator part of problem (3) satisfy to the Hölder condition on variable t with index $0 < \alpha < \frac{1}{2}$.

$$\max_i \sup_{(x,t),(x,t')} \frac{|a_i(t,x) - a_i(t',x)|}{|t-t'|^\alpha} < \infty$$

$$\max_{i,j} \sup_{(x,t),(x,t')} \frac{|\sigma_{i,j}(t,x) - \sigma_{i,j}(t',x)|}{|t-t'|^\alpha} < \infty, \tag{4}$$

here $(t,x), (t,x') \in [0,\tau] \times \bar{Q}$.

We will also assume that operator part of (3) is uniformly parabolic [3, p. 20]. It means that the following condition is executed

$$\nu|\vec{z}|^2 \leq \sum_{i,j} \sigma_{ij}(t,x) z_i z_j \leq \mu |\vec{z}|^2 \quad (t,x) \in [0,\tau] \times \bar{Q}. \tag{5}$$

Here ν, μ there are the fixed positive numbers, and $\vec{z} = (z_1, \cdots, z_d)$ there is an arbitrary real vector.

In what follows, the conditions (2),(4),(5) are executed and region Q is limited by the Lyapunov surface ∂Q of class $C^{(1,\lambda)}$. These suppositions guarantee existence of unique classical solution of problem (3) [3, p. 469].

It is known [2], that $u(\tau, x)$ is equal to probability of remaining in the region Q at time instant τ of a diffusion process from (1), which occurs at the point $(0, x)$ at the initial moment ($\xi(0) = x, \quad x \in Q$).

We denote by $\eta(\tau)$ the number of remaining processes in the region Q at time instant τ.

We introduce the following sets for $y \in [0, 1]$

$$A(\tau, y) = \{x \in Q: \ u(\tau, x) \in [0, y]\}, \quad m_\tau(y) := m(A(\tau, y)).$$

As the function $u(\tau, x) : Q \to [0, 1]$ is continuous at fixed τ, we have that the sets $A(\tau, y)$ are measurable and the definition of the measure $m(\cdot)$ on these sets is correct.

Theorem 1. *If the conditions (2), (4), (5) are fulfilled and the* ∂Q *is Lyapunov surface, then random value* $\eta(\tau)$ *has Poisson distribution:*

$$P(\eta(\tau) = k) = \frac{a^k(\tau)}{k!} e^{-a(\tau)}, \quad k = 0, 1, ..., \quad \tau > 0,$$

with parameter $a(\tau) = \int\limits_0^1 y dm_\tau(y)$.

The analogy theorem was proved in [1] for case if $d = 2$ and if the region Q is circle: $Q = \{(x, y) : x^2 + y^2 \leq R^2\}$. The proof of Theorem 1 repeats the proof theorem from [1] almost word for word.

Thus we have the exact formula of distribution function of $\eta(\tau)$. However, the definition of the sets $A(\tau, y)$ with help analytical formula is difficult problem. We have difficulties in calculation of the function $m_\tau(y)$ in consequence of this. Therefore it would be desirable to obtain an approximation for distribution function of $\eta(\tau)$. Further we will prove such approximate formula for special case. Note that the authors of the article [4] investigated this special case, when the initial number and positions of diffusion processes are defined by the determinate limited measure $N(B, \tau)$, where $N(B, \tau)$ is equal to the number of points γ_k in a set B and $N = N(Q, \tau) < \infty$ for fixed $\tau > 0$.

We consider the following case

$a(t, x) = a = (\underbrace{0, \ldots, 0}_{d}), \quad b_i(t, x) = b_i = (b_{i1}, \ldots, b_{id}), \quad 1 \leq i \leq d;$

We define the matrix $\sigma = B^T B, \quad B = (b_{ij}),\ 1 \le i,j \le d$
$\sigma = (\sigma_{ij}), 1 \le i,j \le d$ and the differential operator $A : \sum\limits_{1 \le i,j \le d} \sigma_{ij} \frac{\partial^2}{\partial x_i \partial x_j}$.
Let σ be a matrix with the following property

$$\sum_{1 \le i,j \le d} \sigma_{ij} z_i z_j \ge \mu |\vec{z}|^2.$$

Here μ, is a fixed positive number, and $\vec{z} = (z_1, \cdots, z_d)$ is an arbitrary real vector. This operator acts in the following space

$$H_A = \{u : u \in L_2(Q) \cap Au \in L_2(Q) \cap u(\partial Q) = 0\}$$

with inner product $(u,v)_A = (Au, v)$. Here $(,)$ is inner product in $L_2(Q)$. The operator A is positive operator.

It is known [5] that the following eigenvalues problem

$$Au = -\lambda u, \quad u(\partial Q) = 0$$

has infinite set of real eigenvalues $\lambda_i \to \infty$ and

$$0 < \lambda_1 < \lambda_2 < \cdots < \lambda_s < \cdots .$$

The corresponding eigenfunctions

$$f_{11}, \ldots, f_{1n_1}, \cdots, f_{s1}, \ldots, f_{sn_s}, \cdots$$

form the complete system of functions both in H_A and $L_2^0(Q) := \{u : u \in L_2(Q) \cap u(\partial Q) = 0\}$. Here the number n_k is equal to multiplicity of eigenvalues λ_k.

We also assume that the σ-additive measure ν is given on the Σ_ν- algebra sets of $Q, \quad \nu(Q) < \infty$. All eigenfunctions $f_{ij} : Q \to R^1$ and all measures $m(\cdot, \tau)$ are (Σ_ν, Σ_Y) measurable. Here Σ_Y is the system of Borel sets of R^1. Let $\Rightarrow$ denotes weak convergence of random values or measures.

The notation is fairly standard. However, for convenience of the reader the following is recalled. We consider probabilistic measures P_τ and P, which are generated by a distribution functions $F_\tau(y) = P(\eta(\tau) \le y)$ and $F(y) = P(\eta \le y)$.

These measures are given on the Σ_ν. They are defined by the following relations uniquely $P_\tau(-\infty, y] = F_\tau(y)$, $P(-\infty, y] = F(y)$. As usual, we define weak convergence of $\eta(\tau)$ and of P_τ in the form $\int f(y) P_\tau(dy) \to \int f(y) P(dy)$ under $\tau \to \infty$ for all bounded continuous functions f on R^1. As we investigate a random values with range in set of numbers $\{0, 1, 2, ...\}$, then weak convergence is equivalent to following convergences under $\tau \to \infty$:

- convergence of generating functions

$Es^{\eta(\tau)} = \sum\limits_{k \ge 0} P(\eta(\tau) = k) s^k \to Es^\eta = \sum\limits_{k \ge 0} P(\eta = k) s^k, \quad 0 \le s \le 1$. The common convention $0^0 = 1$ is used.

- $P_\tau(y) \to P(y)$ for any singleton $\{y\}$.

We assign

$$g(\tau) = \exp\left(-\frac{\tau}{2}\lambda_1\right).$$

Theorem 2. *We suppose that $m(\cdot, \tau)$ holds the condition*

$$\lim_{\tau\to\infty} m(B,\tau)g(\tau) = \nu(B), \quad B \in \Sigma_\nu.$$

Then $\eta(\tau) \Rightarrow \eta$ if $\tau \to \infty$ where η has the Poisson distribution function with the parameter $a = \int_Q F(x)d\nu(x)$ and $F(x) = \sum_{i=1}^{n_1} f_{1i}(x)c_{1i}$, $c_{1i} = \int_Q f_{1i}(x)dx$.

Proof. We consider the following initial-boundary problem

$$\frac{\partial u}{\partial t} = \frac{1}{2}\sum_{1\le i,j\le d} \sigma_{ij}\frac{\partial^2 u}{\partial x_i \partial x_j} \quad x \in Q;$$
$$u(0,x) = 1 \quad \text{if} \quad x \in Q;$$
$$u(t,x) = 0 \quad \text{if} \quad x \in \partial Q, \quad t \ge 0 \tag{6}$$

We designate through $\gamma_k = (x_1^k, \cdots, x_d^k)$ the initial position of k-th process. We define the value of $u(\tau, \gamma_k)$.

We define a particular solution of (6) in the form

$$u(t,x) = u_1(t)u_2(x).$$

The ordinary argumentation leads to definition of joined constant λ:

$$2\frac{1}{u_1}\frac{\partial u_1}{\partial t} = \frac{Au_2}{u_2} = -\lambda.$$

We obtain the following system of problem owing to the latter one

$$Au_2 = -\lambda u_2; \quad u_2(\partial Q) = 0. \tag{7}$$
$$\frac{\partial u_1}{\partial t} = -\frac{\lambda}{2}u_1; \quad u_1(0) = 1 \tag{8}$$

It is clear that $u_1(t,\lambda) = \exp(-\frac{t}{2}\lambda)$ is the solution of (8). The solution of (7) was described above. We assume that the system of functions $\{f_{ij}(x), i \ge 1, 1 \le j \le n_i\}$ is orthonormalized with respect to the space $L_2^0(Q)$.

The general solution of the problem (6) has the following form

$$u(t,x) = \sum_{j=1}^{\infty} \exp(-\frac{t}{2}\lambda_j)\sum_{m=1}^{n_j} c_{jm}f_{jm}(x),$$

where the coefficients c_{jm} are equal to coefficients of decomposition of the initial value (unit) by the system of functions f_{jm}: $c_{jm} = \int_Q f_{jm}(x)dx$. The Parseval - Steklov equality is true for these coefficients:

$$\sum_{j=1}^{\infty}\sum_{m=1}^{n_j} c_{jm}^2 = |Q|. \tag{9}$$

We assign $F(x) = \sum\limits_{m=1}^{n_1} c_{1i} f_{1m}(x)$. The function $F(x)$ is continuous and bounded function on $\bar{Q}$. Since $u(t,x)$ is probability, it is not difficult to show that $F(x) \geq 0$ for all $x \in Q$. Let $M = \sup\limits_{x\in Q} F(x)$. We introduce the following sets

$$B_{k,n} = \left\{ x \in Q : \frac{Mk}{n} < F(x) \leq \frac{M(k+1)}{n} \right\}$$

Here $0 \leq k \leq n-1$ and $n > 1$.

Let us denote by $\zeta_{k,n}(\tau)$, $1 \leq k \leq n$ the number of unabsorbed processes at time instant τ, which occur in the region $B_{k,n}$ at initial time. These values are independent in totality by assumption. As we assume that the diffusion processes are independent, then according to the formula of total probability the distribution function of $\zeta_{k,n}(\tau)$ is defined by the following formula

$$P(\zeta_{k,n}(\tau) = l) = \sum_{d=l}^{\infty} P(\mu(B_{k,n},\tau) = d) \times$$

$$\times \sum_{1\leq i_1,\cdots,i_l\leq d, i_m\neq i_j, m\neq j}\ \prod_{k=1}^{l} u(\tau,\gamma_{i_k}) \prod_{s=l+1, i_s\notin(i_1,\cdots,i_l)}^{d} (1-u(\tau,\gamma_{i_s})), \quad l = 0,1,\ldots.$$

Here $\gamma_{i_j} \in B_{k,n}$. The summation is taken over all collections of l different integer numbers from the set of integer numbers $\{1,2,\ldots,d\}$. We also use common convention $\prod\limits_{k=1}^{0} = \prod\limits_{s=l+1}^{l} = 1$.

We set

$$a_{k,n}(\tau) = \min_{x\in\bar{B}_{k,n}} u(\tau,x), \quad \bar{a}_{k,n}(\tau) = 1 - a_{k,n}(\tau);$$

$$b_{k,n}(\tau) = \max_{x\in\bar{B}_{k,n}} u(\tau,x), \quad \bar{b}_{k,n}(\tau) = 1 - b_{k,n}(\tau).$$

Now

$$J_{k,n}(l,\tau) := \sum_{d=l}^{\infty} \frac{m^d(B_{k,n},\tau)}{d!} \exp(-m(B_{k,n},\tau)) C_d^l a_{k,n}^l(\tau) \bar{b}_{k,n}^{d-l}(\tau) \leq$$

$$\leq P(\zeta_{k,n}(\tau) = l) \leq$$

$$\sum_{d=l}^{\infty} \frac{m^d(B_{k,n},\tau)}{d!} \exp(-m(B_{k,n},\tau)) C_d^l b_{k,n}^l(\tau) \bar{a}_{k,n}^{d-l}(\tau) =: I_{k,n}(l,\tau). \tag{10}$$

Further

$$J_{k,n}(l,\tau)=\frac{\left(m(B_{k,n},\tau)a_{k,n}(\tau)\right)^l}{l!}\exp(-m(B_{k,n},\tau))\sum_{d=l}^{\infty}\frac{\left(\bar{b}_{k,n}(\tau)m(B_{k,n},\tau)\right)^{d-l}}{(d-l)!}=$$

$$=\frac{\left(m(B_{k,n},\tau)a_{k,n}(\tau)\right)^l}{l!}\exp(-b_{k,n}(\tau)m(B_{k,n},\tau));$$

By analogy:

$$I_{k,n}(l,\tau)=\frac{\left(m(B_{k,n},\tau)b_{k,n}(\tau)\right)^l}{l!}\exp(-a_{k,n}(\tau)m(B_{k,n},\tau)). \tag{11}$$

We introduce the following generating functions

$$\varphi(\tau,s)=\sum_{l\geq 0}s^l P(\eta(\tau)=l).$$

$$\varphi_{k,n}(\tau,s)=\sum_{l\geq 0}s^l P(\zeta_{k,n}(\tau)=l),\quad k=\overline{0,n-1},\quad 0\leq s\leq 1.$$

By construction, $\eta(\tau)$ can be represented as $\eta(\tau)=\zeta_{1,n}+\cdots+\zeta_{n-1,n}(\tau)$. Thus,

$$\varphi(\tau,s)=\prod_{k=0}^{n-1}\varphi_{k,n}(\tau,s). \tag{12}$$

Combining (10)-(12), we conclude that

$$\exp\{(sa_{k,n}(\tau)-b_{k,n}(\tau))m(B_{k,n},\tau)\}\leq\varphi_{k.n}(\tau,s)\leq$$
$$\leq\exp\{(sb_{k,n}(\tau)-a_{k,n}(\tau))m(B_{k,n},\tau)\}$$

and

$$\exp\left\{\sum_{k=0}^{n-1}(sa_{k,n}(\tau)-b_{k,n}(\tau))m(B_{k,n},\tau)\right\}\leq\varphi(\tau,s)\leq$$
$$\leq\exp\{\sum_{k=0}^{n-1}(sb_{k,n}(\tau)-a_{k,n}(\tau))m(B_{k,n},\tau)\}. \tag{13}$$

Since the function $u(\tau,x)$ is the continuous function in $x\in Q$, such points x_*, $x^*\in\bar{B}_{k,n}$ exist that the following equalities take place

$$a_{k,n}(\tau)=\exp(-\frac{\tau}{2}\lambda_1)F(x_*)+\sum_{k\geq 2}\exp(-\frac{\tau}{2}\lambda_k)\sum_{m=1}^{n_k}c_{km}f_{km}(x_*),$$

$$b_{k,n}(\tau)=\exp(-\frac{\tau}{2}\lambda_1)F(x^*)+\sum_{k\geq 2}\exp(-\frac{\tau}{2}\lambda_k)\sum_{m=1}^{n_k}c_{km}f_{km}(x^*),$$

here $x_*:=x_*(k,n,\tau)$, $x^*:=x^*(k,n,\tau)$.

Now, we can rewrite the sums in exponent from (13) in the following forms

$$\sum_{k=0}^{n-1}(sF(x_*)-F(x^*))\exp(-\frac{\tau}{2}\lambda_1)m(B_{k,n},\tau)+$$

$$+\sum_{k=0}^{n-1}\exp(-\frac{\tau}{2}\lambda_1)m(B_{k,n},\tau)\sum_{j\geq 2}\exp\left(-\frac{\tau}{2}(\lambda_j-\lambda_1)\right)\sum_{m=1}^{n_j}c_{jm}(sf_{jm}(x_*)-f_{jm}(x^*)), \quad (14)$$

$$\sum_{k=0}^{n-1}(sF(x^*)-F(x_*))\exp(-\frac{\tau}{2}\lambda_1)m(B_{k,n},\tau)+$$

$$+\sum_{k=0}^{n-1}\exp(-\frac{\tau}{2}\lambda_1)m(B_{k,n},\tau)\sum_{j\geq 2}\exp\left(-\frac{\tau}{2}(\lambda_j-\lambda_1)\right)\sum_{m=1}^{n_j}c_{jm}(sf_{jm}(x^*)-f_{jm}(x_*)), \quad (15)$$

We calculate limit of (14) if $\tau\to\infty$. The first sum of (14) converges to the following limit under the condition of theorem

$$\sum_{k=0}^{n-1}sF(x_*)\nu(B_{k.n})-\sum_{k=0}^{n-1}F(x^*)\nu(B_{k.n}).$$

This is difference of two integral sums, which has the following limit under $n\to\infty$ (see [6])

$$(s-1)\int_Q F(x)\nu(dx).$$

We assign

$$s_\tau(x)=\sum_{j\geq 2}\exp\left(-\frac{\tau}{2}(\lambda_j-\lambda_1)\right)\sum_{m=1}^{n_j}c_{km}f_{km}(x).$$

We consider sums of eigenfunctions in the form

$$e(x,\lambda)=\sum_{\lambda_k\leq\lambda}\sum_{m=1}^{n_k}f_{km}^2(x),$$

The following result is proved in the monograph [7, Thm. 17.5.3]

$$\sup_{x\in Q}\sqrt{e(x,\lambda)}\leq C\lambda^{\frac{d}{2}}.$$

Asymptotic characteristic of eigenvalues λ_j under $j\to\infty$ is defined by the following inequalities [5, sec. 18]

$$c_1 j^{\frac{2}{d}}\leq\lambda_j\leq c_2 j^{\frac{2}{d}},\quad \text{where}\quad c_1,\ c_2=const.$$

The latter one, (9) and the Caushy-Bunyakovskii inequality lead to the following convergence under $\tau\to\infty$

$$|s_\tau(x)| \le \sum_{j\ge 2} \exp\left(-\frac{\tau}{2}(\lambda_j-\lambda_1)\right)\sqrt{\sum_{m=1}^{n_j} c_{jm}^2}\sqrt{\sum_{m=1}^{n_j} f_{jm}^2(x)} \le$$
$$\le C\sum_{j\ge 2}\lambda_j^{\frac{d}{2}}\exp(-\frac{\tau}{2}(\lambda_j-\lambda))\sqrt{\sum_{m=1}^{n_j} c_{jm}^2} \le$$
$$\le C\sqrt{\sum_{j\ge 2}\lambda_j^d \exp(-\tau(\lambda_j-\lambda_1))}\sqrt{\sum_{j\ge 2}\sum_{m=1}^{n_j} c_{jm}^2} \to 0.$$

Thus, the second sum of (14) converges to zero.

Similar considerations were applied to (15). Proof is complete. □

Example. Now we apply the general approach to the particular case.

We consider the case if Q is circle $Q = \{(x,y) : x^2+y^2 \le r_0^2\}$. We assume that the diffusion processes occurs at the point $(x_k, y_k) \in Q$ at the initial time.

The processes are described in Q by the following stochastic differential equations

$$d\xi(t) = \sum_{i=1}^{2} b_i dw_i(t) \tag{16}$$
$$\xi(0) = \xi_0 = (x_k, y_k),$$

where $b_1 = (\sigma, 0), b_2 = (0, \sigma)$ and $W(t) = (w_i(t), i = 1, 2)$ is a 2-dimensional Wiener process.

We assume that the equation (16) defines a diffusion process with absorption on the boundary $\partial Q = \{(x, y, z) : x^2 + y^2 = r_0^2\}$.

In follows that $J_0(x), J_1(x)$ are Bessel functions of zero and first order. They are defined as solutions of the following equations

$$\frac{d^2y}{dx^2} + \frac{1}{x}\frac{dy}{dx} + (1 - \frac{n^2}{x^2}) = 0,$$
$$y(x_0) = 0, \quad (x_0 = \sqrt{\lambda} r); \quad |y(0)| < \infty;$$

for $n = 0$ and $n = 1$.

The value of $\mu_m^{(0)}$ is equal to m-th root of the equation $J_0(\mu) = 0$ [8, 9].

Let $mes(\cdot)$ denotes the Lebesgue measure.

We set

$$f(\tau) := \exp\left(-\frac{\tau}{2}\Big(\frac{\sigma\mu_1^{(0)}}{r_0}\Big)^2\right).$$

We suppose that $m(\cdot, \tau)$ holds the condition

$$m(\cdot,\tau)f(\tau) \Rightarrow mes(\cdot) \quad \text{if} \quad \tau \to \infty.$$

In this case the system of problems (7), (8) has the following form

$$\triangle u_2 = -\mu u_2, \quad (x,y) \in C; \quad u_2(x,y) = 0 \quad \text{if} \quad x^2 + y^2 = r_0^2, \tag{17}$$

$$\frac{\partial u_1}{\partial t} = -\frac{\sigma^2}{2}\mu u_1, \quad u_1(0) = 1. \tag{18}$$

According to the general approach for construction of solution $u(t,x,y)$ (see, for example, [8, sec. IV]) we rewrite the problem of (17) in polar coordinates: $u_3(r,\varphi) := u_2(r\cos\varphi, r\sin\varphi)$. The u_3 is the solution the following problem

$$\frac{\partial^2 u_3}{\partial r^2} + \frac{1}{r}\frac{\partial u_3}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u_3}{\partial \varphi^2} + \mu u_3 = 0,$$

$$u_3(r_0, \varphi) = 0.$$

We obtain

$$u(t,x,y) = u(t,r) = \sum_{m=1}^{\infty} c_m J_0\left(\frac{\mu_m^{(0)}}{r_0} r\right) \exp\left(-\frac{t}{2}\left(\frac{\sigma\mu_m^{(0)}}{r_0}\right)^2\right),$$

where $c_m = 2\left(\mu_m^{(0)} J_1(\mu_m^{(0)})\right)^{-1}$.

The function $J_0\left(\frac{\mu_1^{(0)}}{r_0} r\right)$ is a strictly decreasing function if $0 \le r \le r_0$. Thus we can construct the partitions $B_{k,n}$ by the following partitions

$$\tilde{B}_{k,n} = \left\{(x,y) \in C : \frac{r_0 k}{n} < \sqrt{x^2+y^2} \le \frac{r_0(k+1)}{n}\right\}, \quad 0 \le k \le n-1.$$

Now $mes(\tilde{B}_{k,n}) = g(\frac{k+1}{n}) - g(\frac{k}{n})$, where $g(x) = \pi r_0^2 x^2, \quad 0 \le x \le 1.$
Finally, the parameter of Poisson distribution is equal to

$$a = 2\left(\mu_1^{(0)} J_1(\mu_1^{(0)})\right)^{-1} 2\pi r_0^2 \int_0^1 J_0(\mu_1^{(0)} x) x dx = \pi \left(\frac{2r_0}{\mu_1^{(0)}}\right)^2.$$

We used the following known relation $\alpha J_0(\alpha) = [\alpha J_1(\alpha)]'$ [8, p. 466] for calculation of the latter integral.

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