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Lipschitz stability for the electrostatic inverse boundary value problem with piecewise linear conductivities

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A B S T R A C T

We consider the electrostatic inverse boundary value problem also known as electrical impedance tomography (EIT) for the case where the conductivity is a piecewise linear function on a domain $\Omega \subset \mathbb{R}^n$ and we show that a Lipschitz stability estimate for the conductivity in terms of the local Dirichlet-to-Neumann map holds true.

R É S U M É

On s'intéresse au problème électrostatique inverse également connu comme Tomographie d'Impédance Electrique pour le cas où le conductivité est une fonction linéaire par morceaux sur un domaine $\Omega \subset \mathbb{R}^n$. On établit une estimation de stabilité lipschitzienne pour le conductivité en relation avec l'opérateur Dirichlet–Neumann.

1. Introduction

We consider the inverse boundary value problem (IBVP) associated with the elliptic equation for an electric potential, where the objective is to recover electrical resistivity, or conductivity, from partial data. We focus our attention on the stability of this inverse problem, in particular, when the conductivity is isotropic. We obtain a Lipschitz stability result if the conductivity is known to be piecewise linear on a given domain partition.

We let Ω be a bounded domain in $\mathbb{R}^n$, $n \geq 2$. In the absence of internal sources, the electric potential, u, satisfies the elliptic equation

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$$\operatorname{div}(\gamma \nabla u) = 0 \qquad \text{in} \quad \Omega, \tag{1.1}$$

where the function γ signifies the *conductivity* in Ω; γ is a bounded measurable function satisfying the ellipticity condition,

$$0 < \lambda^{-1} \leq \gamma \leq \lambda, \qquad \text{almost everywhere in } \Omega, \tag{1.2}$$

for some positive $\lambda \in \mathbb{R}$. The inverse conductivity problem consists of finding γ when the so-called Dirichlet-to-Neumann (DtoN) map

$$\Lambda_\gamma : \; u|_{\partial\Omega} \in H^{\frac{1}{2}}(\partial\Omega) \longrightarrow \gamma \nabla u \cdot \nu|_{\partial\Omega} \in H^{-\frac{1}{2}}(\partial\Omega) \tag{1.3}$$

is given for any weak solution $u \in H^1(\Omega)$ to (1.1). Here, ν denotes the unit outward normal to $\partial\Omega$. If measurements can be taken on a portion Σ of $\partial\Omega$ only, then the relevant map is referred to as the local DtoN map. And in fact this will be the point of view that we will adopt in this paper. Details of the rigorous definition of a local map are given later on in Definition 2.3.

This inverse problem has different appearances, namely, as electrical impedance tomography (EIT) and direct current (DC) method or electrical resistivity tomography (ERT) in geophysics (belonging to the class of potential field methods). Although the mathematical framework of this paper is the one described by (1.1)–(1.3), the application we have in mind is the determination of the resistivity $\rho = \gamma^{-1}$ in DC or ERT methods corresponding with the following type of experiment or "sounding": a current is injected into the ground through a pair of electrodes at the boundary while the voltage is measured with another pair of electrodes. Thus the data, viewed as an operator, can be identified with the so-called Neumann-to-Dirichlet (NtoD) map. We note that in the mathematical literature the use of the DtoN map as the data is more common. The DtoN map is invertible on its range. Indeed the applied boundary current fluxes must have a vanishing average. We note that the solution is defined up to an additive (grounding potential) constant. Whereas it is well known that, at a theoretical level, the knowledge of either of the two maps is equivalent, matters may be more complicated when, in different applications, the physical settings provide different discrete and noisy samples of such maps. The NtoD map, upon applying it to a particular subset of currents, provides the so-called *apparent resistivity*. To be precise, the apparent resistivity is a geometrical (acquisition) factor multiplied by the ratio of voltage (potential difference) over current.

The first mathematical formulation of this inverse problem is due to Calderón in the context of EIT in [26], where he addressed the problem of whether it is possible to determine the (isotropic) conductivity from the DtoN map. To be precise, Calderón investigated the injectivity of the map

$$Q : \gamma \longrightarrow Q_\gamma,$$

where $Q_\gamma(\phi)$ is the quadratic form associated to Λ_γ, by linearizing the problem. As main contributions in this respect we mention the papers by Kohn and Vogelius [51,52], Sylvester and Uhlmann [76], and Nachman [67]. We wish to recall the uniqueness results of Druskin who, independently from Calderón, dealt directly with the geophysical setting of the problem in [33–35]. We also refer to [22,30] and [77] for an overview of recent developments regarding the issues of uniqueness and reconstruction of the conductivity.

It is well known that the IBVP of determining the conductivity γ from the DtoN map is ill-posed. Indeed, regarding the stability of this inverse problem, Alessandrini [1] proved that, assuming $n \geq 3$ and an *a-priori* bound on γ in $H^s(\Omega)$, for some $s > \frac{n}{2} + 2$, γ depends continuously on Λ_γ with a modulus of continuity of logarithmic type. We also refer to [2,3], which improve the result in [1] for conductivities $\gamma \in W^{2,\infty}(\Omega)$. See also [27] for the most recent advances on minimal regularity assumptions. We refer to [12,13,59] and [31] for the two-dimensional case, where logarithmic type stability estimates have been established too. The common logarithmic type of stability cannot be avoided [4,66]. However, the ill-posed nature of this problem can be modified to be conditionally well-posed by restricting the conductivity to certain function subspaces. Well-posedness is here expressed by Lipschitz stability.

A first result of this kind was established by Alessandrini and Vessella [8], to which we refer, together with [4], for an in-depth description and analysis. The result of [8] is a Lipschitz stability estimate in which the conductivity is *piecewise constant* on a given domain partition.

Such a result was extended to different types of problems, for example, in [17,18] for the Schrödinger and the Helmholtz equations, respectively, in [19] for the inverse conductivity problem with complex conductivity, and in [20,21] for the determination of the Lamé parameters in the elastostatic problem. All of these papers have in common the stable determination of coefficients that are *piecewise constant* on a fixed partition.

We wish to recall, here, that the uniqueness result obtained by Druskin [34] was in the context of piecewise constant conductivities too. Another relevant uniqueness result is due to S.E. Kim [48] who treated piecewise smooth conductivities with polyhedral boundaries.

In the present paper, we achieve Lipschitz stability for conductivities which are *piecewise linear*. This accounts to iteratively determine boundary values and normal derivatives of the conductivity at the various interfaces of the domain partition. In turn, this involves the use of singular solutions of higher order in comparison to those used in [8] and subsequent papers. It may be worth noticing here, that the analysis could be adapted easily to piecewise linear resistivities.

In dimension $n \geq 3$ – which we consider in geophysics – uniqueness has been established by Haberman and Tataru for conductivities in C^1 [46] and more recently for Lipschitz conductivities in [28], both assuming full boundary data. The original uniqueness result by Sylvester and Uhlmann [76] required the conductivity to be C^∞. For the two-dimensional case we refer to [25] and the breakthrough paper [9] where uniqueness has been proven for conductivities that are merely L^∞.

The class of conductivities considered in this paper consists of piecewise linear functions on a given domain partition, which are possibly discontinuous at the interfaces of this partition.

This partition needs to satisfy certain geometric conditions. The allowable partitions include models of layered media and bodies with multiple inclusions.

At the same time the piecewise linear parametrizations tie in well with the finite elements method for computations. In fact, we can use the stability result to estimate and incorporate an approximation error in the DtoN map.

The Lipschitz stability estimate we provide requires a direct proof. (Indeed, the uniqueness result of [28] ($n \geq 3$) does not apply; in fact, in the case of partial data, the result of [9] does not apply either.) We reiterate this estimate is given in terms of the local DtoN map. Note also that with a slight modification, our arguments can apply when the local NtoD map is available instead, see for instance the discussion in [7].

With a Lipschitz stability estimate at hand, we can apply certain iterative methods for reconstruction within a subspace of piecewise linear functions with a starting model at a distance less than the radius of convergence to the unique solution [36,56]. This radius is roughly inversely proportional to the stability constant appearing in the estimate. More importantly, we can iteratively construct the best piecewise linear approximation for a given domain partition. Since the stability constant will grow at least exponentially with the number of subdomains in the partition [72], the radius of convergence shrinks accordingly. One can expect accurate piecewise linear approximations with relatively few subdomains to describe the subsurface, noting that the domain partition need not be uniform and may show a local refinement, and hence our result provides the necessary insight for developing a practical approach with relatively minor prior information. Whether we can recover, also, an unknown domain partition (such as one of tetrahedral type) is a current subject of research.

As we mentioned earlier, the application we have in mind here is the DC acquisition and method, which were introduced by Schlumberger in 1920 [73]. Initial DC deep resistivity studies of Earth's crust were carried out as early as in 1932 [74]. Many studies have followed. We mention, in particular, the experiments and results by Constable, McElhinny and McFadden [32] carried out in central Australia in 1984. For a general description and the history of the DC method (and the closely related induced electrical polarization (IP) method) we refer to the textbooks by Koefoed [49], Zhdanov and Keller [79], and Kaufman and Anderson [50]; for a concise tutorial and review, see Ward [78]. For a finite-element method and solver for and computational studies of the DC method, see Li and Spitzer [56]. Here, we consider isotropic conductivities (and therefore resistivities); however, Earth's materials can certainly be anisotropic, which was already recognized by Mallet and Doll [65]. We refer to [5–7,10,16,38,41,42,57] and [54] for results concerning the anisotropic case.

Through recent decades, electromagnetic methods have been widely used in geothermal prospecting [23]. Amongst different geophysical exploration methods, in geothermal prospecting, resistivity methods have been demonstrated to be the most effective. The reason is that the electrical resistivity of rocks is controlled by important geothermal parameters including temperature, fluid type and salinity, porosity, permeable pathways, fracture zones and faults (structural), the composition of the rocks, and the presence of alteration minerals. In this context, we mention the work of Hersir, Björnsson and Eysteinsson [47] and, more recently, of Flóvenz et al. [40].

We briefly mention how the acquisition – essentially probing the NtoD map – is carried out (see, for example, [14,15]). The original acquisition was designed for two-dimensional configurations ($n = 2$). The Schlumberger array consists of four collinear electrodes. The outer two electrodes are current (boundary source) electrodes and the inner two electrodes are the potential (receiver) electrodes. The potential electrodes are installed at the center of the electrode array with a small separation. The current electrodes are gradually increased to a greater separation during the survey – while the potential electrodes remain in the same position until the observed voltage becomes too small to measure – for the current to probe deeper into the earth. Indeed, the depth resolution of the DC method is sensitive to the separation between current electrodes [69]. There is also the (crossed) square-array acquisition which is designed to be more sensitive to anisotropy than the Schlumberger array [45,44].

There are different types of electrode configuration that are commonly used. In two-dimensional configurations, the dipole-dipole array is widely being used because of the low electromagnetic coupling between the current and potential circuits. In three-dimensional configurations, the pole-pole electrode configuration is commonly used. (In practice, the ideal pole-pole array, with only one current and one potential electrode does not exist. To approximate the pole-pole array, the second current and potential electrodes must be placed at a large distance.) For convenience the electrodes are arranged in a square grid with the same unit electrode spacing in orthogonal (coordinate) directions. (We mention the E-SCAN method [55,37].) It can be very time-consuming to make such a large number of measurements. To reduce the number of measurements required without seriously degrading the resolution, "cross-diagonal survey" method was introduced; here, the potential measurements are only made at the electrodes along two orthogonal directions and the 45 degrees diagonal lines passing through the current electrode (extracted from Loke's tutorial: 2-D and 3-D electrical imaging surveys, [63]).

The inverse problem pertaining to resistivity interpretation was reported as early as the 1930s (e.g. Slichter, 1933; Stevenson, 1934; Ejen, 1938; Pekeris, 1940 [71]). Slichter [75] published a method of interpretation of resistivity data over a planarly layered earth using Hankel's Fourier–Bessel inversion formula. It gives a unique solution if the resistivity is a continuous function of electrode spacings. A substantial number of papers have been written on approaches based on partial boundary data "fitting" or optimization to estimate the resistivity, without knowledge of uniqueness or convergence [80]. Narayan, Dusseault and Nobes [68] give an extensive overview. In the context of data fitting, Parker [70] indicates and illustrates in planarly layered models the ill-posedness of the IBVP. Interestingly, various studies and implementations have resorted to "blocky" (and pseudo-layered) representations of resistivity [64,39,11] and, hence, fit the class of functions for which Lipschitz stability estimates have been obtained. Finally, we mention the complementary frequency-dependent transient electromagnetic (TEM), magnetotelluric (MT) and electroseismic methods. The hybrid inverse problem of electroseismic conversion was analyzed by Chen and De Hoop [29]. The further analysis of TEM/MT [43] is a subject of current research.

In recent years, there has been a renewed and growing interest in the application of electrostatic and diffuse electromagnetic inverse boundary value problems in geophysics driven by the idea of combining different probing fields, including acousto-elastic waves, to identify the (poro-elastic) rock properties in Earth's interior within a particular geological structure in an integrated fashion. These properties certainly will not vary smoothly. We capture the geological structure in a domain partition, let the properties be discontinuous across subdomain boundaries of geological significance, and approximate the parameters, here conductivity in the electrostatic problem, in each subdomain by linear interpolation. (From a rock physics point of view, this interpolation should be obtained from a nonlinear upscaling, which is still an active area of research.) This approach, and the generality of these approximations, analyzed in the context of conditional well-posedness are the novelty of this paper.

The outline of the paper is as follows. Our main assumptions and our main result (Theorem 2.3) are given in section 2. Section 3 contains the proof of the main result, as well as two intermediate results (Theorem 3.2 and Proposition 3.3) needed to build the necessary machinery. Theorem 3.2 provides original asymptotic estimates for the Green's function of the conductivity equation, its gradient and a mixed derivatives, for conductivities that are linear on each domain D_j of a given partition $\{D_j\}$ of Ω. These asymptotic estimates are given at the interfaces between the domains D_j, where the conductivity is discontinuous. Proposition 3.3 provides estimates of unique continuation of the solution to the conductivity equation for piecewise linear conductivities. Section 4 is devoted to various technical proofs. First we prove Theorem 3.2. Next we recall Proposition 3.3. Here the proof is based on arguments introduced in [8, proof of Proposition 4.4], therefore only the main differences in the two proofs are highlighted. Finally we provide a proof of the initial step of Theorem 2.3 which boils down to stability estimates at the boundary for γ and $\frac{\partial \gamma}{\partial \nu}$. These are minor variants of well known results and are provided here for the sake of completeness.

2. Main result

2.1. Notation and definitions

In several places within this manuscript it will be useful to single out one coordinate direction. To this purpose, the following notations for points $x \in \mathbb{R}^n$ will be adopted. For $n \geq 3$, a point $x \in \mathbb{R}^n$ will be denoted by $x = (x', x_n)$, where $x' \in \mathbb{R}^{n-1}$ and $x_n \in \mathbb{R}$. Moreover, given a point $x \in \mathbb{R}^n$, we will denote with $B_r(x), B'_r(x')$ the open balls in $\mathbb{R}^n, \mathbb{R}^{n-1}$ respectively centered at x and x' with radius r and by $Q_r(x)$ the cylinder

$$Q_r(x) = B'_r(x') \times (x_n - r, x_n + r).$$

We will also denote

$$\begin{aligned}
&\mathbb{R}^n_+ = \{(x', x_n) \in \mathbb{R}^n | x_n > 0\}; \quad \mathbb{R}^n_- = \{(x', x_n) \in \mathbb{R}^n | x_n < 0\};\\
&B^+_r = B_r \cap \mathbb{R}^n_+; \quad B^-_r = B_r \cap \mathbb{R}^n_-;\\
&Q^+_r = Q_r \cap \mathbb{R}^n_+; \quad Q^-_r = Q_r \cap \mathbb{R}^n_-,
\end{aligned}$$

where we understand $B_r = B_r(0)$ and $Q_r = Q_r(0)$.

In the sequel, we will make a repeated use of quantitative notions of smoothness for the boundaries of various domains. Let us introduce the following notation and definitions.

Definition 2.1. Let Ω be a domain in $\mathbb{R}^n$. We say that a portion Σ of $\partial\Omega$ is of Lipschitz class with constants $r_0, L > 0$ if there exists $P \in \Sigma$ and there exists a rigid transformation of $\mathbb{R}^n$ under which we have $P = 0$ and

$$\Omega \cap Q_{r_0} = \{x \in Q_{r_0} \,:\, x_n > \varphi(x')\},$$
$$\Sigma \cap Q_{r_0} = \{x \in Q_{r_0} \,:\, x_n = \varphi(x')\},$$

where φ is a Lipschitz function on B'_{r_0} satisfying

$$\varphi(0) = 0; \qquad \|\varphi\|_{C^{0,1}(B'_{r_0})} \leq Lr_0.$$

It is understood that $\partial\Omega$ is of Lipschitz class with constants r_0, L as a special case of Σ, with $\Sigma = \partial\Omega$.

Definition 2.2. Let Ω be a domain in $\mathbb{R}^n$. We say that a portion Σ of $\partial\Omega$ is a flat portion of size r_0 if there exists $P \in \Sigma$ and there exists a rigid transformation of $\mathbb{R}^n$ under which we have $P = 0$ and

$$\begin{aligned}\Sigma \cap Q_{r_0/3} &= \{x \in Q_{r_0/3} | x_n = 0\}\\ \Omega \cap Q_{r_0/3} &= \{x \in Q_{r_0/3} | x_n > 0\}\\ (\mathbb{R}^n \setminus \Omega) \cap Q_{r_0/3} &= \{x \in Q_{r_0/3} | x_n < 0\}.\end{aligned} \tag{2.1}$$

We rigorously define the local D–N map.

Definition 2.3. Let Ω be a domain in $\mathbb{R}^n$ with Lipschitz boundary $\partial\Omega$ and Σ a non-empty (flat) open portion of $\partial\Omega$. We introduce the subspace of $H^{\frac{1}{2}}(\partial\Omega)$

$$H^{\frac{1}{2}}_{co}(\Sigma) = \left\{f \in H^{\frac{1}{2}}(\partial\Omega) \,|\, \operatorname{supp} f \subset \Sigma\right\} \tag{2.2}$$

and its dual $H^{-\frac{1}{2}}_{co}(\Sigma)$. Assume that $\gamma \in L^\infty(\Omega)$ satisfies

$$\lambda^{-1} \leq \gamma(x) \leq \lambda, \text{ for almost every } x \in \Omega, \tag{2.3}$$

then the local Dirichlet-to-Neumann map associated to γ and Σ is the operator

$$\Lambda^{\Sigma}_{\gamma} : H^{\frac{1}{2}}_{co}(\Sigma) \longrightarrow H^{-\frac{1}{2}}_{co}(\Sigma) \tag{2.4}$$

defined by

$$\langle \Lambda^{\Sigma}_{\gamma}\, g,\, \eta \rangle \;=\; \int\limits_{\Omega} \gamma(x) \nabla u(x) \cdot \nabla \phi(x)\, dx, \tag{2.5}$$

for any $g, \eta \in H^{\frac{1}{2}}_{co}(\Sigma)$, where $u \in H^1(\Omega)$ is the weak solution to

$$\begin{cases} \operatorname{div}(\gamma(x)\nabla u(x)) = 0, & \text{in } \Omega,\\ u = g, & \text{on } \partial\Omega, \end{cases}$$

and $\phi \in H^1(\Omega)$ is any function such that $\phi|_{\partial\Omega} = \eta$ in the trace sense. Here we denote by $\langle \cdot, \cdot \rangle$ the $L^2(\partial\Omega)$-pairing between $H^{\frac{1}{2}}_{co}(\Sigma)$ and its dual $H^{-\frac{1}{2}}_{co}(\Sigma)$.

Note that, by (2.5), it is easily verified that Λ_γ^Σ is selfadjoint. We will denote by $\|\cdot\|_*$ the norm on the Banach space of bounded linear operators between $H_{co}^{\frac{1}{2}}(\Sigma)$ and $H_{co}^{-\frac{1}{2}}(\Sigma)$.

Remark 2.1. Note that the space $H_{00}^{\frac{1}{2}}(\Sigma)$ [58, Chapter 1] is the closure of $H_{co}^{\frac{1}{2}}(\Sigma)$ in $H^{\frac{1}{2}}(\partial\Omega)$, therefore the local DN map could be equivalently given by replacing in Definition 2.3 the spaces $H_{co}^{\frac{1}{2}}(\Sigma)$, $H_{co}^{-\frac{1}{2}}(\Sigma)$ with $H_{00}^{\frac{1}{2}}(\Sigma)$ and its dual $H_{00}^{-\frac{1}{2}}(\Sigma)$ respectively and by continuing to use the notation $\langle\cdot,\cdot\rangle$ for the $L^2(\partial\Omega)$-pairing between $H_{00}^{\frac{1}{2}}(\Sigma)$ and $H_{00}^{-\frac{1}{2}}(\Sigma)$.

Definition 2.4. Let N, r_0, L, λ be given positive numbers with $N \in \mathbb{N}$. We will refer to this set of numbers, along with the space dimension n, as to the *a-priori data*. Several constants depending on the *a-priori data* will appear within the paper. In order to simplify our notation, we shall denote by $C, C_1, C_2, \ldots$ any of these constants, avoiding in most cases to point out their specific dependence on the data which may vary from case to case.

2.2. Assumptions

2.2.1. Assumptions about the domain Ω

1. We assume that Ω is a domain in $\mathbb{R}^n$ with boundary of Lipschitz class with constants r_0, L according to Definition 2.1 and satisfying

$$|\Omega| \leq N r_0^n, \tag{2.6}$$

where $|\Omega|$ denotes the Lebesgue measure of Ω.

2. We fix an open non-empty subset Σ of $\partial\Omega$ (where the measurements in terms of the local D–N map are taken).
3.

$$\bar{\Omega} = \bigcup_{j=1}^{N} \bar{D}_j,$$

where D_j, $j = 1, \ldots, N$ are known open sets of $\mathbb{R}^n$, satisfying the conditions below.

(a) D_j, $j = 1, \ldots, N$ are connected and pairwise nonoverlapping polyhedrons.

(b) ∂D_j, $j = 1, \ldots, N$ are of Lipschitz class with constants r_0, L.

(c) There exists one region, say D_1, such that $\partial D_1 \cap \Sigma$ contains a *flat* portion Σ_1 of size r_0 and for every $i \in \{2, \ldots, N\}$ there exists $j_1, \ldots, j_K \in \{1, \ldots, N\}$ such that

$$D_{j_1} = D_1, \qquad D_{j_K} = D_i. \tag{2.7}$$

In addition we assume that, for every $k = 1, \ldots, K$, $\partial D_{j_k} \cap \partial D_{j_{k-1}}$ contains a *flat* portion Σ_k of size r_0 (here we agree that $D_{j_0} = \mathbb{R}^n \setminus \Omega$), such that

$$\Sigma_k \subset \Omega, \quad \text{for every } k = 2, \ldots, K,$$

and, for every $k = 1, \ldots, K$, there exists $P_k \in \Sigma_k$ and a rigid transformation of coordinates under which we have $P_k = 0$ and

$$\begin{aligned} \Sigma_k \cap Q_{r_0/3} &= \{x \in Q_{r_0/3} | x_n = 0\} \\ D_{j_k} \cap Q_{r_0/3} &= \{x \in Q_{r_0/3} | x_n > 0\} \\ D_{j_{k-1}} \cap Q_{r_0/3} &= \{x \in Q_{r_0/3} | x_n < 0\}. \end{aligned} \tag{2.8}$$

2.2.2. A-priori information on the conductivity γ

We will consider a conductivity function γ of type

$$\gamma(x) = \sum_{j=1}^{N} \gamma_j(x)\chi_{D_j}(x), \qquad x \in \Omega, \tag{2.9a}$$

$$\gamma_j(x) = a_j + A_j \cdot x, \tag{2.9b}$$

where $a_j \in \mathbb{R}$, $A_j \in \mathbb{R}^n$ and D_j, $j = 1, \dots, N$ are the given subdomains introduced in section 2.2.1. We also assume that

$$\lambda^{-1} \le \gamma_j \le \lambda, \qquad \text{a.e. in } \Omega, \quad \text{for any } j = 1, \dots n, \tag{2.10}$$

for some positive constant λ.

Remark 2.2. Observe that the class of functions of the form (2.9a)–(2.9b) is a finite dimensional linear space. The L^∞-norm $||\gamma||_{L^\infty(\Omega)}$ is equivalent to the norm

$$|||\gamma||| = \max_{j=1,\dots,N} \{|a_j| + |A_j|\}$$

modulo constants which only depend on the a-priori data.

From now on for simplicity we will write

$$\Lambda_i = \Lambda^{\Sigma}_{\gamma^{(i)}}, \qquad i = 1, 2.$$

Theorem 2.3. *Let Ω, D_j, $j = 1, \dots, N$ and Σ be a domain, N subdomains of Ω and a portion of $\partial\Omega$ as in section 2.2.1 respectively. Let $\gamma^{(i)}$, $i = 1, 2$ be two conductivities satisfying (2.10) and of type*

$$\gamma^{(i)} = \sum_{j=1}^{N} \gamma^{(i)}_j(x)\chi_{D_j}(x), \qquad x \in \Omega, \tag{2.11}$$

where

$$\gamma^{(i)}_j(x) = a^{(i)}_j + A^{(i)}_j \cdot x,$$

with $a^{(i)}_j \in \mathbb{R}$ and $A^{(i)}_j \in \mathbb{R}^n$, then we have

$$||\gamma^{(1)} - \gamma^{(2)}||_{L^\infty(\Omega)} \le C||\Lambda_1 - \Lambda_2||_*, \tag{2.12}$$

where C is a positive constant that depends on the a-priori data only.

Remark 2.4. In this paper we are assuming that the conductivity γ is a piecewise linear function. However, the case where the resistivity function $\rho = \gamma^{-1}$ is piecewise linear, could be treated equally well.

3. Proof of the main result

The proof of our main result (Theorem 2.3) is based on an argument that combines asymptotic type of estimates for the Green's function of the operator

$$L = \mathrm{div}\,(\gamma(x)\nabla) \qquad \text{in} \quad \Omega\ , \tag{3.1}$$

associated to homogeneous Dirichlet boundary condition (Theorem 3.2), with γ satisfying (2.9a)–(2.10), together with a result of unique continuation (Proposition 3.3) for solutions to

$$Lu = 0, \qquad \text{in} \quad \Omega.$$

Our idea in estimating $\gamma^{(1)} - \gamma^{(2)}$ exploits, on one hand, an estimate from below of the blow up of some singular solutions (which we will introduce below) $S_{\mathcal{U}}$ and some of its derivatives if $\gamma^{(1)} - \gamma^{(2)}$ is large at some point. On the other hand, we will use estimates of propagation of smallness to show that $S_{\mathcal{U}}$ needs to be small if $\Lambda_1 - \Lambda_2$ is small. We will give the precise formulation of these results in what follows.

3.1. Singular solutions

We find convenient to introduce Green's function not precisely for the physical domain Ω but for an augmented domain Ω_0.

We recall that by assumption 3(c) of Subsection 2.2.1 we can assume that there exists a point P_1 such that up to a rigid transformation of coordinates we have that $P_1 = 0$ and (2.1) holds with $\Sigma = \Sigma_1$.

Denoting by

$$D_0 = \left\{ x \in (\mathbb{R}^n \setminus \Omega) \cap B_{r_0} \,\middle|\, |x_i| < \frac{2}{3} r_0,\ i = 1, \dots, n-1,\ \left| x_n - \frac{r_0}{6} \right| < \frac{5}{6} r_0 \right\},$$

and recalling that $\partial\Omega$ is of Lipschitz class with constant r_0 and L, as assumed in Definition 2.1, it turns out that the augmented domain $\Omega_0 = \Omega \cup D_0$ is of Lipschitz class with constants $\frac{r_0}{3}$ and $\tilde{L}$, where $\tilde{L}$ depends on L only.

3.1.1. Green's function

We consider the operator L_i given by

$$L_i = \mathrm{div}\left(\tilde{\gamma}^{(i)}(x)\nabla\right) \qquad \text{in} \quad \Omega_0, \quad i = 1, 2, \tag{3.2}$$

where $\tilde{\gamma}^{(i)}$ is the extension on Ω_0 of $\gamma^{(i)}$ obtained by setting $\tilde{\gamma}^{(i)}|_{D_0} = 1$, for $i = 1, 2$.

If L_i is the operator given in (3.2), then for every $y \in \Omega_0$, the Green's function $\tilde{G}_i(\cdot, y)$ is the weak solution to the Dirichlet problem

$$\begin{cases} \mathrm{div}(\tilde{\gamma}^{(i)} \nabla \tilde{G}_i(\cdot, y)) = -\delta(\cdot - y)\ , & \text{in } \Omega_0, \\ \tilde{G}_i(\cdot, y) = 0\ , & \text{on } \partial\Omega_0, \end{cases} \tag{3.3}$$

where $\delta(\cdot - y)$ is the Dirac measure at y. We recall that $\tilde{G}$ satisfies the properties [60]

$$\tilde{G}_i(x, y) = \tilde{G}_i(y, x) \qquad \text{for every } x, y \in \Omega_0, \quad x \neq y, \tag{3.4}$$

$$0 < \tilde{G}_i(x, y) < C|x - y|^{2-n} \qquad \text{for every } x, y \in \Omega, \quad x \neq y, \tag{3.5}$$

where $C > 0$ is a constant depending on λ and n only. Moreover, the following result holds true.

Proposition 3.1. *For any $y \in \Omega_0$ and every $r > 0$ we have that*

$$\int_{\Omega_0 \setminus B_r(y)} |\nabla \tilde{G}_i(\cdot, y)|^2 \leq C r^{2-n} \tag{3.6}$$

where $C > 0$ depends on λ and n only.

Proof. The proof can be obtained in a straightforward fashion by combining Caccioppoli inequality with (3.5). □

3.1.2. The $\tilde{S}_{\mathcal{U}_k}$ singular solutions

For any number $r \in \left(0, \frac{2}{3}r_0\right)$ we also denote

$$(D_0)_r = \{x \in D_0 \,|\, dist(x, \Omega) > r\}.$$

Let $K \in \{1, \ldots, N\}$ be such that the subdomain D_K of Ω satisfies

$$E = \|\gamma^{(1)} - \gamma^{(2)}\|_{L^\infty(\Omega)} = \|\gamma^{(1)} - \gamma^{(2)}\|_{L^\infty(D_K)}. \tag{3.7}$$

Recall that there exist $j_1, \ldots, j_K \in \{1, \ldots, N\}$ with $D_{j_1}, \ldots D_{j_K}$ satisfying assumption 3(c) of Subsection 2.2.1. For simplicity, let us rearrange the indices of these subdomains so that the above mentioned chain is simply denoted by $D_1, \ldots, D_K, K \leq N$. We also denote

$$\mathcal{W}_k = \bigcup_{i=0}^{k} D_i, \qquad \mathcal{U}_k = \Omega_0 \setminus \overline{\mathcal{W}_k}, \quad \text{for } k = 0, \ldots, K \tag{3.8}$$

and for any $y, z \in \mathcal{W}_k$ we define

$$\tilde{S}_{\mathcal{U}_k}(y, z) = \int_{\mathcal{U}_k} (\tilde{\gamma}^{(1)} - \tilde{\gamma}^{(2)}) \nabla \tilde{G}_1(\cdot, y) \cdot \nabla \tilde{G}_2(\cdot, z), \quad \text{for } k = 0, \ldots, K. \tag{3.9}$$

It is a relatively straightforward matter to see that for every $y, z \in \mathcal{W}_k$ with $k = 0, \ldots, K$ we have that $\tilde{S}_{\mathcal{U}_k}(\cdot, z), \tilde{S}_{\mathcal{U}_k}(y, \cdot) \in H^1_{loc}(\mathcal{W}_k)$ are weak solutions to

$$\text{div}\left(\tilde{\gamma}^{(1)}(\cdot) \nabla \tilde{S}_{\mathcal{U}_k}(\cdot, z)\right) = 0, \quad \text{in } \mathcal{W}_k \tag{3.10}$$

$$\text{div}\left(\tilde{\gamma}^{(2)}(\cdot) \nabla \tilde{S}_{\mathcal{U}_k}(y, \cdot)\right) = 0\ , \quad \text{in } \mathcal{W}_k. \tag{3.11}$$

It is expected that $\tilde{S}_{\mathcal{U}_k}(y, z)$ blows up as y, z approach simultaneously one point of $\partial \mathcal{U}_k$.

We will denote with

$$\Gamma(x, y) = \frac{1}{(n-2)\omega_n} |x - y|^{2-n}, \tag{3.12}$$

the fundamental solution of the Laplace operator (here ω_n / n denotes the volume of the unit ball in $\mathbb{R}^n$). If D_i, $i = 1, \ldots, N$ are the domains introduced in section 2.2.1 and L is the operator given by (3.1), we will give asymptotic estimates for the Green's function of L, with respect to (3.12) at the interfaces between the domains D_i, $i = 1, \ldots N$. These estimates are given below. In what follows let $\tilde{G}$ be the Green's function associated to the operator L in Ω_0.

3.1.3. Asymptotics at interfaces

Theorem 3.2. *Let Q_{l+1} be a point such that $Q_{l+1} \in B_{\frac{r_0}{8}}(P_{l+1}) \cap \Sigma_{l+1}$ with $l \in \{1, \dots, N-1\}$. There exist constants $\theta_1, \theta_2, 0 < \theta_1 < 1, 0 < \theta_2 < 1$ and $C > 0$ depending on the a priori data only such that following inequalities hold true for every $\bar{x} \in B_{\frac{r_0}{16}}(Q_{l+1}) \cap D_{j_{l+1}}$ and every $\bar{y} = Q_{l+1} - re_n$, where $r \in (0, \frac{r_0}{16})$*

$$\left|\tilde{G}(\bar{x},\bar{y}) - \frac{2}{\gamma_{j_l}(Q_{l+1}) + \gamma_{j_l+1}(Q_{l+1})}\Gamma(\bar{x},\bar{y})\right| \le C|\bar{x}-\bar{y}|^{3-n}, \tag{3.13}$$

$$\left|\nabla_x\tilde{G}(\bar{x},\bar{y}) - \frac{2}{\gamma_{j_l}(Q_{l+1}) + \gamma_{j_l+1}(Q_{l+1})}\nabla_x\Gamma(\bar{x},\bar{y})\right| \le C|\bar{x}-\bar{y}|^{\theta_1+1-n}, \tag{3.14}$$

$$\left|\nabla_y\nabla_x\tilde{G}(\bar{x},\bar{y}) - \frac{2}{\gamma_{j_l}(Q_{l+1}) + \gamma_{j_l+1}(Q_{l+1})}\nabla_y\nabla_x\Gamma(\bar{x},\bar{y})\right| \le C|\bar{x}-\bar{y}|^{\theta_2-n} . \tag{3.15}$$

3.2. Quantitative unique continuation

We introduce for any number $b > 0$, the concave non-decreasing function $\omega_b(t)$, defined on $(0, +\infty)$,

$$\omega_b(t) = \begin{cases} 2^b e^{-2}|\log t|^{-b}, & t \in (0, e^{-2}), \\ e^{-2}, & t \in [e^{-2}, +\infty). \end{cases}$$

We recall (see (4.34) and (4.35) in [8]) that for any $\beta \in (0,1)$ we have that

$$(0,+\infty) \ni t \to t\omega_b\left(\frac{1}{t}\right) \quad \text{is a nondecreasing function} \tag{3.16}$$

and

$$\omega_b\left(\frac{t}{\beta}\right) \le |\log e\beta^{-1/2}|^b\omega_b(t) \ , \quad \omega_b(t^\beta) \le \left(\frac{1}{\beta}\right)^b \omega_b(t) \ . \tag{3.17}$$

Furthermore, we set $\omega_\alpha^{(0)}(t) = t^\alpha$ with $0 < \alpha < 1$ and we shall denote the iterated compositions

$$\omega_b^{(1)} = \omega_b \ , \qquad \omega_b^{(j)} = \omega_b \circ \omega_b^{(j-1)} \quad j = 2, 3, \dots . \tag{3.18}$$

The following parameters will also be introduced

$$\beta = \arctan\frac{1}{L}, \quad \beta_1 = \arctan\left(\frac{\sin\beta}{4}\right), \quad \lambda_1 = \frac{r_0}{1+\sin\beta_1}$$
$$\rho_1 = \lambda_1 \sin\beta_1, \quad a = \frac{1-\sin\beta_1}{1+\sin\beta_1}$$
$$\lambda_m = a\lambda_{m-1}, \quad \rho_m = a\rho_{m-1}, \quad \text{for every}\, m \ge 2,$$
$$d_m = \lambda_m - \rho_m, \quad m \ge 1.$$

Note in particular that $d_m = r_0 a^m, \ 0 < a < 1$.

For $k = 1, \dots, K$ and a fixed point $\bar{y} \in \Sigma_{k+1}$, denote

$$w_m(\bar{y}) = \bar{y} - \lambda_m\nu(\bar{y}), \qquad \text{for every } m \ge 1, \tag{3.19}$$

where $\nu(\bar{y})$ is the exterior unit normal to ∂D_k.

For a given $r \in (0, d_1]$ we denote

$$\bar{h}(r) = \min\{m \in \mathbb{N} \,|\, d_m \le r\}. \tag{3.20}$$

We notice that $\bar{h}(r)$ is such that

$$\log\left(\frac{r_0}{r}\right)^C \le \bar{h}(r) \le \log\left(\frac{r_0}{r}\right)^C + 1. \tag{3.21}$$

The following estimate for $\tilde{S}_{\mathcal{U}_k}(y,z)$ holds true, for $k = 1, \dots, K$.

Proposition 3.3 *(Estimates of unique continuation). Let $k = 1, \dots, K$. If, for a positive number ε_0, we have*

$$\left|\tilde{S}_{\mathcal{U}_k}(y,z)\right| \le r_0^{2-n}\varepsilon_0, \quad \text{for every } (y,z) \in (D_0)_{\frac{r_0}{4}} \times (D_0)_{\frac{r_0}{4}}, \tag{3.22}$$

then the following inequalities hold true for every $r \in (0, d_1]$

$$\left|\tilde{S}_{\mathcal{U}_k}\left(w_{\bar{h}}(Q_{k+1}), w_{\bar{h}}(Q_{k+1})\right)\right| \le r_0^{-n+2} C_1^{\bar{h}}(E+\varepsilon_0)\left(\omega_{1/C}^{(2k)}\left(\frac{\varepsilon_0}{E+\varepsilon_0}\right)\right)^{(1/C)^{\bar{h}}}, \tag{3.23}$$

$$\left|\partial_{y_j}\partial_{z_i}\tilde{S}_{\mathcal{U}_k}\left(w_{\bar{h}}(Q_{k+1}), w_{\bar{h}}(Q_{k+1})\right)\right| \le r_0^{-n} C_2^{\bar{h}}(E+\varepsilon_0)\left(\omega_{1/C}^{(2k)}\left(\frac{\varepsilon_0}{E+\varepsilon_0}\right)\right)^{(1/C)^{\bar{h}}}, \tag{3.24}$$

for any $i, j = 1, \dots, n$, where $Q_{k+1} \in \Sigma_{k+1} \cap B_{\frac{r_0}{8}}(P_{k+1})$, $w_{\bar{h}(r)}(Q_{k+1}) = Q_{k+1} - \lambda_{\bar{h}(r)}\nu(Q_{k+1})$, λ_m has been introduced above, ν is the exterior unit normal to ∂D_k and $C_1, C_2 > 0$ depend on the a-priori data only.

3.3. Lipschitz stability

Proof of Theorem 2.3. Let D_K be the subdomain of Ω satisfying (3.7) and let $D_1, \dots, D_K$ be the chain of domains satisfying assumption 4(d). For any $k = 1, \dots, K$ we will denote by $D_T f$ and $\partial_\nu f$ the $n-1$ dimensional vector of the tangential partial derivatives of a function f on Σ_k and the normal partial derivative of f on Σ_k respectively. We also simplify our notation by replacing $\Lambda^{\Gamma}_{\gamma^{(i)}}$ with Λ_i, for $i = 1, 2$. We will also denote

$$\varepsilon_0 = ||\Lambda_1 - \Lambda_2||_*, \qquad \delta_l = ||\tilde{\gamma}^{(1)} - \tilde{\gamma}^{(2)}||_{L^\infty(\mathcal{W}_l)}, \quad l = 1, 2, \dots, \qquad \delta_0 = 0.$$

We begin by noticing that for each $l = 1, 2, \dots$ $||\widetilde{\gamma}_l^{(1)} - \widetilde{\gamma}_l^{(2)}||_{L^\infty(D_l)}$ can be evaluated in terms of the quantities

$$||\widetilde{\gamma}_l^{(1)} - \widetilde{\gamma}_l^{(2)}||_{L^\infty(\Sigma_l \cap B_{\frac{r_0}{4}}(P_l))} \tag{3.25}$$

$$\left|\partial_\nu(\widetilde{\gamma}_l^{(1)} - \widetilde{\gamma}_l^{(2)})(P_l)\right|. \tag{3.26}$$

where $r_0 > 0$ is the constant introduced in subsection 2.1. In fact, let us denote

$$\alpha_l + \beta_l \cdot x = (\widetilde{\gamma}_l^{(1)} - \widetilde{\gamma}_l^{(2)})(x), \qquad x \in D_l \tag{3.27}$$

and choose $\{e_j\}_{j=1,\dots,n-1}$ orthonormal vectors starting at P_l and generating the hyperplane containing the flat part of Σ_l. By computing $\widetilde{\gamma}_l^{(1)} - \widetilde{\gamma}_l^{(2)}$ on the points P_l, $P_l + \frac{r_0}{5}e_j$, $j = 1, \dots, n-1$ and taking their differences we obtain

$$|\alpha_l + \beta_l \cdot P_l| + \sum_{j=1}^{n-1} |\beta_l \cdot e_j| \leq C||\widetilde{\gamma}_l^{(1)} - \widetilde{\gamma}_l^{(2)}||_{L^\infty(\Sigma_l \cap B_{\frac{r_0}{4}}(P_l))}. \tag{3.28}$$

Next we notice that

$$|\beta_l \cdot \nu| = \left|\partial_\nu(\gamma_l^{(1)} - \gamma_l^{(2)})(P_l)\right|. \tag{3.29}$$

Hence each of the components of β_l can be estimated and eventually also $|\alpha_l|$. In conclusion

$$|\alpha_l| + |\beta_l| \leq C\left(||\widetilde{\gamma}_l^{(1)} - \widetilde{\gamma}_l^{(2)}||_{L^\infty(\Sigma_l \cap B_{\frac{r_0}{4}}(P_l))} + \left|\partial_\nu(\gamma_l^{(1)} - \gamma_l^{(2)})(P_l)\right|\right). \tag{3.30}$$

Hence our task will be to estimate

$$||\widetilde{\gamma}_l^{(1)} - \widetilde{\gamma}_l^{(2)}||_{L^\infty(\Sigma_l \cap B_{\frac{r_0}{4}}(P_l))} \quad \text{and} \quad \left|\partial_\nu(\gamma_l^{(1)} - \gamma_l^{(2)})(P_l)\right|$$

iteratively with respect to l.

When $l = 1$ this correspond to a stability estimate at the boundary for the conductivity and its normal derivatives. Such estimates are well-known under slightly varying hypotheses [3,6,24,8,7]. Indeed we have

$$||\widetilde{\gamma}_l^{(1)} - \widetilde{\gamma}_l^{(2)}||_{L^\infty(\Sigma_l \cap B_{\frac{r_0}{4}}(P_l))} + \left|\partial_\nu(\gamma_l^{(1)} - \gamma_l^{(2)})(P_l)\right| \leq C(\varepsilon_0 + E)\omega_{1/C}^{(0)}\left(\frac{\varepsilon_0}{\varepsilon_0 + E}\right). \tag{3.31}$$

A proof under the current hypotheses is relegated to section 4. We proceed to estimate δ_2 by proving

$$||\widetilde{\gamma}_2^{(1)} - \widetilde{\gamma}_2^{(2)}||_{L^\infty(\Sigma_2 \cap B_{\frac{r_0}{4}}(P_2))} \leq C(\varepsilon_0 + E)\left(\omega_{1/C}^{(3)}\left(\frac{\varepsilon_0}{\varepsilon_0 + E}\right)\right)^{\frac{1}{C}}, \tag{3.32}$$

$$\left|\partial_\nu(\gamma^{(1)} - \gamma^{(2)})(P_2)\right| \leq C(\varepsilon_0 + E)\left(\omega_{1/C}^{(4)}\left(\frac{\varepsilon_0}{\varepsilon_0 + E}\right)\right)^{\frac{1}{C}}. \tag{3.33}$$

The proofs of (3.32), (3.33) are similar. The one of (3.33) contains some new elements (in comparison to previous results such as [8]) and therefore we concentrate on (3.33) only, assuming (3.32) proven.

We recall that for every $y, z \in D_0$ we have

$$\begin{aligned}&\langle(\Lambda_1 - \Lambda_2)\widetilde{G}_1(\cdot, y), \widetilde{G}_2(\cdot, z)\rangle \\ &= \widetilde{S}_{\mathcal{U}_1}(y, z) + \int_{\mathcal{W}_1} (\widetilde{\gamma}^{(1)} - \widetilde{\gamma}^{(2)})(\cdot)\nabla\widetilde{G}_1(\cdot, y) \cdot \nabla\widetilde{G}_2(\cdot, z)\end{aligned} \tag{3.34}$$

and

$$\begin{aligned}&\langle(\Lambda_1 - \Lambda_2)\partial_{y_n}\widetilde{G}_1(\cdot, y), \partial_{z_n}\widetilde{G}_2(\cdot, z)\rangle \\ &= \partial_{y_n}\partial_{z_n}\widetilde{S}_{\mathcal{U}_1}(y, z) + \int_{\mathcal{W}_1} (\widetilde{\gamma}^{(1)} - \widetilde{\gamma}^{(2)})(\cdot)\partial_{y_n}\nabla\widetilde{G}_1(\cdot, y) \cdot \partial_{z_n}\nabla\widetilde{G}_2(\cdot, z).\end{aligned} \tag{3.35}$$

From (3.34) we obtain

$$\begin{aligned}|\tilde{S}_{\mathcal{U}_1}(y, z)| \leq C \leq \varepsilon_0 ||\tilde{G}_1(\cdot, y)||_{H_{co}^{1/2}(\Sigma)}||\tilde{G}_2(\cdot, z)||_{H_{co}^{1/2}(\Sigma)} \\ + \delta_1||\nabla\tilde{G}_1(\cdot, y)||_{L^2(\mathcal{W}_1)}||\nabla\tilde{G}_2(\cdot, z)||_{L^2(\mathcal{W}_1)} \\ \leq C(\varepsilon_0 + \delta_1)r_0^{2-n}, \qquad \text{for every } y, z \in (D_0)_{r_0/3}.\end{aligned} \tag{3.36}$$

Let $\rho_0 = \frac{r_0}{C}$, where C is the constant introduced in Theorem 3.2, let $r \in (0, d_2)$ and denote

$$w = P_2 + \sigma\nu, \qquad \text{where}\, \sigma = a^{\bar{h}-1}\lambda_1,$$

then

$$\partial_{y_n}\partial_{z_n}\tilde{S}_{\mathcal{U}_0}(w,w) = I_1(w) + I_2(w), \tag{3.37}$$

where

$$I_1(w) = \int\limits_{B_{\rho_0}(P_2)\cap D_2} (\gamma^{(1)} - \gamma^{(2)})(\cdot)\partial_{y_n}\nabla\widetilde{G}_1(\cdot,w)\cdot\partial_{z_n}\nabla\widetilde{G}_2(\cdot,w),$$

$$I_2(w) = \int\limits_{\Omega\setminus(B_{\rho_0}(P_2)\cap D_2)} (\gamma^{(1)} - \gamma^{(2)})(\cdot)\partial_{y_n}\nabla\widetilde{G}_1(\cdot,w)\cdot\partial_{z_n}\nabla\widetilde{G}_2(\cdot,w)$$

and by (3.6)

$$|I_2(w)| \leq CE\rho_0^{-n}. \tag{3.38}$$

We have

$$\begin{aligned}|I_1(w)| \geq &\left|\int\limits_{B_{\rho_0}(P_1)\cap D_2} (\partial_\nu(\gamma_2^{(1)} - \gamma_2^{(2)})(P_2))(x-P_2)_n\partial_{y_n}\nabla\widetilde{G}_1(\cdot,w)\cdot\partial_{z_n}\nabla\widetilde{G}_2(\cdot,w)\right| \\ &- \int\limits_{B_{\rho_0}(P_2)\cap D_2} |(D_T(\gamma_2^{(1)} - \gamma_2^{(2)})(P_2))\cdot(x-P_2)'||\partial_{y_n}\nabla\widetilde{G}_1(\cdot,w)|\,|\partial_{z_n}\nabla\widetilde{G}_2(\cdot,w)| \\ &- \int\limits_{B_{\rho_0}(P_2)\cap D_2} |(\gamma_2^{(1)} - \gamma_2^{(2)})(P_2)||\partial_{y_n}\nabla\widetilde{G}_1(\cdot,w)|\,|\partial_{z_n}\nabla\widetilde{G}_2(\cdot,w)|.\end{aligned}$$

Noticing that up to a transformation of coordinates we can assume that P_2 coincides with the origin 0 of the coordinates system and by Theorem 3.2, this leads to

$$\begin{aligned}|I_1(w)| \geq |\partial_\nu(\gamma_2^{(1)} - \gamma_2^{(2)})|C\Bigg\{&\int\limits_{B_{\rho_0}(0)\cap D_2} |\partial_{y_n}\nabla_x\Gamma(x,w)|^2|x_n| \\ &- E\int\limits_{B_{\rho_0}(0)\cap D_2} |\partial_{y_n}\nabla_x\Gamma(x,w)||x-w|^{-n+\theta_2}|x_n| \\ &- E\int\limits_{B_{\rho_0}(0)\cap D_2} |x-w|^{-2n+2\theta_2}|x_n|\Bigg\} \\ &- \int\limits_{B_{\rho_0}(0)\cap D_2} |D_T(\gamma_2^{(1)} - \gamma_2^{(2)})|\,|x'|\,|\partial_{y_n}\nabla\tilde{G}_1(\cdot,w)|\,|\partial_{z_n}\nabla\tilde{G}_2(\cdot,w)| \\ &- \int\limits_{B_{\rho_0}(0)\cap D_2} |(\gamma_2^{(1)} - \gamma_2^{(2)})(O)|\,|\partial_{y_n}\nabla\tilde{G}_1(\cdot,w)|\,|\partial_{z_n}\nabla\tilde{G}_2(\cdot,w)|,\end{aligned} \tag{3.39}$$

therefore, by combining (4.50) together with (4.48) and (4.49), we obtain

$$
\begin{aligned}
|I_1(w)| \geq |\partial_\nu(\gamma_2^{(1)} - \gamma_2^{(2)})| C\Big\{ & \int\limits_{B_{\rho_0}(0)\cap D_2} |x-w|^{1-2n} \\
& - E \int\limits_{B_{\rho_0}(0)\cap D_2} |x-w|^{1-2n+\theta_2} - E \int\limits_{B_{\rho_0}(0)\cap D_2} |x-w|^{1-2n+2\theta_2} \\
& - (\varepsilon_0 + E)\left(\omega_{1/C}^{(3)}\left(\frac{\varepsilon_0}{\varepsilon_0+E}\right)\right)^{1/C} \int\limits_{B_{\rho_0}(0)\cap D_2} |x-w|^{1-2n} \\
& - (\varepsilon_0 + E)\left(\omega_{1/C}^{(3)}\left(\frac{\varepsilon_0}{\varepsilon_0+E}\right)\right)^{1/C} \int\limits_{B_{\rho_0}(0)\cap D_2} |x-w|^{-2n} \Big\},
\end{aligned}
$$

which leads to

$$
|\partial_\nu(\gamma_1^{(1)} - \gamma_1^{(2)})|\sigma^{1-n} \leq |I_1(w)| + C\left\{(\varepsilon_0+E)\left(\omega_{1/C}^{(3)}\left(\frac{\varepsilon_0}{\varepsilon_0+E}\right)\right)^{1/C}\sigma^{-n} + E\frac{\sigma^{1-n+\theta_2}}{\rho_0^{\theta_2}}\right\}, \tag{3.40}
$$

and

$$
|I_1(w)| \leq |\partial_{y_n}\partial_{z_n}\widetilde{S}_{\mathcal{U}_0}(w,w)| + CE\rho_0^{-n}. \tag{3.41}
$$

Thus by combining the last two inequalities we get

$$
\begin{aligned}
|\partial_\nu(\gamma_1^{(1)} - \gamma_1^{(2)})|\sigma^{1-n} \leq |\partial_{y_n}\partial_{z_n}\widetilde{S}_{\mathcal{U}_0}(w,w)| + C\Big\{ & E\rho_0^{-n} \\
& + (\varepsilon_0+E)\left(\omega_{1/C}^{(3)}\left(\frac{\varepsilon_0}{\varepsilon_0+E}\right)\right)^{1/C}\sigma^{-n} + E\frac{\sigma^{1-n+\theta_2}}{\rho_0^{\theta_2}}\Big\}
\end{aligned} \tag{3.42}
$$

and by recalling that by Proposition 3.3 we have

$$
\left|\partial_{y_j}\partial_{z_i}\tilde{S}_{\mathcal{U}_1}(w,w)\right| \leq r_0^{-n} C^{\bar{h}(r)}(\varepsilon_0+\delta_1+E)\omega_{1/C}^{(2)}\left(\frac{\varepsilon_0+\delta_1}{E+\delta_1+\varepsilon_0}\right)^{(1/C)^{\bar{h}(r)}},
$$

we obtain

$$
\begin{aligned}
|\partial_\nu(\gamma_1^{(1)} - \gamma_1^{(2)})| \leq\ & C^{\bar{h}(r)}(\varepsilon_0+\delta_1+E)\left(\omega_{1/C}^{(2)}\left(\frac{\varepsilon_0+\delta_1}{E+\delta_1+\varepsilon_0}\right)\right)^{(1/C)^{\bar{h}(r)}}\sigma^{n-1} \\
& + \sigma^{-1}(\varepsilon_0+E)\left(\omega_{1/C}^{(3)}\left(\frac{\varepsilon_0}{\varepsilon_0+E}\right)\right)^{1/C} + CE\frac{\sigma^{\theta_2}}{\rho_0^{\theta_2}}.
\end{aligned} \tag{3.43}
$$

We need to estimate $C^{\bar{h}}$ and $\left(\frac{1}{C}\right)^{\bar{h}}$ in terms of r. Recalling (3.21), it turns out that

$$
\left(\frac{d_1}{r}\right)^{C_1} \leq C^{\bar{h}} \leq C_2\left(\frac{d_1}{r}\right)^{C_1}, \tag{3.44}
$$

therefore for any $r \in (0, d_2)$

$$|\partial_\nu(\gamma_2^{(1)}-\gamma_2^{(2)})| \le C(\varepsilon_0+E)\Bigg\{\left(\frac{r}{d_1}\right)^{-C}\left(\omega^{(2)}_{\frac{1}{C}}\left(\frac{\varepsilon_0+\delta_1}{E+\delta_1+\varepsilon_0}\right)\right)^{\left(\frac{r}{d_1}\right)^C} + \left(\frac{r}{d_1}\right)^{-1}\left(\omega^{(3)}_{1/C}\left(\frac{\varepsilon_0}{\varepsilon_0+E}\right)\right)^{1/C} + \left(\frac{r}{d_1}\right)^{\theta_2}\Bigg\}. \tag{3.45}$$

From (3.31) we trivially obtain

$$\frac{\varepsilon_0+\delta_1}{E+\delta_1+\varepsilon_0} \le C\omega^{(0)}_{1/C}\left(\frac{\varepsilon_0}{\varepsilon_0+E}\right) \tag{3.46}$$

and by combining (3.46) together with (3.45) we obtain

$$|\partial_\nu(\gamma_2^{(1)}-\gamma_2^{(2)})| \le C(\varepsilon_0+E)\Bigg\{\left(\frac{r}{d_1}\right)^{-C}\left(\omega^{(3)}_{\frac{1}{C}}\left(\frac{\varepsilon_0}{E+\varepsilon_0}\right)\right)^{\left(\frac{r}{d_1}\right)^C} + \left(\frac{r}{d_1}\right)^{\theta_2}\Bigg\} \tag{3.47}$$

and optimizing with respect to r we obtain

$$|\partial_\nu(\gamma_1^{(1)}-\gamma_1^{(2)})| \le C(\varepsilon_0+E)\left(\omega^{(4)}_{1/C}\left(\frac{\varepsilon_0}{\varepsilon_0+E}\right)\right)^{\frac{1}{C}}. \tag{3.48}$$

Proceeding by iteration to estimate $\gamma_l^{(1)}-\gamma_l^{(2)}$ for $l=2,\dots,K$, we replace (3.34) and (3.35) by

$$\begin{aligned}&\langle(\Lambda_1-\Lambda_2)\widetilde{G}_1(\cdot,y),\widetilde{G}_2(\cdot,z)\rangle\\ &=\widetilde{S}_{\mathcal{U}_{l-1}}(y,z)+\int\limits_{\mathcal{W}_{l-1}}(\widetilde{\gamma}^{(1)}-\widetilde{\gamma}^{(2)})(\cdot)\nabla\widetilde{G}_1(\cdot,y)\cdot\nabla\widetilde{G}_2(\cdot,z)\end{aligned} \tag{3.49}$$

and

$$\begin{aligned}&\langle(\Lambda_1-\Lambda_2)\partial_{y_n}\widetilde{G}_1(\cdot,y),\partial_{z_n}\widetilde{G}_2(\cdot,z)\rangle\\ &=\partial_{y_n}\partial_{z_n}\widetilde{S}_{\mathcal{U}_{l-1}}(y,z)\\ &+\int\limits_{\mathcal{W}_{l-1}}(\widetilde{\gamma}^{(1)}-\tilde{\gamma}^{(2)})(\cdot)\partial_{y_n}\nabla\widetilde{G}_1(\cdot,y)\cdot\partial_{z_n}\nabla\widetilde{G}_2(\cdot,z)\end{aligned} \tag{3.50}$$

respectively. By noticing that (3.49) and Proposition 3.1 imply

$$\begin{aligned}|\widetilde{S}_{\mathcal{U}_l}(y,z)| &\le \varepsilon_0\|\widetilde{G}_1(\cdot,y)\|_{H^{1/2}_{co}(\Sigma)}\|\widetilde{G}_2(\cdot,z)\|_{H^{1/2}_{co}(\Sigma)}\\ &\quad+\delta_{l-1}\|\nabla\widetilde{G}_1(\cdot,y)\|_{L^2(\mathcal{W}_{l-1})}\|\nabla\widetilde{G}_2(\cdot,z)\|_{L^2(\mathcal{W}_{l-1})}\\ &\le C(\varepsilon_0+\delta_{l-1})r_0^{2-n}, \qquad \text{for every } y,z\in(D_0)_{r_0/3},\end{aligned} \tag{3.51}$$

where C depends on L, λ, n and by repeating the same argument applied for the special case $l=2$ and observing that

$$\|\tilde{\gamma}_2^{(1)}-\tilde{\gamma}_2^{(2)}\|_{L^\infty(\Sigma_l\cap B_{\frac{r_0}{4}}(P_l))} \le C(\varepsilon_0+E)\left(\omega^{(2l-1)}_{1/C}\left(\frac{\varepsilon_0}{\varepsilon_0+E}\right)\right)^{\frac{1}{C}}, \tag{3.52}$$

$$\left|\partial_\nu(\gamma^{(1)}-\gamma^{(2)})(P_l)\right| \le C(\varepsilon_0+E)\left(\omega^{(2l)}_{1/C}\left(\frac{\varepsilon_0}{\varepsilon_0+E}\right)\right)^{\frac{1}{C}}, \tag{3.53}$$

$$\delta_l \leq \delta_{l-1} + ||\gamma_l^{(1)} - \gamma_l^{(2)}||_{L^\infty(D_l)},$$

we obtain for every $l = 2, 3, \dots$

$$\delta_l \leq \delta_{l-1} + C(\varepsilon_0 + \delta_{l-1} + E)\left(\omega_{1/C}^{(2l)}\left(\frac{\varepsilon_0 + \delta_{l-1}}{\varepsilon_0 + \delta_{l-1} + E}\right)\right)^{\frac{1}{C}},$$

hence trivially

$$\frac{\varepsilon_0 + \delta_l}{\varepsilon_0 + \delta_l + E} \leq C\left(\omega_{1/C}^{(2l)}\left(\frac{\varepsilon_0 + \delta_{l-1}}{\varepsilon_0 + \delta_{l-1} + E}\right)\right)^{\frac{1}{C}}. \tag{3.54}$$

Using the properties of the logarithmic moduli $\omega_{1/C}$, by (3.46) and using the induction step (3.54) we arrive at

$$||\gamma^{(1)} - \gamma^{(2)}||_{L^\infty(\Omega)} \leq C(\varepsilon_0 + E)\left(\omega_{\frac{1}{C}}^{(K(K+1))}\left(\frac{\varepsilon_0}{\varepsilon_0 + E}\right)\right)^{\frac{1}{C}},$$

therefore

$$E \leq C(\varepsilon_0 + E)\left(\omega_{\frac{1}{C}}^{(K(K+1))}\left(\frac{\varepsilon_0}{\varepsilon_0 + E}\right)\right)^{\frac{1}{C}}. \tag{3.55}$$

Assuming that $E > \varepsilon_0 e^2$ (if this is not the case then the theorem is proven) we obtain

$$E \leq C\left(\frac{E}{e^2} + E\right)\left(\omega_{\frac{1}{C}}^{(K(K+1))}\left(\frac{\varepsilon_0}{E}\right)\right)^{\frac{1}{C}},$$

which leads to

$$\frac{1}{C} \leq \omega_{\frac{1}{C}}^{(K(K+1))}\left(\frac{\varepsilon_0}{E}\right)$$

therefore

$$E \leq \frac{1}{\omega_{\frac{1}{C}}^{(-(K(K+1)))}\left(\frac{1}{C}\right)}\varepsilon_0,$$

where here, with a slight abuse of notation, $\omega_{\frac{1}{C}}^{(-(K(K+1)))}$ denotes the inverse function of $\omega_{\frac{1}{C}}^{(K(K+1))}$. $\square$

4. Technical proofs

4.1. Asymptotic estimates

Theorem 4.1. *Let $r > 0$ be a fixed number. Let $U \in H^1(Q_r)$ be a solution to*

$$div(b(x)\nabla U) = 0 , \tag{4.1}$$

where

$$b(x) = \begin{cases} b^+ + B^+ \cdot x, & x \in Q_r^+, \\ b^- + B^- \cdot x, & x \in Q_r^- , \end{cases}$$

where $b^+, b^- \in \mathbb{R}, B^+, B^- \in \mathbb{R}^n$ and $0 < \bar{b}^{-1} \leq b(x) \leq \bar{b}$. Then, there exist positive constants $0 < \alpha' \leq 1$, $C > 0$ depending on $\bar{b}, r$ and n only, such that for any $\rho \leq \frac{r}{2}$ and for any $x \in Q_{r-2\rho}$, the following estimate holds

$$\begin{aligned}&\|\nabla U\|_{L^\infty(Q_\rho(x))} + \rho^{\alpha'}|\nabla U|_{\alpha', Q_\rho(x)\cap Q_r^+} + \rho^{\alpha'}|\nabla U|_{\alpha', Q_\rho(x)\cap Q_r^-}\\ &\leq \frac{C}{\rho^{1+n/2}}\|U\|_{L^2(Q_{2\rho}(x))} \ .\end{aligned} \tag{4.2}$$

Proof. For the proof we refer to [53, Theorem 16.2, Chap.3], where the authors obtained piecewise $C^{1,\alpha'}$ estimates for solutions to linear second order elliptic equations with piecewise Hölder continuous coefficients and $C^{1,1}$ discontinuity interfaces (see also [62,61] for more recent results under weaker regularity hypotheses). □

Proof of Theorem 3.2. We fix $l \in \{1, \dots, N-1\}$. We set $\gamma_l = \gamma_{j_l}(Q_{l+1})$ and $\gamma_{l+1} = \gamma_{j_{l+1}}(Q_{l+1})$. We will denote $a^+ = \gamma_{l+1}$ and $a^- = \gamma_l$. Furthermore, we observe that up to a transformation of coordinates we can assume that Q_{l+1} coincides with the origin 0 of the coordinates system.

For any $x = (x', x_n)$ we denote $x^* = (x', -x_n)$ and we have that a fundamental solution of the operator $\mathrm{div}_x(a^- + (a^+ - a^-)\chi^+\nabla_x)$ has the following explicit form

$$H(x,y) = \begin{cases} \frac{1}{a^+}\Gamma(x,y) + \frac{a^+-a^-}{a^+(a^++a^-)}\Gamma(x,y^*) \ , & \text{if } x_n, y_n > 0,\\ \frac{2}{a^++a^-}\Gamma(x,y) \ , & \text{if } x_n y_n < 0,\\ \frac{1}{a^-}\Gamma(x,y) + \frac{a^--a^+}{a^-(a^++a^-)}\Gamma(x,y^*) \ , & \text{if } x_n, y_n < 0.\end{cases} \tag{4.3}$$

We then define

$$R(x,y) = \tilde{G}(x,y) - H(x,y) \ . \tag{4.4}$$

We observe that R in (4.4) satisfies

$$\begin{cases} \mathrm{div}_x(\gamma(\cdot)\nabla_x R(\cdot,y)) = -\mathrm{div}_x((\gamma(\cdot) - \gamma_0(\cdot))\nabla_x H(\cdot,y)) \ , & \text{in } \tilde{\Omega} \ ,\\ R(\cdot,y) = -H(\cdot,y) \ , & \text{on } \partial\tilde{\Omega},\end{cases} \tag{4.5}$$

here $\gamma_0 = a^- + (a^+ - a^-)\chi^+$. By the representation formula over $\tilde{\Omega}$ we have that R in (4.4) satisfies

$$R(x,y) = -\int_{\tilde{\Omega}} (\gamma(\zeta) - \gamma_0(\zeta))\nabla_\zeta H(\zeta,y)\cdot\nabla_\zeta\tilde{G}(\zeta,x)d\zeta \tag{4.6}$$

$$+\int_{\partial\tilde{\Omega}} \gamma(\zeta)\partial_\nu\tilde{G}(\zeta,x)H(\zeta,y)d\sigma(\zeta). \tag{4.7}$$

We first treat the boundary term on the right hand side of the above equation. We have that

$$\left|\int_{\partial\tilde{\Omega}} \gamma(\cdot)\partial_\nu\,\tilde{G}(\cdot,x)\,H(\cdot,e_n y_n)\,d\sigma\right| \tag{4.8}$$

$$\leq \bar{\gamma}\,\|\partial_\nu\tilde{G}(\cdot,x)\|_{H^{-\frac{1}{2}}(\partial\tilde{\Omega})}\,\|H(\cdot,e_n y_n)\|_{H^{\frac{1}{2}}(\partial\tilde{\Omega})} \tag{4.9}$$

$$\leq \bar{\gamma}\,\|\tilde{G}(\cdot,x)\|_{H^1(\tilde{\Omega}\setminus B_{r_0/2}(x))}\,\|H(\cdot,e_n y_n)\|_{H^1(\tilde{\Omega}\setminus B_{r_0/2}(e_n y_n))} \ . \tag{4.10}$$

Hence, we deduce that

$$\left| \int_{\partial\tilde{\Omega}} \gamma(\cdot)\partial_\nu \tilde{G}(\cdot,x)\, H(\cdot,e_n y_n)\, d\sigma \right| \leq C, \tag{4.11}$$

where $C > 0$ is a constant depending on the a priori data only.

We observe that

$$|\gamma(\zeta) - \gamma_0(\zeta)| \leq C|\zeta| \tag{4.12}$$

with $C > 0$ constant depending on the a priori data only.

Moreover by combining (4.2) and (3.5) we find the following two pointwise bounds

$$|\nabla_\zeta \tilde{G}(\zeta,x)| \leq C|\zeta - x|^{1-n} \text{ for every } \zeta, x \in Q_{r_0}\ , \tag{4.13}$$

$$|\nabla_\zeta H(\zeta,y)| \leq C|\zeta - y|^{1-n} \text{ for every } \zeta, y \in Q_{r_0}\ , \tag{4.14}$$

which in turn together with (4.12) leads to

$$|\int_{\tilde{\Omega}} (\gamma(\zeta) - \gamma_0(\zeta))\nabla_\zeta H(\zeta,e_n y_n)\cdot \nabla_\zeta \tilde{G}(\zeta,x)\, d\zeta| \leq C_1|x - e_n y_n|^{3-n}\ . \tag{4.15}$$

Combining (4.11) and (4.15) we get

$$|R(x,e_n y_n)| \leq C|x - e_n y_n|^{3-n} \tag{4.16}$$

when $x \in B^+_{r_0}$ and $y_n \in (-r_0, 0)$.

We now focus on the estimate for $\nabla_x R(x, e_n y_n)$. Again arguing as in [8, Claim 4.3], we fix $x \in B^+_{r_0/4}$ and $y_n \in (-r_0/4, 0)$ and let us denote

$$Q = B'_{h/4}(x') \times \left(x_n, x_n + \frac{h}{4}\right)\ , \tag{4.17}$$

where $h = |x - y|$. We observe that $Q \subset Q^+_{\frac{r_0}{2}}$. Moreover, we have that $Q \subset Q_{\frac{h}{2}}(x)$ and $x \in \partial Q$.

By (3.5), Theorem 4.1 and explicit computation on $H(x,y)$ we get

$$|\nabla_x \tilde{G}(\cdot,e_n y_n)|_{\alpha',Q}\ ,\ |\nabla_x H(\cdot,e_n y_n)|_{\alpha',Q} \leq Ch^{-\alpha'+1-n}\ . \tag{4.18}$$

Hence by (4.4) and (4.18) we get

$$|\nabla_x R(\cdot,e_n y_n)|_{\alpha',Q} \leq Ch^{-\alpha'+1-n}\ . \tag{4.19}$$

We recall the following interpolation inequality

$$\|\nabla_x R(\cdot,e_n y_n)\|_{L^\infty(Q)} \leq C\|R(\cdot,e_n y_n)\|^{\alpha'/1+\alpha'}_{L^\infty(Q)}\ |\nabla_x R(\cdot,e_n y_n)|^{1/1+\alpha'}_{\alpha',Q}\ . \tag{4.20}$$

By the above estimate and (4.16) we obtain

$$|\nabla_x R(x,y)| \leq Ch^{\beta+1-n}, \tag{4.21}$$

where $\beta = \frac{\alpha'^2}{1+\alpha}$.

Finally, we study the behavior of $\nabla_y \nabla_x R(x, y)$. We define the cylinder $\hat{Q} = B'_{\frac{h}{8}}(0) \times \left(y_n - \frac{h}{8}, y_n\right)$. As before we have that $\hat{Q} \subset Q^-_{\frac{r_0}{4}}, \hat{Q} \subset Q_{\frac{h}{4}}(y)$. In particular, we have that $x \notin Q_{\frac{h}{4}}(y)$.

Let k be an integer such that $k \in \{1, \ldots, n\}$. We observe that $\partial_{x_k}\Gamma(x, \cdot)$ and $\partial_{x_k}G(x, \cdot)$ are solutions to

$$\Delta_y(\partial_{x_k}\Gamma(x, \cdot)) = 0 \quad \text{in} \quad Q_{\frac{h}{4}}(y) \ , \tag{4.22}$$

$$\mathrm{div}_y(\gamma(\cdot)\nabla_y \partial_{x_k}\tilde{G}(x, \cdot)) = 0 \quad \text{in} \quad Q_{\frac{h}{4}}(y) \tag{4.23}$$

respectively.

Again by applying Theorem 4.1, we have that

$$|\nabla_y \partial_{x_k}\tilde{G}(x, \cdot)|_{\alpha', \hat{Q}} \leq Ch^{-\alpha'-1-\frac{n}{2}} \|\partial_{x_k}\tilde{G}(x, \cdot)\|_{L^2(Q_{\frac{h}{4}}(y))}. \tag{4.24}$$

We now fix $\eta \in Q_{\frac{h}{4}}(y)$ and we notice that $\eta \notin Q_{\frac{h}{16}}(x)$. By Theorem 4.1 we have that

$$\|\nabla_x \tilde{G}(\cdot, \eta)\|_{L^\infty(Q_{\frac{h}{32}}(x))} \leq Ch^{-1-\frac{n}{2}} \|\tilde{G}(\cdot, \eta)\|_{L^\infty(Q_{\frac{h}{16}}(x))} \leq Ch^{1-n}. \tag{4.25}$$

Combining (4.24) and (4.25) we have

$$|\nabla_y \partial_{x_k}\tilde{G}(x, \cdot)|_{\alpha', \hat{Q}} \leq Ch^{-\alpha'-n} \quad . \tag{4.26}$$

By explicit computations we infer that

$$|\nabla_y \partial_{x_k}\Gamma(x, \cdot)|_{\alpha', \hat{Q}} \leq Ch^{-\alpha'-n} \ . \tag{4.27}$$

From (4.26) and (4.27), we have that

$$|\nabla_y \partial_{x_k}R(x, \cdot)|_{\alpha', \hat{Q}} \leq Ch^{-\alpha'-n} \ . \tag{4.28}$$

Moreover, we observe that by analogous arguments of those discussed above, we can infer that

$$\|\partial_{x_k}R(x, \cdot)\|_{L^\infty(\hat{Q})} \leq Ch^{\beta+1-n}, \tag{4.29}$$

where $\beta = \frac{\alpha'^2}{1+\alpha}$.

By the following interpolation inequality

$$\|\nabla_y \partial_{x_k}R(x, \cdot)\|_{L^\infty(\hat{Q})} \leq C\|\partial_{x_k}R(x, \cdot)\|^{\frac{\alpha'}{\alpha'+1}}_{L^\infty(\hat{Q})} |\nabla_y \partial_{x_k}R(x, \cdot)|^{\frac{1}{\alpha'+1}}_{\alpha', \hat{Q}} \tag{4.30}$$

and by (4.29) and (4.28) we have that

$$|\nabla_y \partial_{x_k}R(x, y)| \leq Ch^{-n+\theta}, \tag{4.31}$$

where $\theta = \frac{\beta\alpha'}{1+\alpha'}$. $\square$

4.2. Propagation of smallness

Proof of Proposition 3.3. By repeating the argument in [8, proof of Proposition 4.4] concerning a careful analysis of unique continuation argument across K discontinuity interfaces and based on an iterated use of the three spheres inequality for elliptic equation, we have that for any $y, z \in B_{\rho_{\bar{h}(r)}}(w_{\bar{h}(r)}(Q_{k+1}))$

$$|\tilde{S}_{\mathcal{U}_k}(y,z)| \leq r_0^{-n+2} C^{\bar{h}(r)}(E+\varepsilon_0)\left(\omega_{1/C}^{(2k)}\left(\frac{\varepsilon_0}{E+\varepsilon_0}\right)\right)^{(1/C)^{\bar{h}(r)}}. \tag{4.32}$$

Hence (3.23) trivially follows from (4.32).

We now consider $\tilde{S}_{\mathcal{U}_k}(y,z)$ as a function of $2n$ variables where $(y,z) \in \mathbb{R}^{2n}$, hence by (4.32) we have that

$$|\tilde{S}_{\mathcal{U}_k}(y_1,\ldots,y_n,z_1,\ldots,z_n)| \leq r_0^{-n+2} C^{\bar{h}(r)}(E+\varepsilon_0)\left(\omega_{1/C}^{(2K)}\left(\frac{\varepsilon_0}{E+\varepsilon_0}\right)\right)^{(1/C)^{h(\bar{r})}}, \tag{4.33}$$

for any $x = (y_1,\ldots,y_n,z_1,\ldots,z_n) \in B_{\rho_{\bar{h}(r)}}(w_{\bar{h}(r)}(Q_{k+1})) \times B_{\rho_{\bar{h}(r)}}(w_{\bar{h}(r)}(Q_{k+1}))$.

Now observing that $\tilde{S}_{\mathcal{U}_k}(y_1,\ldots,y_n,z_1,\ldots,z_n)$ is a solution in $D_k \times D_k$ of the elliptic equation

$$\mathrm{div}_y(\gamma^{(1)}(y)\nabla_y \tilde{S}_{\mathcal{U}_k}(y,z)) + \mathrm{div}_z(\gamma^{(2)}(z)\nabla_z \tilde{S}_{\mathcal{U}_k}(y,z)) = 0 \tag{4.34}$$

we have that by Schauder interior estimates that for any $i, j = 1,\ldots,n$ it follows

$$\begin{aligned}&\|\partial_{x_i}\partial_{x_j}\tilde{S}_{\mathcal{U}_k}(x_1,\ldots,x_n,x_{n+1},\ldots,x_{2n})\|_{L^\infty(B_{\frac{\rho_{\bar{h}(r)}}{2}}(w_{\bar{h}(r)}(Q_{k+1}))\times B_{\frac{\rho_{\bar{h}(r)}}{2}}(w_{\bar{h}(r)}(Q_{k+1})))}\\ &\leq \frac{C}{\rho^2_{\bar{h}(r)-1}}\|\tilde{S}_{\mathcal{U}_k}(x_1,\ldots,x_n,x_{n+1},\ldots,x_{2n})\|_{L^\infty(B_{\rho_{\bar{h}(r)}}(w_{\bar{h}(r)}(Q_{k+1}))\times B_{\rho_{\bar{h}(r)}}(w_{\bar{h}(r)}(Q_{k+1})))},\end{aligned}$$

where $x_i = y_i, x_{i+n} = z_i$ for $i = 1,\ldots,n$.

Moreover, we have that being $d_{\bar{h}(r)-1} > r$, hence it follows $r < \frac{d_0}{a\rho_0}\rho_{\bar{h}(r)}$, which in turn leads to

$$\begin{aligned}&\|\partial_{x_i}\partial_{x_j}\tilde{S}_{\mathcal{U}_{\mathcal{K}}}(x_1,\ldots,x_{2n})\|_{L^\infty(\tilde{Q}_{\frac{\rho_{\bar{h}(r)}}{2}}(w_{\bar{h}(r)}(Q_{k+1})))}\\ &\leq \frac{C}{r^2}\|\tilde{S}_{\mathcal{U}_{\mathcal{K}}}(x_1,\ldots,x_{2n})\|_{L^\infty(\tilde{Q}_{\rho_{\bar{h}(r)}}(w_{\bar{h}(r)}(Q_{k+1})))}.\end{aligned} \tag{4.35}$$

Recalling (3.21) we find

$$r^{-2} \leq \left(\frac{a}{r_0}\right)^2\left(\frac{1}{a^2}\right)^{\bar{h}(r)}. \tag{4.36}$$

Finally by combining (4.33), (4.35) and the above inequality we get the desired estimate. □

4.3. Stability at the boundary

Proof of estimate (3.31). We recall that for every $y, z \in D_0$ we have

$$\begin{aligned}\left\langle(\Lambda_1-\Lambda_2)\tilde{G}_1(\cdot,y),\tilde{G}_2(\cdot,z)\right\rangle &= \int_\Omega(\tilde{\gamma}^{(1)}-\tilde{\gamma}^{(2)})(\cdot)\nabla\tilde{G}_1(\cdot,y)\cdot\nabla\tilde{G}_2(\cdot,z)\\ &= \tilde{S}_{\mathcal{U}_0}(y,z),\end{aligned} \tag{4.37}$$

and

$$\left\langle(\Lambda_1-\Lambda_2)\partial_{y_n}\tilde{G}_1(\cdot,y),\partial_{z_n}\tilde{G}_2(\cdot,z)\right\rangle=\int_\Omega(\tilde{\gamma}^{(1)}-\tilde{\gamma}^{(2)})(\cdot)\partial_{y_n}\nabla\tilde{G}_1(\cdot,y)\cdot\partial_{z_n}\nabla\tilde{G}_2(\cdot,z)$$
$$=\partial_{y_n}\partial_{z_n}\tilde{S}_{\mathcal{U}_0}(y,z). \tag{4.38}$$

From (4.37) we obtain

$$\left|\int_\Omega(\widetilde{\gamma}^{(1)}-\widetilde{\gamma}^{(2)})(\cdot)\nabla\widetilde{G}_1(\cdot,y)\cdot\nabla\widetilde{G}_2(\cdot,z)\right|$$
$$\leq\varepsilon_0||\widetilde{G}_1(\cdot,y)||_{H^{1/2}_{co}(\Sigma)}||\widetilde{G}_2(\cdot,z)||_{H^{1/2}_{co}(\Sigma)}$$
$$\leq C\varepsilon_0\left(d(y)d(z)\right)^{1-\frac{n}{2}},\qquad\text{for every } y,z\in D_0, \tag{4.39}$$

where $d(y)$ denotes the distance of y from Ω and C is a constant that depends on L, λ and n. Let $\rho_0=\frac{r_0}{C}$, where C is the constant introduced in Theorem 3.2, let $r\in(0,d_2)$ and denote

$$w=P_1+\sigma\nu,\qquad\text{where } \sigma=a^{\bar{h}-1}\lambda_1.$$

We set $y=z=w$ and obtain

$$\int_\Omega(\tilde{\gamma}^{(1)}-\tilde{\gamma}^{(2)})(\cdot)\nabla\tilde{G}_1(\cdot,w)\cdot\nabla\tilde{G}_2(\cdot,w)$$
$$=\int_{B_{\rho_0}(P_1)\cap D_1}(\gamma^{(1)}-\gamma^{(2)})(\cdot)\nabla\tilde{G}_1(\cdot,w)\cdot\nabla\tilde{G}_2(\cdot,w)$$
$$+\int_{\Omega\backslash(B_{\rho_0}(P_1)\cap D_1)}(\gamma^{(1)}-\gamma^{(2)})(\cdot)\nabla\tilde{G}_1(\cdot,w)\cdot\nabla\tilde{G}_2(\cdot,w), \tag{4.40}$$

which leads to

$$\varepsilon_0\sigma^{2-n}\geq\left|\int_{B_{\rho_0}(P_1)\cap D_1}(\gamma_1^{(1)}-\gamma_1^{(2)})(\cdot)\nabla\tilde{G}_1(\cdot,w)\cdot\nabla\tilde{G}_2(\cdot,w)\right|$$
$$-\left|\int_{\Omega\backslash(B_{\rho_0}(P_1)\cap D_1)}(\gamma^{(1)}-\gamma^{(2)})(\cdot)\nabla\tilde{G}_1(\cdot,w)\cdot\nabla\tilde{G}_2(\cdot,w)\right|. \tag{4.41}$$

Let $x^0\in\overline{\Sigma_1\cap B_{\frac{r_0}{4}(P_1)}}$ such that $\left(\gamma_1^{(1)}-\gamma_1^{(2)}\right)(x^0)=||\widetilde{\gamma}^{(1)}-\widetilde{\gamma}^{(2)}||_{L^\infty(\Sigma_1\cap B_{\frac{r_0}{4}(P_1)})}$ and recall that $\left(\gamma_1^{(1)}-\gamma_1^{(2)}\right)(x)=\alpha_1+\beta_1\cdot x$, therefore by combining this with (3.6) we obtain

$$\varepsilon_0\sigma^{2-n}\geq\int_{B_{\rho_0}(P_1)\cap D_1}(\gamma_1^{(1)}-\gamma_1^{(2)})(x^0)\nabla\widetilde{G}_1(\cdot,w)\cdot\nabla\widetilde{G}_2(\cdot,w)$$
$$-\left|\int_{B_{\rho_0}(P_1)\cap D_1}\beta\cdot(x-x^0)\nabla\widetilde{G}_1(\cdot,w)\cdot\nabla\widetilde{G}_2(\cdot,w)\right|-C\rho_0^{2-n} \tag{4.42}$$

and then

$$||\widetilde{\gamma}_1^{(1)}-\widetilde{\gamma}_1^{(2)}||_{L^\infty(\Sigma_1\cap B_{\frac{r_0}{4}(P_1)})}\int\limits_{B_{\rho_0}(P_1)\cap D_1}\nabla\widetilde{G}_1(\cdot,w)\cdot\nabla\widetilde{G}_2(\cdot,w)$$

$$\leq \int\limits_{B_{\rho_0}(P_1)\cap D_1}|\beta|\,|x-x^0|\,|\nabla\widetilde{G}_1(\cdot,w)||\nabla\widetilde{G}_2(\cdot,w)|+C\rho_0^{2-n}+\varepsilon_0\sigma^{2-n}. \tag{4.43}$$

By combining (4.43) together with (3.14) we obtain

$$||\widetilde{\gamma}_1^{(1)}-\widetilde{\gamma}_1^{(2)}||_{L^\infty(\Sigma_1\cap B_{\frac{r_0}{4}(P_1)})}\int\limits_{B_{\rho_0}(P_1)\cap D_1}|\nabla\Gamma(x-w)|^2\,dx \tag{4.44}$$

$$\leq C\left\{\int\limits_{B_{\rho_0}(P_1)\cap D_1}|x-w|^{2-2n+2\theta_1}\,dx+\int\limits_{B_{\rho_0}(P_1)\cap D_1}|x-w|^{2-2n+\theta_1}\,dx\right\}$$

$$+\int\limits_{B_{\rho_0}(P_1)\cap D_1}|\theta_1|\,|x-w|^{3-2n}\,dx+C\rho_0^{2-n}+\varepsilon_0\sigma^{2-n}, \tag{4.45}$$

which leads to

$$||\widetilde{\gamma}^{(1)}-\widetilde{\gamma}^{(2)}||_{L^\infty(\Sigma_1\cap B_{\frac{r_0}{4}(P_1)})}\leq\sigma^{\theta_1}+\sigma+\sigma^{n-2}+C\varepsilon_0. \tag{4.46}$$

Letting $\sigma\to 0$ we then obtain

$$||\widetilde{\gamma}^{(1)}-\widetilde{\gamma}^{(2)}||_{L^\infty(\Sigma_1\cap B_{\frac{r_0}{4}(P_1)})}\leq C\varepsilon_0. \tag{4.47}$$

From (4.38), setting again $y=z=w$, we obtain

$$\left\langle(\Lambda_1-\Lambda_2)\partial_{y_n}\tilde{G}_1(\cdot,y),\partial_{z_n}\tilde{G}_2(\cdot,z)\right\rangle=I_1(w)+I_2(w), \tag{4.48}$$

where

$$I_1(w)=\int\limits_{B_{\rho_0}(P_1)\cap D_1}(\gamma^{(1)}-\gamma^{(2)})(\cdot)\partial_{y_n}\nabla\tilde{G}_1(\cdot,w)\cdot\partial_{z_n}\nabla\tilde{G}_2(\cdot,w),$$

$$I_2(w)=\int\limits_{\Omega\backslash(B_{\rho_0}(P_1)\cap D_1)}(\gamma^{(1)}-\gamma^{(2)})(\cdot)\partial_{y_n}\nabla\widetilde{G}_1(\cdot,w)\cdot\partial_{z_n}\nabla\widetilde{G}_2(\cdot,w)$$

and by (3.6)

$$|I_2(w)|\leq CE\rho_0^{-n}. \tag{4.49}$$

We have

$$|I_1(w)|\;\geq\left|\int\limits_{B_{\rho_0}(P_1)\cap D_1}(\partial_\nu(\gamma_1^{(1)}-\gamma_1^{(2)})(P_1))(x-P_1)_n\partial_{y_n}\nabla\widetilde{G}_1(\cdot,w)\cdot\partial_{z_n}\nabla\widetilde{G}_2(\cdot,w)\right|$$

$$- \int_{B_{\rho_0}(P_1)\cap D_1} |(D_T(\gamma_1^{(1)} - \gamma_1^{(2)})(P_1)) \cdot (x - P_1)'||\partial_{y_n} \nabla \widetilde{G}_1(\cdot, w)|\, |\partial_{z_n} \nabla \widetilde{G}_2(\cdot, w)|$$
$$- \int_{B_{\rho_0}(P_1)\cap D_1} |(\gamma_1^{(1)} - \gamma_1^{(2)})(P_1)||\partial_{y_n} \nabla \widetilde{G}_1(\cdot, w)|\, |\partial_{z_n} \nabla \widetilde{G}_2(\cdot, w)|.$$

Noticing that up to a transformation of coordinates we can assume that P_1 coincides with the origin O of the coordinates system and by Theorem 3.2, this leads to

$$\begin{aligned} |I_1(w)| \geq{}& |\partial_\nu(\gamma_1^{(1)} - \gamma_1^{(2)})(O)|C \int_{B_{\rho_0}(O)\cap D_1} |\partial_{y_n} \nabla_x \Gamma(x, w)|^2 |x_n| \\ &- C\Big\{E \int_{B_{\rho_0}(O)\cap D_1} |\partial_{y_n} \nabla_x \Gamma(x, w)||x - w|^{-n+\theta_2} |x_n| \\ &- E \int_{B_{\rho_0}(O)\cap D_1} |x - w|^{-2n+\theta_2} |x_n|\Big\} \\ &- \int_{B_{\rho_0}(O)\cap D_1} |D_T(\gamma_1^{(1)} - \gamma_1^{(2)})|\, |x'|\, |\partial_{y_n} \nabla \widetilde{G}_1(\cdot, w)|\, |\partial_{z_n} \nabla \widetilde{G}_2(\cdot, w)| \\ &- \int_{B_{\rho_0}(O)\cap D_1} |(\gamma_1^{(1)} - \gamma_1^{(2)})(O)|\, |\partial_{y_n} \nabla \widetilde{G}_1(\cdot, w)|\, |\partial_{z_n} \nabla \widetilde{G}_2(\cdot, w)|. \end{aligned} \tag{4.50}$$

Therefore, by combining (4.50) together with (4.48) and (4.49), we obtain

$$\begin{aligned} |I_1(w)| \geq{}& |\partial_\nu(\gamma_1^{(1)} - \gamma_1^{(2)})(O)|C \int_{B_{\rho_0}(P_1)\cap D_1} |x - w|^{1-2n} \\ &- C\Big\{E \int_{B_{\rho_0}(P_1)\cap D_1} |x - w|^{1-2n+\theta_2} \\ &- E \int_{B_{\rho_0}(P_1)\cap D_1} |x - w|^{2-2n+\theta_2} \\ &- \varepsilon_0 \int_{B_{\rho_0}(P_1)\cap D_1} |x - w|^{1-2n} \\ &- \varepsilon_0 \int_{B_{\rho_0}(P_1)\cap D_1} |x - w|^{-2n}\Big\}, \end{aligned}$$

which leads to

$$|\partial_\nu(\gamma_1^{(1)} - \gamma_1^{(2)})(O)|\sigma^{1-n} \leq |I_1(w)| + C\Big(\varepsilon_0\sigma^{-n} + E\sigma^{1-n+\theta_2}\Big), \tag{4.51}$$

and

$$|I_1(w)| \leq \big|\big\langle(\Lambda_1 - \Lambda_2)\partial_{y_n}\tilde{G}_1(\cdot, y), \partial_{z_n}\tilde{G}_2(\cdot, z)\big\rangle\big| + CE\rho_0^{-n}. \tag{4.52}$$

Thus by combining together the last two inequalities we get

$$|\partial_\nu(\gamma_1^{(1)} - \gamma_1^{(2)})(O)|\sigma^{1-n} \le C\Big(\varepsilon_0\sigma^{-n} + E\rho_0^{-n} + \varepsilon_0\sigma^{-n} + E\sigma^{1-n+\theta_2}\Big), \tag{4.53}$$

therefore

$$|\partial_\nu(\gamma_1^{(1)} - \gamma_1^{(2)})(O)| \le C\left\{\varepsilon_0\sigma^{-1} + E\sigma^{\theta_2}\right\} \tag{4.54}$$

and by optimizing with respect to σ we get

$$|\partial_\nu(\gamma_1^{(1)} - \gamma_1^{(2)})(O)| \le C\varepsilon_0^{\frac{\theta_2}{\theta_2+1}}(E+\varepsilon_0)^{\frac{1}{1+\theta_2}}. \qquad \square \tag{4.55}$$

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References

[1] G. Alessandrini, Stable determination of conductivity by boundary measurements, Appl. Anal. 27 (1988) 153–172.
[2] G. Alessandrini, Singular solutions of elliptic equations and the determination of conductivity by boundary measurements, J. Differ. Equ. 84 (2) (1990) 252–272.
[3] G. Alessandrini, Determining conductivity by boundary measurements, the stability issue, in: R. Spigler (Ed.), Applied and Industrial Mathematics, Kluwer, 1991, pp. 317–324.
[4] G. Alessandrini, Open issues of stability for the inverse conductivity problem, J. Inverse Ill-Posed Probl. 15 (2007) 1–10.
[5] G. Alessandrini, M.V. de Hoop, R. Gaburro, Uniqueness for the electrostatic inverse boundary value problem with piecewise constant anisotropic conductivities, arxiv preprint, arXiv:1604.02948, 2016.
[6] G. Alessandrini, R. Gaburro, Determining conductivity with special anisotropy by boundary measurements, SIAM J. Math. Anal. 33 (2001) 153–171.
[7] G. Alessandrini, R. Gaburro, The local Calderón problem and the determination at the boundary of the conductivity, Commun. Partial Differ. Equ. 34 (2009) 918–936.
[8] G. Alessandrini, S. Vessella, Lipschitz stability for the inverse conductivity problem, Adv. Appl. Math. 35 (2005) 207–241.
[9] K. Astala, L. Päivärinta, Calderòn's inverse conductivity problem in the plane, Ann. Math. (2006) 265–299.
[10] K. Astala, M. Lassas, L. Päivärinta, Calderòn inverse problem for anisotropic conductivity in the plane, Commun. Partial Differ. Equ. 30 (2005) 207–224.
[11] E. Auken, A. Vest Christiansen, Layered and laterally constrained 2D inversion of resistivity data, Geophysics 69 (3) (2004) 752–761.
[12] J.A. Barceló, T. Barceló, A. Ruiz, Stability of the inverse conductivity problem in the plane for less regular conductivities, J. Differ. Equ. 173 (2001) 231–270.
[13] T. Barceló, D. Faraco, A. Ruiz, Stability of Calderón inverse conductivity problem in the plane, J. Math. Pures Appl. 88 (6) (2007) 522–556.
[14] R.D. Barker, The offset system of electrical resistivity sounding and its use with a multicore cable, Geophys. Prospect. 29 (1981) 128–143.
[15] R.D. Barker, Depth of investigation of collinear symmetrical four-electrode arrays, Geophysics 54 (1989) 1031–1037.
[16] M.I. Belishev, The Calderón problem for two-dimensional manifolds by the BC-method, SIAM J. Math. Anal. 35 (1) (2003) 172–182.

[17] E. Beretta, M. De Hoop, L. Qiu, Lipschitz stability of an inverse boundary value problem for a Schrödinger type equation, SIAM J. Math. Anal. 45 (2) (2013) 679–699.
[18] E. Beretta, M. De Hoop, L. Qiu, O. Scherzer, Inverse boundary value problem for the Helmholtz equation: multi-level approach and iterative reconstruction, arXiv, preprint, arXiv:1406.2391, 2014.
[19] E. Beretta, E. Francini, Lipschitz stability for the electrical impedance tomography problem: the complex case, Commun. Partial Differ. Equ. 36 (2011) 1723–1749.
[20] E. Beretta, E. Francini, S. Vessella, Uniqueness and Lipschitz stability for the identification of Lamé parameters from boundary measurements, Inverse Probl. Imaging 8 (3) (2014) 611–644.
[21] E. Beretta, E. Francini, A. Morassi, E. Rosset, S. Vessella, Lipschitz continuous dependence of piecewise constant Lamé coefficients from boundary data: the case of non-flat interfaces, Inverse Probl. 30 (12) (2014) 125005.
[22] L. Borcea, Electrical impedance tomography, Inverse Probl. 18 (2002) R99–R136.
[23] D. Bruhn, A. Manzella, F. Vuatiaz, J. Faulds, I. Moeck, K. Erbas, Exploration methods, in: E. Huenges, P. Ledru (Eds.), Geothermal Energy Systems: Exploration, Development, and Utilization, John Wiley and Sons, 2011.
[24] R.M. Brown, Recovering the conductivity at the boundary from the local Dirichlet-to-Neumann map: a pointwise result, J. Inverse Ill-Posed Probl. 9 (6) (2001) 567–574.
[25] R.M. Brown, G. Uhlmann, Uniqueness in the inverse conductivity problem with less regular conductivities in two dimensions, Commun. Partial Differ. Equ. 22 (1997) 1009–1027.
[26] A.P. Calderón, On an inverse boundary value problem, in: Seminar on Numerical Analysis and its Applications to Continuum Physics (Rio de Janeiro, 1980), Soc. Brasil. Mat., Rio de Janeiro, 1980, pp. 65–73;
A.P. Calderón, Reprinted in: Comput. Appl. Math. 25 (2–3) (2006) 133–138.
[27] P. Caro, A. Garcia, J.M. Reyes, Stability of the Calderón problem for less regular conductivities, J. Differ. Equ. 254 (2) (2013) 469–492.
[28] P. Caro, K. Rogers, Global uniqueness for the Calderón problem with Lipschitz conductivities, Forum Math. Pi 4 (2016) e2, 28pp.
[29] J. Chen, M. De Hoop, Inverse problem of electroseismic conversion. I: Inversion of Maxwell's equations with internal data, arXiv preprint, arXiv:1406.0367, 2014.
[30] M. Cheney, D. Isaacson, J.C. Newell, Electrical impedance tomography, SIAM Rev. 41 (1) (1999) 85–101.
[31] A. Clop, D. Faraco, A. Ruiz, Stability of the Calderón's inverse conductivity problem in plane for discontinuous conductivities, Inverse Probl. Imaging 4 (1) (2010) 49–91.
[32] S.C. Constable, M.W. McElhinny, P.L. McFadden, Deep Schlumberger sounding and the crustal resistivity structure of central Australia, Geophys. J. Int. 79 (3) (1984) 893–910.
[33] V. Druskin, The unique solution of the inverse problem of electrical surveying and electrical well-logging for piecewise-continuous conductivity, Izv. Earth Phys. 18 (1982) 51–53 (in Russian).
[34] V. Druskin, On uniqueness of the determination of the three-dimensional underground structures from surface measurements with variously positioned steady-state or monochromatic field sources, Sov. Phys.-Solid Earth 21 (1985) 210–214 (in Russian).
[35] V. Druskin, On the uniqueness of inverse problems from incomplete boundary data, SIAM J. Appl. Math. 58 (5) (1998) 1591–1603.
[36] M. De Hoop, L. Qiu, O. Scherzer, Local analysis of inverse problems: Hölder stability and iterative reconstruction, Inverse Probl. 28 (4) (2012) 045001.
[37] R.G. Ellis, D.W. Oldenburg, The pole–pole 3-D Dc-resistivity inverse problem: a conjugate gradient approach, Geophys. J. Int. 119 (1) (1994) 187–194.
[38] D. Faraco, Y. Kurylev, A. Ruiz, G-convergence, Dirichlet to Neumann maps and invisibility, J. Funct. Anal. 267 (7) (2013) 2478–2506.
[39] C.G. Farquharson, D.W. Oldenburg, Constructing piecewise constant models using general measures in nonlinear, minimum-structure inversion algorithms, in: 6th International Symposium of the SEG of Japan, Tokyo, January, 2003, pp. 240–243, Expanded Abstracts, 2224.
[40] Ó.G. Flóvenz, G.P. Hersir, K. Saemundsson, H. Ármannsson, T. Fridriksson, Geothermal energy exploration techniques, in: A. Syigh (Ed.), Comprehensive Renewable Energy, vol. 7, Elsevier, Oxford, 2012, pp. 51–95.
[41] R. Gaburro, W.R.B. Lionheart, Recovering Riemannian metrics in monotone families from boundary data, Inverse Probl. 25 (4) (2009).
[42] R. Gaburro, E. Sincich, Lipschitz stability for the inverse conductivity problem for a conformal class of anisotropic conductivities, Inverse Probl. 31 (2015) 015008.
[43] A.L. Gurasov, On uniqueness of solution of inverse magnetotelluric problem for two-dimensional media, in: V.I. Demitriev (Ed.), Mathematical Models in Geophysics, Moscow State University, 1981, pp. 31–61 (in Russian).
[44] G.M. Habberjam, The effects of anisotropy on square array resistivity measurements, Geophys. Prospect. 20 (1972) 249–266.
[45] G.M. Habberjam, G.E. Watkins, The use of a square configuration in resistivity prospecting, Geophys. Prospect. 15 (1967) 221–235.
[46] B. Haberman, D. Tataru, Uniqueness in Calderón's problem with Lipschitz conductivities, Duke Math. J. 162 (3) (2013) 497–516.
[47] G.P. Hersir, A. Björnsson, H. Eysteinsson, Volcanism and geothermal activity in the Hengill area. Geophysical exploration: Resistivity data. Orkustofnun Report OS-90032/JHD-16 B, 1990, p. 89 (in Icelandic).
[48] S.E. Kim, Calderón's problem for Lipschitz piecewise smooth conductivities, Inverse Probl. 24 (2008) 055016, 16pp.
[49] O. Koefoed, Resistivity Sounding Measurements, Geosounding Principles I, Methods in Geochemistry and Geophysics, vol. 1, Elsevier, Amsterdam, 1979.
[50] A. Kaufman, B. Anderson, Principles of Electric Methods in Surface and Borehole Geophysics, Methods in Geochemistry and Geophysics, vol. 44, Elsevier, 2010, 456 pp.

[51] R. Kohn, M. Vogelius, Determining conductivity by boundary measurements, Commun. Pure Appl. Math. 37 (3) (1984) 289–298.
[52] R. Kohn, M. Vogelius, Determining conductivity by boundary measurements II. Interior results, Commun. Pure Appl. Math. 38 (1985) 643–667.
[53] O.A. Ladyzhenskaya, N.N. Ural'tseva, Linear and Quasilinear Elliptic Equations, Academic Press, New York, 1968.
[54] M. Lassas, G. Uhlmann, On determining a Riemannian manifold from the Dirichlet-to-Neumann map, Ann. Sci. Éc. Norm. Supér. (4) 34 (5) (2001) 771–787.
[55] Y. Li, D.W. Oldenburg, Approximate inverse mappings in DC resistivity problems, Geophys. J. Int. 109 (2) (1992) 343–362.
[56] Y. Li, K. Spitzer, Three-dimensional DC resistivity forward modelling using finite elements in comparison with finite-difference solutions, Geophys. J. Int. 151 (3) (2002) 924–934.
[57] W.R.B. Lionheart, Conformal uniqueness results in anisotropic electrical impedance imaging, Inverse Probl. 13 (1997) 125–134.
[58] J.L. Lions, E. Magenes, Non-homogeneous Boundary Value Problems and Applications 1, Die Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 181, Springer-Verlag, New York, 1972, Translated from French by P. Kenneth, MR0350177 (50 #2670).
[59] L. Liu, Stability estimates for the two-dimensional inverse conductivity problem, PhD thesis, University of Rochester, New York, 1997.
[60] W. Littman, G. Stampacchia, H.W. Weinberger, Regular points for elliptic equations with discontinuous coefficients, Ann. Sc. Norm. Super. Pisa, Cl. Sci. 3 (17) (1963) 43–77.
[61] Y.Y. Li, L. Nirenberg, Estimates for elliptic systems from composite material, Commun. Pure Appl. Math. LVI (2003) 892–925.
[62] Y.Y. Li, M. Vogelius, Gradient estimates for solutions to divergence form elliptic equations with discontinuous coefficients, Arch. Ration. Mech. Anal. 153 (2000) 91–151.
[63] M.H. Loke, Tutorial: 2-D and 3-D electrical imaging surveys, 1996–2001 (revision);
M.H. Loke, R.D. Barker, Practical techniques for 3D resistivity surveys and data inversion, Geophys. Prospect. 44 (1996) 499–523.
[64] M.H. Loke, I. Ackworth, T. Dahlin, A comparison of smooth and blocky inversion methods in 2D electrical imaging surveys, Explor. Geophys. 34 (3) (2003) 182–187.
[65] R. Mallet, H.G. Doll, Sur un théorème relatif aux milieux électriquement anistropes et ses applications à la prospection électrique en courant continu, Ergänzunhefte angew. Geophys. 3 (1932) 109–124.
[66] N. Mandache, Exponential instability in an inverse problem for the Schrödeinger equation, Inverse Probl. 17 (2001) 1435–1444.
[67] A. Nachman, Global uniqueness for a two dimensional inverse boundary value problem, Ann. Math. 143 (2) (1996) 71–96.
[68] S. Narayan, M.B. Dusseault, D.C. Nobes, Inversion techniques applied to resistivity inverse problems, Inverse Probl. 10 (1994) 669–686.
[69] D.W. Oldenburg, J. Li, Estimating depth of investigation in DC resistivity and IP surveys, Geophysics 64 (1999) 403–416.
[70] R.L. Parker, The inverse problem of resistivity sounding, Geophysics 49 (12) (1984) 2143–2158.
[71] C.L. Pekeris, Direct method of interpretation in resistivity prospecting, Geophysics 5 (1940) 31–42.
[72] L. Rondi, A remark on a paper by Alessandrini and Vessella, Adv. Appl. Math. 36 (1) (2006) 67–69.
[73] C. Schlumberger, Etude sur la Prospection Électrique du Sous-Sol, Gauthier Villars, Paris, France, 1920.
[74] M. Schlumberger, C. Schlumberger, Electrical studies of Earth's crust at great depths, Trans. Am. Inst. Min. Metall. Engrs. AIME Geophys. Prosp. (1932) 134–140.
[75] L.B. Slichter, The interpretation of the resistivity prospecting method for horizontal structures, J. Appl. Phys. 4 (1933) 307–322.
[76] J. Sylvester, G. Uhlmann, A global uniqueness theorem for an inverse boundary valued problem, Ann. Math. 125 (1987) 153–169.
[77] G. Uhlmann, Electrical impedance tomography and Calderón's problem (topical review), Inverse Probl. 25 (12) (2009) 123011, http://dx.doi.org/10.1088/0266-5611/25/12/123011.
[78] S.H. Ward, Resistivity and induced polarization methods, Geotech. Environ. Geophys. 1 (1990) 147–189.
[79] M.S. Zhdanov, G.V. Keller, The Geoelectrical Methods in Geophysical Exploration, vol. 31, Elsevier Science Limited, 1994.
[80] A.A.R. Zohdy, A new method for the automatic interpretation of Schlumberger and Wenner sounding curves, Geophysics 54 (1989) 245–253.