ON OPTIMIZATION OF A HIGHLY RE-ENTRANT PRODUCTION SYSTEM

Ciro D'Apice

Department of Information Engineering
Electrical Engineering and Applied Mathematics
University of Salerno
Via Giovanni Paolo II, 132, Fisciano (SA), Italy

Peter I. Kogut

Department of Differential Equations, Dnipropetrovsk National University
Gagarin av., 72, 49010 Dnipropetrovsk, Ukraine

Rosanna Manzo

Department of Information Engineering
Electrical Engineering and Applied Mathematics
University of Salerno
Via Giovanni Paolo II, 132, Fisciano (SA), Italy

Abstract. We discuss the optimal control problem stated as the minimization in the L^2-sense of the mismatch between the actual out-flux and a demand forecast for a hyperbolic conservation law that models a highly re-entrant production system. The output of the factory is described as a function of the work in progress and the position of the switch dispatch point (SDP) where we separate the beginning of the factory employing a push policy from the end of the factory, which uses a quasi-pull policy. The main question we discuss in this paper is about the optimal choice of the input in-flux, push and quasi-pull constituents, and the position of SDP.

1. **Introduction.** The aim of this article is to analyze an optimal control problem (OCP) for a highly re-entrant production system which is described by a scalar nonlinear conservation law. Typically, in high-technological semi-conductor manufacturing, many machines are repeatedly used for similar processing operations. In such production lines, semi-conductor wafers return to the same set of machines many times. So, the product flow has a re-entrant character. Typically, the semi-conductor systems are characterized by a very high volume (number of parts manufactured per unit time) and a very large number of consecutive production steps. This fact motivates to consider the scalar nonlinear conservation laws for the simulation of such processes. Partial differential equations, which are related with nonlinear conservation laws, are rather popular due to their superior analytic properties and availability of efficient numerical tools for simulation. For more detailed discussions of these models we refer to [3, 4, 6, 11, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 29, 30, 31].

2010 *Mathematics Subject Classification.* 35L65, 90B30, 49J20, 76N25.

Key words and phrases. Conservation laws, optimal control, existence result, re-entrant systems.

This is a pre-copy-editing, author-produced PDF of an article accepted for publication in NETWORKS AND HETEROGENEOUS MEDIA following peer review. The definitive publisher-authenticated C. D'APICE, P.I. KOGUT, R. MANZO (2016). On Optimization of a Highly Re-Entrant Production System. NETWORKS AND HETEROGENEOUS MEDIA, 11(3), 415-445, ISSN: 1556-1801, doi: 10.3934/nhm.2016003 is available online at: https://www.aimsciences.org/article/doi/10.3934/nhm.2016003.

From the optimization point of view, in manufacturing systems the natural control input is the in-flux (see, for instance, [15, 17] where the inflow on the output has been studied). Specifically, re-entrant production creates the opportunity to set priority rules for the various stages of production competing for capacity at the same machines. This dispatch policy, as it was indicated in [5], typically allows for two models of operations — the so-called push and pull policies. A push policy, also known as first buffer first step, is typically assigned to the front of the factory. A pull policy gives priority to later or fixed production steps over the earlier production steps. The step where push policy switches to pull policy is called the push-pull point (PPP). Moving the PPP leads not only to a change in dispatch rules, but also it may have an effect on the total output. In view of this it makes s sense to consider the PPP as a control variable.

A modern introduction to the study of hyperbolic conservation laws and especially of the control systems governed by such laws can be found in [9]. Fundamental are questions of wellposedness, regularity properties of solutions, controllability, existence, uniqueness and regularity of optimal controls. Existence of solutions, regularity and wellposedness of nonlinear conservation laws have been widely studied under diverse sets of hypotheses, see e.g. [1, 2, 6, 7, 8, 10, 14, 28, 32, 33] and the references therein. Concerning the manufacturing systems, an optimal control problem related to minimization of the error-signal that is the difference between a given demand forecast and the actual out-flux of manufacturing system, was studied in [17, 34]. For the controllability, exponential stability, and feedback stabilization of highly re-entrant production systems and further results in this field, we refer to [12, 14, 15, 16, 17, 31].

Here we consider an optimal control problem for a PDE model of a re-entrant system governed by nonlinear hyperbolic conservation law for the part density $\rho(t,x)$

$$\partial_t \rho + \partial_x \left(\rho V(\rho)\right) = 0 \quad \text{in} \quad Q = (0,T)\times(0,1), \tag{1}$$

where

$$\begin{aligned} V(\rho) &= H(x-x^*)V_2\left(\int_x^1 \rho(t,y-x+x^*)\,dy\right) + H(x^*-x)V_1\left(\int_0^x \rho(t,y)\,dy\right) \\ &= \begin{cases} V_1\left(\int_0^x \rho(t,y)\,dy\right), & \text{if } x<x^*, \\ V_2\left(\int_x^1 \rho(t,y-x+x^*)\,dy\right), & \text{if } x>x^*, \end{cases} \end{aligned} \tag{2}$$

and $H(x)$ stands for the Heaviside function whose value is zero for negative argument and one for positive argument.

The characteristic feature of OCP, we deal with in this article, is the fact that this model depends explicitly both on the so-called switch dispatch point (SDP) which is located at a priori unknown position x^* and velocity functions V_1 and V_2 which describe different types of policy in the regions $[0,x^*]$ and $[x^*,1]$.

Since the SDP divides the production line $[0,1]$ onto two parts $[0,x^*]$ and $[x^*,1]$ with different type of policies, in general, we cannot represent the right hand side of (2) in the form

$$V(\rho) = \lambda\left(\int_0^1 \rho(t,x)\,dx\right) \quad \text{for a certain function} \quad \lambda\in C^1([0,\infty))$$

which is the main constituent of models considered in [15, 16, 17, 34]. Hence, the wellposedness, uniqueness and regularity properties of solutions of hyperbolic conservation law (1) with a nonlocal speed term (2) requires a separate analysis. In what follows we will show that in many aspects such analysis can be provided in the spirit of recent work [17]. Moreover, as follows from (2), we consider the push policy for the region $[0, x^*]$, whereas the dependency

$$V_2\left(\int_x^1 \rho(t, y - x + x^*)\, dy\right) = V_2\left(\int_{x^*}^1 \rho(t, y)\, dy - \int_{1-x+x^*}^1 \rho(t, y)\, dy\right)$$

on the rest production line $[x^*, 1]$ can be interpreted as a certain version of a pull policy — the so-called quasi-pull policy. However, the right choice of functions V_1 and V_2 is definitely open question (see, for instance, [5, 17, 34]). This fact motivates us to consider the functions V_1 and V_2 as controls too. As a result, we deal with an OCP for the nonlinear conservation law with a nonlocal character of the velocity and with three different control actions — the in-flux, the SDP, and the functions V_1 and V_2.

The paper is organized as follows. In Section 2 we give the precise statement of the OCP for a highly re-entrant production system. The aim of Section 3 is to give some preliminaries and auxiliary results that we make use for our further analysis. In Section 4 we prove the existence of a unique weak solution to the Cauchy problem associated with the re-entrant system under given control functions when the initial and boundary conditions we consider in $L^1(0, T)$ and $L^1(0, 1)$ sense, respectively. We also study the main functional properties of the weak solutions and derive a priori estimates for them. Section 5 is addressed to the solvability of the original OCP. As for the optimality conditions for the given class of OCP, these aspects will be considered in the forthcoming paper.

2. **Statement of the problem.** Let $\alpha_2 > \alpha_1 > 0$ and $\alpha_3 > 0$ be given constants. Let $\mathfrak{A}_{ad}$ be the following subset of $C^1([0, \infty))$

$$\mathfrak{A}_{ad} = \left\{ V \in C^1([0, \infty)) \;\middle|\; \begin{array}{c} 0 \le \alpha_1 \le V(x) \le \alpha_2 \;\forall\, x \in [0, \infty), \\ \|V'\|_{C^0([0,\infty))} \le \alpha_3. \end{array} \right\} \tag{3}$$

Following the concept of the continuous flow model, describing the flow of products through a factory like a fluid flow, we denote $\rho(t, x)$ the product density at the stage $x \in [0, 1]$ and time $t \in [0, T]$. Here, $x = 0$ refers to the point of raw material and $x = 1$ to the finished product.

Definition 2.1. We say that a mapping $F : [0, T] \times [0, 1] \mapsto [0, \infty)$ is the clearing function if there exists a point $x^* \in [0, 1]$ and functions $V_1, V_2 \in \mathfrak{A}_{ad}$ such that

$$F(t, x) := \rho(t, x)\left[H(x^* - x)V_1\left(W_p(t, x)\right) + H(x - x^*)V_2\left(W_q(t, x)\right)\right],$$

where $H(x)$ stands for the Heaviside function and

$$W_q(t, x) = \int_x^1 \rho(t, y - x + x^*)\, dy, \quad W_p(t, x) = \int_0^x \rho(t, y)\, dy.$$

As follows from this definition $F(t, x)$ can be associated with the flux (production rate) at the time $t \in [0, T]$ and stage $x \in [0, 1]$ in the factory, whereas $x^* \in [0, 1]$ is the SDP.

We consider the following statement of OCP for manufacturing system:

$$\text{Minimize } \Big\{ I(u, V_1, V_2, x^*) = \int_0^T |y(t) - y_d(t)|^2\, dt + \|V_1'' - z_{1,d}\|^2_{L^2(0,a_1)} + \|V_2'' - z_{2,d}\|^2_{L^2(0,a_2)} \Big\} \tag{4}$$

subject to the constraints

$$\partial_t \rho(t,x) + \partial_x \left(V\rho(t,x)\right) = 0 \quad \text{in} \quad Q = (0,T)\times(0,1), \tag{5}$$

$$V = H(x-x^*)V_2\left(\int_x^1 \rho(t,y-x+x^*)\,dy\right) + H(x^*-x)V_1\left(\int_0^x \rho(t,y)\,dy\right), \tag{6}$$

$$\rho(0,x) = \rho_0(x) \quad \text{for} \quad x\in[0,1], \quad \rho(t,0)V_1(0) = u, \quad \text{for} \quad t\in[0,T], \tag{7}$$

$$y(t) = \rho(t,1)V_2(0), \tag{8}$$

$$V_1, V_2 \in \mathfrak{A}_{ad}, \quad x^*\in[0,1], \tag{9}$$

$$u \in \mathfrak{U}_{ad} := \left\{ w\in L^2(0,T) \mid \|w\|_{L^2(0,T)} \le \alpha_4,\ w(x)\ge 0 \ \text{ a.e. on } (0,T)\right\}, \tag{10}$$

where

$$a_1 = \sqrt{T}\,\alpha_4 + \|\rho_0\|_{L^\infty(0,1)}, \quad a_2 = \frac{\alpha_2}{\alpha_1}\left(\sqrt{T}\,\alpha_4\|\rho_0\|_{L^\infty(0,1)}\right), \tag{11}$$

$z_{1,d}\in L^2(0,a_1)$, $z_{2,d}\in L^2(0,a_2)$, $\rho_0\in L^2(0,1)$, and $y_d\in L^2(0,T)$ are functions, and $y(t)$ is the out-flux corresponding to the in-flux $u\in L^2_+(0,T)$, functions V_1, V_2, and initial data ρ_0. We also suppose $\rho_0\in L^2(0,1)$ and $y_d\in L^2(0,T)$ are nonnegative almost everywhere, and the constant a in definition of the cost functional is such that $\max\{a_1,a_2\}\le a < +\infty$. As for the second and third terms in the cost functional (4), they are related to the regularity (and, hence, to the solvability) of the original problem.

Hereinafter, a tuple $(u,V_1,V_2,x^*)\in L^2(0,T)\times C^1([0,a_1])\times C^1([0,a_2])\times[0,1]$ we call an admissible control to OCP (4)–(10) if (u,V_1,V_2,x^*) satisfies constraints (9)–(10).

3. **Preliminaries and auxiliary results.** It is easy to see that, for each admissible control (u,V_1,V_2,x^*), the Cauchy problem (5)–(7) can be represented in the form of a coupled system

$$\partial_t \rho_1(t,x) + \partial_x\left(V_1\left(\int_0^x \rho_1(t,y)\,dy\right)\rho_1(t,x)\right) = 0 \quad \text{in} \quad (0,T)\times(0,x^*), \tag{12}$$

$$\rho_1(0,x) = \rho_0(x) \quad \text{for} \quad x\in[0,x^*], \quad \rho_1(t,0)V_1(0) = u(t), \quad \text{for} \quad t\in[0,T], \tag{13}$$

$$\partial_t \rho_2(t,x) + \partial_x\left(V_2\left(\int_x^1 \rho_2(t,y-x+x^*)\,dy\right)\rho_2(t,x)\right) = 0 \quad \text{in} \quad (0,T)\times(x^*,1), \tag{14}$$

$$\rho_2(0,x) = \rho_0(x) \quad \text{for} \quad x\in[x^*,1], \tag{15}$$

$$\rho_2(t,x^*)V_2\left(\int_{x^*}^1 \rho_2(t,y)\,dy\right) = \rho_1(t,x^*)V_1\left(\int_0^{x^*} \rho_1(t,y)\,dy\right), \quad \text{for} \quad t\in[0,T], \tag{16}$$

where the compatibility condition (16) means that the output flux at $x=x^*$ of the push region must be considered as the in-flux for the quasi-pull region.

Remark 1. It is easy to note that the following representation for the solutions to the Cauchy problem (5)–(7)

$$\rho(t,x) = \begin{cases} \rho_1(t,x), & \text{if} \quad t\in[0,T],\ x\in[0,x^*), \\ \rho_2(t,x), & \text{if} \quad t\in[0,T],\ x\in(x^*,1]), \end{cases}$$

holds, where $x=x^*$ is the discontinuity point for the work in progress (wip) profile. However, the continuity assumption of the flux (16) guarantees the smooth solutions later on.

Following [13], we adopt the following definition of a weak solution to the problem (12)–(16).

Definition 3.1. Let $T>0$, $\rho_0\in L^1(0,1)$, $u\in L^1(0,T)$, $x^*\in[0,1]$, and $V_1,V_2\in\mathfrak{A}_{ad}$ be given. We say that a pair $(\rho_1,\rho_2)\in C^0([0,T];L^1(0,x^*)\times L^1(x^*,1))$ is a weak solution to the Cauchy problem (12)–(16) if for every $\tau\in[0,T]$ and every test functions $(\varphi_1,\varphi_2)\in C^1([0,T]\times[0,x^*])\times C^1([0,T]\times[x^*,1])$ such that

$$\varphi_1(\tau,x)=0,\quad \forall\, x\in[0,x^*],\quad \varphi_1(t,x^*)=0,\quad \forall\, t\in[0,\tau],$$
$$\varphi_2(\tau,x)=0,\quad \forall\, x\in[x^*,1],\quad \varphi_2(t,1)=0,\quad \forall\, t\in[0,\tau],$$

the following integral identities hold true

$$\int_0^\tau\int_0^{x^*}\rho_1(t,x)\left[\partial_t\varphi_1(t,x)+V_1\left(\int_0^x\rho_1(t,y)\,dy\right)\partial_x\varphi_1(t,x)\right]dxdt$$
$$+\int_0^\tau u(t)\varphi_1(t,0)\,dt+\int_0^{x^*}\rho_0(x)\varphi_1(0,x)\,dx=0, \tag{17}$$
$$\int_0^\tau\int_{x^*}^1\rho_2(t,x)\left[\partial_t\varphi_2(t,x)+V_2\left(\int_x^1\rho_2(t,y-x+x^*)\,dy\right)\partial_x\varphi_2(t,x)\right]dxdt$$
$$+\int_0^\tau\rho_1(t,x^*)V_1\left(\int_0^{x^*}\rho_1(t,y)\,dy\right)\varphi_2(t,x^*)\,dt+\int_{x^*}^1\rho_0(x)\varphi_2(0,x)\,dx=0. \tag{18}$$

We make use of a few of auxiliary results. In particular, the next Lemmas give existence of characteristics to the original Cauchy problem and their regularity what is a crucial point for our further analysis.

Lemma 3.2. *Let $\rho_0\in L^1(0,1)$ and $u,v\in L^1(0,T)$ be nonnegative functions. Let $x^*\in[0,1]$, $x\in[0,x^*]$, $z\in[x^*,1]$, and $V_1,V_2\in\mathfrak{A}_{ad}$ be given and such that $V_i(s)=V_i(0)$ for all $s<0$. Then there exists $\delta\in[0,T]$ independent of x and y such that the Cauchy problem*

$$\begin{cases}\dfrac{d\xi(t)}{dt}=V_1\left(\displaystyle\int_0^t u(\sigma)\,d\sigma+\int_0^{x-\xi(t)}\rho_0(y)\,dy\right), & t\in[0,\delta],\qquad \xi(0)=0,\\[2ex] \dfrac{d\zeta(t)}{dt}=V_2\left(\displaystyle\int_0^t v(\sigma)\,d\sigma+\int_{x^*}^{1-\zeta(t)+x^*-z}\rho_0(y)\,dy\right), & t\in[0,\delta],\quad \zeta(0)=0,\end{cases} \tag{19}$$

has a unique solution $(\xi_x,\zeta_z)\in\left[C^1([0,\delta])\right]^2$.

Proof. We associate with the Cauchy problem (19) the mapping $(\xi,\zeta)\mapsto F(\xi,\zeta):\Omega_\delta\times\Omega_\delta\to\left[C^0([0,\delta])\right]^2$ such that

$$F(\xi,\zeta)(t)=\begin{bmatrix}\displaystyle\int_0^t V_1\left(\int_0^s u(\sigma)\,d\sigma+\int_0^{x-\xi(s)}\rho_0(y)\,dy\right)ds\\ \displaystyle\int_0^t V_2\left(\int_0^s v(\sigma)\,d\sigma+\int_{x^*}^{1-\zeta(s)+x^*-z}\rho_0(y)\,dy\right)ds\end{bmatrix},\ \forall\, t\in[0,\delta] \tag{20}$$

and

$$\Omega_\delta=\left\{\xi\in C^0([0,\delta])\mid \xi(0)=0,\quad \alpha_1\le\frac{\xi(s)-\xi(t)}{s-t}\le\alpha_2,\ \forall\, s,t\in[0,\delta],\ s>t\right\} \tag{21}$$

where the constants α_1 and α_2 are defined as in (3). It is clear that Ω_δ consists of monotonically increasing functions on $[0,\delta]$.

Let us show that there exists a constant $\kappa\in(0,1)$ such that

$$\|F(\xi_1,\zeta_1)-F(\xi_2,\zeta_2)\|_{[C^0([0,\delta])]^2}\le\kappa\left[\|\zeta_1-\zeta_2\|_{C^0([0,\delta])}+\|\xi_1-\xi_2\|_{C^0([0,\delta])}\right] \tag{22}$$

for all $\xi_i,\zeta_i\in\Omega_\delta$ and $\delta>0$ small enough. Since F maps into Ω_δ provided $\delta<\alpha_2^{-1}$, it follows from (22) that $F(\xi,\zeta):\Omega_\delta\times\Omega_\delta\to\Omega_\delta\times\Omega_\delta$ is a contraction mapping. Then, by the Banach fixed point theorem, there exists a unique pair (ξ_x,ζ_z) such that $F(\xi_x,\zeta_z)=(\xi_x,\zeta_z)$, i.e. (ξ_x,ζ_z) is the unique solution to the Cauchy problem (19). Moreover, as follows from definition of the set $\mathfrak{A}_{ad}$ and the fact that $V_1,V_2\in\mathfrak{A}_{ad}$, the unique fixed pair (ξ_x,ζ_z) for F is in $\left[C^1([0,\delta])\right]^2$.

Let ξ_i,ζ_i $(i=1,2)$ be arbitrary elements of Ω_δ. Then (20) implies the estimate

$$\begin{aligned}|F(\xi_1,\zeta_1)-F(\xi_2,\zeta_2)|_1 &\le \alpha_3\int_0^t\left|\int_{x-\xi_1(s)}^{x-\xi_2(s)}\rho_0(y)\,dy\right|ds\\ &\quad+\alpha_3\int_0^t\left|\int_{1-\zeta_1(s)+x^*-z}^{1-\zeta_2(s)+x^*-z}\rho_0(y)\,dy\right|ds\\ &=\alpha_3\left[J_1(\xi_1,\xi_2)+J_2(\zeta_1,\zeta_2)\right]. \end{aligned}\tag{23}$$

We define $\widehat{\xi},\underline{\xi}\in C^0([0,\delta])$ by

$$\widehat{\xi}(t):=\max\{\xi_1(t),\xi_2(t)\}\ \text{ and }\ \underline{\xi}(t):=\min\{\xi_1(t),\xi_2(t)\}.$$

Since ξ_i are monotonically increasing functions, it follows that the inverse functions $\widehat{\xi}^{-1}$ and $\underline{\xi}^{-1}$ are well defined. Then, changing the order of integrations in (23) and following in many aspects [17], we obtain

$$\begin{aligned}J_1(\xi_1,\xi_2)&=\int_0^t\left|\int_{x-\xi_1(s)}^{x-\xi_2(s)}\rho_0(y)\,dy\right|ds\\ &=\int_{x-\widehat{\xi}(t)}^{x-\underline{\xi}(t)}\rho_0(y)\left(t-\widehat{\xi}^{-1}(x-y)\right)dy\\ &\quad+\int_{x-\underline{\xi}(t)}^{x}\rho_0(y)\left(\underline{\xi}^{-1}(x-y)-\widehat{\xi}^{-1}(x-y)\right)dy\\ &\le\int_{x-\widehat{\xi}(t)}^{x-\underline{\xi}(t)}\rho_0(y)\left(\underline{\xi}^{-1}\left(\underline{\xi}(t)\right)-\widehat{\xi}^{-1}\left(\underline{\xi}(t)\right)\right)dy\\ &\quad+\int_{x-\underline{\xi}(t)}^{x}\rho_0(y)\left(\underline{\xi}^{-1}(x-y)-\widehat{\xi}^{-1}(x-y)\right)dy\\ &\le\int_{x-\widehat{\xi}(t)}^{x}\rho_0(y)\,dy\sup_{0\le y\le\underline{\xi}(t)}\left[\underline{\xi}^{-1}(y)-\widehat{\xi}^{-1}(y)\right],\end{aligned}$$

where for the term $\underline{\xi}^{-1}(y)-\widehat{\xi}^{-1}(y)$ we have the following estimate for each $y\in[0,\underline{\xi}(t)]$ (this inference is based on (21) and the definition of $\underline{\xi}$ and $\widehat{\xi}$)

$$\begin{aligned}0&\le\underline{\xi}^{-1}(y)-\widehat{\xi}^{-1}(y)\\ &=\left(\underline{\xi}^{-1}(y)-\frac{\underline{\xi}^{-1}(y)+\widehat{\xi}^{-1}(y)}{2}\right)+\left(\frac{\underline{\xi}^{-1}(y)+\widehat{\xi}^{-1}(y)}{2}-\widehat{\xi}^{-1}(y)\right)\\ &\le\frac{1}{\alpha_1}\left[y-\underline{\xi}\left(\frac{\underline{\xi}^{-1}(y)+\widehat{\xi}^{-1}(y)}{2}\right)\right]+\frac{1}{\alpha_1}\left[\widehat{\xi}\left(\frac{\underline{\xi}^{-1}(y)+\widehat{\xi}^{-1}(y)}{2}\right)-y\right]\\ &=\frac{1}{\alpha_1}\left[\widehat{\xi}\left(\frac{\underline{\xi}^{-1}(y)+\widehat{\xi}^{-1}(y)}{2}\right)-\underline{\xi}\left(\frac{\underline{\xi}^{-1}(y)+\widehat{\xi}^{-1}(y)}{2}\right)\right]\le\frac{1}{\alpha_1}\|\xi_1-\xi_2\|_{C^0([0,\delta])}.\end{aligned}$$

Combining the above results, we finally get

$$\begin{aligned}J_1(\xi_1,\xi_2)&\le\frac{1}{\alpha_1}\|\xi_1-\xi_2\|_{C^0([0,\delta])}\int_{x-\widehat{\xi}(t)}^{x}\rho_0(y)\,dy\\ &\le\frac{1}{\alpha_1}\|\xi_1-\xi_2\|_{C^0([0,\delta])}\int_{x-\delta\alpha_2}^{x}\rho_0(y)\,dy.\end{aligned}$$

By analogy, it can be shown that

$$J_2(\zeta_1,\zeta_2) \le \frac{1}{\alpha_1}\|\zeta_1-\zeta_2\|_{C^0([0,\delta])}\int_{1-\widehat{\zeta}(t)}^1 \rho_0(y)\,dy$$

$$\le \frac{1}{\alpha_1}\|\zeta_1-\zeta_2\|_{C^0([0,\delta])}\int_{1-\delta\alpha_2}^1 \rho_0(y)\,dy.$$

As a result, the inequality (23) implies

$$|F(\xi_1,\zeta_1)(y)-F(\xi_2,\zeta_2)(y)|_1 \le \frac{\alpha_3}{\alpha_1}\left[\int_{x-\delta\alpha_2}^x \rho_0(y)\,dy + \int_{1-\delta\alpha_2}^1 \rho_0(y)\,dy\right] \times \left[\|\xi_1-\xi_2\|_{C^0([0,\delta])} + \|\zeta_1-\zeta_2\|_{C^0([0,\delta])}\right]. \tag{24}$$

Since $\rho_0 \in L^1(0,1)$, it follows that there exists $\delta\in(0,T)$ small enough such that

$$\int_{x-\delta\alpha_2}^x \rho_0(y)\,dy + \int_{1-\delta\alpha_2}^1 \rho_0(y)\,dy < \frac{\alpha_1}{2\alpha_3}. \tag{25}$$

In view of estimate (24), this immediately leads us to inequality (22). □

Our next intention is to study the properties of the mappings $x\mapsto\xi_x(t)$ and $z\mapsto\zeta_z(t)$.

Lemma 3.3. *Assume that $\rho_0\in L^\infty(0,1)$. Then, for given $u,v\in L^1(0,T)$, $V_1,V_2\in\mathfrak{A}_{ad}$, and $t\in[0,\delta]$, the mappings*

$$x\mapsto\xi_x(t):[0,x^*]\to\mathbb{R}_+ \quad \text{and} \quad z\mapsto\zeta_z(t):[x^*,1]\to\mathbb{R}_+ \tag{26}$$

are uniformly Lipschitz.

Proof. Let $x,y\in[0,x^*]$ be arbitrary points. Then, in view of definition of the class $\mathfrak{A}_{ad}$, we can derive from the first equation of (19) the estimate

$$|\xi_x(t)-\xi_y(t)| \le \alpha_3\int_0^t\left|\int_{y-\xi_y(s)}^{x-\xi_x(s)}\rho_0(\sigma)\,d\sigma\right|ds$$

$$\le \alpha_3\|\rho_0\|_{L^\infty(0,1)}\int_0^t\left(|x-y|+|\xi_x(s)-\xi_y(s)|\right)ds$$

$$\le \alpha_3\|\rho_0\|_{L^\infty(0,1)}\delta|x-y| + \alpha_3\|\rho_0\|_{L^\infty(0,1)}\int_0^t|\xi_x(s)-\xi_y(s)|\,ds.$$

As a result, by Gronwall-Bellman inequality, we see that

$$|\xi_x(t)-\xi_y(t)| \le \alpha_3\|\rho_0\|_{L^\infty(0,1)}\delta|x-y|\exp\left(\alpha_3\|\rho_0\|_{L^\infty(0,1)}t\right) \le C|x-y|, \tag{27}$$

that is, $x\mapsto\xi_x(t):[0,x^*]\to\mathbb{R}_+$ is a uniformly Lipschitz continuous mapping. The same property of $z\mapsto\zeta_z(t)$ can be established in a similar manner. □

Let $x\in[0,x^*]$ and $z\in[x^*,1]$ be fixed. Let $(\xi_x,\zeta_z)\in\left[C^1([0,\delta])\right]^2$ be the corresponding solution of the system (19) on some small time interval $[0,\delta]$. For given $\rho_0\in L^1(0,1)$, $u,v\in L^1(0,T)$, and $x^*\in[0,1]$, we introduce the following couple of functions

$$\widetilde{\rho}_{1,x}(t,y) = \begin{cases} \dfrac{u\left(\xi_x^{-1}\left(\xi_x(t)-y\right)\right)}{\xi_x'\left(\xi_x^{-1}\left(\xi_x(t)-y\right)\right)}, & 0\le y\le\xi_x(t),\\ \rho_0(y-\xi_x(t)), & \xi_x(t)\le y\le x^*, \end{cases} \quad \forall\, t\in[0,\delta], \tag{28}$$

$$\widetilde{\rho}_{2,z}(t,y) = \begin{cases} \dfrac{v\left(\zeta_z^{-1}\left(\zeta_z(t)+x^*-y\right)\right)}{\zeta_z'\left(\zeta_z^{-1}\left(\zeta_z(t)+x^*-y\right)\right)}, & x^*\le y\le x^*+\zeta_z(t),\\ \rho_0(y-\zeta_z(t)), & x^*+\zeta_z(t)\le y\le 1. \end{cases} \quad \forall\, t\in[0,\delta]. \tag{29}$$

Lemma 3.4. *For given $\rho_0 \in L^1(0,1)$, $u, v \in L^1(0,T)$, $x^* \in [0,1]$, $x \in [0,x^*]$, $z \in [x^*,1]$, and $(\xi_x, \zeta_z) \in \left[C^1([0,\delta])\right]^2$, the functions $\widetilde{\rho}_{1,x}$ and $\widetilde{\rho}_{2,z}$, defined by (28)–(29) are such that*

$$\widetilde{\rho}_{1,x} \in C([0,\delta]; L^1(0,x^*)), \qquad \widetilde{\rho}_{2,z} \in C([0,\delta]; L^1(x^*,1)). \tag{30}$$

Proof. We only prove the inclusion $\rho_2 \in C([0,\delta]; L^1(x^*,1))$, since the second one in (30) can be established by analogy. Let $\varepsilon > 0$ be an arbitrary value. Our aim it to show that there exists $\theta = \theta(\varepsilon) > 0$ such that, for arbitrary points $s,t \in [0,\delta]$, we have

$$\|\widetilde{\rho}_{2,z}(s,\cdot) - \widetilde{\rho}_{2,z}(t,\cdot)\|_{L^1(x^*,1)} < \varepsilon, \quad \text{provided} \quad |s-t| < \theta.$$

Indeed, having assumed for simplicity that $s > t$, we have

$$\begin{aligned}
&\int_{x^*}^1 |\widetilde{\rho}_{2,z}(s,y) - \widetilde{\rho}_{2,z}(t,y)|\, dy \\
&\quad \le \int_{x^*}^{x^*+\zeta_z(t)} \left| \frac{v\left(\zeta_z^{-1}\left(\zeta_z(s)+x^*-y\right)\right)}{\zeta_z'\left(\zeta_z^{-1}\left(\zeta_z(s)+x^*-y\right)\right)} - \frac{v\left(\zeta_z^{-1}\left(\zeta_z(t)+x^*-y\right)\right)}{\zeta_z'\left(\zeta_z^{-1}\left(\zeta_z(t)+x^*-y\right)\right)} \right| dy \\
&\quad + \int_{x^*+\zeta_z(t)}^{x^*+\zeta_z(s)} |\widetilde{\rho}_{2,z}(s,y) - \widetilde{\rho}_{2,z}(t,y)|\, dy \\
&\quad + \int_{x^*+\zeta_z(s)}^1 |\rho_0(y-\zeta_z(s)) - \rho_0(y-\zeta_z(t))|\, dy = J_1 + J_2 + J_3.
\end{aligned} \tag{31}$$

Since

$$\begin{aligned}
J_2 &:= \int_{x^*+\zeta_z(t)}^{x^*+\zeta_z(s)} |\widetilde{\rho}_{2,z}(s,y) - \widetilde{\rho}_{2,z}(t,y)|\, dy \le \\
&\le \int_{x^*+\zeta_z(t)}^{x^*+\zeta_z(s)} \widetilde{\rho}_{2,z}(s,y)\, dy + \int_{x^*+\zeta_z(t)}^{x^*+\zeta_z(s)} \widetilde{\rho}_{2,z}(t,y)\, dy \\
&\overset{\text{by } (29)}{=} \int_{x^*+\zeta_z(t)}^{x^*+\zeta_z(s)} \frac{v\left(\zeta_z^{-1}\left(\zeta_z(s)+x^*-y\right)\right)}{\zeta_z'\left(\zeta_z^{-1}\left(\zeta_z(s)+x^*-y\right)\right)}\, dy + \int_{x^*+\zeta_z(t)}^{x^*+\zeta_z(s)} \rho_0(y-\zeta_z(t))\, dy \\
&= \int_0^{\zeta_z^{-1}(\zeta_z(s)-\zeta_z(t))} v(\sigma)\, d\sigma + \int_{x^*}^{x^*+\zeta_z(s)-\zeta_z(t)} \rho_0(\gamma)\, d\gamma
\end{aligned}$$

and $\rho_0 \in L^1(0,1)$ and $v \in L^1(0,T)$, it is easy to conclude from monotonicity property of $\zeta_z \in C^1([0,\delta])$ and condition $\zeta_z(0) = 0$ that there exists a value $\theta_2(\varepsilon) > 0$ such that $J_2 < \varepsilon/3$.

Now we show that the same conclusion can be obtained with respect to the term J_3. Indeed, let $\left\{\rho_0^k\right\}_{k\in\mathbb{N}} \subset C^1([x^*,1])$ be an arbitrary sequence such that $\rho_0^k \to \rho_0$

in $L^1(x^*,1)$ as $k\to\infty$. Then

$$
\begin{aligned}
J_3 &:= \int_{x^*+\zeta_z(s)}^{1} |\rho_0(y-\zeta_z(s)) - \rho_0(y-\zeta_z(t))|\, dy \\
&\le \int_{x^*+\zeta_z(s)}^{1} \left|\rho_0(y-\zeta_z(s)) - \rho_0^k(y-\zeta_z(s))\right| dy \\
&+ \int_{x^*+\zeta_z(s)}^{1} \left|\rho_0^k(y-\zeta_z(s)) - \rho_0^k(y-\zeta_z(t))\right| dy \\
&\quad + \int_{x^*+\zeta_z(s)}^{1} \left|\rho_0^k(y-\zeta_z(t)) - \rho_0(y-\zeta_z(t))\right| dy \\
&\le \int_{x^*}^{1-\zeta_z(s)} \left|\rho_0(y) - \rho_0^k(y)\right| dy \\
&+ \int_{x^*+\zeta_z(s)-\zeta_z(t)}^{1-\zeta_z(t)} \left|\rho_0^k(y) - \rho_0(y)\right| dy + C(k)|\zeta_z(s)-\zeta_z(t)| \\
&\le 2\int_{x^*}^{1} \left|\rho_0(y) - \rho_0^k(y)\right| dy + C(k)|\zeta_z(s)-\zeta_z(t)|,
\end{aligned}
$$

where the constant $C(k)$ depends on $k\in\mathbb{N}$ but does not depend on t and s. Hence, in view of the strong convergence $\rho_0^k\to\rho_0$ in $L^1(x^*,1)$ and monotonicity of $\zeta_z\in C^1([0,\delta])$, there exists a value $\theta_3(\varepsilon)>0$ such that $J_3<\varepsilon/3$.

It remains to estimate the first term in the right hand side of (31). Let $\{v_k\}_{k\in\mathbb{N}}\subset C^1([0,T])$ be a strongly convergent sequence to v in $L^1(0,T)$. Then

$$
J_1 := \int_{x^*}^{x^*+\zeta_z(t)} \left| \frac{v\left(\zeta_z^{-1}\left(\zeta_z(s)+x^*-y\right)\right)}{\zeta_z'\left(\zeta_z^{-1}\left(\zeta_z(s)+x^*-y\right)\right)} - \frac{v\left(\zeta_z^{-1}\left(\zeta_z(t)+x^*-y\right)\right)}{\zeta_z'\left(\zeta_z^{-1}\left(\zeta_z(t)+x^*-y\right)\right)} \right| dy
$$

$$
\begin{aligned}
&\le \int_{x^*}^{x^*+\zeta_z(t)} \left| \frac{v\left(\zeta_z^{-1}\left(\zeta_z(s)+x^*-y\right)\right)}{\zeta_z'\left(\zeta_z^{-1}\left(\zeta_z(s)+x^*-y\right)\right)} - \frac{v_k\left(\zeta_z^{-1}\left(\zeta_z(s)+x^*-y\right)\right)}{\zeta_z'\left(\zeta_z^{-1}\left(\zeta_z(s)+x^*-y\right)\right)} \right| dy \\
&\quad + \int_{x^*}^{x^*+\zeta_z(t)} \left| \frac{v_k\left(\zeta_z^{-1}\left(\zeta_z(s)+x^*-y\right)\right)}{\zeta_z'\left(\zeta_z^{-1}\left(\zeta_z(s)+x^*-y\right)\right)} - \frac{v_k\left(\zeta_z^{-1}\left(\zeta_z(t)+x^*-y\right)\right)}{\zeta_z'\left(\zeta_z^{-1}\left(\zeta_z(t)+x^*-y\right)\right)} \right| dy \\
&\quad + \int_{x^*}^{x^*+\zeta_z(t)} \left| \frac{v_k\left(\zeta_z^{-1}\left(\zeta_z(t)+x^*-y\right)\right)}{\zeta_z'\left(\zeta_z^{-1}\left(\zeta_z(t)+x^*-y\right)\right)} - \frac{v\left(\zeta_z^{-1}\left(\zeta_z(t)+x^*-y\right)\right)}{\zeta_z'\left(\zeta_z^{-1}\left(\zeta_z(t)+x^*-y\right)\right)} \right| dy \\
&= A_1 + A_2 + A_3.
\end{aligned}
$$

Since

$$
\begin{aligned}
A_1 &= \int_{\zeta_z^{-1}(\zeta_z(s)-\zeta_z(t))}^{s} |v(\sigma)-v_k(\sigma)|\, d\sigma \le \|v-v_k\|_{L^1(0,T)}, \\
A_3 &= \int_0^t |v(\sigma)-v_k(\sigma)|\, d\sigma \le \|v-v_k\|_{L^1(0,T)},
\end{aligned}
$$

and

$$\begin{aligned}
A_2 &:= \int_{x^*}^{x^*+\zeta_z(t)} \left| \frac{v_k\left(\zeta_z^{-1}\left(\zeta_z(s)+x^*-y\right)\right)}{\zeta_z'\left(\zeta_z^{-1}\left(\zeta_z(s)+x^*-y\right)\right)} - \frac{v_k\left(\zeta_z^{-1}\left(\zeta_z(t)+x^*-y\right)\right)}{\zeta_z'\left(\zeta_z^{-1}\left(\zeta_z(t)+x^*-y\right)\right)} \right| dy \\
&\le \int_{x^*}^{x^*+\zeta_z(t)} \left| \frac{v_k\left(\zeta_z^{-1}\left(\zeta_z(s)+x^*-y\right)\right) - v_k\left(\zeta_z^{-1}\left(\zeta_z(t)+x^*-y\right)\right)}{\zeta_z'\left(\zeta_z^{-1}\left(\zeta_z(s)+x^*-y\right)\right)} \right| dy \\
&\quad + \int_{x^*}^{x^*+\zeta_z(t)} \left| \frac{v_k\left(\zeta_z^{-1}\left(\zeta_z(t)+x^*-y\right)\right)}{\zeta_z'\left(\zeta_z^{-1}\left(\zeta_z(s)+x^*-y\right)\right)} - \frac{v_k\left(\zeta_z^{-1}\left(\zeta_z(t)+x^*-y\right)\right)}{\zeta_z'\left(\zeta_z^{-1}\left(\zeta_z(t)+x^*-y\right)\right)} \right| dy \\
&\le C(k)|\zeta_z(s)-\zeta_z(t)| \\
&\quad + \widehat{C}(k) \int_{x^*}^{x^*+\zeta_z(t)} \left| \zeta_z'\left(\zeta_z^{-1}\left(\zeta_z(t)+x^*-y\right)\right) - \zeta_z'\left(\zeta_z^{-1}\left(\zeta_z(s)+x^*-y\right)\right) \right| dy
\end{aligned}$$

it follows from definition of function ζ_z (see the Cauchy problem (19)) that

$$\begin{aligned}
A_2 &\le C(k)|\zeta_z(s)-\zeta_z(t)| \\
&\quad + \widehat{C}(k) \int_{x^*}^{x^*+\zeta_z(t)} \left| V_2(\cdot)|_{\zeta_z^{-1}(\zeta_z(t)+x^*-y)} - V_2(\cdot)|_{\zeta_z^{-1}(\zeta_z(s)+x^*-y)} \right| dy \\
&\le C(k)|\zeta_z(s)-\zeta_z(t)| + \widehat{C}(k)\alpha_3 \int_{x^*}^{x^*+\zeta_z(t)} \int_{\zeta_z^{-1}(\zeta_z(t)+x^*-y)}^{\zeta_z^{-1}(\zeta_z(s)+x^*-y)} v(\sigma)\, d\sigma dy \\
&\quad + \widehat{C}(k)\alpha_3 \int_{x^*}^{x^*+\zeta_z(t)} \int_{1-\zeta_z(s)-x^*+y}^{1-\zeta_z(t)-x^*+y} \rho_0(\gamma)\, d\gamma dy. \qquad (32)
\end{aligned}$$

To estimate the right hand side in (32), we change the order of integration. As a result, we obtain

$$\begin{aligned}
\int_{x^*}^{x^*+\zeta_z(t)} \int_{\zeta_z^{-1}(\zeta_z(t)+x^*-y)}^{\zeta_z^{-1}(\zeta_z(s)+x^*-y)} v(\sigma)\, d\sigma dy &= \int_0^{\zeta_z^{-1}(\zeta_z(s)-\zeta_z(t))} \int_{\zeta_z(t)-\zeta_z(\sigma)+x^*}^{x^*+\zeta_z(t)} v(\sigma)\, dy d\sigma \\
&\quad + \int_{\zeta_z^{-1}(\zeta_z(s)-\zeta_z(t))}^{t} \int_{\zeta_z(t)-\zeta_z(\sigma)+x^*}^{\zeta_z(s)-\zeta_z(\sigma)+x^*} v(\sigma)\, dy d\sigma + \int_t^s \int_{x^*}^{\zeta_z(s)-\zeta_z(\sigma)+x^*} v(\sigma)\, dy d\sigma \\
&= \int_0^{\zeta_z^{-1}(\zeta_z(s)-\zeta_z(t))} \zeta_z(\sigma) v(\sigma)\, d\sigma + \int_{\zeta_z^{-1}(\zeta_z(s)-\zeta_z(t))}^{t} (\zeta_z(s)-\zeta_z(t)) v(\sigma)\, d\sigma \\
&\quad + \int_t^s (\zeta_z(s)-\zeta_z(\sigma)) v(\sigma)\, d\sigma. \qquad (33)
\end{aligned}$$

Taking into account that

$$\begin{aligned}
0 \le \zeta_z(\sigma) \le \zeta_z(s)-\zeta_z(t), &\qquad \text{if } 0 \le \sigma \le \zeta_z^{-1}\left(\zeta_z(s)-\zeta_z(t)\right), \\
0 \le \zeta_z(s)-\zeta_z(\sigma) \le \zeta_z(s)-\zeta_z(t), &\qquad \text{if } t \le \sigma \le s,
\end{aligned}$$

we can conclude from (33) the following estimate

$$\begin{aligned}
\int_{x^*}^{x^*+\zeta_z(t)} \int_{\zeta_z^{-1}(\zeta_z(t)+x^*-y)}^{\zeta_z^{-1}(\zeta_z(s)+x^*-y)} v(\sigma)\, d\sigma dy &\le |\zeta_z(s)-\zeta_z(t)| \int_0^s v(\sigma)\, d\sigma \\
&\le \|v\|_{L^1(0,T)} |\zeta_z(s)-\zeta_z(t)|. \qquad (34)
\end{aligned}$$

It remains to estimate the last term in (32). Following in the similar manner, we change the order of integration. As a result, we obtain

$$\begin{aligned}
&\int_{x^*}^{x^*+\zeta_z(t)} \int_{1-\zeta_z(s)-x^*+y}^{1-\zeta_z(t)-x^*+y} \rho_0(\gamma)\, d\gamma dy = \int_{1-\zeta_z(s)}^{1-\zeta_z(t)} \int_{x^*}^{\gamma+\zeta_z(s)+x^*-1} \rho_0(\gamma)\, dy d\gamma \\
&\quad + \int_{1-\zeta_z(t)}^{1-\zeta_z(s)+\zeta_z(t)} \int_{\gamma+\zeta_z(t)+x^*-1}^{\gamma+\zeta_z(s)+x^*-1} \rho_0(\gamma)\, dy d\gamma \\
&\quad + \int_{1-\zeta_z(s)+\zeta_z(t)}^{1} \int_{\gamma+\zeta_z(t)+x^*-1}^{x^*+\zeta_z(t)} \rho_0(\gamma)\, dy d\gamma \\
&= \int_{1-\zeta_z(s)}^{1-\zeta_z(t)} (\gamma+\zeta_z(s)-1)\rho_0(\gamma)\, d\gamma + \int_{1-\zeta_z(t)}^{1-\zeta_z(s)+\zeta_z(t)} (\zeta_z(s)-\zeta_z(t))\rho_0(\gamma)\, d\gamma \\
&\quad + \int_{1-\zeta_z(s)+\zeta_z(t)}^{1} (1-\gamma)\rho_0(\gamma)\, d\gamma. \qquad (35)
\end{aligned}$$

Since

$$\begin{aligned}
0 \le \gamma+\zeta_z(s)-1 \le \zeta_z(s)-\zeta_z(t), &\qquad \text{provided } \ 1-\zeta_z(s) \le \gamma \le 1-\zeta_z(t), \\
0 \le 1-\gamma \le \zeta_z(s)-\zeta_z(t), &\qquad \text{provided } \ 1-\zeta_z(s)+\zeta_z(t) \le \gamma \le 1,
\end{aligned}$$

we deduce from (35) that

$$\begin{aligned}
\int_{x^*}^{x^*+\zeta_z(t)} \int_{1-\zeta_z(s)-x^*+y}^{1-\zeta_z(t)-x^*+y} \rho_0(\gamma)\, d\gamma dy &\le |\zeta_z(s)-\zeta_z(t)| \int_{1-\zeta_z(s)}^{1} \rho_0(\gamma)\, d\gamma \\
&\le \|\rho_0\|_{L^1(0,1)} |\zeta_z(s)-\zeta_z(t)|. \qquad (36)
\end{aligned}$$

Thus, combining the estimates (32), (34), and (36), we get

$$\begin{aligned}
A_3 &\le \left[C(k) + \widehat{C}(k)\alpha_3 \left(\|v\|_{L^1(0,T)} + \|\rho_0\|_{L^1(0,1)}\right)\right] |\zeta_z(s)-\zeta_z(t)| \\
&= D(k)|\zeta_z(s)-\zeta_z(t)||\zeta_z(s)-\zeta_z(t)|,
\end{aligned}$$

and, hence,

$$J_1 \le A_1 + A_2 + A_3 \le 2\|v-v_k\|_{L^1(0,T)} + D(k)|\zeta_z(s)-\zeta_z(t)||\zeta_z(s)-\zeta_z(t)|, \quad (37)$$

where the constant $D(k)$ depends on $k \in \mathbb{N}$ but does not depend on t and s. As follows from (37), for $k \in \mathbb{N}$ large enough there exists a value $\theta_1(\varepsilon) > 0$ such that $J_1 < \varepsilon/3$. As a result, we arrive at the following conclusion: for a given $\varepsilon > 0$ and all $t, s \in [0,\delta]$ such that $|s-t| < \theta = \min\{\theta_1(\varepsilon), \theta_2(\varepsilon), \theta_3(\varepsilon)\}$, the estimate

$$\|\widetilde{\rho}_{2,z}(s,\cdot) - \widetilde{\rho}_{2,z}(t,\cdot)\|_{L^1(x^*,1)} \le J_1 + J_2 + J_3 < \varepsilon$$

holds true. □

As a consequence of Lemma 3.3, we have the following important property.

Corollary 1. *If, in additional to the assumptions of Lemma 3.4,* $\rho_0 \in L^\infty(0,1)$, *then the mappings*

$$x \mapsto \|\widetilde{\rho}_{1,x}(t,\cdot)\|_{L^1(0,x^*)} : [0,x^*] \to \mathbb{R}_+ \ \text{and} \ z \mapsto \|\widetilde{\rho}_{2,z}(t,\cdot)\|_{L^1(x^*,1)} : [x^*,1] \to \mathbb{R}_+$$

Are continuous for each $t \in [0,\delta]$.

Proof. It is easy to check that the following relations

$$\int_0^{x^*} \widetilde{\rho}_{1,x}(t,y)\, dy = \int_0^t u(\sigma)\, d\sigma + \int_0^{x^*-\xi_x(t)} \rho_0(y)\, dy, \qquad (38)$$

$$\int_{x^*}^1 \widetilde{\rho}_{2,z}(t,y)\, dy = \int_0^t v(\sigma)\, d\sigma + \int_{x^*}^{1-\zeta_z(t)} \rho_0(y)\, dy \qquad (39)$$

hold true for each $x \in [0, x^*]$, $z \in [x^*, 1]$. As a result, for any $x, y \in [0, x^*]$, we have

$$\Big| \|\widetilde{\rho}_{1,x}(t,\cdot)\|_{L^1(0,x^*)} - \|\widetilde{\rho}_{1,y}(t,\cdot)\|_{L^1(0,x^*)} \Big| = \left| \int_0^{x^*} \widetilde{\rho}_{1,x}(t,\sigma)\, d\sigma - \int_0^{x^*} \widetilde{\rho}_{1,y}(t,\sigma)\, d\sigma \right|$$

$$\overset{\text{by (38)}}{=} \left| \int_{x^*-\xi_x(t)}^{x^*-\xi_y(t)} \rho_0(\sigma)\, d\sigma \right| \le \|\rho_0\|_{L^\infty(0,1)} \Big[|\xi_x(t) - \xi_y(t)| \Big]$$

$$\overset{\text{by (27)}}{\le} \|\rho_0\|_{L^\infty(0,1)} C|x-y|.$$

The continuity of the mapping $z \mapsto \|\widetilde{\rho}_{2,z}(t,\cdot)\|_{L^1(x^*,1)}$ can be shown in a similar way. □

By Lemma 3.3, the following limits

$$\lim_{y\to x} \int_0^y \widetilde{\rho}_{1,y}(t,\gamma)\, d\gamma \overset{\text{by (28)}}{=} \lim_{y\to x} \left[\int_0^t u(\sigma)\, d\sigma + \int_0^{x-\xi_y(t)} \rho_0(y)\, dy \right],$$

$$\lim_{z\to x} \int_z^1 \widetilde{\rho}_{2,z}(t,\gamma + x^* - z)\, d\gamma \overset{\text{by (29)}}{=} \lim_{z\to x} \left[\int_0^t v(\sigma)\, d\sigma + \int_{x^*}^{1-\zeta_z(t)-z+x^*} \rho_0(y)\, dy \right] \tag{40}$$

are well defined provided $t \in [0, \xi_x^{-1}(x)]$ in $(40)_1$ and $t \in [0, \zeta_x^{-1}(1-x+x^*)]$ in $(40)_2$. In view of this, we make use of the following notations

$$\begin{aligned} \int_0^x \rho_1(t,\gamma)\, d\gamma &:= \lim_{y\to x} \int_0^y \widetilde{\rho}_{1,y}(t,\gamma)\, d\gamma, \\ \int_x^1 \rho_2(t,\gamma + x^* - x)\, d\gamma &:= \lim_{z\to x} \int_z^1 \widetilde{\rho}_{2,z}(t,\gamma + x^* - z)\, d\gamma. \end{aligned} \tag{41}$$

Then relations (38)–(39) and Lemma 3.3 imply the following integral representation for the limit functions ρ_1 and ρ_2

$$\int_0^x \rho_1(t,\gamma)\, d\gamma = \int_0^t u(\sigma)\, d\sigma + \int_0^{x-\xi_x(t)} \rho_0(\gamma)\, d\gamma, \quad \forall\, t \in [0, \min\{\delta, \xi_x^{-1}(x)\}], \tag{42}$$

$$\int_x^1 \rho_2(t,\gamma + x^* - x)\, d\gamma = \int_0^t v(\sigma)\, d\sigma + \int_{x^*}^{1-\zeta_x(t)-x+x^*} \rho_0(\gamma)\, d\gamma,$$
$$\forall\, t \in [0, \min\{\delta, \zeta_x^{-1}(1-x+x^*)\}]. \tag{43}$$

4. **Existence of weak solutions to the Cauchy Problem (5)–(7).** We begin this section with the following result.

Theorem 4.1. *For given $\rho_0 \in L^\infty(0,1)$, $u \in L^1(0,T)$, $V_1, V_2 \in \mathfrak{A}_{ad}$, and $x^* \in [0,1]$, let $\rho_1 = \rho_1(t,x)$ be defined by (42), and let $\rho_2 = \rho_2(t,x)$ be defined by (43) with*

$$v(t) := \rho_1(t,x^*) V_1 \left(\int_0^{x^*} \rho_1(t,y)\, dy \right). \tag{44}$$

Then $(\rho_1, \rho_2) \in C([0,\delta]; L^1(0,x^*)) \times C([0,\delta]; L^1(x^*,1))$ and

$$\rho(t,x) = \begin{cases} \rho_1(t,x), & \text{if } t \in [0,\delta],\ x \in [0,x^*), \\ \rho_2(t,x), & \text{if } t \in [0,\delta],\ x \in (x^*,1]) \end{cases} \tag{45}$$

is a weak solution to the Cauchy problem (5)–(7) in the strip

$$\Pi_\delta := \{(t,x) \ :\ t \in [0,\delta],\ x \in [0,1]\}. \tag{46}$$

Proof. In view of Remark 1, a weak solution to the Cauchy problem (5)–(7) in the strip (46) can be defined in the sense of Definition 3.1. Following this definition, we fix an arbitrary $\tau \in [0,\delta]$ and a couple of test functions $(\varphi_1,\varphi_2) \in C^1([0,\tau] \times [0,x^*]) \times C^1([0,\tau]\times[x^*,1])$ such that

$$\begin{aligned} &\varphi_1(\tau,x)=0, \quad \forall\, x\in[0,x^*], \quad \varphi_1(t,x^*)=0, \quad \forall\, t\in[0,\tau],\\ &\varphi_2(\tau,x)=0, \quad \forall\, x\in[x^*,1], \quad \varphi_2(t,1)=0, \quad \forall\, t\in[0,\tau]. \end{aligned} \tag{47}$$

Then, direct computations show that

$$\begin{aligned} A :=& \int_0^\tau\int_0^{x^*} \rho_1(t,x)\left[\partial_t\varphi_1(t,x) + V_1\left(\int_0^x \rho_1(t,s)\,ds\right)\partial_x\varphi_1(t,x)\right] dxdt\\ \overset{\text{by (41)}}{=}& \lim_{y\to x}\int_0^\tau\int_0^{x^*} \widetilde{\rho}_{1,y}(t,x)\left[\partial_t\varphi_1(t,x) + V_1\left(\int_0^y \rho_1(t,s)\,ds\right)\partial_x\varphi_1(t,x)\right] dxdt\\ \overset{\text{by Lemma 3.3 and Corollary 1}}{=}& \lim_{y\to x}\int_0^\tau\int_0^{\xi_y(t)} \frac{u\left(\xi_y^{-1}\left(\xi_y(t)-x\right)\right)}{\xi_y'\left(\xi_y^{-1}\left(\xi_y(t)-x\right)\right)}\,\partial_t\varphi_1(t,x)\,dxdt\\ =& \lim_{y\to x}\int_0^\tau\int_0^{\xi_y(t)} \frac{u\left(\xi_y^{-1}\left(\xi_y(t)-x\right)\right)}{\xi_y'\left(\xi_y^{-1}\left(\xi_y(t)-x\right)\right)}\,V_1\left(\int_0^y \rho_1(t,s)\,ds\right)\partial_x\varphi_1(t,x)\,dxdt\\ &+\lim_{y\to x}\int_0^\tau\int_{\xi_y(t)}^{x^*} \rho_0(x-\xi_y(t))\partial_t\varphi_1(t,x)\,dxdt\\ &+\lim_{y\to x}\int_0^\tau\int_{\xi_y(t)}^{x^*} \rho_0(x-\xi_y(t))V_1\left(\int_0^y \rho_1(t,s)\,ds\right)\partial_x\varphi_1(t,x)\,dxdt \end{aligned}$$

$$\begin{aligned} =& \lim_{y\to x}\int_0^\tau\int_0^t u(\sigma)\partial_t\varphi_1(t,\xi_y(t)-\xi_y(\sigma))\,d\sigma dt\\ &+\lim_{y\to x}\int_0^\tau\int_0^t u(\sigma)V_1\left(\int_0^{\xi_y(t)-\xi_y(\sigma)} \rho_1(t,s)\,ds\right)\partial_x\varphi_1(t,\xi_y(t)-\xi_y(\sigma))\,d\sigma dt\\ &+\lim_{y\to x}\int_0^\tau\int_0^{x^*-\xi_y(t)} \rho_0(\sigma)\partial_t\varphi_1(t,\sigma+\xi_y(t))\,d\sigma dt\\ &+\lim_{y\to x}\int_0^\tau\int_0^{x^*-\xi_y(t)} \rho_0(\sigma)V_1\left(\int_0^{\sigma+\xi_y(t)} \rho_1(t,s)\,ds\right)\partial_x\varphi_1(t,\sigma+\xi_y(t))\,d\sigma dt. \end{aligned}$$

Using again Lemma 3.3 and Corollary 1, we can pass to the limit as $y \to x-0$. Therefore,

$$\begin{aligned} A =& \int_0^\tau\int_0^t u(\sigma)\partial_t\varphi_1(t,\xi_x(t)-\xi_x(\sigma))\,d\sigma dt\\ &+\int_0^\tau\int_0^t u(\sigma)V_1\left(\int_0^{\xi_x(t)-\xi_x(\sigma)} \rho_1(t,s)\,ds\right)\partial_x\varphi_1(t,\xi_x(t)-\xi_x(\sigma))\,d\sigma dt\\ &+\int_0^\tau\int_0^{x^*-\xi_x(t)} \rho_0(\sigma)\partial_t\varphi_1(t,\sigma+\xi_x(t))\,d\sigma dt\\ &+\int_0^\tau\int_0^{x^*-\xi_x(t)} \rho_0(\sigma)V_1\left(\int_0^{\sigma+\xi_x(t)} \rho_1(t,s)\,ds\right)\partial_x\varphi_1(t,\sigma+\xi_x(t))\,d\sigma dt. \end{aligned}$$

As a result, making use of relations (19) and (38), we arrive at the following relation

$$\begin{aligned} A &= \int_0^\tau \int_0^t u(\sigma) \frac{d\varphi_1(t, \xi_x(t) - \xi_x(\sigma))}{dt}\, d\sigma dt \\ &+ \int_0^\tau \int_0^{x^* - \xi_x(t)} \rho_0(\sigma) \frac{d\varphi_1(t, \sigma + \xi_x(t))}{dt}\, d\sigma dt \\ &= \int_0^\tau u(\sigma) \left(\int_\sigma^\tau \frac{d\varphi_1(t, \xi_x(t) - \xi_x(\sigma))}{dt}\, dt \right) d\sigma \\ &\quad + \int_0^{x^* - \xi_x(\tau)} \rho_0(\sigma) \left(\int_0^\tau \frac{d\varphi_1(t, \sigma + \xi_x(t))}{dt}\, dt \right) d\sigma \\ &\quad + \int_{x^* - \xi_x(\tau)}^{x^*} \rho_0(\sigma) \left(\int_0^{\xi_x^{-1}(x^* - \sigma)} \frac{d\varphi_1(t, \sigma + \xi_x(t))}{dt}\, dt \right) d\sigma \\ &\overset{\text{by (47)}}{=} - \int_0^\tau u(\sigma) \varphi_1(\sigma, 0)\, d\sigma - \int_0^{x^* - \xi_x(\tau)} \rho_0(\sigma) \varphi_1(0, \sigma)\, d\sigma \\ &- \int_{x^* - \xi_x(\tau)}^{x^*} \rho_0(\sigma) \varphi_1(0, \sigma)\, d\sigma \\ &= - \int_0^\tau u(t) \varphi_1(t, 0)\, dt - \int_0^{x^*} \rho_0(x) \varphi_1(0, x)\, dx, \end{aligned}$$

which immediately yields the integral identity (17).

Following the similar scheme, it can be shown that

$$\begin{aligned} B &:= \int_0^\tau \int_{x^*}^1 \rho_2(t, x) \left[\partial_t \varphi_2(t, x) + V_2 \left(\int_x^1 \rho_2(t, y + x^* - x)\, dy \right) \partial_x \varphi_2(t, x) \right] dx dt \\ &\overset{\text{by (41)}}{=} \lim_{z \to x} \int_0^\tau \int_{x^*}^1 \rho_2(t, x) \left[\partial_t \varphi_2(t, x) + V_2 \left(\int_z^1 \rho_2(t, y)\, dy \right) \partial_x \varphi_2(t, x) \right] dx dt \\ &= \int_0^\tau \int_0^t v(\sigma) \frac{d\varphi_2(t, \zeta_x(t) - \zeta_x(\sigma) + x^*)}{dt}\, d\sigma dt \\ &+ \int_0^\tau \int_{x^*}^{1 - \zeta_x(t)} \rho_0(\sigma) \frac{d\varphi_2(t, \sigma + \zeta_x(t))}{dt}\, d\sigma dt \\ &= \int_0^\tau v(\sigma) \left(\int_\sigma^\tau \frac{d\varphi_2(t, \zeta_x(t) - \zeta_x(\sigma) + x^*)}{dt}\, dt \right) d\sigma \\ &\quad + \int_{x^*}^{1 - \zeta_x(\tau)} \rho_0(\sigma) \left(\int_0^\tau \frac{d\varphi_2(t, \sigma + \zeta_x(t))}{dt}\, dt \right) d\sigma \\ &\quad + \int_{1 - \zeta_x(\tau)}^1 \rho_0(\sigma) \left(\int_0^{\zeta_x^{-1}(1 - \sigma)} \frac{d\varphi_2(t, \sigma + \zeta_x(t))}{dt}\, dt \right) d\sigma \\ &= - \int_0^\tau v(t) \varphi_2(t, x^*)\, dt + \int_{x^*}^1 \rho_0(x) \varphi_2(0, x)\, dx \end{aligned}$$

for all $\varphi_2 \in C([0, \tau] \times [x^*, 1])$ with properties (47), where the input-flux $v(t)$ is defined by (44). Since the inclusion $(\rho_1, \rho_2) \in C([0, \delta]; L^1(0, x^*)) \times C([0, \delta]; L^1(x^*, 1))$ is a consequence of Lemma 3.4 and representations (42)–(43), the existence result to the Cauchy problem (5)–(7) in the strip (46) is established. □

Theorem 4.2. *Under assumptions of Theorem 4.1, a weak solution to the Cauchy problem (5)–(7) in the strip* (46) *is unique.*

Proof. In order to show that the distribution (45) defined in Theorem 4.1 is the unique solution to this problem, we make use of some ideas from [17, Theorem 3.2].

Let us assume, by contraposition, that there exists another distribution

$$\widehat{\rho}(t,x)=\begin{cases}\widehat{\rho}_1(t,x), & \text{if } t\in[0,\delta],\ x\in[0,x^*),\\ \widehat{\rho}_2(t,x), & \text{if } t\in[0,\delta],\ x\in(x^*,1])\end{cases}$$

such that $\rho(t,x)\neq\widehat{\rho}(t,x)$. It is worth to emphasize that, in general, it is unknown whether this function can be represented in the form like (42)–(43), because in this case Lemma 3.2 immediately leads to the conclusion

$$\int_0^x (\rho_1(t,\gamma)-\widehat{\rho}_1(t,\gamma))\,d\gamma=0,\qquad \int_y^1 (\rho_2(t,\gamma+x^*-y)-\widehat{\rho}_2(t,\gamma+x^*-y))\,d\gamma=0,$$

for all $t\in[0,T]$, almost all $x\in[0,x^*]$ and $y\in[x^*,1]$, and, therefore, $\rho_1(t,\gamma)=\widehat{\rho}_1(t,x)$ and $\rho_2(t,\gamma)=\widehat{\rho}_2(t,x)$ almost everywhere in the corresponding domains.

In view of this, we assume that $\widehat{\rho}(t,x)$ is merely a weak solution to the Cauchy problem (5)–(7) in the sense of Definition 3.1. For each $\tau\in[0,\delta]$, $\varepsilon\in(0,\tau)$, and a test function $(\varphi_1,\varphi_2)\in C^1([0,\tau]\times[0,x^*])\times C^1([0,\tau]\times[x^*,1])$ with properties (see for comparison (47))

$$\varphi_1(t,x^*)=0\quad\text{and}\quad\varphi_2(t,1)=0,\quad\forall\, t\in[0,\tau],\tag{48}$$

we set $\varphi_{1,\varepsilon}(t,x):=\eta_\varepsilon(t)\varphi_1(t,x)$ and $\varphi_{2,\varepsilon}(t,x):=\eta_\varepsilon(t)\varphi_2(t,x)$, where

$$\eta_\varepsilon(\tau)=0\ \text{ and }\ \eta_\varepsilon(t)=1,\quad\forall\, t\in[0,\tau-\varepsilon]\ \text{ and }\ \eta'_\varepsilon(t)\le 0,\ \forall\, t\in[0,\tau].\tag{49}$$

It is clear that, in this case, the new test function $(\varphi_{1,\varepsilon},\varphi_{2,\varepsilon})$ satisfies properties (47). Hence, by Definition 3.1, we have the equalities

$$\begin{aligned}&\int_0^\tau\int_0^{x^*}\widehat{\rho}_1(t,x)\left[\partial_t\varphi_{1,\varepsilon}(t,x)+V_1\left(\int_0^x\widehat{\rho}_1(t,y)\,dy\right)\partial_x\varphi_{1,\varepsilon}(t,x)\right]dxdt\\&\qquad+\int_0^\tau u(t)\varphi_{1,\varepsilon}(t,0)\,dt+\int_0^{x^*}\rho_0(x)\varphi_{1,\varepsilon}(0,x)\,dx=0,\\&\int_0^\tau\int_{x^*}^1\widehat{\rho}_2(t,x)\left[\partial_t\varphi_{2,\varepsilon}(t,x)+V_2\left(\int_x^1\widehat{\rho}_2(t,y+x^*-x)\,dy\right)\partial_x\varphi_{2,\varepsilon}(t,x)\right]dxdt\\&\qquad+\int_0^\tau\widehat{v}(t)\varphi_{2,\varepsilon}(t,x^*)\,dt+\int_{x^*}^1\rho_0(x)\varphi_{2,\varepsilon}(0,x)\,dx=0,\end{aligned}$$

where

$$\widehat{v}(t):=\widehat{\rho}_1(t,x^*)V_1\left(\int_0^{x^*}\widehat{\rho}_1(t,y)\,dy\right).$$

In view of (49), these relations can be rewritten as follows

$$\begin{aligned}&\int_0^\tau\int_0^{x^*}\widehat{\rho}_1(t,x)\left[\partial_t\varphi_1(t,x)+V_1\left(\int_0^x\widehat{\rho}_1(t,y)\,dy\right)\partial_x\varphi_1(t,x)\right]dxdt\\&\qquad+\int_0^\tau u(t)\varphi_1(t,0)\,dt+\int_0^{x^*}\rho_0(x)\varphi_1(0,x)\,dx\\&=\int_{\tau-\varepsilon}^\tau\int_0^{x^*}(1-\eta_\varepsilon)\widehat{\rho}_1(t,x)\left[\partial_t\varphi_1(t,x)+V_1\left(\int_0^x\widehat{\rho}_1(t,y)\,dy\right)\partial_x\varphi_1(t,x)\right]dxdt,\\&\qquad+\int_{\tau-\varepsilon}^\tau(1-\eta_\varepsilon)u(t)\varphi_1(t,0)\,dt-\int_{\tau-\varepsilon}^\tau\eta'_\varepsilon(t)\left(\int_0^{x^*}\widehat{\rho}_1(t,x)\varphi_1(t,x)\,dx\right)dt,\end{aligned}\tag{50}$$

$$\begin{aligned}
&\int_0^\tau \int_{x^*}^1 \widehat{\rho}_2(t,x)\left[\partial_t\varphi_2(t,x) + V_2\left(\int_x^1 \widehat{\rho}_2(t,y+x^*-x)\,dy\right)\partial_x\varphi_2(t,x)\right]\,dxdt\\
&\quad + \int_0^\tau \widehat{v}(t)\varphi_2(t,x^*)\,dt + \int_{x^*}^1 \rho_0(x)\varphi_2(0,x)\,dx\\
&= \int_{\tau-\varepsilon}^\tau \int_{x^*}^1 (1-\eta_\varepsilon)\widehat{\rho}_2(t,x)\,[\partial_t\varphi_2(t,x)\\
&\quad + V_2\left(\int_x^1 \widehat{\rho}_2(t,y+x^*-x)\,dy\right)\partial_x\varphi_2(t,x)\Big]\,dxdt\\
&\quad + \int_{\tau-\varepsilon}^\tau (1-\eta_\varepsilon)\widehat{v}(t)\varphi_2(t,x^*)\,dt - \int_{\tau-\varepsilon}^\tau \eta'_\varepsilon(t)\left(\int_{x^*}^1 \widehat{\rho}_2(t,x)\varphi_2(t,x)\,dx\right)dt. \quad (51)
\end{aligned}$$

Since $\widehat{\rho} \in C([0,\delta];L^1(0,x^*)) \times C([0,\delta];L^1(x^*,1))$, $(\varphi_1,\varphi_2) \in C^1([0,\tau]\times[0,x^*]) \times C^1([0,\tau]\times[x^*,1])$, and $V_1, V_2 \in \mathfrak{A}_{ad} \subset C^1([0,\infty))$, it follows that there exists a constant D independent of ε such that

$$\left|\int_{\tau-\varepsilon}^\tau \int_0^{x^*} (1-\eta_\varepsilon)\widehat{\rho}_1(t,x)\left[\partial_t\varphi_1(t,x) + V_1\left(\int_0^x \widehat{\rho}_1(t,y)\,dy\right)\partial_x\varphi_1(t,x)\right]\,dxdt\right| \le D\varepsilon,$$

$$\left|\int_{\tau-\varepsilon}^\tau \int_{x^*}^1 (1-\eta_\varepsilon)\widehat{\rho}_2(t,x)\left[\partial_t\varphi_2(t,x) + V_2\left(\int_x^1 \widehat{\rho}_2(t,y+x^*-x)\,dy\right)\partial_x\varphi_2(t,x)\right]\,dxdt\right| \le D\varepsilon,$$

$$\left|\int_{\tau-\varepsilon}^\tau (1-\eta_\varepsilon)u(t)\varphi_1(t,0)\,dt\right| \le D\varepsilon, \quad \left|\int_{\tau-\varepsilon}^\tau (1-\eta_\varepsilon)\widehat{v}(t)\varphi_2(t,x^*)\,dt\right| \le D\varepsilon.$$

At the same time, the last terms in (50)–(51) possess the following properties

$$\begin{aligned}
\int_{\tau-\varepsilon}^\tau \eta'_\varepsilon(t)\left(\int_0^{x^*} \widehat{\rho}_1(t,x)\varphi_1(t,x)\,dx\right)dt &\overset{\varepsilon\to 0}{\to} -\int_0^{x^*} \widehat{\rho}_1(\tau,x)\varphi_1(\tau,x)\,dx,\\
\int_{\tau-\varepsilon}^\tau \eta'_\varepsilon(t)\left(\int_{x^*}^1 \widehat{\rho}_2(t,x)\varphi_2(t,x)\,dx\right)dt &\overset{\varepsilon\to 0}{\to} -\int_{x^*}^1 \widehat{\rho}_2(\tau,x)\varphi_2(\tau,x)\,dx.
\end{aligned}$$

Thus, passing to the limit in (50)–(51), we arrive at the extended integral identities for the weak solution $\widehat{\rho} \in C([0,\delta];L^1(0,x^*)) \times C([0,\delta];L^1(x^*,1))$:

$$\begin{aligned}
&\int_0^\tau \int_0^{x^*} \widehat{\rho}_1(t,x)\left[\partial_t\varphi_1(t,x) + V_1\left(\int_0^x \widehat{\rho}_1(t,y)\,dy\right)\partial_x\varphi_1(t,x)\right]\,dxdt\\
&\quad + \int_0^\tau u(t)\varphi_1(t,0)\,dt + \int_0^{x^*} \rho_0(x)\varphi_1(0,x)\,dx - \int_0^{x^*} \widehat{\rho}_1(\tau,x)\varphi_1(\tau,x)\,dx = 0,
\end{aligned} \quad (52)$$

$$\begin{aligned}
&\int_0^\tau \int_{x^*}^1 \widehat{\rho}_2(t,x)\left[\partial_t\varphi_2(t,x) + V_2\left(\int_x^1 \widehat{\rho}_2(t,y+x^*-x)\,dy\right)\partial_x\varphi_2(t,x)\right]\,dxdt\\
&\quad + \int_0^\tau \widehat{v}(t)\varphi_2(t,x^*)\,dt + \int_{x^*}^1 \rho_0(x)\varphi_2(0,x)\,dx - \int_{x^*}^1 \widehat{\rho}_2(\tau,x)\varphi_2(\tau,x)\,dx = 0. \quad (53)
\end{aligned}$$

We are now in a position to specify the choice of test functions $(\varphi_1,\varphi_2) \in C^1([0,\tau]\times[0,x^*]) \times C^1([0,\tau]\times[x^*,1])$ in (52)–(53) with properties (48). With that in mind, for given $\rho_0 \in L^\infty(0,1)$, $u \in L^1(0,T)$, $V_1, V_2 \in \mathfrak{A}_{ad}$, $x^* \in [0,1]$, and arbitrary $y \in [0,x^*]$ and $z \in [x^*,1]$, we define functions $(\widehat{\xi}_y(t), \widehat{\zeta}_z(t))$ by the rule

$$\widehat{\xi}_y(t) := \int_0^t V_1\left(\int_0^y \widehat{\rho}_1(s,\sigma)\,d\sigma\right)ds, \quad \widehat{\zeta}_z(t) := \int_0^t V_2\left(\int_z^1 \widehat{\rho}_2(s,\sigma+x^*-z)\,d\sigma\right)ds. \quad (54)$$

It is clear that these functions are monotonically increasing, $(\widehat{\xi}_y, \widehat{\zeta}_z) \in \left[C^1([0,\delta])\right]^2$, and the mappings

$$y \mapsto \widehat{\xi}_y(t) : [0,x^*] \to \mathbb{R}_+ \quad \text{and} \quad z \mapsto \widehat{\zeta}_z(t) : [x^*,1] \to \mathbb{R}_+ \tag{55}$$

are continuous. Moreover, direct computations show that

$$\begin{aligned}
\frac{\partial}{\partial y}\widehat{\xi}_y(t) &:= \int_0^t V_1'\left(\int_0^y \widehat{\rho}_1(s,\sigma)\,d\sigma\right)\widehat{\rho}_1(s,y)\,ds,\\
\frac{\partial}{\partial z}\widehat{\zeta}_z(t) &:= -\int_0^t V_2'\left(\int_z^1 \widehat{\rho}_2(s,\sigma+x^*-z)\,d\sigma\right)\widehat{\rho}_2(s,x^*)\,ds
\end{aligned}$$

and, therefore, the mappings

$$y \mapsto \frac{\partial}{\partial y}\widehat{\xi}_y(t) : [0,x^*] \to \mathbb{R}_+ \quad \text{and} \quad z \mapsto \frac{\partial}{\partial z}\widehat{\zeta}_z(t) : [x^*,1] \to \mathbb{R}_+$$

are measurable and integrable. Thus, the mappings (55) are absolutely continuous.

Let $(\psi_1,\psi_2) \in C_0^1([0,x^*]) \times C_0^1([x^*,1])$ be arbitrary functions. As a result, we define the test functions (φ_1,φ_2) for (52)–(53) as follows: $(\varphi_1,\varphi_2) = (\varphi_1^y, \varphi_2^z)$, where

$$\varphi_1^y(t,x) := \begin{cases} \psi_1\left(\widehat{\xi}_y(\tau) - \widehat{\xi}_y(t) + x\right), & 0 \le x \le \widehat{\xi}_y(t) - \widehat{\xi}_y(\tau) + x^*,\\ 0, & \widehat{\xi}_y(t) - \widehat{\xi}_y(\tau) + x^* \le x \le x^*, \end{cases} \quad t \in [0,\tau], \tag{56}$$

$$\varphi_2^z(t,x) := \begin{cases} \psi_2\left(\widehat{\zeta}_z(\tau) - \widehat{\zeta}_z(t) + x\right), & x^* \le x \le 1 - \widehat{\zeta}_z(\tau) + \widehat{\zeta}_z(t),\\ 0, & 1 - \widehat{\zeta}_z(\tau) + \widehat{\zeta}_z(t) \le x \le 1, \end{cases} \quad t \in [0,\tau]. \tag{57}$$

It is clear now that $(\varphi_1^y, \varphi_2^z) \in C^1([0,\delta]\times[0,x^*]) \times C^1([0,\delta]\times[x^*,1])$ and for each $y \in [0,x^*]$ and $z \in [x^*,1]$ these functions satisfy the Cauchy problems

$$\begin{cases} \partial_t \varphi_1^y(t,x) + V_1\left(\int_0^y \widehat{\rho}_1(t,\sigma)\,d\sigma\right)\partial_x \varphi_1^y(t,x) = 0, & (t,x) \in (0,\delta)\times(0,x^*),\\ \varphi_1^y(\tau,x) = \psi_1(x), & x \in [0,x^*],\\ \varphi_1^y(t,x^*) = 0, & t \in [0,\delta], \end{cases} \tag{58}$$

and

$$\begin{cases} \partial_t \varphi_2^z(t,x) + V_2\left(\int_z^1 \widehat{\rho}_2(t,\sigma+x^*-z)\,d\sigma\right)\partial_x \varphi_2^z(t,x) = 0, & (t,x) \in (0,\delta)\times(x^*,1),\\ \varphi_2^z(\tau,x) = \psi_2(x), & x \in [x^*,1],\\ \varphi_2^z(t,1) = 0, & t \in [0,\delta], \end{cases} \tag{59}$$

respectively.

As immediately follows from (56)–(57) and properties (55), the mapping

$$y \mapsto \varphi_1^y(t,x) : [0,x^*] \to \mathbb{R} \quad \text{and} \quad z \mapsto \varphi_2^z(t,x) : [x^*,1] \to \mathbb{R},$$

$$y \mapsto \partial_t\varphi_1^y(t,x) : [0,x^*] \to \mathbb{R} \quad \text{and} \quad z \mapsto \partial_t\varphi_2^z(t,x) : [x^*,1] \to \mathbb{R},$$

$$y \mapsto V_1\left(\int_0^y \widehat{\rho}_1(t,\sigma)\,d\sigma\right)\partial_x\varphi_1^y(t,x) : [0,x^*] \to \mathbb{R} \quad \text{and}$$

$$z \mapsto V_2\left(\int_z^1 \widehat{\rho}_2(t,\sigma+x^*-z)\,d\sigma\right)\partial_x\varphi_2^z(t,x) : [x^*,1] \to \mathbb{R}$$

are continuous. Hence, the limit passage in (58)–(59) as $y \to x$ and $z \to x$ yields

$$\begin{cases} \partial_t \varphi_1^x(t,x) + V_1\left(\int_0^x \widehat{\rho}_1(t,\sigma)\,d\sigma\right)\partial_x \varphi_1^x(t,x) = 0, & (t,x) \in (0,\delta)\times(0,x^*),\\ \varphi_1^x(\tau,x) = \psi_1(x), & x\in[0,x^*],\\ \varphi_1^x(t,x^*) = 0, & t\in[0,\delta], \end{cases} \tag{60}$$

$$\begin{cases} \partial_t \varphi_2^x(t,x) + V_2\left(\int_x^1 \widehat{\rho}_2(t,\sigma+x^*-x)\,d\sigma\right)\partial_x \varphi_2^x(t,x) & = 0,\\ & (t,x)\in(0,\delta)\times(x^*,1),\\ \varphi_2^x(\tau,x) = \psi_2(x+\widehat{z}-x^*), & x\in[x^*,1+x^*-\widehat{z}],\\ \varphi_2^x(\tau,x) = 0, & x\in[1+x^*-\widehat{z},1],\\ \varphi_2^x(t,1) = 0, & t\in[0,\delta]. \end{cases} \tag{61}$$

As a result, we deduce from (52)–(53) that

$$\begin{aligned} 0 &= \lim_{y\to x}\int_0^\tau\int_0^{x^*}\widehat{\rho}_1(t,x)\left[\partial_t\varphi_1^y(t,x) + V_1\left(\int_0^x\widehat{\rho}_1(t,y)\,dy\right)\partial_x\varphi_1^y(t,x)\right]dxdt\\ &\quad + \lim_{y\to x}\int_0^\tau u(t)\varphi_1^y(t,0)\,dt + \lim_{y\to x}\int_0^{x^*}\rho_0(x)\varphi_1^y(0,x)\,dx\\ &\quad - \lim_{y\to x}\int_0^{x^*}\widehat{\rho}_1(\tau,x)\varphi_1^y(\tau,x)\,dx\\ &\overset{\text{by (60),(56)}}{=} \lim_{y\to x}\int_0^\tau u(t)\psi_1\left(\widehat{\xi}_y(\tau)-\widehat{\xi}_y(t)\right)\,dt\\ &\quad + \lim_{y\to x}\int_0^{x^*-\widehat{\xi}_y(\tau)}\rho_0(x)\psi_1\left(\widehat{\xi}_y(\tau)+x\right)\,dx - \int_0^{x^*}\widehat{\rho}_1(\tau,x)\psi_1(x)\,dx\\ &= \lim_{y\to x}\left[\int_0^{\widehat{\xi}_y(\tau)}\frac{u\left(\widehat{\xi}_y^{-1}(\widehat{\xi}_y(\tau)-\sigma)\right)}{\widehat{\xi}_y'\left(\widehat{\xi}_y^{-1}(\widehat{\xi}_y(\tau)-\sigma)\right)}\psi_1(\sigma)\,d\sigma + \int_{\widehat{\xi}_y(\tau)}^{x^*}\rho_0(\sigma-\widehat{\xi}_y(\tau))\psi_1(\sigma)\,d\sigma\right]\\ &\quad - \int_0^{x^*}\widehat{\rho}_1(\tau,x)\psi_1(x)\,dx = -\int_0^{x^*}\widehat{\rho}_1(\tau,x)\psi_1(x)\,dx\\ &\quad + \int_0^{\widehat{\xi}_x(\tau)}\frac{u\left(\widehat{\xi}_x^{-1}(\widehat{\xi}_x(\tau)-\sigma)\right)}{\widehat{\xi}_x'\left(\widehat{\xi}_x^{-1}(\widehat{\xi}_x(\tau)-\sigma)\right)}\psi_1(\sigma)\,d\sigma + \int_{\widehat{\xi}_x(\tau)}^{x^*}\rho_0(\sigma-\widehat{\xi}_x(\tau))\psi_1(\sigma)\,d\sigma, \end{aligned} \tag{62}$$

and

$$\begin{aligned} 0 &= \lim_{z\to x}\int_0^\tau\int_{x^*}^1\widehat{\rho}_2(t,x)\left[\partial_t\varphi_2^z(t,x) + V_2\left(\int_z^1\widehat{\rho}_2(t,y+x^*-z)\,dy\right)\partial_x\varphi_2^z(t,x)\right]dxdt\\ &\quad + \lim_{z\to x}\int_0^\tau\widehat{v}(t)\varphi_2^z(t,x^*)\,dt + \lim_{z\to x}\int_{x^*}^1\rho_0(x)\varphi_2^z(0,x)\,dx\\ &\quad - \lim_{z\to x}\int_{x^*}^1\widehat{\rho}_2(\tau,x)\varphi_2^z(\tau,x)\,dx\\ &\overset{\text{by (61),(57)}}{=} \lim_{z\to x}\int_0^\tau\widehat{v}(t)\psi_2\left(\widehat{\zeta}_z(\tau)-\widehat{\zeta}_z(t)+x^*\right)\,dt - \int_{x^*}^1\widehat{\rho}_2(\tau,x)\psi_2(x)\,dx\\ &\quad + \lim_{z\to x}\int_{x^*}^{1-\widehat{\zeta}_z(\tau)}\rho_0(x)\psi_2\left(\widehat{\zeta}_z(\tau)+x\right)\,dx \end{aligned}$$

$$= \lim_{z\to x} \int_{x^*}^{x^*+\widehat{\zeta}_z(\tau)} \frac{\widehat{v}\left(\widehat{\zeta}_z^{-1}\left(\widehat{\zeta}_z(\tau)+x^*-x\right)\right)}{\widehat{\zeta}_z'\left(\widehat{\zeta}_z^{-1}\left(\widehat{\zeta}_z(\tau)+x^*-x\right)\right)} \psi_2(x)\,dx$$

$$+ \lim_{z\to x} \int_{x^*+\widehat{\zeta}_z(\tau)}^{1} \rho_0(x-\widehat{\zeta}_z(\tau))\psi_2(x)\,dx - \int_{x^*}^{1} \widehat{\rho}_2(\tau,x)\psi_2(x)\,dx. \tag{63}$$

Taking into account the continuity result (26) and the fact that functions $(\psi_1,\psi_2) \in C_0^1([0,\widehat{x}]) \times C_0^1([\widehat{z},1])$ and parameter $\tau \in [0,\delta]$ were arbitrary, after localization, we can conclude from (62)–(63) the following relations

$$\int_0^x \widehat{\rho}_1(t,\sigma)\,d\sigma := \int_0^{\widehat{\xi}_x(t)} \frac{u\left(\widehat{\xi}_x^{-1}(\widehat{\xi}_x(t)-x)\right)}{\widehat{\xi}_x'\left(\widehat{\xi}_x^{-1}(\widehat{\xi}_x(t)-x)\right)}\,dx + \int_{\widehat{\xi}_x(t)}^{x} \rho_0(\sigma-\widehat{\xi}_x(t))\,d\sigma$$

$$= \int_0^t u(s)\,ds + \int_0^{x-\widehat{\xi}_x(t)} \rho_0(\sigma)\,d\sigma, \tag{64}$$

$$\int_{x^*}^1 \widehat{\rho}_2(t,\sigma)\,d\sigma := \lim_{z\to x} \int_{x^*}^{x^*+\widehat{\zeta}_z(t)} \frac{\widehat{v}\left(\widehat{\zeta}_z^{-1}\left(\widehat{\zeta}_z(t)+x^*-x\right)\right)}{\widehat{\zeta}_z'\left(\widehat{\zeta}_z^{-1}\left(\widehat{\zeta}_z(t)+x^*-x\right)\right)}\,dx$$

$$+ \lim_{z\to x} \int_{x^*+\widehat{\zeta}_z(t)}^{1} \rho_0(\sigma-\widehat{\zeta}_z(t))\,d\sigma = \int_0^t \widehat{v}(s)\,ds + \int_{x^*}^{1-\widehat{\zeta}_x(t)} \rho_0(\sigma)\,d\sigma \tag{65}$$

which evidently hold true in $C([0,\delta];L^1(0,x^*))$ and $C([0,\delta];L^1(x^*,1))$, respectively. Moreover, as immediately follows from (65), we have

$$\int_x^1 \widehat{\rho}_2(t,\gamma+x^*-x)\,d\gamma = \int_0^t \widehat{v}(\sigma)\,d\sigma + \int_{x^*}^{1-\widehat{\zeta}_x(t)-x+x^*} \rho_0(\gamma)\,d\gamma. \tag{66}$$

Then, combining relations (54), (66), and (64), we see that the functions $(\widehat{\xi}_y(t),\widehat{\zeta}_z(t))$ satisfy the Cauchy problem (19). Since, by Lemma 3.2, this problem has a unique solution, it follows that $\widehat{\xi}_y(t)\equiv\xi_y(t)$ and $\widehat{\zeta}_z(t)\equiv\zeta_z$ as elements of $C^1([0,\delta])$. Hence, $\rho=\widehat{\rho}$ by comparing (66) and (64) with (42) and (43). Thus, a weak solution to the Cauchy problem (5)–(7) is unique for small time. □

As a consequence of Theorem 4.1, we have the following hidden regularity property of the weak solutions.

Corollary 2. *Let $\rho=(\rho_1,\rho_2)\in C([0,\tau];L^1(0,x^*))\times C([0,\tau];L^1(x^*,1))$ be a weak solution to the Cauchy problem (5)–(7) for some $\tau\in(0,T]$. Then for given $\rho_0\in L^\infty(0,1)$, $u\in L^1(0,T)$, $V_1,V_2\in\mathfrak{A}_{ad}$, and $x^*\in[0,1]$, we have*

$$(\rho_1,\rho_2)\in C([0,x^*];L^1(0,\tau))\times C([x^*,1];L^1(0,\tau)). \tag{67}$$

Proof. Let $x\in(0,x^*)$ be an arbitrary point. Then, by the first mean value theorem for integration, we get

$$A(x) := \int_0^\tau \rho_1(t,x)\,dt = \lim_{\Delta\to 0}\frac{1}{2\Delta}\int_0^\tau \left(\int_{x-\Delta}^{x+\Delta}\rho_1(t,y)\,dy\right)dt = \{\text{by } (41)\}$$

$$= \lim_{\Delta\to 0}\lim_{z\to x}\frac{1}{2\Delta}\int_0^\tau \left(\int_{x-\Delta}^{x+\Delta}\widetilde{\rho}_{1,z}(t,y)\,dy\right)dt. \tag{68}$$

As follows from (28), we have the following representation

$$
\begin{aligned}
&\int_{x-\Delta}^{x+\Delta} \widetilde{\rho}_{1,z}(t,y)\,dy \\
&= \begin{cases}
\int_{x-\Delta}^{x+\Delta} \rho_0(y-\xi_z(t))\,dy, & 0<t<\xi_z^{-1}(x-\Delta),\\
\int_{x-\Delta}^{\xi_z(t)} \frac{u\left(\xi_z^{-1}\left(\xi_z(t)-y\right)\right)}{\xi_z'\left(\xi_z^{-1}\left(\xi_z(t)-y\right)\right)}\,dy & \\
\qquad\qquad + \int_{\xi_z(t)}^{x+\Delta} \rho_0(y-\xi_z(t))\,dy, & \xi_z^{-1}(x-\Delta)<t<\xi_z^{-1}(x+\Delta),\\
\int_{x-\Delta}^{x+\Delta} \frac{u\left(\xi_z^{-1}\left(\xi_z(t)-y\right)\right)}{\xi_z'\left(\xi_z^{-1}\left(\xi_z(t)-y\right)\right)}\,dy, & \xi_z^{-1}(x+\Delta)<t<\tau,
\end{cases}\\
&= \begin{cases}
\int_{x-\Delta-\xi_z(t)}^{x+\Delta-\xi_z(t)} \rho_0(\sigma)\,d\sigma, & 0<t<\xi_z^{-1}(x-\Delta),\\
\int_0^{\xi_z^{-1}(\xi_z(t)-x+\Delta)} u(s)\,ds + \int_0^{x+\Delta-\xi_z(t)} \rho_0(\sigma) & d\sigma,\\
 & \xi_z^{-1}(x-\Delta)<t<\xi_z^{-1}(x+\Delta),\\
\int_{\xi_z^{-1}(\xi_z(t)-x+\Delta)}^{\xi_z^{-1}(\xi_z(t)-x-\Delta)} u(s)\,ds, & \xi_z^{-1}(x+\Delta)<t<\tau.
\end{cases}
\end{aligned}
\tag{69}
$$

In view of (68), we can conclude from (69) that

$$
\begin{aligned}
A(x) = \lim_{\Delta\to 0}\lim_{z\to x}\frac{1}{2\Delta}\Bigg[&\int_0^{\xi_z^{-1}(x-\Delta)}\int_{x-\Delta-\xi_z(t)}^{x+\Delta-\xi_z(t)} \rho_0(\sigma)\,d\sigma dt\\
&+\int_{\xi_z^{-1}(x-\Delta)}^{\xi_z^{-1}(x+\Delta)}\int_0^{\xi_z^{-1}(\xi_z(t)-x+\Delta)} u(s)\,dsdt + \int_{\xi_z^{-1}(x-\Delta)}^{\xi_z^{-1}(x+\Delta)}\int_0^{x+\Delta-\xi_z(t)} \rho_0(\sigma)\,d\sigma dt\\
&+\int_{\xi_z^{-1}(x+\Delta)}^{\tau}\int_{\xi_z^{-1}(\xi_z(t)-x+\Delta)}^{\xi_z^{-1}(\xi_z(t)-x-\Delta)} u(s)\,ds\,dt\Bigg] = A_1(x)+A_2(x)+A_3(x)+A_4(x).
\end{aligned}
\tag{70}
$$

Changing the order of integration in each terms of (70), we arrive at the following relations

$$
\begin{aligned}
A_1(x) &= \lim_{\Delta\to 0}\lim_{z\to x}\frac{1}{2\Delta}\Bigg[\int_0^{2\Delta}\left(\int_{\xi_z^{-1}(x-\Delta-\sigma)}^{\xi_z^{-1}(x-\Delta)} dt\right)\rho_0(\sigma)\,d\sigma\\
&\quad+\int_{2\Delta}^{x-\Delta}\left(\int_{\xi_z^{-1}(x-\Delta-\sigma)}^{\xi_z^{-1}(x+\Delta-\sigma)} dt\right)\rho_0(\sigma)\,d\sigma + \int_{x-\Delta}^{x+\Delta}\left(\int_0^{\xi_z^{-1}(x+\Delta-\sigma)} dt\right)\rho_0(\sigma)\,d\sigma\Bigg]\\
&= \lim_{\Delta\to 0}\lim_{z\to x}\Bigg[\int_0^{2\Delta}\frac{\xi_z^{-1}\left(x-\Delta\right)-\xi_z^{-1}\left(x-\Delta-\sigma\right)}{2\Delta}\rho_0(\sigma)\,d\sigma\\
&\qquad+\int_{2\Delta}^{x-\Delta}\frac{\xi_z^{-1}\left(x+\Delta-\sigma\right)-\xi_z^{-1}(x-\Delta-\sigma)}{2\Delta}\,\rho_0(\sigma)\,d\sigma
\end{aligned}
$$

$$+ \int_{x-\Delta}^{x+\Delta} \frac{\xi_z^{-1}(x+\Delta-\sigma)}{2\Delta} \rho_0(\sigma)\, d\sigma \Bigg] \overset{\text{by Lemmas 3.2 and 3.3}}{=}$$

$$= \lim_{\Delta\to 0} \int_{2\Delta}^{x-\Delta} \frac{\xi_z^{-1}(x+\Delta-\sigma) - \xi_z^{-1}(x-\Delta-\sigma)}{2\Delta} \rho_0(\sigma)\, d\sigma$$

$$= \int_0^x \frac{d\xi_x^{-1}(y)}{dy}\Big|_{y=x-\sigma} \rho_0(\sigma)\, d\sigma, \tag{71}$$

$$A_2(x) = \lim_{\Delta\to 0} \lim_{z\to x} \frac{1}{2\Delta} \int_0^{2\Delta} \left(\int_{\xi_z^{-1}(x-\Delta)}^{\xi_z^{-1}(\xi_z(s)+x-\Delta)} dt \right) u(s)\, ds$$

$$= \lim_{\Delta\to 0} \lim_{z\to x} \frac{1}{2\Delta} \int_0^{2\Delta} \left(\xi_z^{-1}(\xi_z(s)+x-\Delta) - \xi_z^{-1}(x-\Delta) \right) u(s)\, ds = 0, \tag{72}$$

$$A_3(x) = \lim_{\Delta\to 0} \lim_{z\to x} \frac{1}{2\Delta} \int_0^{2\Delta} \left(\int_{\xi_z^{-1}(x-\Delta)}^{\xi_z^{-1}(x+\Delta-\sigma)} dt \right) \rho_0(\sigma)\, d\sigma$$

$$= \lim_{\Delta\to 0} \lim_{z\to x} \frac{1}{2\Delta} \int_0^{2\Delta} \left(\xi_z^{-1}(x+\Delta-\sigma) - \xi_z^{-1}(x-\Delta) \right) \rho_0(\sigma)\, d\sigma = 0, \tag{73}$$

$$A_4(x) = \lim_{\Delta\to 0} \lim_{z\to x} \frac{1}{2\Delta} \left[\int_0^{\xi_z^{-1}(2\Delta)} \left(\int_{\xi_z^{-1}(\xi_z(s)+x-\Delta)}^{\xi_z^{-1}(x+\Delta)} dt \right) u(s)\, ds \right.$$

$$\left. + \int_{\xi_z^{-1}(2\Delta)}^{\xi_z^{-1}(\xi_z(\tau)-x-\Delta)} \left(\int_{\xi_z^{-1}(\xi_z(s)+x-\Delta)}^{\xi_z^{-1}(\xi_z(s)+x+\Delta)} dt \right) u(s)\, ds \right]$$

$$= \lim_{\Delta\to 0} \frac{1}{2\Delta} \left[\int_{\xi_x^{-1}(2\Delta)}^{\xi_x^{-1}(\xi_x(\tau)-x-\Delta)} \left(\int_{\xi_x^{-1}(\xi_x(s)+x-\Delta)}^{\xi_x^{-1}(\xi_x(s)+x+\Delta)} dt \right) u(s)\, ds \right]$$

$$= \int_0^{\xi_z^{-1}(\xi_z(\tau)-x)} \frac{d\xi_x^{-1}(y)}{dy}\Big|_{y=\xi_x(s)+x} u(s)\, ds. \tag{74}$$

Combining results (70)–(74), we finally get

$$\int_0^\tau \rho_1(t,x)\, dt = \int_0^x \frac{d\xi_x^{-1}(y)}{dy}\Big|_{y=x-\sigma} \rho_0(\sigma)\, d\sigma$$

$$+ \int_0^{\xi_z^{-1}(\xi_z(\tau)-x)} \frac{d\xi_x^{-1}(y)}{dy}\Big|_{y=\xi_x(s)+x} u(s)\, ds, \quad x \in [0, x^*]. \tag{75}$$

By analogy, it can be shown that

$$\int_0^\tau \rho_2(t,x)\, dt = \int_{x^*}^x \frac{d\zeta_x^{-1}(y)}{dy}\Big|_{y=x-x^*-\sigma} \rho_0(\sigma)\, d\sigma$$

$$+ \int_0^{\zeta_z^{-1}(\zeta_z(\tau)-x+x^*)} \frac{d\zeta_x^{-1}(y)}{dy}\Big|_{y=\zeta_x(s)+x-x^*} v(s)\, ds, \quad x \in [x^*, 1]. \tag{76}$$

It is worth to note that $\xi_x^{-1} \in C^1([0, \xi_x(\tau)])$ and $\zeta_x^{-1} \in C^1([0, \zeta_x(\tau)])$ because $(\xi_x, \zeta_z) \in \left[C^1([0,\delta])\right]^2$ are monotonically increasing functions. Hence, to conclude the proof, it remains to apply the arguments of Lemma 3.4 to relations (75)–(76). □

Remark 2. Taking into account the fact that

$$\left(\xi_x^{-1}(y)\right)' = \frac{1}{\xi_x'\left(\xi_x^{-1}(y)\right)}, \quad \left(\zeta_x^{-1}(y)\right)' = \frac{1}{\zeta_x'\left(\zeta_x^{-1}(y)\right)}, \tag{77}$$

and by Lemma 3.2 and relations (42)-(43)

$$\begin{aligned} \xi_x'\left(\xi_x^{-1}(y)\right) &= V_1\left(\int_0^x \rho_1(\xi_x^{-1}(y),\gamma)\,d\gamma\right), \\ \zeta_x'\left(\zeta_x^{-1}(y)\right) &= V_2\left(\int_x^1 \rho_2(\zeta_x^{-1}(y),\gamma+x^*-x)\,d\gamma\right), \end{aligned} \tag{78}$$

it is easy to deduce from definition of the set $\mathfrak{A}_{ad}$ and representations (75)–(76) the following estimates

$$\|\rho_1(\cdot,x)\|_{L^1(0,\tau)} := \int_0^\tau \rho_1(t,x)\,dt \le \alpha_1^{-1}\left[\|u\|_{L^1(0,\tau)} + \|\rho_0\|_{L^1(0,x^*)}\right] \quad \forall\, x\in[0,x^*], \tag{79}$$

$$\|\rho_2(\cdot,x)\|_{L^1(0,\tau)} := \int_0^\tau \rho_2(t,x)\,dt \le \alpha_1^{-1}\left[\|v\|_{L^1(0,\tau)} + \|\rho_0\|_{L^1(x^*,1)}\right] \quad \forall\, x\in[x^*,1]. \tag{80}$$

We are now in a position to prove the main result of this section.

Theorem 4.3. *Let $\rho_0\in L^\infty(0,1)$, $u\in L^1(0,T)$, $V_1,V_2\in\mathfrak{A}_{ad}$, and $x^*\in[0,1]$ be given. Then the Cauchy problem (5)–(7) admits a unique global solution*

$$\rho(t,x) = \begin{cases} \rho_1(t,x), & \text{if } t\in[0,T],\ x\in[0,x^*), \\ \rho_2(t,x), & \text{if } t\in[0,T],\ x\in(x^*,1]) \end{cases} \tag{81}$$

such that

$$\begin{aligned} &(\rho_1,\rho_2)\in C([0,T];L^1(0,x^*))\times C([0,T];L^1(x^*,1)), \\ &(\rho_1,\rho_2)\in C([0,x^*];L^1(0,T))\times C([x^*,1];L^1(0,T)). \end{aligned} \tag{82}$$

Proof. As follows from Theorem 4.1, there exists a value $\delta\in(0,T]$ such that the Cauchy problem (5)–(7) is uniquely solvable in the strip $(t,x)\in[0,\delta]\times(0,1)$. Moreover, in view of representation (42)–(43), we have the following estimates for the weak solution $(\rho_1,\rho_2)\in C([0,\delta];L^1(0,x^*))\times C([0,\delta];L^1(x^*,1))$

$$0\le \int_0^{x^*}\rho_1(t,\gamma)\,d\gamma = \int_0^t u(\sigma)\,d\sigma + \int_0^{x^*-\xi_{x^*}(t)}\rho_0(\gamma)\,d\gamma \le \|u\|_{L^1(0,T)} + \|\rho_0\|_{L^\infty(0,1)}$$
$$0\le \int_{x^*}^{1}\rho_2(t,\gamma)\,d\gamma = \int_0^t v(\sigma)\,d\sigma + \int_{x^*}^{1-\zeta_{x^*}(t)}\rho_0(\gamma)\,d\gamma \le \|v\|_{L^1(0,\delta)} + \|\rho_0\|_{L^\infty(0,1)}$$

for all $t\in[0,\delta]$. In order to estimate the term

$$\|v\|_{L^1(0,\delta)} := \int_0^\delta \rho_1(t,x^*)V_1\left(\int_0^{x^*}\rho_1(t,y)\,dy\right)dt,$$

we apply the inequality (79). Then

$$\|v\|_{L^1(0,\delta)} \le \alpha_2\int_0^\delta \rho_1(t,x^*)\,dt \le \frac{\alpha_2}{\alpha_1}\left[\|u\|_{L^1(0,T)} + \|\rho_0\|_{L^1(0,x^*)}\right].$$

Since $\|\rho_0\|_{L^1(0,x^*)}\le\|\rho_0\|_{L^\infty(0,1)}$, we finally get

$$\|\rho_1\|_{C([0,\delta];L^1(0,x^*))} \le \|u\|_{L^1(0,T)} + \|\rho_0\|_{L^\infty(0,1)}, \tag{83}$$

$$\|\rho_2\|_{C([0,\delta];L^1(x^*,1))} \le \frac{2\alpha_2}{\alpha_1}\left[\|u\|_{L^1(0,T)} + \|\rho_0\|_{L^\infty(0,1)}\right]. \tag{84}$$

Since both the a priori estimates for weak solution (ρ_1,ρ_2) and the choice rule (25) does not depend on δ, it follows that the weak solution $(\rho_1,\rho_2)\in C([0,\delta];L^1(0,x^*))\times C([0,\delta];L^1(x^*,1))$ can be extended to the next time interval $[\delta,2\delta]\cap[0,T]$. Hence, following this iterative procedure, we finally find a unique global solution

$$(\rho_1,\rho_2)\in C([0,T];L^1(0,x^*))\times C([0,T];L^1(x^*,1)).$$

It remains to note that inclusion $(82)_2$ is a direct consequence of Corollary 2. □

5. **Existence of optimal solutions.** In this section we focus on solvability of OCP (4)–(10). To begin with we note that unknown control functions V_1 and V_2 in (4)–(9) are supposed to be defined on domains $[0,a_1]$ and $[0,a_2]$, respectively, with constants a_1 and a_2 given by (11). As follows now from Theorem 4.3, the reason to define the constant a_i in the way (11) comes from a priori estimates (83)–(84) and the fact that $\|u\|_{L^1(0,T)} \le \sqrt{T}\|u\|_{L^2(0,T)}$.

Definition 5.1. We say that a tuple (u, V_1, V_2, x^*, ρ) is an admissible solution to OCP (4)–(10) if (u, V_1, V_2, x^*) satisfies constraints (9)–(10), the function $\rho(t,x)$ with properties (81)–(82) is the corresponding weak solution to the Cauchy problem (5)–(7), and $I(u, V_1, V_2, x^*) < \infty$.

We denote by Ξ the set of all admissible solutions for the OCP (4)–(10).

Remark 3. As follows from (3) and Definition 5.1, if $(u, V_1, V_2, x^*, \rho) \in \Xi$, then $V_i \in W^{2,2}(0,a_i)$ $(i = 1,2)$. Indeed, since $I(u, V_1, V_2, x^*) < +\infty$, it follows that $V_i'' \in L^2(0,a_i)$. Hence, by Sobolev Embedding Theorem, we have $W^{2,2}(0,a_i) \hookrightarrow C^{1,1/2}([0,a_i])$, where $\frac{1}{2}$ stands for the Hölder exponent. Therefore, as a direct consequence of Arzelà-Ascoli Theorem, we have a compact inclusion $C^{1,1/2}([0,a_i]) \hookrightarrow C^1([0,a_i])$.

We say that a tuple $(u^0, V_1^0, V_2^0, x^{*,0}, \rho^0)$ is an optimal solution to (4)–(10) if

$$(u^0, V_1^0, V_2^0, x^{*,0}, \rho^0) \in \Xi \quad \text{and} \quad I(u^0, V_1^0, V_2^0, x^{*,0}) = \inf_{(u,V_1,V_2,x^*,\rho)\in\Xi} I(u, V_1, V_2, x^*).$$

We are now in a position to give the existence result for OCP (4)–(10).

Theorem 5.2. *For arbitrary* $z_{1,d} \in L^2(0,a_1)$, $z_{2,d} \in L^2(0,a_2)$, $\rho_0 \in L^\infty(0,1)$, $y_d \in L^2(0,T)$, $\alpha_i > 0$, $(i = 1,\dots,4)$*, such that* $\alpha_2 > \alpha_1 > 0$*, and constants* a_i, $(i = 1,2)$ *given by* (11)*, the OCP* (4)–(10) *admits at least one optimal solution* $(u^0, V_1^0, V_2^0, x^{*,0}, \rho^0) \in \Xi$.

Proof. Since the cost functional $I : \Xi \to \overline{\mathbb{R}}$ is bounded from below and $\Xi \neq \emptyset$, it follows that there exists a sequence $\{(u_k, V_{1,k}, V_{2,k}, x_k^*)\}_{k\in\mathbb{N}} \subset L^2(0,T) \times W^{2,2}([0,a_1]) \times W^{2,2}([0,a_2]) \times [0,1]$ such that

$$\lim_{k\to\infty} I(u_k, V_{1,k}, V_{2,k}, x_k^*) = \inf_{(u,V_1,V_2,x^*,\rho)\in\Xi} I(u, V_1, V_2, x^*) \ge 0. \tag{85}$$

Hence, $\sup_{k\in\mathbb{N}} I(u_k, V_{1,k}, V_{2,k}, x_k^*) \le C$, where the constant C is independent of k. Without lost of generality we can suppose that the sequence $\{x_k^*\}_{k\in\mathbb{N}} \subset [0,1]$ is monotone. For the simplicity, we assume that this sequence is monotonically increasing. The case of a monotonically decreasing sequence can be considered in a similar manner. Then, in view of (3) and (10), we have

$$\|u_k\|_{L^2(0,T)} \le \alpha_4, \ \|V_{i,k}\|_{W^{2,2}(0,a_i)} \le \widetilde{C}, \ i = 1,2, \ \|y_k\|_{L^2(0,T)} \le \widetilde{C}, \ \forall\, k \in \mathbb{N} \tag{86}$$

for some constant $\widetilde{C}$ independent of k, where $y_k = y_k(t)$ is the out-flux of system (5)–(9) corresponding to the in-flux $u_k \in \mathfrak{U}_{ad}$, functions $V_{1,k}, V_{2,k}$, and initial data ρ_0. Hence, by Banach-Alaoglu Theorem and Remark 3, there exist functions $u^0 \in L^2(0,T)$, $V_1^0 \in C^1([0,a_1])$, $V_2^0([0,a_2])$, $y^0 \in L^2(0,T)$, and a value $x^{*,0} \in [0,1]$ such that (up to subsequences)

$$\begin{gathered} V_{1,k}'' \to \left(V_1^0\right)'' \ \text{ in } \ L^2([0,a_1]), \quad V_{2,k}'' \to \left(V_2^0\right)'' \ \text{ in } \ L^2([0,a_2]), \\ u_k \rightharpoonup u^0 \ \text{ in } \ L^2(0,T), \quad V_{1,k} \to V_1^0 \ \text{ in } \ C^1([0,a_1]), \quad V_{2,k} \to V_2^0 \ \text{ in } \ C^1([0,a_2]), \\ y_k \rightharpoonup y^0 \ \text{ in } \ L^2(0,T), \quad \text{and} \quad x_k^* \to x^{*,0} \quad \text{as} \ \ k \to \infty. \end{gathered} \tag{87}$$

We may always suppose that functions $V_i^0 \in C^1([0,a_i])$, $(i = 1,2)$, are extended to $\mathbb{R}_+$ in such way that $V_i^0 \in \mathfrak{A}_{ad}$ for $i = 1,2$.

Before proceeding further, we note that the set

$$\mathfrak{U}_{ad} := \left\{ w \in L^2(0,T) \mid \|w\|_{L^2(0,T)} \le \alpha_4,\ w(x) \ge 0 \ \text{ a.e. on } (0,T) \right\}$$

is sequentially closed with respect to the weak convergence in $L^2(0,T)$. Hence, (87) and the admissibility condition $u_k \in \mathfrak{U}_{ad}$ for all $k \in \mathbb{N}$ imply that $u^0 \in \mathfrak{U}_{ad}$.

Let $\rho_k = (\rho_{1,k}, \rho_{2,k})$ be the weak solution to the Cauchy problem (5)–(7) with $u = u_k$, $V_i = V_{i,k}$, and $x^* = x_k^*$. Let

$$W_{1,k} : [0,T] \times [0, x_k^*] \to \mathbb{R}_+, \quad W_{2,k} : [0,T] \times [x_k^*, 1] \to \mathbb{R}_+,$$
$$\xi_k : [0,T] \times [0, x_k^*] \to \mathbb{R}_+, \quad \text{ and } \quad \zeta_k : [0,T] \times [x_k^*, 1] \to \mathbb{R}_+$$

be defined by

$$W_{1,k}(t,x) = \int_0^x \rho_{1,k}(t,\gamma)\, d\gamma, \quad W_{2,k}(t,x) = \int_x^1 \rho_{2,k}(t, \gamma + x_k^* - x)\, d\gamma, \tag{88}$$

$$\xi_k(t,x) = \int_0^t V_{1,k}\left(W_{1,k}(s,x)\right) ds, \quad \zeta_k(t,x) = \int_0^t V_{2,k}\left(W_{2,k}(s,x)\right) ds, \quad \forall\, k \in \mathbb{N}. \tag{89}$$

Then by (42)–(43), we have

$$\xi_k(t,x) = \int_0^t V_{1,k}\left(\int_0^s u_k(\sigma)\, d\sigma + \int_0^{x - \xi_k(s,x)} \rho_0(\gamma)\, d\gamma \right) ds, \tag{90}$$
$$x \in [0, x_k^*],\ t \in [0, t_{1,k}],$$

$$\zeta_k(t,x) = \int_0^t V_{2,k}\left(\int_0^s v_k(\sigma)\, d\sigma + \int_{x_k^*}^{1 - \zeta_k(s,x) - x + x_k^*} \rho_0(\gamma)\, d\gamma \right) ds, \tag{91}$$
$$x \in [x_k^*, 1],\ t \in [0, t_{2,k}],$$

where

$$t_{1,k} = \min\{t_{1,k}^*, T\},\ \xi_k(t_{1,k}^*, x_k^*) = x_k^*,\ t_{2,k} = \min\{t_{2,k}^*, T\},\ \zeta_k(t_{2,k}^*, 1) = 1 - x_k^*, \tag{92}$$

and

$$v_k(t) := \rho_{1,k}(t, x_k^*) V_{1,k}\left(\int_0^{x_k^*} \rho_{1,k}(t,y)\, dy \right). \tag{93}$$

By (93) and Corollary 2, the sequence $\{v_k\}_{k\in\mathbb{N}}$ is uniformly bounded in $L^1(0,T)$. Moreover, as immediately follows from (3), Corollary 2, and a priori estimate (79), this sequence is equi-integrable. Indeed, for any $\tau_1, \tau_2 \in [0,T]$ ($\tau_1 < \tau_2$), we have

$$\begin{aligned}
\int_{\tau_1}^{\tau_2} |v_k(t)|\, dt &\le \alpha_2 \int_{\tau_1}^{\tau_2} \rho_{1,k}(t, x_k^*)\, dt \\
&\overset{\text{by } (75)}{=} \alpha_2 \int_{\xi^{-1}_{k,x_k^*}(\xi_{k,x_k^*}(\tau_1) - x_k^*)}^{\xi^{-1}_{k,x_k^*}(\xi_{k,x_k^*}(\tau_2) - x_k^*)} \left. \frac{d\xi^{-1}_{k,x_k^*}(y)}{dy} \right|_{y = \xi_{k,x_k^*}(s) + x_k^*} u_k(s)\, ds \overset{\text{by } (77)\text{–}(78)}{=} \\
&\le \alpha_2 \alpha_1^{-1} \int_{\xi^{-1}_{k,x_k^*}(\xi_{k,x_k^*}(\tau_1) - x_k^*)}^{\xi^{-1}_{k,x_k^*}(\xi_{k,x_k^*}(\tau_2) - x_k^*)} u_k(s)\, ds \\
&\overset{\text{by } (86)_1}{\le} \frac{\alpha_2 \alpha_4}{\alpha_1} \sqrt{\xi^{-1}_{k,x_k^*}(\xi_{k,x_k^*}(\tau_2) - x_k^*) - \xi^{-1}_{k,x_k^*}(\xi_{k,x_k^*}(\tau_1) - x_k^*)} \le \frac{\alpha_2 \alpha_4 C}{\alpha_1} |\tau_2 - \tau_1|
\end{aligned}$$

by Lipschitz continuity of the functions $\xi^{-1}_{k,x_k^*} \in C^1([0, x_0])$ (see the proof of Corollary 2). Hence, by Dunford-Pettis Criterion, the sequence $\{v_k\}_{k\in\mathbb{N}}$ is weakly compact in $L^1(0,T)$. It means that there exists a function $v^0 \in L^1(0,T)$ such that, up to a subsequence, we have

$$\int_0^T v_k(t)\varphi(t)\,dt \to \int_0^T v^0(t)\varphi(t)\,dt, \quad \forall\,\varphi \in L^\infty(0,T). \tag{94}$$

In view of (86), (67), Lemma 3.4, and (88)–(89), we can derive from (90)–(91) and Lemma 3.3 that

$$\|W_{i,k}\|_{C([0,T]\times[0,1]} \le C,\ \|\xi_k\|_{C^1([0,T]\times[0,x_k^*]} \le C,\ \|\zeta_k\|_{C^1([0,T]\times[x_k^*,1]} \le C,\ \forall\,k\in\mathbb{N}.$$

Moreover, in view of definition of the class $\mathfrak{A}_{ad}$, ξ_k' and ζ_k' are uniformly bounded from above and below:

$$0<\alpha_1 \le \xi_k'(t,x) \le \alpha_2, \quad \forall (t,x)\in[0,T]\times[0,x_k^*],\ \forall\,k\in\mathbb{N},$$
$$0<\alpha_1 \le \zeta_k'(t,x) \le \alpha_2, \quad \forall (t,x)\in[0,T]\times[x_k^*,1],\ \forall\,k\in\mathbb{N}.$$

Then it follows from Arzelà-Ascoli Theorem that there exist functions $\xi^0(t,x)$ and $\zeta^0(t,x)$ such that, up to subsequences,

$$\xi_k \to \xi^0 \quad \text{in}\ \ C([0,T]\times[0,x^{*,0}]) \quad \text{and} \quad \zeta_k \to \zeta^0 \quad \text{in}\ \ C([0,T]\times[x^{*,0},1]). \tag{95}$$

Since $\xi_k(t,x) := \xi_{k,x}(t)$ and $\xi^0(t,x) = \xi^0_x(t)$ in the notations of the previous sections, and

$$0 \overset{\text{by (3) and (89)}}{<} \alpha_1 |\xi_{k,x}^{-1}(y) - (\xi_x^0)^{-1}(y)| \le \left|\xi_{k,x}\left(\xi_{k,x}^{-1}(y)\right) - \xi_{k,x}\left((\xi_x^0)^{-1}(y)\right)\right|$$
$$= \left|y - \xi_{k,x}\left((\xi_x^0)^{-1}(y)\right)\right| = \left|\xi_x^0\left((\xi_x^0)^{-1}(y)\right) - \xi_{k,x}\left((\xi_x^0)^{-1}(y)\right)\right| \to 0 \ \text{ as }\ k\to\infty$$

uniformly for x,y by (95), it follows that

$$\xi_{k,x}^{-1} \to (\xi_x^0)^{-1} \quad \text{in}\ \ C([0,x^0]) \quad \text{for any}\ \ x,x^0\in[0,x^{*,0}]. \tag{96}$$

A similar conclusion can be made with respect to the functions $\zeta_{k,x}^{-1}$. Hence, we can pass to the limit in (92) as $k\to\infty$. As a result, we have

$$t_{1,k} \to t_1 = \min\{t_1^*,T\}, \quad \xi^0(t_1^*,x^{*,0}) = x^{*,0}, \quad t_{2,k}\to t_2 = \min\{t_2^*,T\},$$
$$\zeta^0(t_2^*,1) = 1 - x^{*,0},$$

and, therefore, passing to the limit in $(88)_1$, $(89)_1$, and (90) as $k\to\infty$, we arrive at the following relation

$$\xi^0(t,x) = \int_0^t V_1^0\left(\int_0^s u^0(\sigma)\,d\sigma + \int_0^{x-\xi^0(s,x)} \rho_0(\gamma)\,d\gamma\right) ds,\ x\in[0,x^{*,0}],\ t\in[0,t_1]. \tag{97}$$

By Lemma 3.2 and Theorem 4.3, the function $\xi^0(t,x)$ satisfying equation (97) is uniquely defined and such that $\xi^0(\cdot,x)\in C^1([0,T])$ for each $x\in[0,x^{*,0}]$. Moreover, this function is strictly related with the weak solution of the following Cauchy problem

$$\partial_t\rho_1^0(t,x) + V_1^0\left(\int_0^x \rho_1^0(t,y)\,dy\right)\partial_x\rho_1^0(t,x) = 0 \quad \text{in}\ \ Q=(0,T)\times(0,x^{*,0}),$$
$$\rho_1^0(0,x) = \rho_0(x) \quad \text{for}\ \ x\in[0,x^{*,0}], \quad \rho_1^0(t,0)V_1^0(0) = u^0(t), \quad \text{for}\ \ t\in[0,T],$$

namely, (for the details we refer to (42))

$$\int_0^x \rho_1^0(t,\gamma)\,d\gamma = \int_0^t u^0(\sigma)\,d\sigma + \int_0^{x-\xi^0(t,x)} \rho_0(\gamma)\,d\gamma, \quad \forall\,t\in[0,t_1],\ \forall\,x\in[0,x^{*,0}].$$

Taking this fact into account, let us show that, in fact, the limit function $v^0\in L^1(0,T)$ in (94) is such that

$$v^0(t) := \rho_1^0(t,x_k^*)V_1^0\left(\int_0^{x^{*,0}} \rho_1^0(t,y)\,dy\right). \tag{98}$$

In this case, we could reiterate the previous arguments in order to pass to the limit in $(88)_2$, $(89)_2$, and (91) as $k \to \infty$, and conclude the same properties for the limit function $\zeta^0(t,x)$ and its relation with the weak solution to (14)–(15) under $v(t) = v^0(t)$.

With that in mind, we assume that $\xi^0(T, x^{*,0}) \le x^{*,0}$. Then, by (95) and monotonicity of $\{x_k^*\}_{k\in\mathbb{N}} \subset [0,1]$, we have $\xi_k(T, x_k^*) \le x^{*,0}$ for k large enough. Therefore, for an arbitrary test function $\varphi \in L^\infty(0,T)$, we get

$$
\begin{aligned}
\int_0^T v_k(t)\varphi(t)\,dt &= \int_0^T \rho_{1,k}(t,x_k^*)V_{1,k}\left(\int_0^{x_k^*} \rho_{1,k}(t,y)\,dy\right)\varphi(t)\,dt \\
&= \int_0^T \rho_0(x_k^* - \xi_{x_k^*}(t))V_{1,k}\left(\int_0^{x_k^*} \rho_{1,k}(t,y)\,dy\right)\varphi(t)\,dt \\
&= \int_{x_k^* - \xi_{x_k^*}(T)}^{x_k^*} \rho_0(y)\varphi\left(\xi_{x_k^*}^{-1}\left(x_k^* - y\right)\right)\,dy \\
&\overset{\text{by (87),(95), and (96)}}{\longrightarrow} \int_{x^{*,0} - \xi^0_{x^{*,0}}(T)}^{x^{*,0}} \rho_0(y)\varphi\left((\xi^0_{x^{*,0}})^{-1}\left(x^{*,0} - y\right)\right)\,dy \\
&= \int_0^T \rho_0(x^{*,0} - \xi^0_{x^{*,0}}(t))V_1^0\left(\int_0^{x^{*,0}} \rho_1^0(t,y)\varphi(t)\,dy\right)\,dt \\
&= \int_0^T \rho_1^0(t,x^{*,0})V_1^0\left(\int_0^{x^{*,0}} \rho_1^0(t,y)\,dy\right)\varphi(t)\,dt =: \int_0^T \widehat{v}(t)\varphi(t)\,dt.
\end{aligned}
\tag{99}
$$

Hence, $v_k \rightharpoonup \widehat{v}$ in $L^1(0,T)$ as $k \to \infty$. Then, in view of condition (94), this implies that $v^0(t) = \widehat{v}(t)$ and we arrive at the representation (98).

It remains to consider the case when $\xi^0(T, x^{*,0}) > x^{*,0}$. Then $\xi_k(T, x_k^*) > x^{*,0}$ for k large enough. Since

$$
\int_0^T v_k(t)\varphi(t)\,dt = \int_0^{(\xi^0_{x^{*,0}})^{-1}(x^{*,0})} v_k(t)\varphi(t)\,dt + \int_{(\xi^0_{x^{*,0}})^{-1}(x^{*,0})}^T v_k(t)\varphi(t)\,dt, \tag{100}
$$

for each $\varphi \in L^\infty(0,T)$, it follows from the previous case that we need only to treat the last term in (100). With that in mind, we assume that $(\xi_{k,x_k^*})^{-1}(x_k^*) < (\xi^0_{x^{*,0}})^{-1}(x^{*,0})$ (the case $(\xi_{k,x_k^*})^{-1}(x_k^*) > (\xi^0_{x^{*,0}})^{-1}(x^{*,0})$ can be considered in a similar way). Then we get from (28) that

$$
\begin{aligned}
\int_{(\xi^0_{x^{*,0}})^{-1}(x^{*,0})}^T v_k(t)\varphi(t)\,dt &= \int_{(\xi^0_{x^{*,0}})^{-1}(x^{*,0})}^T u_k\left(\xi_{k,x_k^*}^{-1}(\xi_{k,x_k^*}(t) - x_k^*)\right)\varphi(t)\,dt \\
&= \int_{\xi_{k,x_k^*}^{-1}(\xi_{k,x_k^*}((\xi^0_{x^{*,0}})^{-1}(x^{*,0})) - x_k^*)}^{\xi_{k,x_k^*}^{-1}(\xi_{k,x_k^*}(T) - x_k^*)} \frac{u_k(\sigma)\varphi\left(\xi_{k,x_k^*}^{-1}(\xi_{k,x_k^*}(\sigma) + x_k^*)\right)}{\xi'_{k,x_k^*}\left(\xi_{k,x_k^*}^{-1}(\xi_{k,x_k^*}(\sigma) + x_k^*)\right)}\,d\sigma \\
&= \int_{\tau_k\left((\xi^0_{x^{*,0}})^{-1}(x^{*,0})\right)}^{\tau_k(T)} \frac{u_k(\sigma)\varphi\left(\eta_k(\sigma)\right)}{\xi'_{k,x_k^*}\left(\eta_k(\sigma)\right)}\,d\sigma,
\end{aligned}
\tag{101}
$$

where

$$
\tau_k(t) := \xi_{k,x_k^*}^{-1}(\xi_{k,x_k^*}(t) - x_k^*), \quad \eta_k(t) := \xi_{k,x_k^*}^{-1}(\xi_{k,x_k^*}(t) + x_k^*).
$$

By continuity property (26) of functions $\xi_{k,x_k^*}(t) = \xi_k(t, x_k^*)$ and conditions (95) and (87), we have

$$
\begin{array}{ll}
\tau_k(t) \to \tau^0(t) := (\xi^0_{x^{*,0}})^{-1}(\xi^0_{x^{*,0}}(t) - x^{*,0}), & \text{in } C([0,T]), \\
\eta_k(t) \to \eta^0(t) := (\xi^0_{x^{*,0}})^{-1}(\xi^0_{x^{*,0}}(t) + x^{*,0}), & \text{in } C([0,\tau^0(T)]), \\
\xi'_{k,x_k^*}\left(\eta_k(\sigma)\right) \to (\xi^0_{x^{*,0}})'\left(\eta_0(\sigma)\right), & \text{in } C([0,T]).
\end{array}
\tag{102}
$$

Then we can conclude from (101) that

$$\int_{\tau_k\left((\xi^0_{x^*,0})^{-1}(x^{*,0})\right)}^{\tau_k(T)} \frac{u_k(\sigma)\varphi\left(\eta_k(\sigma)\right)}{\xi'_{k,x^*_k}\left(\eta_k(\sigma)\right)}\,d\sigma = \int_{\tau^0\left((\xi^0_{x^*,0})^{-1}(x^{*,0})\right)}^{\tau^0(T)} \frac{u_k(\sigma)\varphi\left(\eta_k(\sigma)\right)}{\xi'_{k,x^*_k}\left(\eta_k(\sigma)\right)}\,d\sigma$$
$$+ \int_{\tau_k\left((\xi^0_{x^*,0})^{-1}(x^{*,0})\right)}^{\tau^0\left((\xi^0_{x^*,0})^{-1}(x^{*,0})\right)} \frac{u_k(\sigma)\varphi\left(\eta_k(\sigma)\right)}{\xi'_{k,x^*_k}\left(\eta_k(\sigma)\right)}\,d\sigma + \int_{\tau^0(T)}^{\tau_k(T)} \frac{u_k(\sigma)\varphi\left(\eta_k(\sigma)\right)}{\xi'_{k,x^*_k}\left(\eta_k(\sigma)\right)}\,d\sigma$$
$$= J_1 + J_2 + J_3$$

where $J_1 \to \int_{\tau^0\left((\xi^0_{x^*,0})^{-1}(x^{*,0})\right)}^{\tau^0(T)} \frac{u^0(\sigma)\varphi\left(\eta^0(\sigma)\right)}{(\xi^0_{x^*,0})'\left(\eta^0(\sigma)\right)}\,d\sigma$ as $k \to \infty$ as a product of strongly convergent sequence in $C([0,T])$ and weakly convergent one in $L^2(0,T)$, and

$$J_2 \le \alpha_1^{-1}\|\varphi\|_{L^\infty(0,T)} \left| \int_{\tau_k\left((\xi^0_{x^*,0})^{-1}(x^{*,0})\right)}^{\tau^0\left((\xi^0_{x^*,0})^{-1}(x^{*,0})\right)} u_k(\sigma)\,d\sigma \right| \overset{\text{by (102)}}{\longrightarrow} 0 \quad \text{as} \quad k \to \infty,$$
$$J_3 \le \alpha_1^{-1}\|\varphi\|_{L^\infty(0,T)} \left| \int_{\tau^0(T)}^{\tau_k(T)} u_k(\sigma)\,d\sigma \right| \overset{\text{by (102)}}{\longrightarrow} 0 \quad \text{as} \quad k \to \infty.$$

Since

$$\int_{\tau^0\left((\xi^0_{x^*,0})^{-1}(x^{*,0})\right)}^{\tau^0(T)} \frac{u^0(\sigma)\varphi\left(\eta^0(\sigma)\right)}{(\xi^0_{x^*,0})'\left(\eta^0(\sigma)\right)}\,d\sigma$$
$$\overset{\text{by (101)}}{=} \int_{(\xi^0_{x^*,0})^{-1}(x^{*,0})}^{T} u^0\left((\xi^0_{x^*,0})^{-1}(\xi^0_{x^*,0}(t) - x^{*,0})\right)\varphi(t)\,dt$$
$$= \int_{(\xi^0_{x^*,0})^{-1}(x^{*,0})}^{T} \rho_1^0(t,x^{*,0})V_1^0\left(\int_0^{x^{*,0}} \rho_1^0(t,y)\,dy\right)\varphi(t)\,dt$$
$$=: \int_{(\xi^0_{x^*,0})^{-1}(x^{*,0})}^{T} \widehat{v}(t)\varphi(t)\,dt,$$

it follows from (99) and (100) that $v_k \rightharpoonup \widehat{v}$ in $L^1(0,T)$ as $k\to\infty$. Hence, $v^0(t) = \widehat{v}(t)$ by (94). Thus, the representation (98) for the limit function $v^0(t)$ holds true, and the tuple $(u^0, V_1^0, V_2^0, x^{*,0}, \rho^0)$ is admissible for the OCP (4)–(10).

It remains to show that $(u^0, V_1^0, V_2^0, x^{*,0}, \rho^0)$ is an optimal solution. With that in mind, we note that following the similar reasoning as we applied to the convergence (94), it can be shown that the same property is inherent to the out-flux functions $y_k(t)$, that is, there exists an element $\widehat{y} \in L^1(0,T)$ such that $y_k \rightharpoonup \widehat{y}$ in $L^1(0,T)$ as $k \to \infty$, where $\widehat{y}(t) = \rho_2^0(t,1)V_2^0(0)$. However, the weak L^2-convergence $y_k \rightharpoonup y^0$ (see (87)) implies that $y^0(t) = \widehat{y}(t)$. As a result, we finally obtain from (85)

$$\inf_{(u,V_1,V_2,x^*,\rho)\in\Xi} I(u,V_1,V_2,x^*) = \lim_{k\to\infty} I(u_k, V_{1,k}, V_{2,k}, x^*_k)$$
$$= \lim_{k\to\infty}\left[\int_0^T |y_k(t) - y_d(t)|^2\,dt + \|V''_{1,k} - z_{1,d}\|^2_{L^2(0,a_1)} + \|V''_{2,k} - z_{2,d}\|^2_{L^2(0,a_2)}\right]$$
$$\overset{\text{by (87)}}{\ge} \int_0^T |y^0(t) - y_d(t)|^2\,dt + \|\left(V_1^0\right)'' - z_{1,d}\|^2_{L^2(0,a_1)} + \|\left(V_2^0\right)'' - z_{2,d}\|^2_{L^2(0,a_2)}$$
$$= I(u^0, V_1^0, V_2^0, x^{*,0}).$$

Thus, $(u^0, V_1^0, V_2^0, x^{*,0}, \rho^0)$ is an optimal solution to the problem (4)–(10). □

REFERENCES

[1] A. Adimurthi, S. S. Ghoshal and G. D. Veerappa Gowda, Exact controllability of scalar conser- vation laws with strict convex flux, *Math. Control Relat. Fields,* **4** (2014), 401–449.

[2] F. Ancona and G. M. Coclite, On the attainable set for Temple class systems with boundary controls, *SIAM J. Control Optim.,* **43** (2005), 2166–2190.

[3] D. Armbruster, P. Degond and C. Ringhofer, A model for the dynamics of large queuing networks and supply chains, *SIAM J. Appl. Math.,* **66** (2006), 896–920.

[4] D. Armbruster, D. Marthaler, C. Ringhofer, K. Kempf and T. C. Jo, A continuum model for a re-entrant factory, *Oper. Res.,* **54** (2006), 933–950.

[5] D. Armbruster, M. Herty, X. Wang and L. Zhao, Integrating release and dispatch policies in production models, *Networks and Heterogeneous Media,* **10** (2015), 511–526.

[6] D. Armbruster and C. Ringhofer, Thermalized kinetic and fluid models for reentrant supply chains, *Multiscale Model. Simul.,* **3** (2005), 782–800.

[7] A. Bressan and G. M. Coclite, On the boundary control of systems of conservation laws, *SIAM J. Control Optim.,* **41** (2002), 607–622.

[8] P. Baiti, P. LeFloch and B. Piccoli, Uniqueness of classical and nonclassical solutions for nonlinear hyperbolic systems, *J. Differential Equations,* **172** (2001), 59–82.

[9] A. Bressan, *Hyperbolic Systems of Conservation Laws*, Oxford Lecture Series in Mathematics and its Applications, Vol. 20, Oxford University Press, Oxford, 2000.

[10] A. Bressan, G. Crasta and B. Piccoli, Well-posedness of the Cauchy problem for n systems of conservation laws, *Mem. Amer. Math. Soc.,* **146** (2000), viii+134pp.

[11] G. Bretti, C. D'Apice, R. Manzo and B. Piccoli, A continuum-discrete model for supply chains dynamics, *Networks and Heterogeneous Media,* **2** (2007), 661–694.

[12] R. Colombo, M. Herty and M. Mercier, Control of the continuity equation with a non local flow, *ESAIM: COCV,* **17** (2011), 353–379.

[13] J.-M. Coron, *Control and Nonlinearity,* Mathematical Surveys and Monographs, Vol. 136, American Mathematical Society, Providence, RI, 2007.

[14] J.-M. Coron, O. Glass and Z. Wang, Exact boundary controllability for 1-D quasilinear hyperbolic systems with a vanishing characteristic speed, *SIAM J. Control Optim.,* **48** (2009), 3105–3122.

[15] J.-M. Coron and Z. Wang, Output feedback stabilization for a scalar conservation law with a nonlocal velocity, *SIAM J. Math. Analysis.,* **45** (2013), 2646–2665.

[16] J.-M. Coron and Z. Wang, Controllability for a scalar conservation law with nonlocal velocity, *J. Differential Equations,* **252** (2012), 181–201.

[17] J.-M. Coron, M. Kawski and Z. Wang, Analysis of a conservation law modelling a highly re-entrant manufacturing system, *Discrete and Continuous Dynamical Systems, Series B,* **14** (2010), 1337–1359.

[18] C. D'Apice, S. Goettlich, H. Herty and B. Piccoli, *Modeling, Simulation and Optimization of Supply Chains,* SIAM, Philadelphia PA, USA, 216 pages, 2010.

[19] C. D'Apice, P. I. Kogut and R. Manzo, On Approximation of Entropy Solutions for One System of Nonlinear Hyperbolic Conservation Laws with Impulse Source Terms, *Journal of Control Science and Engineering,* **2010** (2010), Article ID 982369, 10 pp.

[20] C. D'Apice, P. I. Kogut and R. Manzo, On relaxation of state constrained optimal control problem for a PDE-ODE model of supply chains, *Networks and Heterogeneous Media*, **9** (2014), 501–518.

[21] C. D'Apice and R. Manzo, A fluid-dynamic model for supply chains, *Networks and Heterogeneous Media,* **1** (2006), 379–398.

[22] C. D'Apice, R. Manzo and B. Piccoli, Modelling supply networks with partial differential equations, *Quarterly of Applied Mathematics*, **67** (2009), 419–440.

[23] C. D'Apice, R. Manzo and B. Piccoli, Existence of solutions to Cauchy problems for a mixed continuum-discrete model for supply chains and networks, *Journal of Mathematical Analysis and Applications*, **362** (2010), 374–386.

[24] C. D'Apice, R. Manzo and B. Piccoli, Optimal input flows for a PDE-ODE model of supply chains, *Communications in Mathematical Sciences,* **10** (2012), 1225–1240.

[25] C. D'Apice, R. Manzo and B. Piccoli, Numerical schemes for the optimal input flow of a supply-chain, *SIAM Journal on Numerical Analysis,* **51** (2013), 2634–2650.

[26] P. Degond, S. Gottlich, M. Herty and A. Klar, A network model for supply chains with multiple policies, *Multiscale Model. Simul.,* **6** (2007), 820–837.

[27] M. Herty, A. Klar and B. Piccoli, Existence of solutions for supply chain models based on partial differential equations, *SIAM J. Math. Anal.*, **39** (2007), 160–173.

[28] T. Horsin, On the controllability of the Burgers equation, *ESAIM Control Optim. Calc. Var.,* **3** (1998), 83–95.

[29] P. I. Kogut and R. Manzo, Efficient controls for one class of fluid dynamic models, *Far East Journal of Applied Mathematics,* **46** (2010), 85–119.

[30] P. I. Kogut and R. Manzo, On vector-valued approximation of state constrained optimal control problems for nonlinear hyperbolic conservation laws, *Journal of Dynamical and Control Systems,* **19** (2013), 381–404.

[31] M. La Marca, D. Armbruster, M. Herty and C. Ringhofer, Control of continuum models of production systems, *IEEE Trans. Automat. Control,* **55** (2010), 2511–2526.

[32] T. Li, *Controllability and Observability for Quasilinear Hyperbolic Systems,* AIMS Ser. Appl. Math. 3, American Institute of Mathematical Sciences, Springfield, MO, 2010.

[33] V. Perrollaz, AExact controllability of scalar conservation laws with an additional control and in the context of entropy solutions, *SIAM J. Control Optim.,* **50** (2012), 2025–2045.

[34] P. Shang and Z. Wang, Analysis and control of a scalar conservation law modelling a highly re-entrant manufacturing system, *J. Differential Equations,* **250** (2011), 949–982.

E-mail address: cdapice@unisa.it
E-mail address: p.kogut@i.ua
E-mail address: rmanzo@unisa.it