On Spinors of Zero Nullity

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Abstract

We present a necessary and sufficient condition for a spinor ω to be of nullity zero, i.e. such that for any null vector v, $v\omega \neq 0$. This dives deeply in the subtle relations between a spinor ω and ω_c, the (complex) conjugate of ω belonging to the *same* spinor space.

1 Introduction

101 years ago Élie Cartan [6, 7] introduced spinors that were later thoroughly investigated by Claude Chevalley [8] in the mathematical frame of Clifford algebra; in this work spinors were identified as elements of minimal left ideals of the algebra. The interplay between spinors and null (also: isotropic) vectors, pioneered by Cartan, and thus sometimes called the Cartan map, is central and have been visited many times since then, see e.g. [5, 9, 3] and references therein. This relation is pivotal to many fields of physics, the Weyl equation being just one prominent application.

Let the *nullity* $N(\omega)$ of spinor ω be the dimension of the subspace of null vectors that annihilate ω i.e. those vectors v such that $v\omega = 0$. Simple (also: pure) spinors are the spinors with maximum nullity. Nullity provides a coarse classification of spinors that have been studied in detail: see [11] and references therein. In this paper we investigate the properties of a family of spinors complementary to simple spinors: the spinors of zero nullity i.e. those spinors that are not annihilated by any null vector.

We will investigate these spinors in $\mathbb{C}^{2m}$ and $\mathbb{R}^{2m}$ with signature (m, m), a common choice in these studies [5, 11], exploiting the Extended Fock Basis (EFB) of Clifford algebra [1, 2], recalled in section 2. With this basis *any* element of the algebra can be expressed in terms of simple spinors: from scalars to vectors and multivectors. Section 3 present vector and spinor spaces of the algebra and reports some needed results [3]. Section 4 is dedicated to spinors and at the end brings the main result: a necessary and sufficient condition for a spinor to be of zero nullity; with respect to the previous study of this problem [11] that tackled Weyl spinors here the results hold for any spinor. A part from exceptional cases a spinor of zero nullity can be seen as the sum of a spinor of positive nullity with its (complex) conjugate. With this result it is easy to build a basis of spinor space made entirely of spinors of zero nullity.

For the convenience of the reader we tried to make this paper as elementary and self-contained as possible.

2 Clifford algebra and its 'Extended Fock Basis'

We start summarizing the essential properties of the EFB introduced in [1] and [2]. We consider Clifford algebras [8] over field $\mathbb{F}$, with an even number of generators $\gamma_1, \gamma_2, \ldots, \gamma_{2m}$, a vector space $\mathbb{F}^{2m} := V$ and a scalar product g: these are simple, central, algebras of dimension 2^{2m}. As usual

$$2g(\gamma_i, \gamma_j) = \gamma_i\gamma_j + \gamma_j\gamma_i := \{\gamma_i, \gamma_j\}$$

and we concentrate to $\mathbb{F} = \mathbb{C}$ or $\mathbb{F} = \mathbb{R}$ with signature $V = \mathbb{R}^{m,m}$; $g(\gamma_i, \gamma_j) = \delta_{ij}(-1)^{i+1}$ i.e.

$$\begin{cases} \gamma_{2i-1}^2 & = & 1 \\ \gamma_{2i}^2 & = & -1 \end{cases} \qquad i = 1, \ldots, m \quad . \tag{1}$$

Given the $\mathbb{R}^{m,m}$ signature we indicate the Clifford algebra with $\mathcal{C}\ell_{m,m}(g)$. The Witt, or null, basis of the vector space V is defined:

$$\begin{cases} p_i & = & \frac{1}{2}(\gamma_{2i-1} + \gamma_{2i}) \\ q_i & = & \frac{1}{2}(\gamma_{2i-1} - \gamma_{2i}) \end{cases} \Rightarrow \begin{cases} \gamma_{2i-1} & = & p_i + q_i \\ \gamma_{2i} & = & p_i - q_i \end{cases} \qquad i = 1, 2, \ldots, m \tag{2}$$

that, with $\gamma_i\gamma_j = -\gamma_j\gamma_i$, easily gives

$$\{p_i, p_j\} = \{q_i, q_j\} = 0 \qquad \{p_i, q_j\} = \delta_{ij} \tag{3}$$

showing that all p_i, q_i are mutually orthogonal, also to themselves, that implies $p_i^2 = q_i^2 = 0$, at the origin of the name "*null*" given to these vectors.

Following Chevalley we define spinors as elements of a minimal left ideal we will indicate with S. Simple spinors are those elements of S that are annihilated by a null subspace of V of maximal dimension.

The EFB of $\mathcal{C}\ell_{m,m}(g)$ is given by the 2^{2m} different sequences

$$\psi_1\psi_2 \cdots \psi_m := \Psi \qquad \psi_i \in \{q_i p_i, p_i q_i, p_i, q_i\} \qquad i = 1, \ldots, m$$

in which each ψ_i is either a vector or a bi–vector and we will reserve Ψ for EFB elements. The main characteristics of EFB is that all its elements are *simple* spinors [1, 2].

The EFB essentially extends to the entire algebra the Fock basis [5] of its spinor spaces and, making explicit the construction $\mathcal{C}\ell_{m,m}(g) \cong \overset{m}{\otimes}\mathcal{C}\ell_{1,1}(g)$, allows one to prove in $\mathcal{C}\ell_{1,1}(g)$ many properties of $\mathcal{C}\ell_{m,m}(g)$.

A classical results that we will need in what follows exploits the isomorphism (of vector spaces) $\mathcal{C}\ell_{m,m}(g) \cong \Lambda V$ with the Grassmann algebra and leads [8] to the following useful formula for the Clifford product $v\mu$ of any two elements $v \in V, \mu \in \mathcal{C}\ell_{m,m}(g)$

$$v\mu := v \lrcorner \mu + v \wedge \mu \tag{4}$$

where $v \lrcorner \mu$ represents the *contraction* of v with μ (if also $\mu \in V$ then $2v \lrcorner \mu = \{v, \mu\}$) and $v \wedge \mu$ is the *exterior* or *wedge product*.

3 Properties of vector V and spinor S spaces

With the Witt basis (2) it is easy to see that the null vectors $\{p_i\}$ can build vector subspaces made only of null vectors that we call Totally Null Planes (TNP, also: isotropic planes) of dimension at maximum m [7]. Moreover the vector space V is easily seen to be the direct sum of two of these maximal TNP P and Q respectively:

$$V = P \oplus Q \qquad \begin{cases} P & := & \mathrm{Span}\,(p_1, p_2, \ldots, p_m) \\ Q & := & \mathrm{Span}\,(q_1, q_2, \ldots, q_m) \end{cases}$$

since $P \cap Q = \{0\}$ each vector $v \in V$ may be expressed in the form $v = \sum_{i=1}^{m} (\alpha_i p_i + \beta_i q_i)$ with $\alpha_i, \beta_i \in \mathbb{F}$. Using (3) it is easy to derive the anticommutator of two generic vectors v and $u = \sum_{i=1}^{m} (\gamma_i p_i + \delta_i q_i)$

$$\{v, u\} = \sum_{i=1}^{m} \alpha_i \delta_i + \beta_i \gamma_i \quad \in \mathbb{F} \quad \Rightarrow \quad \frac{1}{2}\{v, v\} = v^2 = \sum_{i=1}^{m} \alpha_i \beta_i \ . \tag{5}$$

We define

$$V_0 = \{v \in V : v^2 = 0\} \qquad V_1 = \{v \in V : v^2 \neq 0\}$$

clearly $V = V_0 \cup V_1$ and $V_0 \cap V_1 = \emptyset$ but neither V_0 nor V_1 are subspaces of V which is simple to see. Nevertheless V_0 contains subspaces of dimension m, e.g. Q, and, similarly, $V_1 \cup \{0\}$ contains subspaces of dimension m, e.g. $\mathrm{Span}\,(\gamma_1, \ldots, \gamma_{2k-1}, \ldots, \gamma_{2m-1})$. In [3] it is proved that for any nonzero vector v and spinor ω

$$v\omega = 0 \quad \Longleftrightarrow \quad v \in V_0 \tag{6}$$

and thus, for all $v \in V_1$, $v\omega \neq 0$.

3.1 Conjugation in V

When $\mathbb{F} = \mathbb{C}$ complex conjugation in vector space V is given by

$$v = \sum_{i=1}^{m} \alpha_i p_i + \beta_i q_i \quad \Rightarrow \quad \overline{v} = \sum_{i=1}^{m} \overline{\beta}_i p_i + \overline{\alpha}_i q_i \tag{7}$$

that with (5) gives $\overline{v}^2 = \overline{v^2}$. For $\mathbb{F} = \mathbb{R}$, since $\overline{\alpha}_i = \alpha_i$, the conjugation is obtained by exchanging basis vectors p_i and q_i (or, identically, exchanging coefficients α_i and β_i) and in both cases conjugation defines an involutive automorphism on V since $\overline{\overline{v}} = v$;

For $\mathbb{F} = \mathbb{R}$ we can go further: by (5) $\overline{v}^2 = v^2$ and this conjugation is an isometry on V that lifts uniquely to an automorphism on the entire algebra and since our algebra is central simple all its automorphisms are inner. So there must exist an element C such that $\overline{v} = CvC^{-1}$.

To find its explicit form let $\Delta_\pm = (p_1 \pm q_1) \cdots (p_m \pm q_m)$ and with (2) it is easy to see that $\Delta_+ = \gamma_1 \cdots \gamma_{2k-1} \cdots \gamma_{2m-1}$ whereas Δ_- is the product of the even, spacelike, γ's. With (1) one easily finds $\Delta_\pm^2 = (-1)^{\frac{m(m\mp1)}{2}}$ and defining

$$C = \begin{cases} \Delta_+ \\ \Delta_- \end{cases} \qquad C^{-1} = \begin{cases} (-1)^{\frac{m(m-1)}{2}} \Delta_+ & \text{for } m \text{ odd} \\ (-1)^{\frac{m(m+1)}{2}} \Delta_- & \text{for } m \text{ even} \end{cases} \tag{8}$$

we can prove that $\overline{v} = CvC^{-1}$: it suffices to write v in the Witt basis and make the simple exercise of proving that $Cp_iC^{-1} = q_i$. One easily verifies

$$\overline{\overline{v}} = CCvC^{-1}C^{-1} = CC^{-1}vCC^{-1} = v \quad .$$

Returning to the case $\mathbb{F} = \mathbb{C}$, also in this case C is defined and $Cp_iC^{-1} = q_i$ so that, indicating with $v^\star$ the vector v with complex conjugate field coefficients, we can write (7) as

$$\overline{v} = Cv^\star C^{-1}$$

that holds also for $\mathbb{F} = \mathbb{R}$ since in this case $v^\star = v$ and thus from now on we will stick to this form for (complex) conjugation. It is an easy exercise to verify that this form generalizes to any element of the algebra ω giving

$$\overline{\omega} = C\omega^\star C^{-1}$$

and that, for both $\mathbb{F} = \mathbb{C}$ and $\mathbb{R}$,

$$v^2 = 0 \iff \overline{v}^2 = 0$$

and one can prove [3]:

Proposition 1. *Given nonzero vector v and $\omega \in S$ such that $v\omega = 0$ it follows $\overline{v}\omega \neq 0$, conversely $\overline{v}\omega = 0$ implies $v\omega \neq 0$.*

3.2 Some results for spinor space S

Given the spinor space S we can build its Fock basis Ψ_a where the index a takes 2^m values and can be thought expressed in binary form as a string of m "bits" taking values ± 1 that represent the $h-$signature of Ψ_a [5, 3]. The generic element of S is expressed by the simple spinor expansion:

$$\omega \in S \qquad \omega = \sum_a \xi_a \Psi_a \ . \tag{9}$$

For each nonzero spinor $\omega \in S$ we define its associated TNP as:

$$M(\omega) := \{v \in V : v\omega = 0\} \qquad \text{and} \qquad N(\omega) = \dim_{\mathbb{F}} M(\omega)$$

and the spinor is *simple* iff the TNP is of maximal dimension, i.e. iff $N(\omega) = m$. A standard result [4] says that given $u_1, u_2, \ldots, u_k \in V_0$ they form a TNP of dimension k with $0 < k \leq m$ if and only if

$$u_1 u_2 \cdots u_k = u_1 \wedge u_2 \wedge \cdots \wedge u_k \neq 0 \tag{10}$$

that implies also $\{u_i, u_j\} = 0 \ \ \forall i, j = 1, \ldots, k$ and thus that all vectors in $M(\omega)$ are mutually orthogonal and it's easy to see that $M(\omega)$ is a vector subspace of V contained in V_0.

There is also a result [3] complementary to that of proposition 1:

Proposition 2. *For any nonzero vector v and $\omega \in S$ such that $v\omega = 0$ it follows $v\overline{\omega} \neq 0$, conversely $v\overline{\omega} = 0$ implies $v\omega \neq 0$.*

We remark that given $\omega \in S$, in general $\overline{\omega} = C\omega^\star C^{-1}$ belongs to a different spinor space $SC \neq S$, see [3], [10]. Since S is a minimal left ideal one can define the "projection" of $\overline{\omega}$ in the same spinor space of ω as

$$\omega_c := C\omega^\star \tag{11}$$

and for any $\omega \in S$ with (9) it is simple to get [3]:

$$\omega_c = C\omega^\star = \sum_a \overline{\xi}_a C \Psi_a = \sum_a s(a) \overline{\xi}_a \Psi_{-a} \tag{12}$$

where $s(a) = \pm 1$ is a sign, quite tedious to calculate exactly [2] and Ψ_{-a} is the Fock basis element with $h-$signature opposite to that of Ψ_a. A significative difference with $\overline{\omega}$ is that while $\overline{\overline{\omega}} = \omega$, $(\omega_c)_c = C^2\omega = (-1)^{\frac{m(m-1)}{2}}\omega$. Previous result on $\overline{\omega}$ can be extended [3] to ω_c:

Proposition 3. *For any nonzero $v \in V_0$, given nonzero $\omega \in S$ such that $v\omega = 0$ it follows $v\omega_c \neq 0$, conversely $v\omega_c = 0$ implies $v\omega \neq 0$.*

A useful consequence of this result is:

$$M(\omega) \cap M(\omega_c) = \{0\} \ . \tag{13}$$

In [3] is proved the

Proposition 4. *Given $k \leq m$ nonzero $v_1, v_2, \ldots, v_k \in V_0$ forming a TNP of dimension k, any spinor that annihilates $v_1, v_2, \ldots, v_k$ may be written as*

$$\omega = u_1 u_2 \cdots u_k \Phi \tag{14}$$

for an appropriate choice of $\Phi \in S$ whereas the choice of the null vectors u_i is completely free provided they span the same TNP.

We are now ready to prove the technical

Lemma 1. *Given a nonzero spinor ω with $M(\omega) = Span\left(u_1, u_2, \ldots, u_{N(\omega)}\right)$, then given nonzero $v \in V_0$ such that $v\omega := \omega' \neq 0$ then $N(\omega') \geq N(\omega)$ and, more precisely*

$$\begin{array}{lcl} N(\omega') \geq N(\omega) + 1 & \Longleftrightarrow & \left\{ \begin{array}{l} N(\omega) = 0 \quad \text{or} \\ \{v, u_i\} = 0 \quad \forall i = 1, \ldots, N(\omega) \end{array} \right. \\ N(\omega') = N(\omega) & \Longleftrightarrow & \{v, u_i\} \neq 0 \quad \text{for at least one } i = 1, \ldots, N(\omega) \ . \end{array}$$

Proof. Spinors are member of a minimal left ideal and thus ω' is a spinor and $v \in M(\omega')$ and this is enough to prove the case $N(\omega) = 0$. By proposition 4 we may write $\omega = u_1 u_2 \cdots u_{N(\omega)} \Phi$ for an appropriate choice of Φ and since $v\omega \neq 0$ it follows $v u_1 u_2 \cdots u_{N(\omega)} \neq 0$ and with (10) $u_1 u_2 \cdots u_{N(\omega)} = u_1 \wedge u_2 \wedge \cdots \wedge u_{N(\omega)} \neq 0$ and with (4) we can write

$$v u_1 u_2 \cdots u_{N(\omega)} = v \lrcorner \left(u_1 \wedge u_2 \wedge \cdots \wedge u_{N(\omega)}\right) + v \wedge u_1 \wedge u_2 \wedge \cdots \wedge u_{N(\omega)} \neq 0$$

and at least one of the two terms must be nonzero. Necessarily $v \wedge u_1 \wedge u_2 \wedge \cdots \wedge u_{N(\omega)} \neq 0$ because otherwise $v \in \text{Span}\left(u_1, u_2, \ldots, u_{N(\omega)}\right)$ that would give $v u_1 u_2 \cdots u_{N(\omega)} = 0$ against hypothesis.

With the recurrence relation

$$v \lrcorner (u_1 \wedge u_2 \wedge \cdots \wedge u_{N(\omega)}) = \frac{1}{2} \{v, u_1\} \, u_2 \wedge \cdots \wedge u_{N(\omega)} - u_1 \wedge [v \lrcorner (u_2 \wedge \cdots \wedge u_{N(\omega)})]$$

the first term expands in a sum containing all $\{v, u_i\}$ and there are two possibilities; the first is $\{v, u_i\} = 0$ for all $i = 1, \ldots, N(\omega)$: this implies that

$$v u_1 u_2 \cdots u_{N(\omega)} = v \wedge u_1 \wedge u_2 \wedge \cdots \wedge u_{N(\omega)} \neq 0$$

and with (10) this is sufficient to get that $\text{Span}\left(v, u_1, u_2, \ldots, u_{N(\omega)}\right) \subseteq M(\omega')$ and $N(\omega') \geq N(\omega) + 1$. The second possibility is that $\{v, u_i\} \neq 0$ for some $i = 1, \ldots, N(\omega)$ and we can assume that there is only one vector u_i for which this holds (if this is not the case it is always possible to make a proper rotation in Span $\left(u_1, u_2, \ldots, u_{N(\omega)}\right)$ to get it). So without loss of generality we suppose $\{v, u_1\} \neq 0$, $\{v, u_i\} = 0$ for $i = 2, \ldots, N(\omega)$ and so

$$\begin{aligned} v\omega' &= v^2\omega = 0 \\ u_1\omega' &= u_1 v\omega = u_1 v u_1 u_2 \cdots u_{N(\omega)}\Phi = \{v, u_1\} u_1 u_2 \cdots u_{N(\omega)}\Phi = \{v, u_1\}\omega \neq 0 \\ u_i\omega' &= u_i v\omega = -v u_i \omega = 0 \qquad i = 2, \ldots, N(\omega) \end{aligned}$$

so that $M(\omega') = \text{Span}\left(v, u_2, \ldots, u_{N(\omega)}\right)$ and $N(\omega') \geq N(\omega)$. We conclude showing that $N(\omega') > N(\omega)$ is forbidden in this case; supposing the contrary $M(\omega')$ should contain, beyond v and $N(\omega)-1$ of the null vectors of $M(\omega)$, at least one null vector z that would give $zv\omega = 0$. But this new vector would necessarily be orthogonal to all previous vectors and, by the hypothesis on $N(\omega)$, would also give $z\omega \neq 0$ and thus also $z\{v, u_1\}\omega = zu_1v\omega \neq 0$. But $\{z, u_1\} = 0$ and one would get the contradiction $0 \neq zu_1v\omega = -u_1zv\omega = 0$.

Along this proof, using the expression $\omega' = v\omega = vu_1u_2\cdots u_{N(\omega)}\Phi$, we have seen that, in all cases, there are at least $N(\omega)$ null vectors in $M(\omega')$ thus we can conclude that, in full generality, $N(v\omega) \geq N(\omega)$. $\square$

In summary multiplication $v\omega$ either 'adds' v to $M(\omega)$ or 'removes' the vector with which v had a nonzero scalar product, neat examples are:

$$\begin{aligned} \omega' &= u_{k+1}\omega = (-1)^k u_1u_2\cdots u_k u_{k+1}\Phi \\ \omega' &= \overline{u}_j\omega = u_1u_2\cdots u_{j-1}\overline{u}_j u_{j+1}\cdots u_k\Phi' \end{aligned}$$

moreover, in the first case, it is easy to exhibit examples for which $N(v\omega) > N(\omega) + 1$.

4 Spinors of zero nullity

We start from the following observation: if $\mathbb{F} = \mathbb{R}$ and a spinor ω is such that

$$\omega_c = C\omega = \alpha\omega \qquad \alpha \in \mathbb{R} - \{0\} \tag{15}$$

then it is simple to see that for any nonzero $v \in V_0$, $v\omega \neq 0$: supposing the contrary would violate (13). This introduces the spinors of zero nullity for which $N(\omega) = 0$ i.e. $M(\omega) = \{0\}$.

Before characterizing them we observe that (15) implies $\alpha^2 = C^2 = (-1)^{\frac{m(m-1)}{2}}$ and thus

$$\alpha = \pm\begin{cases} 1 & \iff \quad m \equiv 0, 1 \pmod 4 \\ i & \iff \quad m \equiv 2, 3 \pmod 4 \end{cases}$$

and in the second case the problem $C\omega = \alpha\omega$ has solution only if $\mathbb{F} = \mathbb{C}$.[1]

[1] it is a simple exercise to show that $\alpha = \pm i$ also for real spaces of Lorentzian signature $\mathbb{R}^{2m-1,1}$

On the other hand if $\mathbb{F} = \mathbb{C}$ then ω_c can never be equal to $\alpha\omega$ since $(\cdot)_c : S \to S$ given by $\omega_c = C\omega^\star$ is $\mathbb{C}$-semilinear while $\alpha\omega$ is $\mathbb{C}$-linear and there can be equality in $\mathbb{C}$ only if $\omega = 0$. So, in the complex case, ω_c and ω are always linearly independent. We have thus proved:

Proposition 5. *Any nonzero spinor $\omega \in S$ is linearly independent from ω_c with the exception of $\mathbb{F} = \mathbb{R}$ and $m \equiv 0, 1 \pmod 4$ when there exist cases in which $\omega_c = C\omega = \pm\omega$.*

We continue showing that for all spinors $N(\omega) = N(\omega_c)$:

Proposition 6. *For any nonzero spinor $\omega \in S$, $N(\omega) = N(\omega_c)$ and if $M(\omega) = Span(v_1, v_2, \ldots, v_k)$ then $M(\omega_c) = Span(\overline{v}_1, \overline{v}_2, \ldots, \overline{v}_k)$.*

Proof. Let's suppose first $N(\omega) > 0$, for any $v \in M(\omega)$ one has

$$0 = v\omega = v^\star\omega^\star = v^\star C^{-1}C\omega^\star = Cv^\star C^{-1}C\omega^\star = \overline{v}C\omega^\star = \overline{v}\omega_c$$

that implies $N(\omega_c) \geq N(\omega)$. In turn from $v \in M(\omega_c)$ one has $(C^\star = C)$

$$0 = v\omega_c = vC\omega^\star = v^\star C\omega = C^{-1}v^\star C\omega = \overline{v}\omega$$

that implies $N(\omega) \geq N(\omega_c)$ and thus $N(\omega) = N(\omega_c)$. This argument proves also the part on the composition of TNP's $M(\omega)$ and $M(\omega_c)$.

It remains the case $N(\omega) = 0$: since now $v\omega \neq 0$ for any $v \in V_0$ it follows also $\overline{v}\omega = C^{-1}v^\star C\omega \neq 0$ and this relation can be multiplied by C, that, being a product of non null vectors, by (6), keeps the result different from zero, thus for any $v \in V_0$ also $v^\star C\omega \neq 0$ and $vC\omega^\star = v\omega_c \neq 0$ and thus $N(\omega_c) = 0$. $\square$

With this proposition applied to (14) we get, for any $0 \leq k \leq m$

$$\omega = u_1 u_2 \cdots u_k \Phi \quad \Longleftrightarrow \quad \omega_c = \overline{u}_1\overline{u}_2 \cdots \overline{u}_k C\Phi^\star := \overline{u}_1\overline{u}_2 \cdots \overline{u}_k \Phi_c \ . \quad (16)$$

This result together with (13) gives a first characterization of spinors of zero nullity since it is now simple to prove that

$$N(\omega) = 0 \quad \Longleftrightarrow \quad M(\omega) = M(\omega_c)$$

and clearly, for $\mathbb{F} = \mathbb{R}$, (15) implies $M(\omega) = M(\omega_c)$, not viceversa.

The spinors that are eigenvectors of C are the exception, rather than the rule, for spinors of nullity zero. In the general case spinors φ_c and φ are linearly independent and we will show that, under proper conditions, any linear combination of φ and φ_c is a spinor with nullity zero; for example $\omega = \alpha q_1 q_2 q_3 + \beta p_1 q_1 p_2 q_2 p_3 q_3$ has $N(\omega) = 0$ for any $\alpha\beta \neq 0$. To proceed we need some technical results holding for both $\mathbb{F} = \mathbb{R}$ and $\mathbb{F} = \mathbb{C}$:

Lemma 2. *For any nonzero spinor φ linearly independent from φ_c let*

$$\omega = \alpha\varphi + \beta\varphi_c \qquad \alpha, \beta \in \mathbb{F} - \{0\} \tag{17}$$

then $v \in V_0$ is such that $v\omega = 0$ if and only if

$$\alpha v \varphi = -\beta v \varphi_c \neq 0 \tag{18}$$

this in turn requires $0 \leq N(\varphi) \leq 2$. For $N(\varphi) > 0$ necessarily $m > 1$ and, defining $M(\varphi) = Span\left(u_1, u_2, \ldots, u_{N(\varphi)}\right)$, then $\{v, u_i\} \neq 0$ and $\{v, \overline{u}_j\} \neq 0$ for at least one i and one j; $i, j = 1, \ldots, N(\varphi)$.

Proof. Given the form of ω, by proposition 3, neither $v\varphi$ nor $v\varphi_c$ can be zero if one wants $v\omega = 0$ that thus can hold only if (18) holds.

To prove the bounds on $N(\varphi)$ we show that outside these bounds a necessary condition for (18) does not hold. Let's define spinors $\varphi' := \alpha v \varphi$ and $\varphi'' := -\beta v \varphi_c$ with which (18) reads $\varphi' = \varphi''$ that obviously implies

$$M(\varphi') = M(\varphi'') \Rightarrow N(\varphi') = N(\varphi'') \tag{19}$$

moreover $v \in M(\varphi')$.

If $N(\varphi) = 0$ we have seen that by lemma 1 that $N(\varphi'), N(\varphi'') \geq 1$ and if e.g. $M(\varphi') = M(\varphi'') = \mathrm{Span}\,(v)$ then (19) can be satisfied.

For $N(\varphi) > 0$ with lemma 1 there are four possibilities for $N(\varphi')$ and $N(\varphi'')$ but the two in which $N(\varphi') \neq N(\varphi'')$ are immediately ruled out. There remain either $N(\varphi') = N(\varphi'') \geq N(\varphi) + 1$ or $N(\varphi') = N(\varphi'') = N(\varphi)$. The condition $M(\varphi') = M(\varphi'')$ with proposition 6 rules out the first case since clearly $\mathrm{Span}\left(v, u_1, u_2, \ldots, u_{N(\varphi)}\right) \neq \mathrm{Span}\left(v, \overline{u}_1, \overline{u}_2, \ldots, \overline{u}_{N(\varphi)}\right)$ for any $N(\varphi) > 0$ so the only remaining possibility is to have $N(\varphi') = N(\varphi'') = N(\varphi)$ that implies, by quoted lemma, $\{v, u_i\} \neq 0$ and $\{v, \overline{u}_j\} \neq 0$ for at least one $i, j \in \{1, \ldots, N(\varphi)\}$.

We show with an example that if $N(\varphi) = 2$ a solution of (18) can't be excluded: let $\varphi = u_1 u_2 \Phi$, $\varphi_c = \overline{u}_1 \overline{u}_2 \Phi_c$ and $v = u_1 + \overline{u}_2$, clearly $v \in V_0$ and $M(\varphi') = M(\varphi'') = \mathrm{Span}\,(u_1, \overline{u}_2)$ and (19) could be satisfied.

Supposing $N(\varphi) > 2$ with lemma 1, since one can always reduce to the case in which $\{v, u_i\} \neq 0$ and $\{v, \overline{u}_j\} \neq 0$ for exactly one $i, j \in \{1, \ldots, N(\varphi)\}$, we would have that in $M(\varphi')$ necessarily remains at least one u_i that appears as $\overline{u}_i$ in $M(\varphi'')$ and thus (19) can never be realized with which we proved that necessarily $0 \leq N(\varphi) \leq 2$.

For $m = 1$ the maximum dimension of a TNP is 1 but to satisfy $N(\varphi') = N(\varphi'') = 1$ with lemma 1 one should have $\{v, u_1\} \neq 0$ and $\{v, \overline{u}_1\} \neq 0$ that would imply $v^2 \neq 0$ against initial hypothesis of $v \in V_0$ so for $N(\varphi) > 0$ we must necessarily have $m > 1$. $\square$

Corollary 7. *For any spinor φ with $N(\varphi) > 2$ then any $\omega \in Span\,(\varphi, \varphi_c)$ has $N(\omega) = 0$.*

Proof. We start remarking that $N(\varphi) > 2$ implies $m > 2$ and that φ is linearly independent from φ_c since, otherwise, $N(\varphi) = 0$. Supposing by absurdum that $N(\alpha\varphi+\beta\varphi_c) > 0$ by lemma 2 this would require $0 \leq N(\varphi) \leq 2$ against hypothesis. □

4.1 The subspace S_ω

We show that every $\omega \in S$ defines uniquely a 2-dimensional subspace $S_\omega \subseteq S$ that corresponds usually to Span (ω, ω_c). Given nonzero $\omega \in S$ let

$$S_\omega = \begin{cases} \text{Span}\,(\omega, \omega_c) & \iff \quad \omega \text{ and } \omega_c \text{ are linearly independent} \\ \text{Span}\,(\omega_+, \omega_-) & \iff \quad \omega_c = \pm\omega \text{ (see below)} \end{cases} \tag{20}$$

and in the first case it is fairly obvious that S_ω is a two dimensional subspace of S. In the second case necessarily $\mathbb{F} = \mathbb{R}$ and $m \equiv 0, 1 \pmod 4$; let e.g. $\omega_c = C\omega = \omega := \omega_+$, then with (12) one obtains that $(\omega_+)_c = \sum_a s(a)\xi_a\Psi_{-a}$ and to have $C\omega_+ = \omega_+$ one must have $\xi_a = s(-a)\xi_{-a}$. Choosing instead $\xi_a = -s(-a)\xi_{-a}$ one defines ω_- such that $C\omega_- = -\omega_-$ that thus always exists and that, by eigenvector properties, is linearly independent from ω_+ and they thus form, also in this case, the two dimensional subspace S_ω containing the initial spinor ω. An example for $\mathbb{F} = \mathbb{R}$ and $m = 1$ is:

$$\begin{cases} C(q + pq) = (p + q)(q + pq) = (q + pq) \\ C(q - pq) = -(q - pq) \end{cases}$$

and for any ω, $S_\omega = S$. A simple property of $S_\omega$is

Proposition 8. *Given nonzero ω and its S_ω, given any $\varphi \in S_\omega$ also $\varphi_c \in S_\omega$.*

Proof. For any $\varphi = \alpha\omega + \beta\omega_c$, $\alpha, \beta \in \mathbb{F}$, then $\varphi_c = C^2\overline{\beta}\omega + \overline{\alpha}\omega_c$; the other definition of S_ω is proved similarly. □

Proposition 9. *Given any nonzero ω and its S_ω, there always exist $\omega_0, \omega_{0c} \in S_\omega$ such that $N(\omega_0) = N(\omega_{0c}) > 0$.*

This proposition is proved in detail in the Appendix but one can get an intuition of this result from an interesting property of S_ω. The spinor ω is nonzero, so let us suppose that in its Fock basis expansion (9) appears the term $\xi_a\Psi_a$. Moving now to the spinor space S' of $g-$signature $-a$, then here Ψ_a is a primitive idempotent [2]. It is not difficult to see that in this spinor space the spinors $\omega, \omega_c, \omega_c C^{-1}(= \overline{\omega})$ and ωC^{-1} (the last two are in $S'C^{-1}$) form a sub algebra of $\mathcal{C}\ell(m, m)$ that is isomorphic to $\mathcal{C}\ell(1, 1)$. So it is always possible to "rotate" the minimal left ideal formed by ω, ω_c, combining them linearly, to build a Fock basis of $\mathcal{C}\ell(1, 1)$ made of two spinors of positive nullity.

We will call the spinors (ω_0, ω_{0c}) the Fock basis of S_ω; a useful consequence is:

Corollary 10. *Given nonzero ω and S_ω, any $\varphi \in S_\omega$ can be expressed $\varphi = \alpha\omega_0 + \beta\omega_{0c}$, $\alpha, \beta \in \mathbb{F}$, with $N(\omega_0) = N(\omega_{0c}) > 0$.*

For the next proposition, that brings the main result, we need a different form for the generic spinor $\omega \in S$ that exploits the properties of the Fock basis expansion (9). If $m \geq 2$ one can collect all terms with identical first two components of (9) and any ω may be written as

$$\omega = q_1 q_2 \Phi_{qq} + q_1 p_2 q_2 \Phi_{qp} + p_1 q_1 q_2 \Phi_{pq} + p_1 q_1 p_2 q_2 \Phi_{pp} \tag{21}$$

where the spinors Φ_{xy} belong to a spinor space S' of dimension 2^{m-2} and contain all the field coefficients ξ_a of (9). We remark the subtle difference with (16): whereas there $\Phi \in S$ and the relation works since S is a minimal left ideal, here $\Phi_{xy} \in S'$ and we are exploiting the properties of Fock basis expansion (9). The difference emerges when we calculate ω_c: writing from (8) $C = \left(p_1 + (-1)^{m-1} q_1\right) \cdots \left(p_m + (-1)^{m-1} q_m\right)$, we find from (11)

$$\omega_c \;=\; C q_1^\star q_2^\star \Phi_{qq}^\star + C q_1^\star p_2^\star q_2^\star \Phi_{qp}^\star + C p_1^\star q_1^\star q_2^\star \Phi_{pq}^\star + C p_1^\star q_1^\star p_2^\star q_2^\star \Phi_{pp}^\star$$

and we observe that $q_i^\star = q_i$ because they all have field coefficients 1 (all field coefficients that are not 1 are actually buried in Φ_{xy}) and defining $C' := \left(p_3 + (-1)^{m-1} q_3\right) \cdots \left(p_m + (-1)^{m-1} q_m\right)$ the conjugation operator of the spinor space S' we find (obviously $(-1)^{m-3} = (-1)^{m-1}$)

$$\begin{aligned}
\omega_c \;&=\; C q_1 q_2 \Phi_{qq}^\star + C q_1 p_2 q_2 \Phi_{qp}^\star + C p_1 q_1 q_2 \Phi_{pq}^\star + C p_1 q_1 p_2 q_2 \Phi_{pp}^\star = \\
&=\; -\left(p_1 + (-1)^{m-1} q_1\right) q_1 \left(p_2 + (-1)^{m-1} q_2\right) q_2 C' \Phi_{qq}^\star + \\
&\quad +(-1)^{m-1} \left(p_1 + (-1)^{m-1} q_1\right) q_1 \left(p_2 + (-1)^{m-1} q_2\right) p_2 q_2 C' \Phi_{qp}^\star + \\
&\quad +(-1)^{m-2} \left(p_1 + (-1)^{m-1} q_1\right) p_1 q_1 \left(p_2 + (-1)^{m-1} q_2\right) q_2 C' \Phi_{pq}^\star + \\
&\quad + \left(p_1 + (-1)^{m-1} q_1\right) p_1 q_1 \left(p_2 + (-1)^{m-1} q_2\right) p_2 q_2 C' \Phi_{pp}^\star = \\
&=\; q_1 q_2 \Phi_{pp_c} - q_1 p_2 q_2 \Phi_{pq_c} + p_1 q_1 q_2 \Phi_{qp_c} - p_1 q_1 p_2 q_2 \Phi_{qq_c} \;.
\end{aligned} \tag{22}$$

4.2 The case of $N(\omega_0) \leq 2$

Given ω and its S_ω we give now sufficient conditions for having spinors of nullity zero also in the case that the Fock basis of S_ω has $N(\omega_0) \leq 2$:

Proposition 11. *Given nonzero ω and its S_ω (20) with its Fock basis (ω_0, ω_{0c}) and $m > 2$, then for $\mathbb{F} = \mathbb{C}$ for any $\varphi = \alpha\omega_0 + \beta\omega_{0c}$, $\alpha, \beta \in \mathbb{F}$ and $\alpha\beta \neq 0$, then $N(\varphi) = 0$.*
For $\mathbb{F} = \mathbb{R}$ with $m \equiv 2, 3 \pmod 4$ and $N(\omega_0) \leq 2$ additional conditions on ω_0 are needed:

- *if $N(\omega_0) = 2$ that the $\Phi_{xy} \in S'$ of its expression (21) is such that $\Phi_{xy_c} \neq \pm\Phi_{xy}$;*

- *if $N(\omega_0) = 1$ that do not hold that: both of the $\Phi_{xy} \in S'$ of its expression (21) are such that $\Phi_{xy_c} = \pm\Phi_{xy}$ and $\alpha = \pm\beta$.*

Proof. If $N(\omega_0) = N(\omega_{0c}) > 2$ we already know, by corollary 7, that any $\varphi = \alpha\omega_0 + \beta\omega_{0c}$ with $\alpha\beta \neq 0$ has nullity zero; we prove now that this also holds for $N(\omega_0) = N(\omega_{0c}) = 1, 2$ for $\mathbb{F} = \mathbb{C}$ and for $\mathbb{F} = \mathbb{R}$ with additional conditions. First of all we note that since $N(\omega_0) > 0$, by lemma 2, $m \geq 2$.

Let's consider first $N(\omega_0) = N(\omega_{0c}) = 2$ and let $M(\omega_0) = \mathrm{Span}\,(u_1, u_2)$ for some $u_1, u_2 \in V_0$ and, since they form a TNP, necessarily, $\{u_1, u_2\} = 0$. To avoid unnecessary complications and a heavy notation throughout this proof we will assume, without loss of generality, that $\{u_i, \overline{u}_i\} = 1$ so that $u_1, u_2, \overline{u}_1, \overline{u}_2$ can be seen as 4 elements of a Witt basis of V; and this basis is then also used to build a Fock basis of S so that we will write, with (21) and (22), in full generality and renaming $u_i := q_i$ and $\overline{u}_i := p_i$

$$\omega_0 = q_1 q_2 \Phi_{qq} \qquad \omega_{0c} = p_1 q_1 p_2 q_2 \Phi_{qq_c} \ .$$

We proceed by absurdum supposing that there exists $v \in V_0$ such that $v(\alpha\omega_0 + \beta\omega_{0c}) = 0$. By necessary conditions of lemma 1 we must have $\{v, q_i\} \neq 0$ and $\{v, p_j\} \neq 0$ with $1 \leq i, j \leq 2$ and there are two possibilities: the first is $i = j$; in this case we may always write in full generality

$$v = q_i + \xi p_i + v' \qquad 1 \leq i, j \leq 2, \;\; \xi \in \mathbb{F}$$

with $\{v', q_i\} = \{v', p_i\} = 0$ and $v'^2 = -(q_i + \xi p_i)^2 = -\xi$ and, since we can always obtain that v has nonzero scalar product with just one q_i and one p_i we can conclude that also for the other coordinate $\{v', q_j\} = \{v', p_j\} = 0$. It is easy to see that in this case, supposing e.g. $i = 1$, $M(v\omega_0) = \mathrm{Span}\,(v, q_2)$ while $M(v\omega_{0c}) = \mathrm{Span}\,(v, p_2)$ that violates necessary conditions (19) and so in this case $v(\alpha\omega_0 + \beta\omega_{0c}) \neq 0$. The second possibility is that $i \neq j$ and let e.g. $\{v, q_2\} \neq 0$ and $\{v, p_1\} \neq 0$; it follows that we may write

$$v = q_1 + \xi p_2 + v' \qquad \xi \in \mathbb{F}$$

and again $\{v', q_1\} = \{v', p_1\} = \{v', q_2\} = \{v', p_2\} = 0$ and in this case $v'^2 = 0$; we get now

$$(q_1 + \xi p_2 + v')(\alpha\omega_0 + \beta\omega_{0c}) = \alpha\xi p_2\omega_0 + \alpha v'\omega_0 + \beta q_1\omega_{0c} + \beta v'\omega_{0c}$$

and since $v'\omega_0 \neq 0$ and $v'\omega_{0c} \neq 0$ by the hypothesis $N(\omega_0) = 2$ we must conclude that, to satisfy the relation, one must necessarily have $v' = 0$ because there are no other ways that the terms $\alpha v'\omega_0$ and $\beta v'\omega_{0c}$ can cancel out. So the relation reduces to $\alpha\xi p_2\omega_0 + \beta q_1\omega_{0c} = 0$ where both terms are again nonzero and it is easy to see that

$$\alpha\xi p_2\omega_0 + \beta q_1\omega_{0c} = q_1 p_2 q_2(\alpha\xi\Phi_{qq} - \beta\Phi_{qq_c}) = 0$$

and we observe that $q_1p_2q_2 \neq 0$ and the term in parenthesis is a spinor in S' that can't be zeroed by any of the null vectors that precedes it. So this expression can be zero only if $\Phi_{qq_c} = \frac{\alpha\xi}{\beta}\Phi_{qq}$ in S' spinor space.

We remark that if $m = 2$ this expression involves only field coefficients and can thus always be solved to zero; this shows that there are no spinors of zero nullity in this case, an anticipation of a more general result proved later.

If $m > 2$ then, by proposition 5, is impossible to satisfy this expression in $\mathbb{F} = \mathbb{C}$ and so we must conclude that for $N(\omega_0) = N(\omega_{0c}) = 2$ the nullity of all spinors $\alpha\omega_0 + \beta\omega_{0c}$ with $\alpha\beta \neq 0$ is zero. For $\mathbb{F} = \mathbb{R}$, by the same proposition, the part in parenthesis can have solution only for $m - 2 \equiv 0, 1 \pmod 4$ i.e. $m \equiv 2, 3 \pmod 4$ with the necessary condition $\alpha\xi = \pm\beta$ that shows that for any α, β the vector $v = q_1 \pm \frac{\beta}{\alpha}p_2$ annihilates $\alpha\omega_0 + \beta\omega_{0c}$. So in $\mathbb{F} = \mathbb{R}$, to have $N(\alpha\omega_0 + \beta\omega_{0c}) = 0$ we must add the additional condition that Φ_{qq} is linearly independent from Φ_{qq_c} (that is automatically satisfied if e.g. $N(\Phi_{qq}) > 0$ that happens, for example, when $N(\omega_0) > 2$).

We go now to the case $N(\omega_0) = N(\omega_{0c}) = 1$ and, by same hypothesis of previous case, we can assume $M(\omega_0) = q_1$ and we can write, with (21) and (22) and in full generality

$$\begin{aligned} \omega_0 &= q_1q_2\Phi_{qq} + q_1p_2q_2\Phi_{qp} \\ \omega_{0c} &= p_1q_1q_2\Phi_{qp_c} - p_1q_1p_2q_2\Phi_{qq_c} \ . \end{aligned}$$

We proceed again by absurdum supposing that there exists $v \in V_0$ such that $v(\alpha\omega_0 + \beta\omega_{0c}) = 0$. By necessary conditions of lemma 1 we must have $\{v, q_1\} \neq 0$ and $\{v, p_1\} \neq 0$ so that we may always write in full generality

$$v = q_1 + \xi p_1 + v' \qquad \xi \in \mathbb{F}$$

with $\{v', q_1\} = \{v', p_1\} = 0$ and since v is null we must have $v'^2 = -(q_1 + \xi p_1)^2 = -\xi$ so that

$$v\varphi = (q_1 + \xi p_1 + v')(\alpha\omega_0 + \beta\omega_{0c}) = \alpha(\xi p_1 + v')\omega_0 + \beta(q_1 + v')\omega_{0c}$$

and we observe that $(\xi p_1 + v')^2 = (q_1 + v')^2 = v'^2 = -\xi$ and thus, by (6), both terms in the equality are nonzero so that, to satisfy $v\varphi = 0$, one must have

$$\omega_{0c} = \frac{\alpha}{\beta\xi}(q_1 + v')(\xi p_1 + v')\omega_0 = \cdots = \frac{\alpha}{\beta}v'p_1\omega_0 \ .$$

We observe now that the only request made on v' is that it must be orthogonal to the subspace $\mathrm{Span}\,(q_1, p_1)$ so that it is always possible to make a proper rotation in V basis to obtain, without loss of generality, that

$$v' = q_2 - \xi p_2$$

with which at last the necessary condition becomes:

$$\begin{aligned} p_1q_1q_2\Phi_{qp_c} - p_1q_1p_2q_2\Phi_{qq_c} &= \frac{\alpha}{\beta}(q_2 - \xi p_2)p_1(q_1q_2\Phi_{qq} + q_1p_2q_2\Phi_{qp}) = \\ &= \frac{\alpha}{\beta}(p_1q_1q_2\Phi_{qp} - \xi p_1q_1p_2q_2\Phi_{qq}) \end{aligned}$$

that to be satisfied needs that the two equations are separately satisfied

$$\begin{aligned} p_1q_1q_2(\Phi_{qp_c} - \frac{\alpha}{\beta}\Phi_{qp}) = 0 \\ p_1q_1p_2q_2(\Phi_{qq_c} - \frac{\alpha}{\beta}\xi\Phi_{qq}) = 0 \end{aligned}$$

and again for $m > 2$ these equations cannot be satisfied in S' if $\mathbb{F} = \mathbb{C}$. If $\mathbb{F} = \mathbb{R}$ again they can be satisfied only for $m \equiv 2, 3 \pmod 4$ and in this case, if $\alpha = \pm\beta$ it is always possible to find v such that $v\varphi = 0$ and it is sufficient that either Φ_{qq} is linearly independent from Φ_{qq_c} or Φ_{qp} from Φ_{qp_c} or that $\alpha \neq \pm\beta$ to have $N(\alpha\omega_0 + \beta\omega_{0c}) = 0$ also in $\mathbb{F} = \mathbb{R}$. $\square$

4.3 The main result

We resume all previous results in the following characterization of spinors of zero nullity:

Theorem 1. *In $\mathcal{C}\ell(m, m)$ with $m \neq 2$ a nonzero spinor $\omega \in S$ has $N(\omega) = 0$ if and only if it can be written in the Fock basis (ω_0, ω_{0c}) of S_ω (20) as*

$$\omega = \alpha\omega_0 + \beta\omega_{0c} \quad \alpha, \beta \in \mathbb{F} - \{0\} \ .$$

For $\mathbb{F} = \mathbb{R}$ with $m \equiv 2, 3 \pmod 4$ and $N(\omega_0) \leq 2$ additional conditions on ω_0 are needed:

- *if $N(\omega_0) = 2$ that the $\Phi_{xy} \in S'$ of its expression (21) is such that $\Phi_{xy_c} \neq \pm\Phi_{xy}$;*
- *if $N(\omega_0) = 1$ that do not hold that: both of the $\Phi_{xy} \in S'$ of its expression (21) are such that $\Phi_{xy_c} = \pm\Phi_{xy}$ and $\alpha = \pm\beta$.*

The case $m = 2$ is exceptional since there are no spinors of zero nullity for both $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$.

Proof. Proposition 11 proves the forward part of the theorem for $m > 2$. We now suppose $N(\omega) = 0$: we can define S_ω with its Fock basis (ω_0, ω_{0c}) and obviously $\omega = \alpha\omega_0 + \beta\omega_{0c}$ with $\alpha\beta \neq 0$ because otherwise one would contradict the hypothesis $N(\omega) = 0$. In the particular case $\mathbb{F} = \mathbb{R}$ with $m \equiv 2, 3 \pmod 4$ and $N(\omega_0) \leq 2$ then at least one of the $\Phi_{xy} \in S'$ of its expression (21) is linearly independent from its conjugate Φ_{xy_c} because otherwise, as pointed out in the proof of proposition 11, there always exists a null vector that annihilates $\alpha\omega_0 + \beta\omega_{0c}$ that would contradict our initial hypothesis.

The case $m = 1$ cannot be derived by proposition 11 but it can be proved directly solving $v\omega = 0$ for the generic null vector and the generic spinor

$$v\omega = (\alpha p + \beta q)(\xi_1 q + \xi_2 pq) = \beta\xi_2 q + \alpha\xi_1 pq = 0 \qquad \alpha\beta = 0$$

that can be solved only if $\xi_1\xi_2 = 0$.

In the case $m = 2$ we already saw in the proof of proposition 11 that there are no spinors of 0 nullity but also in this case we can give a direct proof; we can write the generic spinor (9) as

$$\omega = \xi_1 q_1 q_2 + \xi_2 q_1 p_2 q_2 + \xi_3 p_1 q_1 q_2 + \xi_4 p_1 q_1 p_2 q_2$$

and it is a simple exercise to check that the vector[2]

$$v = \xi_3\xi_4 p_1 - \xi_1\xi_2 q_1 - \xi_2\xi_4 p_2 - \xi_1\xi_3 q_2$$

is null and such that $v\omega = 0$. $\square$

An interesting offspring of this result is that one can build a basis of spinor space(s) S made entirely of spinors of zero nullity since, trivially from (9) one can write

$$\begin{aligned}
\omega &= \sum_{a>0} \xi_a \Psi_a + \xi_{-a}\Psi_{-a} = \\
&= \sum_{a>0} \frac{\xi_a + \xi_{-a}}{2}(\Psi_a + \Psi_{-a}) + \frac{\xi_a - \xi_{-a}}{2}(\Psi_a - \Psi_{-a})
\end{aligned}$$

and for $m \neq 2$ the basis $\{\Psi_a + \Psi_{-a}, \Psi_a - \Psi_{-a} : a > 0\}$ is made entirely of spinors of zero nullity, each element being the sum of two simple spinors. Moreover any nonzero ω with $N(\omega_0) = N(\omega_{0c}) > 0$ can be written, not uniquely, as a linear combination of two zero nullity spinors taken from its S_ω.

These results show the complementary roles of ω and ω_c and that their span contains all spinors of zero nullity but for two "directions", those of the Fock basis of S_ω (apart from pathological cases when $\mathbb{F} = \mathbb{R}$). This situation is very similar to the spinor space S of $\mathcal{C}\ell(1,1)$ that has two directions, q and pq, of nullity 1 (by the way in this case these are also the simple spinors of S) while all other directions are of zero nullity.

[2] this is the solution when $\xi_i \neq 0$, $\forall i$, in other cases it takes slightly different forms.

Appendix

To prove proposition 9 we need some preliminary results.

Proposition 12. *Given any nonzero ω with $N(\omega) = 0$ and any $v \in V_0$ it is always possible to write ω as:*

$$\omega = v\Phi_v + \overline{v}\Phi_{\overline{v}} \tag{23}$$

where $\Phi_v, \Phi_{\overline{v}} \in S$ and are both nonzero.

Proof. Around any couple of null vectors $v, \overline{v}$ it is possible to build a Witt basis and a Fock basis of S and the written expansion follows immediately. Since $N(\omega) = 0$ clearly $v\omega \neq 0$ and $\overline{v}\omega \neq 0$ and if either of $\Phi_v, \Phi_{\overline{v}}$ would be zero this would contradict $N(\omega) = 0$. □

Proposition 13. *Given a maximal TNP $V_a \subset V_0$ and its corresponding simple spinor Ψ_a, i.e. such that $M(\Psi_a) = V_a$, then $\omega \in S$ is such that*

$$\omega = v\omega' \qquad \forall v \in V_a,\ \omega' \in S$$

if and only if $\omega = \xi_a \Psi_a$.

Proof. Since from any maximal TNP we can build a Witt basis of V naming its null vectors q_i, without loss of generality we suppose $V_a = Q = \mathrm{Span}\,(q_1, \ldots, q_m)$ and $\Psi_a = q_1 q_2 \cdots q_m$.

Supposing first $\omega = \xi_a \Psi_a$, for any $v = \sum_{i=1}^m \alpha_i q_i \in Q$ we have

$$\omega = \xi_a \Psi_a = \frac{\xi_a}{m} \left(\sum_{i=1}^m \alpha_i q_i \right) \sum_{i=1}^m \frac{s(i)}{\alpha_i} \Psi_{a(i)} \qquad \Psi_{a(i)} = q_1 q_2 \cdots p_i q_i \cdots q_m$$

where $s(i) = \pm 1$ and such that $s(i) q_i \Psi_{a(i)} = \Psi_a$ and we have supposed, for simplicity, that all $\alpha_i \neq 0$ (the formula can be easily adapted to other cases).

Viceversa let's suppose that $\omega = v\omega'$ for any $v \in Q$, it follows that for any $v \in Q$ one has $v\omega = 0$ that means that ω is a simple spinor and, by proposition 6 of [3], $\omega = \xi_a \Psi_a$ for some ξ_a. □

This result can be generalized from the case of a simple spinor Ψ_a to the case of a spinor that *contains* Ψ_a in its Fock basis expansion (9)

Corollary 14. *Given a maximal TNP $V_a \subset V_0$ and its corresponding simple spinor Ψ_a then $\omega \in S$ is such that*

$$\omega = v\omega' + \omega'' \qquad \forall v \in V_a,\ \ \omega', \omega'' \in S,\ \ \omega' \neq 0$$

if and only if $\omega = \xi_a \Psi_a + \omega'''$ for some $\omega''' \in S$.

Proof. Supposing $\omega = \xi_a \Psi_a + \omega'''$ previous proposition gives the result. Viceversa let $\omega = v\omega' + \omega''$ for any $v \in Q$ (as before we take $V_a = Q$); in this case we proceed by induction on the dimension m: for $m = 1$ the most general spinor takes the form $\omega = \xi_1 q + \xi_2 pq$ and the proof is simple. Let's now suppose the proposition true for $m-1$ and let's move to m: with self explanatory notation in this case the most general spinor has the form $\omega = q_1 \Phi_q + p_1 q_1 \Phi_p$ and any null vector of Q may be written as $v = \alpha q_1 + \beta q'$ where q' is a null vector of the $m-1$ dimensional maximal TNP Q'. By the induction hypothesis for any null vector $q' \in V'$ we can write $\Phi_q = q' \Phi'_q + \Phi''_q$ and the first term contains the simple spinor $\xi q_2 \cdots q_m$. It follows that our spinor of the case m can be written

$$\omega = q_1 \Phi_q + p_1 q_1 \Phi_p = q_1(q'\Phi'_q + \Phi''_q) + p_1 q_1 \Phi_p = (\alpha q_1 + \beta q')\frac{1}{\alpha} q' \Phi'_q + q_1 \Phi''_q + p_1 q_1 \Phi_p$$

and thus in the term $q_1 \Phi_q$ appears the simple spinor $\xi q_1 q_2 \cdots q_m$. □

Proposition 15. *Given any nonzero ω with $N(\omega) = 0$ for any $\xi_a \neq 0$ in its expansion (9) necessarily also $\xi_{-a} \neq 0$*

Proof. Given any $\xi_a \neq 0$ we write $\omega = \xi_a \Psi_a + \omega'$ and since, by proposition 12, for any null vector $v \in M(\Psi_a)$ we can write ω as in (23) where in $v\Phi_v$ certainly appears the term $\xi_a \Psi_a$ (and possibly other terms). By previous corollary applied to the term $\omega' = \overline{v}\Phi_{\overline{v}} + \omega''$ (ω'' can be zero) it must contain $\xi_{-a}\Psi_{-a}$. □

We are now ready to give the proof of proposition 9

Proof. If $N(\omega) > 0$ then $\omega_0 := \omega$ and we are done so let's suppose that $N(\omega) = 0$, in this case we can write with slightly modified (9) and (12)

$$\begin{aligned} \omega &= \textstyle\sum_{a>0} \xi_a \Psi_a + \xi_{-a}\Psi_{-a} \\ \omega_c &= \textstyle\sum_{a>0} s(-a)\overline{\xi}_{-a}\Psi_a + s(a)\overline{\xi}_a \Psi_{-a} \end{aligned}$$

and let $\xi_b \neq 0$; by previous proposition necessarily also $\xi_{-b} \neq 0$ so that choosing $\omega_0 := s(b)\overline{\xi}_b \omega - \xi_{-b}\omega_c$ we get:

$$\omega_0 = \left(s(b)\xi_b \overline{\xi}_b - s(-b)\xi_{-b}\overline{\xi}_{-b}\right)\Psi_b + \sum_{a>0, a\neq b} \cdots$$

where the field coefficient of Ψ_{-b} is 0. If $\left(s(b)\xi_b\overline{\xi}_b - s(-b)\overline{\xi}_{-b}\xi_{-b}\right) \neq 0$, this violates the necessary condition for a spinor to be of zero nullity and thus $N(\omega_0) > 0$. If $\left(s(b)\xi_b\overline{\xi}_b - s(-b)\overline{\xi}_{-b}\xi_{-b}\right) = 0$ one can repeat the procedure starting from the newly defined ω_0 and ω_{0c} that must be nonzero because otherwise the initial spinors ω and ω_c wouldn't be linearly independent. This linear independence guarantees also that this iterative procedure must terminate with the zeroing of just one term because, otherwise, again, the initial spinors would be linearly dependent. □

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