An efficient and fast parallel method for Volterra Integral Equations of Abel type.

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Abstract

In this paper we present an efficient and fast parallel waveform relaxation method for Volterra Integral equations of Abel type, obtained by reformulating a non–stationary waveform relaxation method for systems of equations with linear coefficient constant kernel. To this aim we consider the Laplace transform of the equation and here we apply the recurrence relation given by the Chebyshev polynomial acceleration for algebraic linear systems. Back in the time domain, we obtain a three term recursion which requires, at each iteration, the evaluation of convolution integrals, where only the Laplace transform of the kernel is known. For this calculation we can use a fast convolution algorithm. Numerical experiments have been done also on problems where it isn't possible to use the original non–stationary method, obtaining good results in terms of improvement of the rate of convergence with respect the stationary method.

Key words: Abel equations, parallel methods, waveform relaxation methods, Chebyshev polynomial, Talbot contours.
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1 Introduction

Parallel Waveform Relaxation (WR) methods for Volterra Integral Equations (VIEs) with weakly singular kernels (of Abel type) have been introduced recently [2] [4].

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A large system of VIEs of Abel type has the form:

$$y(t) = f(t) + \int_0^t \frac{k(t,s,y(s))}{(t-s)^\alpha} ds, \quad t \in [0,T], \quad 0 < \alpha < 1, \qquad (1.1)$$
$$y, f, k \in R^d, \quad d >> 1,$$

and arises directly in many applications as, for example, in reaction–diffusion problems in small cells [6] or indirectly as well as by the semidiscretization in space of Volterra-Fredholm integral equations with weakly singular kernels [1], and of Abel partial integral or integro–differential equations (such as models of an omalous diffusion processes, wave propagation in viscoelastic materials [5]).

The high computational cost for computing the numerical solution of these problems requires high performances numerical methods. A fully parallel WR method satisfies such expectations by decoupling the system into independent equations. In fact, if $\mathcal{G} = \mathcal{G}(t,s,u,v)$ is a suitable function such that $\mathcal{G}(t,s,u,u) = k(t,s,u)$, we introduce the "waveforms" $\{y^{(i)}(t)\}_{i\in N}$ from:

$$y^{(i+1)}(t) = f(t) + \int_0^t \frac{\mathcal{G}(t,s,y^{(i)}(s),y^{(i+1)}(s))}{(t-s)^\alpha} ds \qquad (1.2)$$
$$t \in [0,T], \quad i = 0,1,\ldots$$
$$y^{(0)}(t) = f(t).$$

Obviously, if the sequence $\{y^{(i)}(t)\}_{i\in N}$ is convergent, its limit is the solution of (1.1). In particular, if the function $\mathcal{G}$ is such that the system (1.2) is decoupled into independent subsystems that can be solved in parallel, the corresponding WR method is a parallel method.

The convergence properties as well as the computational cost of each waveform depend heavily on the choice of the function $\mathcal{G}$. Unfortunately, fully parallel WR methods are usually slowly convergent. So, with the aim to develop fully parallel fast convergent WR methods, in [4], have been introduced Non Stationary WR (NSWR) methods. There, it was considered a sequence of functions $\{\mathcal{G}_i(t,s,u,v)\}_{i\in N}$ depending on the iterate i, such that $\mathcal{G}_i(t,s,u,u) = k(t,s,u)$ for each i, and it was obtained the corresponding NSWR method

$$y^{(i+1)}(t) = f(t) + \int_0^t \frac{\mathcal{G}_i(t,s,y^{(i)}(s),y^{(i+1)}(s))}{(t-s)^\alpha} ds\,. \qquad (1.3)$$

In section 2 we will recall in particular the fast Non Stationary WR Richardson method (see [3] [4]), for which

$$\mathcal{G}_i(t,s,u,v) = \mu_i I u - \mu_i I v + k(t,s,v), \qquad i = 0,\ldots,\nu, \qquad (1.4)$$

where I is the identity matrix of order d and the μ_i are the Chebyshev zeros of order ν, opportunately translated.

A limitation in the application of this method is that, in order to use the Chebyshev zeros as acceleration parameters, we have to fix *a priori* the number of iterations (i.e. ν in (1.4)). To overcome this problem, in section 3 we formulate the Chebyshev–Richardson WR method using a three term recurrence relation and we prove that this new method satisfies the same error bound.

To construct the method we consider the Laplace transform of the equation (1.1), limiting for the moment our analysis to the case of linear constant kernel, and here we apply the recurrence relation given by the Chebyshev polynomial acceleration for algebraic linear systems. Back in the time domain, we have constructed a sequence $\left\{z^{(i)}(t)\right\}_{i\in N}$ in which we can obtain $z^{(i+1)}(t)$, $\forall i$, from $z^{(i)}(t)$ and $z^{(i-1)}(t)$ by computing the convolution integrals

$$\int_0^t f_0^{(i)}(t-\tau)z^{(i-1)}(\tau)d\tau \quad \text{and} \quad \int_0^t f_1^{(i)}(t-\tau)z^{(i)}(\tau)d\tau,$$

where we only know the Laplace transform of the functions $f_0^{(i)}$ and $f_1^{(i)}$. So, for this calculation, we use a fast convolution algorithm given in [9]. A further improvement with respect the Richardson method, as it will be described in detail in section 3, is that Chebyshev–Richardson method is applicable to all linear constant VIEs whose kernel has singularities lying in the open left half complex plane.

In section 4, in order to evaluate the performances of the new method, we construct an analogous WR Stationary method and in section 5 we present the numerical experiments, in which we measure the number of iterations the two methods need to obtain a required accuracy. From these tests we can find out the problems for which the Chebyshev–Richardson method has the best performances.

All the numerical experiments we report are done on the test problem

$$y(t) = f(t) + \int_0^t \frac{A}{(t-s)^\alpha} y(s)ds\,, \qquad t \in [0,T], \qquad A \in R^{d\times d}\,, \quad d >> 1, \tag{1.5}$$

where we suppose that the matrix A has eigenvalues lying in the open left half complex plane.

2 Non Stationary Richardson waveform relaxation method

In this section we recall how the acceleration parameters are chosen in Richardson waveform relaxation method [3] [4].

The non stationary fully parallel Richardson waveform relaxation method has the form:

$$y^{(i+1)}(t) = f(t) + \int_0^t \frac{\mu_i}{(t-s)^\alpha} I y^{(i+1)}(s) ds + \\ + \int_0^t \frac{(A - \mu_i I)}{(t-s)^\alpha} y^{(i)}(s) ds, \tag{2.1}$$

where $\mu_i \in R_-$.

If put $e^{(i)}(t) = y(t) - y^{(i)}(t)$, from [4] we have the error bound:

$$\left\| e^{(i)} \right\|_T \leq \|P_i(A)\| \frac{T^{i(1-\alpha)} \Gamma(1-\alpha)^i}{\Gamma[i(1-\alpha)+1]} \left\| e^{(0)} \right\|_T, \tag{2.2}$$

where $P_i(z) = \prod_{j=0}^{i-1} (z - \mu_j)$ is a monic polynomial of degree i.

We can choose the parameters $\{\mu_i\}_{i=0,\ldots,\nu-1}$ in order to obtain a sequence

$$\left\{ {}^v y^{(i)}(t) \right\}_{i=0}^{v} \tag{2.3}$$

which minimizes the error $e^{(\nu)}$ at the iterate ν (with ν fixed), in the sense that it minimizes the $\|P_\nu(A)\|$. The dependence of this sequence on the fixed v has been put in evidence by the superscript v in (2.3). We now describe the choice of the parameters $\{\mu_i\}_{i=0,\ldots,\nu-1}$.

We remind that if D is a convex region of the complex plane which contains the eigenvalues of A the *virtual spectral radius* of $P_\nu(A)$ with respect to D is defined to be: $S_D(P_\nu(A)) = \max_{z \in D} |P_\nu(z)|$ (see [7]).

Let us assume that the spectrum of A can be enclosed in the region E delimited by the ellipse:

$$\left(\frac{x - \delta}{a} \right)^2 + \left(\frac{y}{b} \right)^2 = 1,\ a, b \in R, \tag{2.4}$$

belonging to the family $F(\delta, c)$ (i.e. centered at $\delta \in R_-$ and with foci at $\delta + c$ and $\delta - c$), where c is given by $c^2 = a^2 - b^2$ and with

$$a + \delta < 1. \tag{2.5}$$

We find (see [7] [10] [11]) that, when $c \in R$, the monic polynomial $P_\nu(z)$ of degree ν which minimizes the *virtual spectral radius* of $P_\nu(A)$ with respect to the ellipse E is the monic translated Chebyshev polynomial of degree ν :

$$P_\nu(z) = \frac{c^\nu}{2^{\nu-1}} T_\nu((z-\delta)/c).$$

So, the parameters $\{\mu_i\}_{i=0\ldots\nu-1}$ which minimize the error at the ν-th iterate are the zeros of $P_\nu(z)$, given by:

$$\mu_i = \delta + c\xi_i, \tag{2.6}$$

where $\xi_i = \cos\left[(2i+1)\pi/2\nu\right]$, $i = 0, ..., \nu-1$, denote the zeros of T_ν.

Obviously the spectrum of the matrix A can be enclosed in many different ellipses: in fact, given any family of ellipses $F(\delta, c)$ there is some member of the family that contains the spectrum of A in its interior. We can choose the family $F(\delta, c)$ so that the convergence is optimal in some sense. Let

$$r(\lambda) = \lim_{n\to\infty} \left| P_n(\lambda)^{\frac{1}{n}} \right|$$

be the *asymptotic convergence factor* of $P_n(\lambda)$ at the point λ. In particular, each eigenvalue λ_k, $k = 1, ..., d$ is associated with the convergence factor $r(\lambda_k)$. One way to optimize the choice of δ and c (see [10] [11]) is to make the maximum $r(\lambda_k)$ as small as possible, so they will satisfy the mini-max problem:

$$\min_{\delta,c} \max_{\lambda_k} r(\lambda_k). \tag{2.7}$$

3 Chebyshev–Richardson waveform relaxation method

A disadvantage in the implementation of the method (2.1) with parameters (2.6) is that, in order to choose the "optimal" parameters, we have to fix the number of iterates ν. Moreover this method is applicable only when the $c \in R$, that is when the "optimal" ellipse containing the spectrum of A has real foci.

To overcome these difficulties we have formulated the Richardson-Chebyshev WR method, through the construction of a new sequence $\{z^i(t)\}_{i\in N}$ which minimizes the iteration error, and for which these minimal properties do not depend on the choice *a priori* of the number of iterations. In order to construct this new sequence we first look at Richardson WR method from another point of view.

Let us consider the Laplace transform of the integral equation (1.5):

$$\hat{y}(s) = \hat{f}(s) + \frac{A\Gamma(1-\alpha)}{s^{1-\alpha}}\hat{y}(s). \tag{3.1}$$

For each s this is a system of linear equations, and we can consider the iterative method

$$\hat{y}^{(i+1)}(s) = \hat{f}(s) + \frac{A\Gamma(1-\alpha)}{s^{1-\alpha}}\hat{y}^{(i)}(s), \tag{3.2}$$

which, back in the time domain, corresponds to Picard iteration. We can accelerate the method (3.2) by considering a split of the matrix A:

$$\hat{y}^{(i+1)}(s) = \hat{f}(s) + \frac{\mu_i\Gamma(1-\alpha)}{s^{1-\alpha}}\hat{y}^{(i+1)}(s) + \frac{(A-\mu_i I)\Gamma(1-\alpha)}{s^{1-\alpha}}\hat{y}^{(i)}(s). \tag{3.3}$$

This method corresponds to the Laplace transform of Richardson method (2.1), and the expression of the error is:

$$\hat{e}^{(i)}(s) = \hat{y}^{(i)}(s) - \hat{y}(s) = \frac{(A-\mu_{i-1}I)\Gamma(1-\alpha)}{s^{1-\alpha}-\mu_{i-1}\Gamma(1-\alpha)}\hat{e}^{(i-1)}(s)\,.$$

Therefore, if we put $B(s) = A\Gamma(1-\alpha)/\left(s^{1-\alpha}\right)$, we obtain:

$$\hat{e}^{(i)}(s) = Q_i(s, B(s))\hat{e}^{(0)}(s) \tag{3.4}$$

with $Q_i(s,z) = \prod_{j=0}^{i-1}\left(z - \frac{\mu_j\Gamma(1-\alpha)}{s^{1-\alpha}}\right)/\left(1 - \frac{\mu_j\Gamma(1-\alpha)}{s^{1-\alpha}}\right)$. If we choose the optimal parameters $\{\mu_i\}_{i=0,\dots,\nu-1}$ of Richardson method, given by (2.6), from (3.4) we find that the error at the ν-th iterate is given by:

$$\hat{e}^{(\nu)}(s) = Q_\nu(s, B(s))\hat{e}^{(0)}(s), \quad \text{with } Q_\nu(s,z) = T_\nu(\tfrac{z-\bar{\delta}(s)}{\bar{c}(s)})/T_\nu(\tfrac{1-\bar{\delta}(s)}{\bar{c}(s)}) \tag{3.5}$$

where

$$\bar{\delta}(s) = \tfrac{\delta\Gamma(1-\alpha)}{s^{(1-\alpha)}}, \qquad \bar{c}(s) = \tfrac{c\Gamma(1-\alpha)}{s^{(1-\alpha)}} \tag{3.6}$$

represent the center and the focal length of an ellipse $\bar{E} \in F(\bar{\delta}(s), \bar{c}(s))$ containing the spectrum of $B(s)$. So $Q_\nu(s,z)$ is the translated Chebyshev polynomial of degree ν satisfying the condition $Q_\nu(s,1) = 1$. The presence of this Chebyshev polynomial minimizes the error (3.5) at the iterate ν, in the sense that, being $\bar{\delta}(s)$ and $\bar{c}(s)$ complex numbers, this polynomial asymptotically minimizes the virtual spectral radius of the matrix $Q_\nu(s, B(s))$ over the ellipse $\bar{E}$, and the asymptotic convergence factor is achieved very quickly ([10]). Moreover it can be easily seen that, if δ and c solve the mini-max problem (2.7), the parameters chosen according to (3.6) are the solution of the same mini-max problem, with λ_k eigenvalues of $B(s)$.

Now, using the previous observations, we can prove the following:

Theorem 3.1 *Let* $\left\{z^{(i)}(t)\right\}_{i\in N}$ *be the sequence given by the iteration:*

$$z^{(i+1)}(t) = z^{(i-1)}(t) - \int_0^t f_0^{(i)}(t-\tau)z^{(i-1)}(\tau)d\tau + [A-\delta I]\cdot \tag{3.7}$$
$$\cdot \int_0^t f_1^{(i)}(t-\tau)z^{(i)}(\tau)d\tau + \int_0^t f_2^{(i)}(t-\tau)f(\tau)d\tau,$$

where the functions $f_0^{(i)}(t)$, $f_1^{(i)}(t)$ *and* $f_2^{(i)}(t)$ *have Laplace transforms given by:*

$$\widehat{f_0^{(i)}}(s) = \rho_i(s),\ \widehat{f_1^{(i)}}(s) = \rho_i(s)\beta(s),\ \widehat{f_2^{(i)}}(s) = \rho_i(s)\beta(s)\tfrac{s^{1-\alpha}}{\Gamma(1-\alpha)},$$

with

$$\rho_i(s) = \begin{cases} 1 & \text{if} \quad i=1 \\ \left(1-\frac{1}{2}\sigma^2(s)\right)^{-1} & \text{if} \quad i=2 \\ \left(1-\frac{1}{4}\sigma^2(s)\rho_{i-1}(s)\right)^{-1} & \text{if} \quad i\geqslant 3\,, \end{cases} \tag{3.8}$$

where $\sigma^2(s) = c^2\beta^2(s)$ *and* $\beta(s) = \frac{\Gamma(1-\alpha)}{s^{(1-\alpha)}-\delta\Gamma(1-\alpha)}$. *Then the error*

$$\varepsilon^{(i)}(t) = z^{(i)}(t) - y(t)$$

is minimal for each $i\in N$, *in the sense that it satisfies the bound:*

$$\left\|\varepsilon^{(i)}\right\|_T \leq \|P_i(A)\|\frac{T^{i(1-\alpha)}\Gamma(1-\alpha)^i}{\Gamma[i(1-\alpha)+1]}\left\|\varepsilon^{(0)}\right\|_T \tag{3.9}$$

where $P_i(z) = c^i/2^{i-1}\cdot T_i((z-\delta)/c)$ *is the monic translated Chebyshev polynomial of degree* i, *for each* $i\in N$.

Proof. Let us consider the Chebyshev polynomial acceleration of the iterative method (3.2) and in this way we construct a new sequence $\left\{\hat{z}^{(i)}(s)\right\}_{i\in N}$ such that

$$\hat{\varepsilon}^{(i)}(s) = \hat{z}^{(i)}(s) - \hat{y}(s) = Q_i(s,B(s))\hat{\varepsilon}^{(0)}(s) \tag{3.10}$$

with

$$Q_i(s,z) = \frac{T_i(\frac{z-\bar{\delta}(s)}{\bar{c}(s)})}{T_i(\frac{1-\bar{\delta}(s)}{\bar{c}(s)})} \tag{3.11}$$

Chebyshev polynomial of degree i, for each $i\in N$. The iterates of this polynomial method can be expressed (see [7]) in the three term form:

$$\hat{z}^{(i+1)}(s) = \rho_i(s)\left[\gamma(s)B(s) + (1-\gamma(s))I\right]\hat{z}^{(i)}(s) + (1-\rho_i(s))\hat{z}^{(i-1)}(s) +$$
$$+\rho_i(s)\gamma(s)\hat{f}(s)$$

where $\gamma(s) = 1/(1-\delta(s)) = s^{(1-\alpha)}/\left(s^{(1-\alpha)} - \delta\Gamma(1-\alpha)\right)$ and $\rho_i(s) = 2(1-\bar{\delta}(s))/\bar{c}(s) \cdot T_{i-1}\left((1-\bar{\delta}(s))/\bar{c}(s)\right)/T_i\left((1-\bar{\delta}(s))/\bar{c}(s)\right)$, from which, with easy computation, we obtain

$$\hat{z}^{(i+1)}(s) = [A - \delta I]\,\rho_i(s)\beta(s)\hat{z}^{(i)}(s) + (1-\rho_i(s))\hat{z}^{(i-1)}(s) + \\ +\rho_i(s)\beta(s)\frac{s^{1-\alpha}}{\Gamma(1-\alpha)}\hat{f}(s). \tag{3.12}$$

where $\beta(s) = \Gamma(1-\alpha)/\left(s^{(1-\alpha)} - \delta\Gamma(1-\alpha)\right)$.

Back in the time domain the iteration (3.12) corresponds to the iteration (3.7). Moreover, making use of Chebyshev polynomial recurrence relation, we can express the parameters $\rho_i(s)$ in the more computationally convenient form (3.8). We can easily verify that, by construction of the sequence $\left\{\hat{z}^{(i)}(s)\right\}_{i\in N}$, for each $i \in N$ the function $z^{(i)}(t)$ is equal to the $y^{(i)}(t)$ given by (2.1) if we fix $\nu = i$ and consider the parameters (2.6): it corresponds to the function ${}^{i}y^{(i)}(t)$ given in (2.3). It obviously follows the expression (3.9) for the error. ■

Following the strategy of [4] we can come to a more significant estimate of the error:

$$\|\varepsilon^{(\nu)}\|_T \leq \gamma\left\{\frac{1}{2^\nu}\left[(a+b)^\nu + |a-b|^\nu\right] + \epsilon\right\}\frac{T^{\nu(1-\alpha)}\Gamma(1-\alpha)^\nu}{\Gamma[\nu(1-\alpha)+1]}\|\varepsilon^{(0)}\|_T$$

where γ and ϵ are positive constants. This is an immediate consequence of the:

Theorem 3.2 *Let $P_i(z) = c^i/2^{i-1} \cdot T_i((z-\delta)/c)$ be the monic translated Chebyshev polynomial of degree i. Then*

$$S_E\left(P_i(A)\right) \leq \frac{1}{2^i}\left[(a+b)^i + |a-b|^i\right] \tag{3.13}$$

where a and b are the semiaxes of the ellipse E given by (2.4).

Proof. Let us consider the virtual spectral radius of $P_i(A)$ with respect to the ellipse E:

$$S_E\left(P_i(A)\right) = \max_{z\in E}|P_i(z)| = \max_{z\in E}\left|\frac{c^i}{2^{i-1}}\right||T_i((z-\delta)/c)| = \max_{z\in E_1}\left|\frac{c^i}{2^{i-1}}\right||T_i(z)|$$

where $E_1 \in F(0,1)$ has the real semiaxe $a_1 = a/c$ if c is real, or $a_1 = b/|c|$ if c is purely immaginary. Now, using the fact that a maximum of $|T_i(z)|$ over

the ellipse E_1 occurs at $z = a_1$, we obtain, after some algebraic manipulation, the relation (3.13). ■

The iterative method (3.7), which we call Richardson-Chebyshev method, requires the evaluation of temporal convolutions of the type

$$\int_0^t f(t-\tau)g(\tau)d\tau$$

where it is the Laplace transform $\widehat{f}(s)$ of the kernel $f(t)$, rather than the kernel itself, which is known *a priori* and can be evaluated easily. So we can apply the fast convolution algorithm given in [9], which uses evaluations only of $\widehat{f}(s)$ and $g(\tau)$, and only requires $\mathcal{O}(N_t \log N_t)$ to compute the convolution on the grid $t = 0, \Delta t, 2\Delta t, ..., T = N_t \Delta t$ with stepsize Δt.

We remind that, in order to apply the convolution algorithm, we have to choose some *Talbot contours* (see [8] [9] [13] [14]) such that the singularities of $\widehat{f}(s)$ lie to the left of the contours. The convolution algorithm, in fact, approximates the kernel $f(t)$ by sum of exponentials *locally* on a sequence of fast growing intervals $\mathbf{I}_l$ covering $[\Delta t, T]$:

$$\mathbf{I}_l = \left[B^{l-1}\Delta t, \left(2B^l - 1\right)\Delta t\right].$$

The approximation of $f(t)$ on $\mathbf{I}_l$ results from the approximation of the contour integral for the inverse Laplace transform:

$$f(t) = \tfrac{1}{2\pi i}\int_{\Gamma_l} \hat{f}(\lambda)e^{t\lambda}d\lambda \approx \sum_{j=-N}^{N} \omega_j^{(l)} \widehat{f}(\lambda_j^{(l)})e^{t\lambda_j^{(l)}} \;\; t \in I_l \tag{3.14}$$

obtained applying the trapezoidal rule to a parametrization of this integral, where the complex Talbot contours Γ_l are of the form:

$$(-\pi, \pi) \longrightarrow \Gamma_l \;\; \vartheta \longmapsto \sigma + \mu_l \left(\vartheta \cot(\vartheta) + i\nu\vartheta\right) .$$

We set $\sigma_0 = 0$, $\mu_0 = 8$, $\mu_l = \mu_0/(\left(2B^l - 1\right)\Delta t)$, $\nu_0 = 0.6$. The parameter μ_l depends on l, whereas the parameters ν and σ depend on the singularities of the kernel and have to be chosen in a way such that those singularities lie to the left of the contours. The parameters μ_0 and ν_0 are obtained by minimizing the error in the approximation of the inverse Laplace transform (3.14) (see [8] [9] for details). In our case, if we consider $\alpha = 1/2$, we observe that $\rho_i(s)$ and $\gamma(s)$ have no singularities in the complex plane, except than a diramation point at $s = 0$. So we can choose the Talbot contours putting $\sigma = \sigma_0 = 0$ and $\nu = \nu_0 = 0.6$:

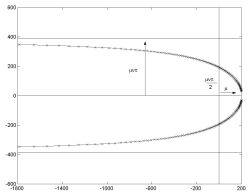

Fig. 1. Talbot contour

4 Stationary method

In numerical tests we will compare the Chebyshev-Richardson WR method with the stationary acceleration of Picard method (3.2), which has the form:

$$\hat{y}^{(i+1)}(s) = \hat{f}(s) + \frac{\mu\Gamma(1-\alpha)}{s^{1-\alpha}}\hat{y}^{(i+1)}(s) + \frac{(A-\mu I)\Gamma(1-\alpha)}{s^{1-\alpha}}\hat{y}^{(i)}(s). \qquad (4.1)$$

where the parameter μ must be chosen in order to accelerate the convergence.

Back in the time domain the iteration (4.1) becomes:

$$y^{(i+1)}(t) = [A-\mu I]\int_0^t f_1(t-\tau)y^{(i)}(\tau)d\tau + \int_0^t f_2(t-\tau)f(\tau)d\tau \qquad (4.2)$$

where the functions f_1 and f_2 have Laplace transforms given by: $\widehat{f_1}(s) = \Gamma(1-\alpha)/\left(s^{(1-\alpha)} - \mu\Gamma(1-\alpha)\right)$, $\widehat{f_2}(s) = s^{(1-\alpha)}/\left(s^{(1-\alpha)} - \mu\Gamma(1-\alpha)\right)$.

We know from [4] that if A has real eigenvalues the best stationary method is obtained choosing μ as the mean value between the minimum and maximum eigenvalue of A. When A has complex eigenvalues, in analogy to the real case, we can choose the parameter μ to be the center δ of the ellipse E in which such eigenvalues are supposed to be contained. We can also observe that choosing $\mu = \delta$, the non stationary method (3.12) becomes the stationary one (4.1) if we take $\rho_i(s) = \rho_1(s) = 1\ \forall\nu \geqslant 0$.

Theorem 4.1 *Let E be an ellipse of the form (2.4) containing all the eigenvalues of A.*

If $a > b$ the stationary method (4.1) with parameter $\mu = \delta$ is convergent $\forall\ \delta$.

If $a < b$ and $\delta < 1 - b$ the stationary method (4.1) with parameter $\mu = \delta$ is convergent.

Proof. Let λ_j, $j = 1, ..., d$ be the eigenvalues of A. From (4.6) in ([12], p.139)

we obtain that the stationary method (4.1) is convergent with parameter $\mu = \delta$

if and only if

$$\delta < \frac{1-|\lambda_j|^2}{2-2\,\mathrm{Re}(\lambda_j)} \qquad \forall j = 1, ..., d. \tag{4.3}$$

Let us consider the real function of two real variables:

$$f(x,y) = \tfrac{1-x^2-y^2}{2-2x} \ (x,y) \in E.$$

If δ satisfies the relation:

$$\delta < \min_{(x,y)\in F} f(x,y) =: m \tag{4.4}$$

in particular the relation (4.3) will be satisfied (because $\lambda_j \in E \ \forall j$), and the stationary method (4.1) will be convergent. The function $f(x,y)$ has no critical points in the interior of the ellipse E, and so takes its minimum on the boundary ∂E. Let us consider a parametrization of ∂E given by:

$$\begin{cases} x(\theta) = \delta + a\cos(\theta) \\ y(\theta) = b\sin(\theta) \end{cases} \quad \theta \in [0, 2\pi]\,.$$

We have that $m = \min_{\theta\in[0,2\pi]} f(x(\theta), y(\theta))$. The derivative of the real function $f(\theta) = f(x(\theta), y(\theta))$ is given by:

$$\begin{aligned} f'(\theta) = \{-\sin(\theta) \ \Big\{ & a(a^2-b^2)\cos^2(\theta) + 2(a^2-b^2)(\delta-1)\cos(\theta) + \\ & + a\left[(\delta-1)^2 - b^2\right]\Big\} \,/\, \Big\{2[\delta - 1 + a\cos(\theta)]^2\Big\}\,, \end{aligned}$$

and a further easy computation shows that:

- If $a > b$, the relation (4.4) is an immediate consequence of (2.5).
- If $a < b$, the relation (4.4) is verified, under he hypothesis (2.5), if and only if $\delta < 1 - b$.

■

5 Numerical results

In this section we want to show the results on the rate of convergence obtained by numerically solving the problem (1.5) using Chebyshev-Richardson WR method (3.7). Moreover, to compare the numerical results with theoretical expectations we have evaluated the *rate of convergence* $R_{E,\infty}$ of Chebyshev–Richardson iterative method. Using the definition (see [7]), if a and b are the

semiaxes of the ellipse E containing the eigenvalues of A, c its focal length, and δ its center, we have

$$R_{E,\infty} = -\log\left(\frac{a+b}{1-\delta+\sqrt{(1-\delta)^2-c^2}}\right), \tag{5.1}$$

The Richardson method (2.1) described in [4] represents an improvement with respect the stationary method (4.2), with a limitation of its applicability only in cases when c is real (i.e. $b < a$). The Chebyshev-Richardson method (3.7) leads, in cases when $b < a$, to the same improvement of Richardson method (2.1) with respect the stationary one (4.2) in terms of number of iterations, but with a considerably reduced computational effort, overcoming the problem of the choice *a priori* of the number of iterations. In fact, if we underestimate the degree ν of the Chebyshev polynomial, then the Richardson method (2.1) will not satisfy the required tolerance in ν iterations, and so we have to reintegrate the problem with a larger ν. If we overestimate the degree ν, we need as much more iterations as ν is larger than the right degree for the required tolerance thus leading to superflous iterations.

Moreover, the Chebyshev-Richardson method (3.7) mantains a good improvement also in the case when Richardson method (2.1) is not applicable (i.e. $b > a$). This is shown in tables below, where we report the direct comparison between the Chebyshev-Richardson method (3.7) and the stationary one (4.2).

The numerical experiments differ from each other for the position of the eigenvalues of A in the open left half complex plane. The matrix A has been constructed by forcing it to have the chosen eigenvalues, and the minimal ellipse has been found using the algorithm given in [10] [11]. The numerous numerical experiments show, according to (5.1), that the rate of convergence (and thus the number of iterations), does not depend neither on the dimension d of the matrix A (we tested for d from 100 to 1000), nor on the entries of the matrix A, but only on the values of the center δ of the minimal ellipse and its semi-axes a and b.

In the tables we report, for each value of the parameters δ, a and b, the number of iterations of each method for a required tolerance of 10^{-6} and the percentage of improvement of the Chebyshev-Richardson WR method with respect the stationary one.

In table 1 we consider the case $0 \leq a+b \leq 1$, with center near the origin ($\delta = -0.6$, $\delta = -10.1$) and far from the origin ($\delta = -100$) and in table 2 the case $10 \leq a+b \leq 20$, with center near the origin ($\delta = -10.1$) and far from the origin ($\delta = -100$). In both the tables, for every single method, there are several columns, each corresponding to a different value of δ. In table 3 we show an example of what happens when we start from a circle ($a = b$), and

we increase b until it exceeds the limit value $1-\delta$ (see theorem (4.1)).

Table 1: $0 \leq a+b \leq 1$, $\quad \delta = -0.6,\ -10.1,\ -100.$

a	b	$Chebyshev-Richardson$	$Stationary$	$improvement(\%)$
0.5	0	10 6 4	16 8 5	37,5 25 20
0.5	0.2	12 6 4	16 8 5	25 25 20
0.5	0.5	15 7 4	16 8 5	6,25 12,5 20
0.2	0.5	11 6 4	15 7 5	26,66 14,2 20
0	0.5	9 6 4	15 7 5	40 14,2 20

We can note we have the best results in terms of improvement with respect the stationary method when the ellipse is flattened ($a >> b$ or $b >> a$). We have little improvement when the ellipse become a circle ($a = b$). Moreover the Chebyshev-Richardson method is faster when δ is large and $a+b$ is small, while the rate of convergence of the stationary method in this case does not change with $a+b$. When δ is sufficiently large ($\delta = -10.1$, $\delta = -100$), we have few improvements because the number of iterations is already small for the stationary method.

In table 2, as in table 1, the improvements are very good when the ellipse is flattened ($a = 10$, $b = 0$ and $a = 0$, $b = 10$) and when δ is smaller, they are negligible when the ellipse becomes a circle ($a = b$). Chebyshev–Richardson method is again faster when δ is larger and $a+b$ is smaller, while the rate of convergence of the stationary method little depends on $a+b$: comparing table 2 with table 1 it seems to depend on the maximum between a and b (i.e. it is faster when this maximum is smaller).

In table 3, as before when $a = b$, we have very little improvement, but when b tends to its limit for the convergence of the stationary method, as this value is very far with respect to the fixed a, we have a very flattened ellipse and so the improvements is very very good. Naturally, this depends on the deteriorated performances of the stationary method: the iterations for the Chebyshev–Richardson grow up only twice, from 12 to 29, for the stationary ten times from 14 to 1454! The last row of this table reports the case in which b becomes bigger than $1-\delta$: the stationary method looses its convergence.

Table 2: $10 \leq a+b \leq 20, \qquad \delta = -10.1, -100.$

a	b	$Chebyshev-Richardson$		$Stationary$		$improvement(\%)$	
10	0	44	8	150	11	$70,66$	$27,27$
10	4	79	8	150	11	$47,33$	$27,27$
10	8	124	9	150	11	$17,33$	$18,18$
10	10	150	10	151	11	$0,66$	$9,09$
8	10	75	9	144	11	$47,91$	$18,18$
4	10	35	8	144	11	$75,69$	$27,27$
0	10	21	8	144	11	$85,41$	$27,27$

Table 3: $\delta = -100$, $b > a = 10$, $1 - \delta = 101$

a	b	$Chebyshev-Richardson$	$Stationary$	$improvement(\%)$
10	10	10	11	$9,09$
10	20	12	14	$14,28$
10	50	18	32	$43,75$
10	100	29	1454	98
10	105	30	–	

6 Concluding remarks and future work

We have constructed an efficient and fast fully parallel WR method for Volterra integral equations of Abel type, which we called the Chebyshev-Richardson method, characterized by an efficient computation of each waveform (through a fast convolution algorithm) and by the fast convergence of the sequence to the solution. The numerical experiments lead to the following remarks:

- The method is faster when $a+b$ is small and when the ellipse is far from the origin (δ is large), while the improvement with respect the stationary method is bigger when $a >> b$ or $b >> a$, and when the ellipse is near to the origin.
- If we fix $a+b$ we have a faster convergence when $b > a$. In fact in this case c is purely immaginary and the denominator in (5.1) becomes bigger

(compare for example the rows 2 and 6 of table 2).

- When we increase b until the limit value $1-\delta$, the rate of convergence decreases, according to (5.1), while there is a rise in the improvement with respect the stationary method (table 3).
- For the values of $b > 1-\delta$ the Chebyshev-Richardson method converges and has good results, while the stationary one looses its convergence (last row of table 3).

All these results are in perfect agreement to what we expected from the expression (5.1) for the rate of convergence and from the theorem (4.1).

Actually, we are working to apply the Chebyshev-Richardson method to the more general case of VIEs (1.1) with linear convolution kernels and to VIEs with regular kernels.

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