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Non-Singular Method of Fundamental Solutions for Biomedical Stokes Flow Problems

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SUMMARY

The purpose of the present paper is development of a Non-singular Method of Fundamental Solutions (NMFS) for Stokes flow problems, widely applicable in biomedical engineering. The NMFS is based on the classical Method of Fundamental Solutions (MFS) with regularization of the singularities. The Stokes problem is decomposed into three coupled Laplace problems. The solution is structured by collocating the pressure and the velocity field boundary conditions by the Laplace fundamental solution. The regularization is achieved by replacement of the concentrated point sources by distributed sources over the disks around the singularity of fundamental solution. The NMFS solution is compared to MFS solution and analytical solution (a.s.) in case of simple 2D duct flow. The described developments represent a first use of NMFS for Stokes problems. The method requires the discretization on the boundary only and is easily applicable in 3D, thus representing an ideal candidate for solving complex biomedical engineering free and moving boundary flow problems in the future.

Key Words: *non-singular method of fundamental solutions, Stokes decomposition, blood flow.*

1. INTRODUCTION

The modeling of vascular systems with its contained blood represents a complex multiscale and multiphysics fluid-structure interaction problem. The blood rheology is depending on the scale considered. For example, the particulate flow has to be considered when taking into account the interactions of the blood cells on the micro level, and the non-Newtonian turbulent flow has to be considered on the macro level. For an exhaustive treatment of these topics as well as biomedical motivations we refer to [1,2,3] and the references therein. Due to the complexity of the spectra of the problems appearing in vascular systems, there is a substantial need to apply novel numerical methods to related problems. For this purpose, the method of fundamental solutions (MFS) [4] appears to be an ideal candidate, since it is a meshless boundary collocation technique, particularly suitable for tracking moving and free boundary problems. The method has similar coding complexity in 2 or 3D. The main drawback of the method represents the fact that it is straightforwardly suitable only for problems with known fundamental solution, and that when using singular fundamental solution, there is a need to include an artificial boundary, positioned outside the physical boundary in order to make the collocation possible. The optimal location of the artificial boundary is a delicate issue. In general, it can be observed that if the artificial boundary is too close to the physical one, then the accuracy of the problem is poor. On the other hand, if the artificial boundary is too far then the problem becomes ill-posed. Recent advances of the method for fluid [5], porous media flow with moving boundaries [6] and for solid mechanics [7], which involve regularization of the singularities, permit to omit the artificial boundary. We shall refer to the latter as the non-singular method of fundamental solutions (NMFS) and demonstrate its use for a class of vascular systems that reduce to a simple Stokes flow. One of the simplest related examples is that of a straight, uniform rigid duct with a steady rate of a laminar liquid flowing through it. In order to demonstrate the NMFS for such problems, we use the regularized Laplace fundamental solution, as suggested by Liu [8], combined with the decomposition of the 2D Stokes problem into three Laplacian problems [9]. First, we intend to describe briefly the underlying mathematical formulation of our method in a quite general framework. Second, we will show the applicability of our novel technique by considering the flow in a duct.

2. GOVERNING EQUATIONS

Let Ω be a connected two-dimensional domain with boundary Γ. We consider Cartesian coordinate system with base vectors $\mathbf{i}_x$ and $\mathbf{i}_y$ and coordinates x and y. The velocity field $\mathbf{q} = u\mathbf{i}_x + v\mathbf{i}_y$ is solution of the following Stokes equation (1.a) and satisfies the incompressibility condition (1.b)

$$\mu\Delta\mathbf{q}(x,y) = \nabla P(x,y) \text{ in } \Omega, \quad \nabla\cdot\mathbf{q} = 0 \quad \text{in } \Omega, \qquad (1.a, 1.b)$$

with μ representing the viscosity and P the pressure. Moreover, arguing as in [9] one can prove that (1.a) and (1.b) are equivalent to

$$\Delta f(x,y) = 0 \;\; \Delta g(x,y) = 0, \;\; \Delta P(x,y) = 0 \text{ in } \Omega, \;\; \partial_x u + \partial_y v = 0 \text{ in } \partial\Omega \qquad (2.a, 2.b, 2.c, 2.d)$$

provided the components u and v of the velocity vector $\mathbf{q}$ satisfy

$$\mu u(x,y) = f(x,y) + \frac{x}{2}P(x,y), \;\; \mu v(x,y) = g(x,y) + \frac{y}{2}P(x,y) \text{ in } \Omega. \qquad (3.a, 3.b)$$

In this paper, for simplicity, we assume that the two components (u,v) of the velocity field satisfy the Dirichlet conditions on the boundary $\partial\Omega$

$$u = \overline{u} \text{ in } \partial\Omega, \;\; v = \overline{v} \text{ in } \partial\Omega \qquad (4.a, 4.b)$$

where $\overline{u}$ and $\overline{v}$ are two given sufficiently smooth functions. Extension to other types of boundary conditions is straightforward.

2. SOLUTION PROCEDURE AND NUMERICAL EXAMPLE

The underlying idea of both the methods employed in the present paper, namely the MFS and NMFS, consists in representing the three harmonic functions f, g, and P, appearing in (2.a-c) as a linear combination of N global approximating functions with unknown coefficients, determined through collocation with the boundary conditions. We take the form ϕ of the approximation functions for MFS (fundamental solution of Laplace equation (5.a)) and $\tilde{\phi}$ for NMFS (desingularized fundamental solution (5.b))

$$\phi(\mathbf{p},\mathbf{s}) = \frac{1}{2\pi}\log(|\mathbf{p}-\mathbf{s}|^{-1}),\ \tilde{\phi}(\mathbf{p},\mathbf{s}) = \begin{cases} \phi(\mathbf{p},\mathbf{s}); & \mathbf{p} \neq \mathbf{s} \\ \dfrac{1}{\pi R^2}\displaystyle\int_{A(\mathbf{s},R)} \phi(\mathbf{p},\mathbf{s})\, dA = \dfrac{1}{2\pi}\log\left(\dfrac{1}{R}\right) + \dfrac{1}{4\pi}; & \mathbf{p} = \mathbf{s} \end{cases} \qquad (5.a, 5.b)$$

where $\mathbf{p}$ denote points on the physical boundary, and $\mathbf{s}$ denote the source points lying on the artificial boundary in case of MFS and on the physical boundary in case of NMFS. $A(\mathbf{s},R)$ represents a circular disk with radius R centered at $\mathbf{s}$. We consider N source points $\mathbf{s}_j$, $j=1,\dots,N$ and N collocation points $\mathbf{p}_i$, $i=1,\dots,N$ and we discretize the problem by representing f, g, and P as

$$f(\mathbf{p}_i) = \sum_{j=1}^{N} a_j\phi(\mathbf{p}_i,\mathbf{s}_j),\ g(\mathbf{p}_i) = \sum_{j=1}^{N} b_j\phi(\mathbf{p}_i,\mathbf{s}_j),\ P(\mathbf{p}_i) = \sum_{j=1}^{N} c_j\phi(\mathbf{p}_i,\mathbf{s}_j) \qquad (6.a, 6.b, 6.c)$$

where ϕ has to be replaced with $\tilde{\phi}$ when dealing with the NMFS method. We then plug the above choices into (3.a-b), (2.d) and (4.a, 4.b). Respectively, we reformulate our boundary value problem (1.a, 1.b, 4.a-b) to the solution of a linear system of $3N$ algebraic equations $\mathbf{Ax}=\mathbf{b}$ where the $3N\times 3N$ full matrix $\mathbf{A}$ has to be meant as a discretization of the inherent partial differential equation, the $3N\times 1$ vector $\mathbf{x}$ collects the unknown coefficients a_j, b_j, c_j in (6.a-c) and the $3N\times 1$ vector $\mathbf{b}$ contains the given information of the Dirichlet boundary condition (6.a, 6.b). We solve the Stokes problem (1.a) and (1.b) in a rectangular domain $\Omega = (-x_0, x_0)\times(-y_0, y_0);\ x_0 = 2,\ y_0 = 0.5$, for laminar flow between two plates. The following a.s. [9] is considered $u(x,y) = \dfrac{1}{2\mu}(y^2 - y_0^2)\partial_x P;\ \mu = 1, \partial_x P = 6$, $v(x,y) = 0$,

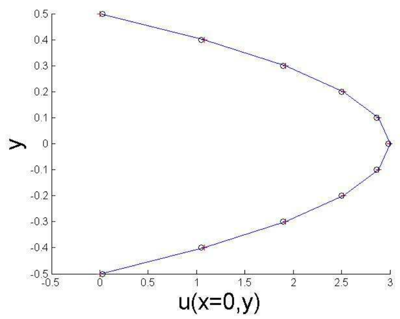

Fig.1a. The profile of *u* at *x=0* along *y* axis.

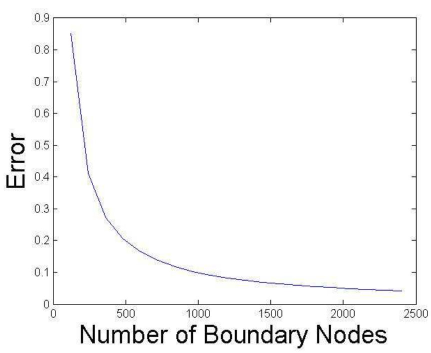

Fig.1b. Error as a function of the discretization.

where the Dirichlet boundary conditions in (4.a) and (4.b) are derived directly from the a.s. We solve the problem with MFS and NMFS. We use 1200 equidistant boundary nodes picking the source points at a distance 0.0417 from the physical boundary in MFS method. On the other hand the numerical solution computed with the NMFS has been obtained with the same number of boundary nodes and by choosing R in (6.b) equal to 0.0033. In Fig. 1.a the first component of velocity field along $x=0$ is presented (+:MFS, o:MMFS, -:a.s.). Moreover, in Fig. 1.b we show the error computed as the Euclidean norm of the difference between the a.s. and the MMFS solution, measured along the profile $x=0$ in 11 equidistant points on the interval $\left[-y_0, y_0\right]$.

3. CONCLUSIONS

The Laplace decomposition technique combined with NMFS is for the first time applied for solving a simple Stokes problem where both components of fluid velocity are specified on the boundary of the solution domain. The accuracy and efficiency of the new method is validated by considering a simple test example arising in steady flow problems. Moreover, we show that the NMFS solution converges to the a.s. with the increasing number of the nodes. The future extensions of the presented work will be focused on flow in axisymmetry and 3D, as well as free and moving boundary problems. The developed method, with its boundary only character of discretization, can potentially be used in effective simulation of a broad spectrum of involved biomedical problems. Indeed, even though we are aware that for a general computational haemodynamics problems the non-linear convective term of the Navier Stokes equation cannot be dropped out, we believe that the adopted linear model could find application in the study of small scale problems, such as the blood flow in capillars.

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