# $m$-ary Balanced Codes with Parallel Decoding

**Danilo Pelusi**

Dip. di Scienze della Comunicazione, Università degli Studi di Teramo, Coste San Agostino, 64100 Teramo. Italy. E-mail: `dpelusi@unite.it`.

**Samir Elmougy**

Dept. of Computer Science, King Saud University, Riyadh 1153 - Saudi Arabia. E-mail: `mougy@ksu.edu.sa`.

**Luca G. Tallini**

Dip. di Scienze della Comunicazione, Università degli Studi di Teramo, Coste San Agostino, 64100 Teramo. Italy. E-mail: `ltallini@unite.it`.

**Bella Bose**

School of EECS, Oregon State University, Corvallis, OR 97331. USA. E-mail: `bose@eecs.orst.edu`.

**Abstract**

An $m$-ary block code, $m = 2, 3, 4, \ldots$, of length $n \in \mathbb{N}$ is called balanced if, and only if, every codeword is balanced; that is, the real sum of the codeword components, or weight, is equal to $\lfloor (m-1)n/2 \rfloor$. This paper presents efficient encoding schemes to $m$-ary balanced codes with parallel (hence, fast) decoding. In fact, the decoding time complexity is $O(1)$ digit operations. These schemes are a generalization to the $m$-ary alphabet of Knuth's complementation method with parallel decoding. Let $\binom{n}{w}_m$ indicate the number of $m$-ary words of length $n$ and weight $w \in \{0, 1, \ldots, (m-1)n\}$. For any $m \in \mathbb{N}$, $m \geq 2$, a simple implementation of the method is given which uses $r \in \mathbb{N}$ check digits to balance $k \leq \left\{ \binom{r}{\lfloor (m-1)r/2 \rfloor}_m - \{m \bmod 2 + [(m-1)k] \bmod 2\} \right\} / (m-1)$ information digits with an encoding time complexity of $O(mk \log_m k)$ digit operations. A refined implementation of the parallel decoding method is also given with $r$ check digits and $k \leq (m^r - 1)/(m-1)$ information digits, where the encoding time complexity is $O(k\sqrt{\log_m k})$. Thus, the proposed codes are less redundant than the $m$-ary balanced codes with parallel decoding found in the literature and yet maintain the same complexity.

**Index Terms**

Balanced codes, $m$-ary alphabet, Knuth's complementation method, parallel decoding scheme, unidirectional error detection, optical and magnetic recording.

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## I. **Introduction**

Let $\mathbb{Z}_m = \{0, 1, \ldots, m-1\}$ indicate the $m$-ary alphabet, $m \geq 2$. Given $n \in \mathbb{N}$, the word $X = x_1 x_2 \ldots x_n$ over $\mathbb{Z}_m$ of length $n$ is called $m$-ary balanced (or briefly, balanced) if, and only if, the weight of $X$, $w(X) = \sum_{i=1}^{n} x_i = \lfloor (m-1)n/2 \rfloor$ (or, equivalently $\lceil (m-1)n/2 \rceil$), where the sum is over the real field. For example, when $n = 8$ and $m = 3$, the word $X = 11210102 \in \mathbb{Z}_3^8$ is balanced. An $m$-ary balanced code is a block code of length $n$ such that each codeword is balanced. The $m$-ary balanced codes can be used to detect unidirectional errors, to design error control codes in general, to reject the low frequencies in digital communication systems, and so on [10], [4], [8], [12]. The code design problem is to convert the information words into balanced words using minimum possible redundancy. This minimum redundancy is $r_{min}(m, k) \simeq (1/2)\log_m[(m-1)k] + (1/2)\log_m[(m+1)\pi/6]$ check digits for a $k$ digit information word over the $m$-ary alphabet [10]. Also, the conversion should be done so that the encoding and decoding processes are computationally as simple as possible. For the first time Knuth gave an efficient method to solve this problem for the binary case [6]. Given a $k \in \mathbb{N}$ bit information word $X \in \mathbb{Z}_2^k$, Knuth's idea is to complement some first $h \in \mathbb{N}$ bits of $X$ until a word $X^{(h)}$ of a certain weight is reached. Then, an $r \in \mathbb{N}$ bit check symbol $C = C(X) \in \mathbb{Z}_2^r$ is appended to obtain the $n = k + r$ bit codeword $\mathcal{E}(X) = X^{(h)} C(X) \in \mathbb{Z}_2^n$ as encoding of $X$. The check $C$ is chosen so that 1) the codeword $\mathcal{E}(X)$ is balanced (that is, $w(\mathcal{E}(X)) = w\left(X^{(h)}\right) + w(C) = \lfloor n/2 \rfloor$), and 2) the original information word $X$ can be recovered from $X^{(h)}$ and $C$. This Knuth's complementation method works because the "random walk" sequence $\left\{ w\left(X^{(h)}\right) : h = 0, 1, \ldots, k \right\}$ always meets any natural number $w \in [\min\{w(X), k - w(X)\}, \max\{w(X), k - w(X)\}]$. In particular, there always exists at least one index $h_b \in [0, (k-1) + k \bmod 2]$ such that $w\left(X^{(h_b)}\right) = \lfloor k/2 \rfloor$. Such indices $h_b = h_b(X)$ are sometimes referred to as the balancing indices of $X$ [11]. Many researchers have given various efficient implementations of this complementation method for both binary and $m$-ary cases [2], [1], [9], [10], [11], [13], [8], [5]. In the parallel decoding implementation of Knuth's complementation method, the check symbol $C$ directly indicates the number $h_b$ of bits of $X$ complemented. In other words, among all possible $k + k \bmod 2$ different functions $\langle C_h \rangle (X) \stackrel{\text{def}}{=} X^{(h)}$, $h \in [0, (k-1) + k \bmod 2]$, used in the code design, the check $C$ encodes the function that is actually used to encode $X$. Such functions $\langle C_h \rangle$'s are sometimes referred to as the balancing functions of the code design [2], [11]. Hence, decoding can be done very fast in parallel once $h_b$ is recovered (say, with a table look-up of size $O(k \log k)$ memory bits) from $C = C_{h_b(X)}$. This implies that the decoding time complexity is $O(1)$ bit operations.

In [8], a generalization to the $m$-ary case of Knuth's complementation method with parallel decoding is given. Here, two balanced code design methods for symbols over $\mathbb{Z}_m$ with parallel decoding are described. In the first method (the simple scheme of Section II), the checks are also balanced (as in [8]) whereas in the second (the refined scheme of Section III) this restriction is not needed, resulting in much less redundant codes. Let the $m$-nomial coefficients be defined as

$$\binom{k}{w}_m \stackrel{\text{def}}{=} \left| \left\{ X \in \mathbb{Z}_m^k : w(X) = w \right\} \right| = \sum_{x \in \mathbb{Z}_m} \binom{k-1}{w-x}_m ,$$

for all $k \in \mathbb{N}$ and $w \in [0, (m-1)k]$. Using $r$ check digits, the first parallel decoding scheme can balance information words with length (in the following integer expressions, we let $x \bmod 2$ indicate the integer equal to $0$ if the integer $x$ is even and $1$ if $x$ is odd)

$$k \leq \frac{1}{(m-1)} \left\{ \binom{r}{\lfloor (m-1)r/2 \rfloor}_m - \{m \bmod 2 + [(m-1)k] \bmod 2\} \right\}. \tag{1}$$

Note that, with the same $r$ check digits, the parallel decoding balanced codes given in [8] can only have

$$k \leq \left\{ \binom{r}{\lfloor (m-1)r/2 \rfloor}_m \right\} / m.$$

With $r \in \mathbb{N}$ check digits, the proposed second scheme improves the redundancy of the simple schemes as it can balance

$$k \leq \frac{m^r - 1}{m-1} \tag{2}$$

information digits. With regard to the complexity, there may be many ways to implement the coding system which may depend on the applications. However, assuming to have a table look-up of size $O(mk \log_m k)$ memory $m$-ary digits, all the above balanced codes can be implemented easily in $O(mk \log_m k)$ $m$-ary digit operations to encode and $O(1)$ $m$-ary digit operations to decode. For the simple scheme, Weber and Immink [13] and Swart and Weber [8] proposed to transmit extra auxiliary data by exploiting the degree of freedom of selecting from more than one possible balanced encoding of a given information word. Section IV shows some experimental results which indicate that some extra $\delta k = (1/2) \log_m k + \Theta(\log \log k)$ information digits can be balanced with this technique applied to the codes proposed here, for all $m \in \mathbb{N}$, $m \geq 2$.

The proposed codes are designed based on the generalized complementation scheme, referred as "$m$-ary complementation in stages" [10]. Given the integer $m \geq 2$ and $k, r, n \in \mathbb{N}$, in the following we let

$$\begin{aligned} K &\stackrel{\text{def}}{=} (m-1)k, \\ R &\stackrel{\text{def}}{=} (m-1)r, \\ N &\stackrel{\text{def}}{=} (m-1)n = K + R. \end{aligned}$$

## II. The simple scheme

As mentioned earlier, in this scheme, the checks are also balanced words as in [8]. However, with the same number of check digits the codes in this section can balance $k/(m-1)$ more extra information digits with respect to the number, $k$, of information digits of the $m$-ary balanced codes in [8]. Let the radix of the code be $m \geq 2$ and $r \in \mathbb{N}$ be the number of check digits. Let $\mathcal{CS} \stackrel{\text{def}}{=} \{C_0, C_1, \ldots, C_{p-1}\}$, $p \in \mathbb{N}$, be the lexicographic ordered set of the first $r$ digit balanced words of weight $\lfloor R/2 \rfloor + (K \bmod 2) \cdot (R \bmod 2)$. For example, if $m = 3$ and $r = 3$ then the weight is $2 \cdot 3/2 = 3$ and there are $\binom{3}{3}_3 = 7$ balanced words. These words in lexicographic order with their indices are $012$ - $0$, $021$ - $1$, $102$ - $2$, $111$ - $3$, $120$ - $4$, $201$ - $5$ and $210$ - $6$. If these words are used as the checks of the proposed balanced code then the index of the balanced check word directly indicates the number of steps used to complement the information word. Thus, every information word $X \in \mathbb{Z}_m^k$ of length $k \in \mathbb{N}$ information digits is encoded as

$\mathcal{E}(X) = \langle C_{h_b}\rangle(X)C_{h_b}$, where $h_b = h_b(X)$ is an index such that $w(\langle C_{h_b}\rangle(X)) = \lfloor K/2 \rfloor$. Note that the codeword $\mathcal{E}(X)$ is an $m$-ary word of length $n = k + r$ and weight

$$w(\mathcal{E}(X)) = w(\langle C_{h_b}\rangle(X)) + w(C_{h_b}) =$$
$$\left\lfloor \frac{K}{2} \right\rfloor + \left\lfloor \frac{R}{2} \right\rfloor + (K \bmod 2) \cdot (R \bmod 2) = \left\lfloor \frac{N}{2} \right\rfloor.$$

On receiving $YC_h \in \mathbb{Z}_m^n$, the decoder simply computes $\mathcal{D}(YC_h) = \mathcal{E}^{-1}(YC_h) = \langle C_h\rangle^{-1}(Y)$. A lookup table (of size $O(p)$) or enumerative encoding [3] method can be used to encode and decode the balancing index $h_b \in [0, p-1]$ in and from the check symbol $C_{h_b} \in \mathcal{CS}$ respectively.

Now we consider a suitable $m$-ary generalization of the complementation method. The complement of a digit $x \in \mathbb{Z}_m$ is $\overline{x} \stackrel{\text{def}}{=} [(m-1) - x] \in \mathbb{Z}_m$. Thus, if $X \in \mathbb{Z}_m^k$ then $w(X) = k(m-1) - w(X) = K - w(X)$. In the following, we develop a general $m$-ary complementation scheme so that the weight of the information word can reach every number in the range $[w(X), K - w(X)]$. In this way, at some point the weight of the word after certain number of complementation steps is guaranteed to reach the value of $\lfloor K/2 \rfloor$.

A digit is complemented in $l \in \mathbb{N}$ stages using a function

$$f : \mathbb{Z}_m \times [0, l] \rightarrow \mathbb{Z}_m \tag{3}$$

as in [10]. However, in this case the function $f$ must satisfy the following three properties to be a good/correct $m$-ary complementation function.

**Complementation property**: for all $x \in \mathbb{Z}_m$, $f(x, 0) = x$ and $f(x, l) = \overline{x} \in \mathbb{Z}_m$; that is, at the end of the last stage $l$ the digit is complemented; (4)

**Connectedness property**: for all $x \in \mathbb{Z}_m$ and $y \in [\min\{x, \overline{x}\}, \max\{x, \overline{x}\}]$ there exists $j \in [0, l]$ such that $f(x, j) = y$; that is, when complementing the digit $x$ in $l$ stages we should get all the integers in the range $[\min\{x, \overline{x}\}, \max\{x, \overline{x}\}]$; and, (5)

**Invertibility property**: for all $j \in [0, l]$ and $x_1, x_2 \in \mathbb{Z}_m$, $x_1 \neq x_2 \Longrightarrow f(x_1, j) \neq f(x_2, j)$; that is, $(f(0, j), f(1, j), \ldots, f(m-1, j))$ is a permutation of $(0, 1, \ldots, m-1)$ (this property is needed because if $y_j = f(x_1, h) = f(x_2, h)$ for $x_1 \neq x_2$, $j \in [1, k]$ and some $h \in [0, l]$, then, while decoding, it may not be clear whether to decode the digit $y_j$ to $x_i$ or $x_j$). (6)

For example, when $m = 3$ and $l = 3$ ($= (m-1) + m \bmod 2$) the function $f : \mathbb{Z}_3 \times [0, 3] \rightarrow \mathbb{Z}_3 = \{0, 1, 2\}$ defined as

$$\begin{array}{llll} f(0,0) = 0, & f(0,1) = 1, & f(0,2) = 2, & f(0,3) = 2, \\ f(1,0) = 1, & f(1,1) = 2, & f(1,2) = 0, & f(1,3) = 1, \\ f(2,0) = 2, & f(2,1) = 0, & f(2,2) = 1, & f(2,3) = 0 \end{array} \tag{7}$$

satisfies the properties (4), (5) and (6). If instead $m = 4$ and $l = 3$ ($= (m-1) + m \bmod 2$) the function $f : \mathbb{Z}_4 \times [0, 3] \rightarrow \mathbb{Z}_4$ defined as

$$\begin{array}{llll} f(0,0) = 0, & f(0,1) = 1, & f(0,2) = 2, & f(0,3) = 3, \\ f(1,0) = 1, & f(1,1) = 0, & f(1,2) = 3, & f(1,3) = 2, \\ f(2,0) = 2, & f(2,1) = 3, & f(2,2) = 0, & f(2,3) = 1, \\ f(3,0) = 3, & f(3,1) = 2, & f(3,2) = 1, & f(3,3) = 0 \end{array} \tag{8}$$

satisfies the properties (4), (5) and (6). For notational convenience, let us also represent the $m$-ary complementation function $f$ with the $m\times(l+1)$ matrix $f=(f(x,j):\ x=0,1,\dots,m-1,\ j=0,1,\dots,l)$. For example, the 3-ary complementation function (7) is also represented by

$$f=\begin{pmatrix}0&1&2&2\\1&2&0&1\\2&0&1&0\end{pmatrix}. \tag{9}$$

Note that the above matrix is the operation table of the group $(\mathbb{Z}_3,+\bmod 3)$ with the addition of the last column so that to assure that the property (4) is satisfied. Whereas, the 4-ary complementation function (8) is also represented by

$$f=\begin{pmatrix}0&1&2&3\\1&0&3&2\\2&3&0&1\\3&2&1&0\end{pmatrix}. \tag{10}$$

This matrix is the operation table of the Klein group $(\mathbb{Z}_2\times\mathbb{Z}_2,+\bmod 2)$, where $0=00$, $1=01$, $2=10$ and $3=11$.

Now, a suitable definition of the random walk for this simple $m$-ary scheme is defined. Given $k\in\mathbb{N}$ and $X=x_1x_2\dots x_k\in\mathbb{Z}_m^k$, let $X^{(0,l-m \bmod 2)}\stackrel{\text{def}}{=}X^{(f;0,l-m \bmod 2)}\stackrel{\text{def}}{=}X$, and

$$X^{(i,j)}\stackrel{\text{def}}{=}X^{(f;i,j)}\stackrel{\text{def}}{=}\overline{x_1x_2\dots x_{i-1}}f(x_i,j)x_{i+1}\dots x_{k-1}x_k, \tag{11}$$

for all $i\in[1,k+m \bmod 2]$ and $j\in[1,l-m \bmod 2]$ ($X^{(i,j)}$ is the word obtained when the first $i-1$ digit of $X$ are complemented and the $i$-th digit is at the $j$-th stage of complementation). The property (6) of $f$ implies that if $X^{(i,j)}=Y=y_1y_2\dots y_k\in\mathbb{Z}_m^k$, with

$$(i,j)\in\{(0,l-m \bmod 2)\}\cup\{[1,k]\times[1,l-m \bmod 2]\}\cup\{(k+m \bmod 2,1)\}$$

then

$$Y^{)i,j(}\stackrel{\text{def}}{=}\overline{y_1y_2\dots y_{i-1}}f^{-1}(y_i,j)y_{i+1}\dots y_{k-1}y_k=X. \tag{12}$$

In other words, the inverse function of $c_{i,j}(X)\stackrel{\text{def}}{=}X^{(i,j)}$ is exactly $c_{i,j}^{-1}(X)\stackrel{\text{def}}{=}Y^{)i,j(}$, for all $(i,j)\in\{(0,l-m \bmod 2)\}\cup\{[1,k]\times[1,l-m \bmod 2]\}\cup\{(k+m \bmod 2,1)\}$.

At this point, the random walk sequence is defined as follows. For all

$$h\in[0,(l-m \bmod 2)k+m \bmod 2]$$

define

$$X^{(h)}\stackrel{\text{def}}{=}X^{(f;h)}\stackrel{\text{def}}{=}X^{(f;i(h),j(h))}; \tag{13}$$

where,

$$\begin{cases} i(h)\stackrel{\text{def}}{=}\lceil h/(l-m \bmod 2)\rceil\in[1,k+m \bmod 2],\\ j(h)\stackrel{\text{def}}{=}(h-1) \bmod (l-m \bmod 2)+1\in[1,l-m \bmod 2].\end{cases} \tag{14}$$

Note that the above two component function $(i(h),j(h))$ from the integer interval $[0,(l-m \bmod 2)k+m \bmod 2]$ to $\{(0,l-m \bmod 2)\}\cup\{[1,k]\times[1,l-m \bmod 2]\}\cup\{(k+m \bmod 2,1)\}$ is a bijection with inverse

$$h(i,j)\stackrel{\text{def}}{=}(i-1)(l-m \bmod 2)+j\in[0,(l-m \bmod 2)k+m \bmod 2]. \tag{15}$$

For this simple scheme the random walk to be considered is $\left\{w\left(X^{(h)}\right) : h = 0, 1, \ldots, p-1\right\}$, with $p \stackrel{\text{def}}{=} (l - m \bmod 2)k + m \bmod 2 + 1$, where the balancing functions are the $p$ bijective functions defined as $\langle C_h \rangle(X) \stackrel{\text{def}}{=} X^{(h)}$, for all $h \in [0, p-1]$. Note that, depending on whether $m$ is odd or even the random walk sequence definition differs.

For example, if $m = 3$ (odd), $l = 3$, $f : \mathbb{Z}_3 \times [0,3] \to \mathbb{Z}_3$ is defined as in (7), $k = 9$ and $X = 201001210 \in \mathbb{Z}_3^9$ then $K = (m-1)k = 18$, $\lfloor K/2 \rfloor = 9$, $p = 2k + 1 + 1 = 20$, $h \in [0, 19]$ and the random walk sequence is

$$\begin{aligned}
X = X^{(0)} = X^{(0,2)} &= 201001201, & w(X^{(0)}) &= 7,\\
X^{(1)} = X^{(1,1)} &= \underline{0}01001201, & w(X^{(1)}) &= 5,\\
X^{(2)} = X^{(1,2)} &= \underline{1}01001201, & w(X^{(2)}) &= 6,\\
X^{(3)} = X^{(2,1)} &= \underline{01}1001201, & w(X^{(3)}) &= 6,\\
X^{(4)} = X^{(2,2)} &= 0\underline{2}1001201, & w(X^{(4)}) &= 7,\\
X^{(5)} = X^{(3,1)} &= 0\underline{22}001201, & w(X^{(5)}) &= 8,\\
X^{(6)} = X^{(3,2)} &= 02\underline{0}001201, & w(X^{(6)}) &= 6,\\
X^{(7)} = X^{(4,1)} &= 02\underline{11}01201, & w(X^{(7)}) &= 8,\\
X^{(8)} = X^{(4,2)} &= 021\underline{2}01201, & w(X^{(8)}) &= 9, \leftarrow\\
X^{(9)} = X^{(5,1)} &= 021\underline{21}1201, & w(X^{(9)}) &= 10,\\
X^{(10)} = X^{(5,2)} &= 0212\underline{2}1201, & w(X^{(10)}) &= 11,\\
X^{(11)} = X^{(6,1)} &= 0212\underline{22}201, & w(X^{(11)}) &= 12,\\
X^{(12)} = X^{(6,2)} &= 02122\underline{0}201, & w(X^{(12)}) &= 10,\\
X^{(13)} = X^{(7,1)} &= 02122\underline{10}01, & w(X^{(13)}) &= 9, \leftarrow\\
X^{(14)} = X^{(7,2)} &= 021221\underline{1}01, & w(X^{(14)}) &= 10,\\
X^{(15)} = X^{(8,1)} &= 021221\underline{01}1, & w(X^{(15)}) &= 10,\\
X^{(16)} = X^{(8,2)} &= 0212210\underline{2}1, & w(X^{(16)}) &= 11,\\
X^{(17)} = X^{(9,1)} &= 0212210\underline{22}, & w(X^{(17)}) &= 12,\\
X^{(18)} = X^{(9,2)} &= 02122102\underline{0}, & w(X^{(18)}) &= 10,\\
\overline{X} = X^{(19)} = X^{(10,1)} &= 02122102\underline{1}, & w(X^{(19)}) &= 11.
\end{aligned}$$

Note that there are two balancing indices of $X$: $h_b(X) = 8$ and 13 (see the "$\leftarrow$" above). Since there are $p = 20 = |[0,19]| < \binom{5}{5}_3 = 51$ different possible balancing indices, it is possible to choose $r = 5$. However, since $K$ is even, it is possible to use only the first $p-1 = 19 = |[0,18]| = \binom{4}{4}_3$ balancing function, and so $r = 4$ can be actually chosen. In this case, the encoding of $X$ is $\mathcal{E}(X) = \langle C_8 \rangle(X) C_8 = X^{(8)} C_8 = 021201210\,1102$ (however, a different encoding could be $\mathcal{E}(X) = \langle C_{13} \rangle(X) C_{13} = X^{(13)} C_{13} = 021221010\,2002$). On receiving $YC = 021201210\,1102$, the decoder computes the balancing index $h_b(X) = 8$ from $C = 1102 = C_8$. Then using (14) it computes $i = i(8) = \lceil 8/2 \rceil = 4$ and $j = j(8) = 7 \bmod 2 + 1 = 2$. So, using (12), it decodes $YC$ as

$$\mathcal{D}(YC) = \langle C_8 \rangle^{-1}(Y) = Y^{)4,2(} = \overline{y_1 y_2 y_3} f^{-1}(y_4, 2) y_5 y_6 y_7 y_8 y_9 =$$

$$\overline{021} f^{-1}(2,2)01210 = \underline{201}\mathbf{0}01210 = X$$

(please see (7) and note that $f(0,2) = 2$, $f(1,2) = 0$ and $f(2,2) = 1$; hence, $f^{-1}(2,2) = 0$). In this way we have given a design example of a 3-ary balanced code with $k = 9$ information digits and $r = 4$ check digits of length $n = 13$. With the coding scheme in [8], $r \geq 5$ check digits are required to make $k = 9$ information digits 3-ary balanced.

Any information word $X$ has its own encoding. In fact, after a certain number of proposed complementation steps it is guaranteed to make the weight of the modified word to be $\lfloor K/2 \rfloor$. This is because when the $i$-th digit $x_i$ of $X$ is complemented using $f(x_i, j)$, with $j = 0, 1, \ldots, l$, every digit in the range $[\min\{x_i, \overline{x_i}\}, \max\{x_i, \overline{x_i}\}]$ occurs (see (5)). Thus, the random walk of the weight of $X$ defined by the proposed complementation scheme reaches every integer in the range $[\min\{w(X), w(\overline{X}) = K - w(X)\}, \max\{w(X), w(\overline{X})\}]$, and so, it reaches the integer $\lfloor K/2 \rfloor$. The following theorem describes how to find the best (that is, "shortest") balancing function $f$.

***Theorem** 1:* For any $m \geq 2$ there exists a complementation function $f$ defined as in (3) and satisfying the properties (4), (5) and (6) with $l = (m-1)+m \bmod 2$. So, the number of balancing functions used by this simple parallel decoding scheme is $p = K + m \bmod 2 + K \bmod 2$, where $k \in \mathbb{N}$ is the number of information digits and $K = (m-1)k$.

*Proof:* In general, the number of balancing functions is $p = (l - m \bmod 2)k + m \bmod 2 + K \bmod 2$. So, if $l = (m-1) + m \bmod 2$ then $p = (m-1)k + m \bmod 2 + K \bmod 2 = K + m \bmod 2 + K \bmod 2$. Now we show that for any $m \geq 2$ it is possible to define complementation functions $f$ as in (3) satisfying the properties (4), (5) and (6) with $l = m$ if $m$ is odd and $l = (m-1)$ if $m$ is even. If $m$ is odd, the desired function $f = f_m$ can be defined in matrix representation by considering the operation table of any right-cancellative groupoid with the property (5) (for example, a group) of order $m$ and adjoining the transposed of the vector $(m-1, \ldots, 1, 0)$ as the last column to satisfy the property (4) (see (9) for example). If $m$ is even, the function $f = f_m$ can be defined from the operation table of any right-cancellative groupoid of order $m$ satisfying the properties (4) and (5). For example, such very general algebric structures can be obtained as follows. Given an $m \times m$ matrix $t = (t_{i,j} : i, j = 0, 1, \ldots, m-1)$ define the $m \times m$ matrices

$$\begin{cases} \bar{t} & = ((m-1) - t_{i,j} : i, j = 0, 1, \ldots, m-1) \,, \\ t^{(rx)} & = \Big(t_{(m-1)-i,j} : i, j = 0, 1, \ldots, m-1\Big) \,, \\ t^{(ry)} & = \Big(t_{i,(m-1)-j} : i, j = 0, 1, \ldots, m-1\Big) \,. \end{cases}$$

Note that $\bar{t}$ is the matrix obtained by complementing each element of $t$. Also, the matrix $t^{(rx)}$ (or $t^{(ry)}$) is obtained by reversing the order of the rows (or columns, respectively) of $t$. Now, if $f_{m/2} : \mathbb{Z}_{m/2} \times [0, m/2-1] \rightarrow \mathbb{Z}_{m/2}$ is (represented by) the operation table of any cancellative groupoid (for example, a group) then

$$f_m \stackrel{\text{def}}{=} \left( \begin{array}{c|c} \Big(f_{m/2}\Big) & \Big(\overline{f_{m/2}}\Big)^{(ry)} \\ \hline \Big(\overline{f_{m/2}}\Big)^{(rx)} & \Big(f_{m/2}\Big)^{(rx)(ry)} \end{array} \right). \tag{16}$$

Note that the above $f_m$ satisfies the properties (4), (5), (6) and $l = m - 1$. For example, the matrix in (10) for $m = 4$ and the matrix

$$f = \left(\begin{array}{ccc|ccc} 0 & 1 & 2 & 3 & 4 & 5 \\ 1 & 2 & 0 & 5 & 3 & 4 \\ 2 & 0 & 1 & 4 & 5 & 3 \\ \hline 3 & 5 & 4 & 1 & 0 & 2 \\ 4 & 3 & 5 & 0 & 2 & 1 \\ 5 & 4 & 3 & 2 & 1 & 0 \end{array}\right)$$

for $m = 6$ are obtained with (16). ■

Note that Theorem 1 implies that using this simple parallel decoding scheme an $m$-ary balanced code can be designed provided that the number of balancing function is no more than the number of balanced check symbols; that is, $p = K + m \bmod 2 + K \bmod 2 \leq \binom{r}{\lfloor R/2 \rfloor}_m$. This relation implies (1). With regard to the complexity, let us assume we have a table look-up of size $O(mk \log_m k)$ $m$-ary digits to encode and decode the balancing index $h_b \in [0, K]$. In each of the $K$ encoding steps $O(\log_m k)$ $m$-ary digits need to be computed (such as, the $m$-ary digits representing the integer $w(X^{(h)})$). So, a total of $O(mk \log_m k)$ $m$-ary digits operations are needed. While decoding, a parallel circuit of size $O(mk \log_m k)$ can output from $h_b \in [0, K]$ a length $k$ vector to be "added (according to the complementation function used in the design)" component-wise to the received information part and obtain the original information word. So, a total of $O(1)$ $m$-ary digit operations are needed to decode.

## III. The refined scheme

In this scheme, the check symbols do not need to be balanced words and possibly more than one check symbol indicate the same number of digits complemented in stages. In this way, we reduce the redundancy with respect to the codes given in Section II. Let $m \geq 2$, $k, r, n = k + r \in \mathbb{N}$, $p \in \mathbb{N}$ and $\mathcal{CS} \stackrel{\text{def}}{=} \{\Gamma_0, \Gamma_1, \ldots, \Gamma_{p-1}\}$, be a sequence of mutually disjoint non-empty subsets of $r$ digit $m$-ary check symbols such that the following property holds.

**Symmetric saturation property of $\mathcal{CS}$**: any $\Gamma = \Gamma_h \in \mathcal{CS}$ is a symmetric saturated set; that is, for all natural $v$ in the symmetric (with respect to $R/2 \in \mathbb{R}$) interval

$$I = I_h \stackrel{\text{def}}{=} \left[ \left\lceil \frac{R}{2} \right\rceil - \left\lfloor \frac{|\Gamma|}{2} \right\rfloor, \left\lfloor \frac{R}{2} \right\rfloor + \left\lfloor \frac{|\Gamma|}{2} \right\rfloor \right] \tag{17}$$

there exists exactly one check $C \in \Gamma$ such that $w(C) = v$.

For example, when $m = 3$ and $r = 3$ the following is a symmetric saturated sequence of subsets of $\mathbb{Z}_3^3$ (it is actually of maximal size because it is a partition of $\mathbb{Z}_3^3$).

$$\begin{aligned} \Gamma_0 &\stackrel{\text{def}}{=} \{000, 001, 002, 012, 022, 122, 222\}, \\ \Gamma_1 &\stackrel{\text{def}}{=} \{010, 020, 120, 220, 221\}, \\ \Gamma_2 &\stackrel{\text{def}}{=} \{100, 200, 201, 202, 212\}, \\ \Gamma_3 &\stackrel{\text{def}}{=} \{011, 021, 121\}, \\ \Gamma_4 &\stackrel{\text{def}}{=} \{101, 102, 112\}, \\ \Gamma_5 &\stackrel{\text{def}}{=} \{110, 210, 211\}, \\ \Gamma_6 &\stackrel{\text{def}}{=} \{111\}. \end{aligned} \tag{18}$$

Every element within a set $\Gamma_h$ indicates the same number $d_h$ (to be defined below) and this $d_h$ represents the number of information digits complemented in "stages" to make a word balanced. As in [1] and [11], the $p \leq \binom{r}{\lfloor R/2 \rfloor}_m$ natural numbers $d_0$, $d_1$, ..., $d_{p-1}$ are defined as

$$d_h \stackrel{\text{def}}{=} \begin{cases} 0 & \text{if } h = 0, \\ d_{h-1} + \lfloor |\Gamma_{h-1}|/2 \rfloor + \lceil |\Gamma_h|/2 \rceil & \text{if } h \in [1, p-1]. \end{cases} \tag{19}$$

For the example given in (18), we have $d_0 = 0$, $d_1 = 0 + 3 + 3 = 6$, $d_2 = 6 + 2 + 3 = 11$, $d_3 = 15$, $d_4 = 18$, $d_5 = 21$ and $d_6 = 23$. Now, given $k \in \mathbb{N}$, define the functions

$$\langle \Gamma_h \rangle (X) \stackrel{\text{def}}{=} X^{(d_h)} \stackrel{\text{def}}{=} X^{(\varphi_i; d_h)} \tag{20}$$

where $X^{(0)} \stackrel{\text{def}}{=} X = x_1 x_2 \ldots x_k \in \mathbb{Z}_m^k$, and, for all $d \in [1, K]$,

$$X^{(d)} \stackrel{\text{def}}{=} X^{(\varphi_i; d)} \stackrel{\text{def}}{=} X^{(\varphi_i; i, j)} \stackrel{\text{def}}{=} \overline{x_1 x_2 \ldots x_{i-1}} \varphi_i(x_i, j) x_{i+1} \ldots x_{k-1} x_k \in \mathbb{Z}_m^k, \tag{21}$$

$$i = i(d) \stackrel{\text{def}}{=} \left\lceil \frac{d}{m-1} \right\rceil \in [1, k], \tag{22}$$

$$j = j(d) \stackrel{\text{def}}{=} (d-1) \bmod (m-1) + 1 \in [1, m-1], \tag{23}$$

and where $\varphi_i : \mathbb{Z}_m \times [0, m-1] \to \mathbb{Z}_m$ are **possibly different** $m$-ary complementation functions yet to be defined for all $i \in [1, k]$. Note that the two component function $(i(d), j(d))$ from the integer interval $[0, K]$ to $\{(0, m-1)\} \cup ([1, k] \times [1, m-1])$ is a bijection with inverse

$$d(i, j) \stackrel{\text{def}}{=} (i-1)(m-1) + j \in [0, K]. \tag{24}$$

To make a word to be a balanced codeword, every information word $X \in \mathbb{Z}_m^k$ is encoded as $\mathcal{E}(X) = \langle \Gamma_{h_b} \rangle (X) C_{h_b}$, where $h_b = h_b(X)$ is an index (referred as a balancing index) such that there exists a (**possibly unbalanced**) check symbol $C_{h_b} \in \Gamma_{h_b}$ which makes $w(\mathcal{E}(X)) = w(\langle \Gamma_{h_b} \rangle (X)) + w(C_{h_b}) = \lfloor N/2 \rfloor$ . On receiving $YC_h \in \mathbb{Z}_m^n$, the decoder simply computes $\mathcal{D}(YC_h) = \mathcal{E}^{-1}(YC_h) = \langle \Gamma_h \rangle^{-1}(Y)$. In this scheme, each balancing function $\langle \Gamma_h \rangle$ can be thought as the component wise addition with a constant vector of length $k$; where the addition is made modulo possibly many complementation matrices defined one for each digit position $i \in [1, k]$. In particular, note that the complementation function may differ from digit to digit. Namely, we may assume that the first, say, 10 digits of the information word being encoded are complemented using a complementation function $\varphi_1$, the second, say, 13 digits are complemented using a complementation function $\varphi_2$, with possibly $\varphi_1 \neq \varphi_2$, and so on. So, let $\mathcal{F} \stackrel{\text{def}}{=} \{\varphi_i : \ i \in [1, k]\}$ be a family of $m$-ary functions to be used in the code design such that

**Complementation property of** $\mathcal{F}$: any $f \in \mathcal{F}$ is a complementation function; that is, for all $x \in \mathbb{Z}_m$, $f(x, 0) = x$ and $f(x, m-1) = \overline{x} \in \mathbb{Z}_m$. In this way, $X^{(0)} = X$, $X^{(K)} = \overline{X}$ and $w\left(X^{(0)}\right) = K - w\left(X^{(K)}\right)$. (25)

**Smoothness property of** $\mathcal{F}$: any $f \in \mathcal{F}$ is smooth; that is, for all $x \in \mathbb{Z}_m$ and $j \in [0, m-2]$, $|f(x, j+1) - f(x, j)| \leq 1$. In this way, $\left| w\left(X^{(d+1)}\right) - w\left(X^{(d)}\right) \right| \leq 1$, for all $d = 0, 1, \ldots, K$. (26)

Then, the "random walk" sequence $\left\{w\left(X^{(d)}\right):\ d=0,1,\ldots,K\right\}$ satisfies

$$\begin{gathered}\text{for all } X\in\mathbb{Z}_m^k, \text{ there exists } d\in[0,K+K \bmod 2-1] \text{ such that}\\ w\left(X^{(d)}\right)=\left\lfloor\frac{K}{2}\right\rfloor\ \left(\text{or }\left\lceil\frac{K}{2}\right\rceil\right),\end{gathered}\tag{27}$$

and

$$\begin{gathered}\text{for all } X\in\mathbb{Z}_m^k \text{ and } d_1,d_2\in[0,K+K \bmod 2-1],\\ w\left(X^{(d_2)}\right)\in\left[w\left(X^{(d_1)}\right)-|d_1-d_2|,w\left(X^{(d_1)}\right)+|d_1-d_2|\right].\end{gathered}\tag{28}$$

Now, the properties (27) and (28) are sufficient conditions for the following theorem to hold.

***Theorem 2:*** Let $m\geq 2$, $k,r,n=k+r,p\in\mathbb{N}$, $\mathcal{CS}\stackrel{\text{def}}{=}\{\Gamma_0,\Gamma_1,\ldots,\Gamma_{p-1}\}$ be **any** sequence of mutually disjoint non-empty symmetric saturated subsets of $\mathbb{Z}_m^r$ (as in (17)) and

$$K\leq\sum_{h=1}^{p}|\Gamma_h|-[(K+R+KR) \bmod 2].\tag{29}$$

If $\mathcal{B}\stackrel{\text{def}}{=}\{\langle\Gamma_h\rangle:\ h\in[0,p-1]\}$ is the set of functions defined in (20) where $\mathcal{F}=\{\varphi_i:\ i\in[1,k]\}$ is a set of $m$-ary smooth complementation functions (as in (25) and (26)) then for any information word $X\in\mathbb{Z}_m^k$ there exists a balancing index $h_b=h_b(X)$. Furthermore, if $m=2$ then every function in $\mathcal{B}$ is one-to-one (that is, $\mathcal{B}$ is a well defined set of balancing functions).

*Proof:* Relation (27) and (28) follow from (25) and (26). Now, by using (27) and (28) the proof of the existence of a balancing index $h_b(X)$ follows from the hypothesis (29) exactly as the proof of Theorem 4 in [11], where $k$ and $r$ are replaced with $K$ and $R$ respectively. Note that, if $m=2$ then every complementation function satisfying the properties (25) and (26) is equal to the usual bit complementation function represented by $\begin{pmatrix}0 & \mathbf{1}\\ 1 & \mathbf{0}\end{pmatrix}$. Since the bit complementation function is a bijection of $\mathbb{Z}_2$, every function in $\mathcal{B}$ is one-to-one when $m=2$. ■

Note that the maximum value of the rightmost expression in (29) is reached when $\mathcal{CS}$ is a partition of $\mathbb{Z}_m^r$. In this case $p=\binom{r}{\lfloor R/2\rfloor}_m$ and (29) becomes $K\leq|\mathbb{Z}_m^r|-[(K+R+KR) \bmod 2]$, which is equivalent to $k\leq(m^r-1)/(m-1)$ because $k$ is an integer. So, if $k\leq(m^r-1)/(m-1)$ then the hypothesis of Theorem 2 can be easily satisfied and if $\mathcal{CS}$ is rearranged properly then $k$ up to $(m^r-1)/(m-1)$ digits can be balanced. This implies (2).

Obviously, we wish the set $\mathcal{B}$ defined in Theorem 2 to be a set of **one-to-one** balancing functions for **all** integer $m\geq 2$. However, this may not be true in general. For example, when $m=3$, the only complementation functions satisfying (25) and (26) are represented by the $3\times 3$ matrices,

$$f=\begin{pmatrix}0&1&\mathbf{2}\\1&1&\mathbf{1}\\2&1&\mathbf{0}\end{pmatrix},\qquad f'=\begin{pmatrix}0&1&\mathbf{2}\\1&0&\mathbf{1}\\2&1&\mathbf{0}\end{pmatrix},\quad\text{or}\quad f''=\begin{pmatrix}0&1&\mathbf{2}\\1&2&\mathbf{1}\\2&1&\mathbf{0}\end{pmatrix}\tag{30}$$

Note that in the above matrices, all the columns are a permutation of the first column except the middle one (that is, the column whose index is 1). So, if for all $\langle\Gamma_h\rangle\in\mathcal{B}$, $j(d_h)\neq 1$, where $j(d)$ is defined in (23), then $\mathcal{B}$ is a set of **one-to-one** balancing functions. On the other hand, if there exists $\langle\Gamma_{\tilde{h}}\rangle\in\mathcal{B}$ such that $\tilde{j}\stackrel{\text{def}}{=}j(d_{\tilde{h}})=1$ then $\langle\Gamma_{\tilde{h}}\rangle$ may not be one-to-one. This is because, depending on which one among the three matrices mentioned above is chosen for complementing the $\tilde{i}$-th digit of $X$, with $\tilde{i} \stackrel{\text{def}}{=} i(d_{\tilde{h}})$ defined in (22), the receiver, on receiving $y_{\tilde{i}} = 1 \in \mathbb{Z}_3$ and knowing $\tilde{j} = 1$, is uncertain on whether $x_{\tilde{i}} = 0$ or $2 \in \mathbb{Z}_3$ for $f'$ and $f''$ above; or is uncertain whether $x_{\tilde{i}} = 0, 1$ or $2 \in \mathbb{Z}_3$ for $f$. This means that decoding may be umbiguous and so $\mathcal{B}$ may not be a good set of balancing functions. When $m = 3$, one way to solve this problem is to choose the sequence $\mathcal{CS}$ (and hence, the sequence of the $d_h$'s) so that $j(d_h) \neq 1$, for all $h \in [0, p-1]$ ( $\Longleftrightarrow$ the $d_h$'s are all even integers). So, we readly see that the sequence $\mathcal{CS}$ in (18) is not a good choice because it implies the odd integers $d_2 = 11$, $d_3 = 15$, $d_5 = 21$ and $d_6 = 23$. By rearranging the sequence $\mathcal{CS}$ in (18), the following example for $r = 3$ and $k = 13$ fixes this problem. For all $i \in [1, k]$, let $\varphi_i = f : \mathbb{Z}_3 \times [0, 2] \to \mathbb{Z}_2$ be defined by the $3 \times 3$ matrix $f$ in (30) (but, we could have chosen $f'$ or $f''$). The code design is defined by the $p = 7$ balancing functions (please see (20) and (19)),

$$\begin{aligned}
&\Big\langle \Gamma_0 \stackrel{\text{def}}{=} \{000, 001, 002, 012, 022, 122, 222\} \Big\rangle(X) \stackrel{\text{def}}{=} X^{(d_0)} = X^{(0)}, \\
&\Big\langle \Gamma_1 \stackrel{\text{def}}{=} \{010, 020, 120, 220, 221\} \Big\rangle(X) \stackrel{\text{def}}{=} X^{(d_1)} = X^{(6)}, \\
&\Big\langle \Gamma_2 \stackrel{\text{def}}{=} \{011, 021, 121\} \Big\rangle(X) \stackrel{\text{def}}{=} X^{(d_2)} = X^{(10)}, \\
&\Big\langle \Gamma_3 \stackrel{\text{def}}{=} \{111\} \Big\rangle(X) \stackrel{\text{def}}{=} X^{(d_3)} = X^{(12)}, \\
&\Big\langle \Gamma_4 \stackrel{\text{def}}{=} \{110, 210, 211\} \Big\rangle(X) \stackrel{\text{def}}{=} X^{(d_4)} = X^{(14)}, \\
&\Big\langle \Gamma_5 \stackrel{\text{def}}{=} \{100, 200, 201, 202, 212\} \Big\rangle(X) \stackrel{\text{def}}{=} X^{(d_5)} = X^{(18)}, \\
&\Big\langle \Gamma_6 \stackrel{\text{def}}{=} \{101, 102, 112\} \Big\rangle(X) \stackrel{\text{def}}{=} X^{(d_6)} = X^{(22)}.
\end{aligned} \tag{31}$$

If $X = 1000022021010 \in \mathbb{Z}_3^{13}$ is an information word then

$$\begin{array}{lllll}
\langle\Gamma_0\rangle(X) = X = X^{(0)} & = X^{(0,2)} & = 1000022021010, & w(X^{(0)}) = 9, \\
\langle\Gamma_1\rangle(X) \quad = X^{(6)} & = X^{(3,2)} & = 12\underline{2}0022021010, & w(X^{(6)}) = 13, \leftarrow \\
\langle\Gamma_2\rangle(X) \quad = X^{(10)} & = X^{(5,2)} & = 1222\underline{2}22021010, & w(X^{(10)}) = 17, \\
\langle\Gamma_3\rangle(X) \quad = X^{(12)} & = X^{(6,2)} & = 12222\underline{0}2021010, & w(X^{(12)}) = 15, \\
\langle\Gamma_4\rangle(X) \quad = X^{(14)} & = X^{(7,2)} & = 122220\underline{0}021010, & w(X^{(14)}) = 13, \leftarrow \\
\langle\Gamma_5\rangle(X) \quad = X^{(18)} & = X^{(9,2)} & = 12222002\underline{0}1010, & w(X^{(18)}) = 13, \leftarrow \\
\langle\Gamma_6\rangle(X) \quad = X^{(22)} & = X^{(11,2)} & = 1222200201\underline{2}10, & w(X^{(22)}) = 15.
\end{array}$$

Hence, the/an encoding of $X$ is $\mathcal{E}(X) = \langle\Gamma_1\rangle(X)120 = X^{(6)}120 = 1220022021010\,120$. On receiving $YC = 1220022021010\,120$, the decoder computes the balancing index $h_b(X) = 1$ from $C = 120 \in \Gamma_1$. Since $d_1 = 6$ it computes $i = i(6) = \lceil 6/2 \rceil = 3$ and $j = j(6) = 5 \bmod 2 + 1 = 2$ (indeed, when $m = 3$ there is no need to compute $j$ because the $d_h$ is chosen so that $j(d_h) = 2$, for all $h \in [0, p-1]$). Hence, it computes

$$\mathcal{D}(YC) = \langle\Gamma_1\rangle^{-1}(Y) = Y^{)3,2(} = \overline{y_1 y_2} f^{-1}(y_3, 2) y_4 y_5 y_7 y_8 y_9 y_{10} y_{11} y_{12} y_{13} =$$

$$\overline{12} f^{-1}(2, 2)0022021010 = \underline{10}\mathbf{0}0022021010 = X.$$

(please note that the rightmost column of $f$ in (30) defines $f(0, 2) = 2$, $f(1, 2) = 1$ and $f(2, 2) = 0$; hence, $f^{-1}(2, 2) = 0$). At this point, let us say that a function is

$$\begin{array}{l}
j\text{-}\textbf{step invertible} \text{ if, and only if, for all } x_1, x_2 \in \mathbb{Z}_m,\ x_1 \neq x_2 \Longrightarrow f(x_1, j) \neq f(x_2, j) \\
\text{(that is, the } j\text{-th column of } f \text{ is a permutation of the 0-th column of } f).
\end{array} \tag{32}$$

For example, all the functions in (30) are 2-step invertible and not 1-step invertible. Obviously, any complementation function is 0-step invertible and $(m-1)$-step invertible.

Given **any** integer $m \geq 2$, our scheme would work easily if the following four competing properties can be satisfied: 1) the number of columns $l+1$ (or, let us say, length) of the matrices/functions in $\mathcal{F}$ is the smallest possible value given by $m$ (otherwise we get more redundant code designs); 2) the complementation property (25) must hold; 3) the smoothness property (26) must hold (if 2) or 3) do not hold then the hypothesis of Theorem 2 may not hold and the balancing index existence may not be guaranteed) and 4) for all $i \in [1, k]$, the $i$-th digit complementation function $\varphi_i \in \mathcal{F}$ is $j$-step invertible for all $j \in [0, m-1]$. Actually, there would be no need to use many different complementation functions (as in the simple scheme of Section II). However, we readly see that when $m$ is odd, no function exists which satisfies property 1), 2), 3) and is $[(m-1)/2]$-step invertible (as we have shown in (30) for $m = 3$). In general, no smooth complement function of length $m$ exists which is $j$-step invertible for all $j = 0, 1, \ldots, m-1$. We were only able to find a systematic way to obtain a family of $m$-ary smooth complementation functions of length $m$ which are $j$-step invertible only for one integer value of $j \in [0, m-1] - \{(m-1)/2\}$ of our choice (the functions in (36) below). Actually, in our code design we only use the functions in (36) which are $j$-step invertible with one integer value for $j \in [\lceil m/2 \rceil, m-1]$ (note that $(m-1)/2 \notin [\lceil m/2 \rceil, m-1]$). However, this choice of functions is successful only if the matching property (34) given below is satisfied. So, to circumvent the invertibility problem shown in (30) for $m = 3$, we use the smooth complementation functions given in (37) as elements of $\mathcal{F}$ and a symmetric saturated sequence $\mathcal{CS}$ of subsets of check symbols such that property (34) holds. Since we can choose only one value of $j \in [0, m-1] - \{(m-1)/2\}$ for which any $i$-th digit complementation function $\varphi_i \in \mathcal{F}$ is $j$-step invertible, in our case, the matching property (34) can be surely assured if

for all $h_1, h_2 \in [0, p-1]$, $i(d_{h_1}) \neq i(d_{h_1})$; where $d_h$ is defined in (19) and $i(d)$ in (22) (that is, for distinct balancing functions the digit positions being complemented are distinct). (33)

Our strategy is to make the above property (33) true.

In this way, example (31) works because of the following general theorem.

***Theorem** 3:* Assume the same hypothesis of Theorem 2 for $\mathcal{CS} \stackrel{\text{def}}{=} \{\Gamma_0, \Gamma_1, \ldots, \Gamma_{p-1}\}$ and $\mathcal{F} = \{\varphi_i : i \in [1, k]\}$. If $\mathcal{CS}$ and $\mathcal{F}$ satisfy the property

**Matching property for $\mathcal{CS}$ and $\mathcal{F}$**: for all $h \in [0, p-1]$, the function $\varphi_i \in \mathcal{F}$ is $j$-step invertible (as in (32)) for $i = i(d_h)$ and $j = j(d_h)$; where $d_h$ is defined in (19), $i(d)$ in (22) and $j(d)$ in (23); (34)

then $\mathcal{B} \stackrel{\text{def}}{=} \{\langle \Gamma_h \rangle : h \in [0, p-1]\}$ as defined in (20) is a set of **one-to-one** balancing functions.

*Proof:* From Theorem 2, we only need to show that any function in $\mathcal{B}$ is one-to-one. Let $h \in [0, p-1]$, $X = x_1 x_2 \ldots x_k \in \mathbb{Z}_m^k$ and $Y = y_1 y_2 \ldots y_k \in \mathbb{Z}_m^k$ be such that $\langle \Gamma_h \rangle (X) = \langle \Gamma_h \rangle (Y)$. Then, from (20) and (21), it follows,

$$X^{(d_h)} = \overline{x_1 x_2 \ldots x_{i-1}} \varphi_i(x_i, j) x_{i+1} \ldots x_{k-1} x_k = \overline{y_1 y_2 \ldots y_{i-1}} \varphi_i(y_i, j) y_{i+1} \ldots y_{k-1} y_k = Y^{(d_h)},$$

with $i = i(d_h)$ and $j = j(d_h)$ defined in (22) and (23) respectively. The above equation implies $x_1 x_2 \ldots x_{i-1} = y_1 y_2 \ldots y_{i-1}$, $x_{i+1} \ldots x_{k-1} x_k = y_{i+1} \ldots y_{k-1} y_k$ and $\varphi_i(x_i, j) = \varphi_i(y_i, j)$. The last equation implies $x_i = y_i$ because $\varphi_i \in \mathcal{F}$ is $j$-step invertible (property (34)). Hence, $X = Y$. This implies that any $\langle \Gamma_h \rangle \in \mathcal{B}$ is one-to-one. ■

So, the general problem is reduced in solving the following non-trivial combinatorial problem.

> **Code design problem**: Given **any** $m \geq 2$, the number of information digits $k \in \mathbb{N}$ and the number of check digits $r \in \mathbb{N}$ which satisfy (29) for some $p \in \mathbb{N}$, find a sequence $\mathcal{CS}$ of mutually disjoint non-empty symmetric saturated subsets of check symbols (as in (17)) and find a sequence $\mathcal{F}$ of $m$-ary smooth complementation functions (as in (25) and (26)) that match (that is, satisfying (34)). (35)

Let us focus our attention on the $m$-ary complementation functions first. Given $m \geq 2$, for all $c = 0, 1, 2, \ldots, \lfloor m/2 \rfloor - 1$, consider any matrix of type,

$$f_m(c) \stackrel{\text{def}}{=} \left( \begin{array}{c|c} A_m(c) & B_m(c) \\ \hline C_m(c) & D_m(c) \\ \hline E_m(c) & F_m(c) \end{array} \right) = \left( \begin{array}{cccc|cccc} 0 & 1 & \ldots & \mathbf{c} & c+1 & c+2 & \ldots & \overline{0} \\ 1 & & \ldots & \mathbf{c-1} & c & c+1 & \ldots & \overline{1} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ c & c-1 & \ldots & \mathbf{0} & 1 & 2 & \ldots & \overline{c} \\ \hline c+1 & & \ldots & \pi_{\mathbf{1}}(\mathbf{c+1}) & \pi_2(c+1) & & \ldots & \overline{c+1} \\ c+2 & & \ldots & \pi_{\mathbf{1}}(\mathbf{c+2}) & \pi_2(c+2) & & \ldots & \overline{c+2} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \overline{c+1} & & \ldots & \pi_{\mathbf{1}}(\overline{\mathbf{c+1}}) & \pi_2(\overline{c+1}) & & \ldots & c+1 \\ \hline \overline{c} & \overline{c-1} & \ldots & \overline{\mathbf{0}} & \overline{1} & \overline{2} & \ldots & c \\ \overline{c-1} & & \ldots & \overline{\mathbf{1}} & \overline{2} & \overline{3} & \ldots & c-1 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \overline{0} & \overline{1} & \ldots & \overline{\mathbf{c}} & \overline{c+1} & \overline{c+2} & \ldots & 0 \end{array} \right), \quad (36)$$

where

$$A_m(c) \stackrel{\text{def}}{=} \begin{pmatrix} 0 & 1 & \ldots & \mathbf{c} \\ 1 & & \ldots & \mathbf{c-1} \\ \vdots & \vdots & \ddots & \vdots \\ c & c-1 & \ldots & \mathbf{0} \end{pmatrix}, \quad B_m(c) \stackrel{\text{def}}{=} \begin{pmatrix} c+1 & c+2 & \ldots & \overline{0} \\ c & c+1 & \ldots & \overline{1} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 2 & \ldots & \overline{c} \end{pmatrix}, \quad C_m(c) \stackrel{\text{def}}{=} \begin{pmatrix} c+1 & \ldots & \pi_{\mathbf{1}}(\mathbf{c+1}) \\ c+2 & \ldots & \pi_{\mathbf{1}}(\mathbf{c+2}) \\ \vdots & \vdots \ddots & \vdots \\ \overline{c+1} & \ldots & \pi_{\mathbf{1}}(\overline{\mathbf{c+1}}) \end{pmatrix},$$

$$D_m(c) \stackrel{\text{def}}{=} \begin{pmatrix} \pi_2(c+1) & \ldots & \overline{c+1} \\ \pi_2(c+2) & \ldots & \overline{c+2} \\ \vdots & \vdots \ddots & \vdots \\ \pi_2(\overline{c+1}) & \ldots & c+1 \end{pmatrix}, \qquad E_m(c) \stackrel{\text{def}}{=} [\overline{A_m(c)}]^{(rx)} \qquad \text{and} \qquad F_m(c) \stackrel{\text{def}}{=} [\overline{B_m(c)}]^{(rx)},$$

and where, $\pi_1, \pi_2 : \{c+1, c+2, \ldots, \overline{c+1}\} \rightarrow \{c+1, c+2, \ldots, \overline{c+1}\}$ are functions such that $\pi_1$ is any bijection (for example, the identity function), and $\pi_2$ is any function such that $|\pi_2(x) - \pi_1(x)| \leq 1$, for all $x \in \{c+1, c+2, \ldots, \overline{c+1}\}$. On the other hand, for all $c = \lceil m/2 \rceil, \lceil m/2 \rceil + 1, \ldots, m-1$, define the matrices

$$f_m(c) \stackrel{\text{def}}{=} [f_m(m-1-c)]^{(rx)(ry)} \quad (37)$$

from the above ones. We do note that any matrix $f_m(c)$ defined with (36) or (37) exists and represents an $m$-ary smooth complementation function (see (25) and (26)) which is $c$-step invertible (as in (32)); this, for all $c\in[0,m-1]$ if $m$ is even, and for all $c\in[0,m-1]-\{(m-1)/2\}$ if $m$ is odd. For example, when $m=4$ one choice of 4-ary complementation matrices is

$$f_m(0)=\left(\begin{array}{c|ccc}\mathbf{0}&1&2&3\\\hline \mathbf{1}&2&2&2\\ \mathbf{2}&1&1&1\\\hline \mathbf{3}&2&1&0\end{array}\right),\qquad f_m(1)=\left(\begin{array}{cc|cc}0&\mathbf{1}&2&3\\ 1&\mathbf{0}&1&2\\\hline\hline 2&\mathbf{3}&2&1\\ 3&\mathbf{2}&1&0\end{array}\right);$$

if $c=0,1$; and so,

$$f_m(2)=\left(\begin{array}{cc|cc}0&1&\mathbf{2}&3\\ 1&2&\mathbf{3}&2\\\hline\hline 2&1&\mathbf{0}&1\\ 3&2&\mathbf{1}&0\end{array}\right),\qquad f_m(3)=\left(\begin{array}{ccc|c}0&1&2&\mathbf{3}\\\hline 1&1&1&\mathbf{2}\\ 2&2&2&\mathbf{1}\\\hline 3&2&1&\mathbf{0}\end{array}\right). \tag{38}$$

for $c=2,3$. When $m=5$ one choice of 5-ary complementation matrices is

$$f_m(0)=\left(\begin{array}{c|cccc}\mathbf{0}&1&2&3&4\\\hline \mathbf{1}&2&3&3&3\\ \mathbf{2}&2&2&2&2\\ \mathbf{3}&2&1&1&1\\\hline \mathbf{4}&3&2&1&0\end{array}\right),\qquad f_m(1)=\left(\begin{array}{cc|ccc}0&\mathbf{1}&2&3&4\\ 1&\mathbf{0}&1&2&3\\\hline 2&\mathbf{2}&2&2&2\\\hline 3&\mathbf{4}&3&2&1\\ 4&\mathbf{3}&2&1&0\end{array}\right);$$

if $c=0,1$; and so,

$$f_m(3)=\left(\begin{array}{ccc|cc}0&1&2&\mathbf{3}&4\\ 1&2&3&\mathbf{4}&3\\\hline 2&2&2&\mathbf{2}&2\\\hline 3&2&1&\mathbf{0}&1\\ 4&3&2&\mathbf{1}&0\end{array}\right),\qquad f_m(4)=\left(\begin{array}{cccc|c}0&1&2&3&\mathbf{4}\\\hline 1&1&1&2&\mathbf{3}\\ 2&2&2&2&\mathbf{2}\\ 3&3&3&2&\mathbf{1}\\\hline 4&3&2&1&\mathbf{0}\end{array}\right).$$

If instead, $m=6$ one choice of 6-ary complementation matrices is

$$f_m(0)=\left(\begin{array}{c|ccccc}\mathbf{0}&1&2&3&4&5\\\hline \mathbf{1}&2&3&4&4&4\\ \mathbf{2}&3&3&3&3&3\\ \mathbf{3}&2&2&2&2&2\\ \mathbf{4}&3&2&1&1&1\\\hline \mathbf{5}&4&3&2&1&0\end{array}\right),\quad f_m(1)=\left(\begin{array}{cc|cccc}0&\mathbf{1}&2&3&4&5\\ 1&\mathbf{0}&1&2&3&4\\\hline 2&\mathbf{2}&3&3&3&3\\ 3&\mathbf{3}&2&2&2&2\\\hline 4&\mathbf{5}&4&3&2&1\\ 5&\mathbf{4}&3&2&1&0\end{array}\right),\quad f_m(2)=\left(\begin{array}{ccc|ccc}0&1&\mathbf{2}&3&4&5\\ 1&1&\mathbf{1}&2&3&4\\ 2&1&\mathbf{0}&1&2&3\\\hline\hline 3&4&\mathbf{5}&4&3&2\\ 4&4&\mathbf{4}&3&2&1\\ 5&4&\mathbf{3}&2&1&0\end{array}\right);$$

if $c=0,1,2$; and so,

$$f_m(3)=\left(\begin{array}{ccc|ccc}0&1&2&\mathbf{3}&4&5\\ 1&2&3&\mathbf{4}&4&4\\ 2&3&4&\mathbf{5}&4&3\\\hline\hline 3&2&1&\mathbf{0}&1&2\\ 4&3&2&\mathbf{1}&1&1\\ 5&4&3&\mathbf{2}&1&0\end{array}\right),\quad f_m(4)=\left(\begin{array}{cccc|cc}0&1&2&3&\mathbf{4}&5\\ 1&2&3&4&\mathbf{5}&4\\\hline 2&2&2&2&\mathbf{3}&3\\ 3&3&3&3&\mathbf{2}&2\\\hline 4&3&2&1&\mathbf{0}&1\\ 5&4&3&2&\mathbf{1}&0\end{array}\right),\quad f_m(5)=\left(\begin{array}{ccccc|c}0&1&2&3&4&\mathbf{5}\\\hline 1&1&1&2&3&\mathbf{4}\\ 2&2&2&2&2&\mathbf{3}\\ 3&3&3&3&3&\mathbf{2}\\ 4&4&4&3&2&\mathbf{1}\\\hline 5&4&3&2&1&\mathbf{0}\end{array}\right); \tag{39}$$

for $c=3,4,5$. Indeed, in our code design, we only need the $m$-ary complementation functions, say, for $c=\lceil m/2\rceil, \lceil m/2\rceil+1,\ldots,m-1$ (in the above examples, when $m=5$, we only need $f_m(3)$ and $f_m(4)$; when $m=6$, we only need $f_m(3)$, $f_m(4)$ and $f_m(5)$).

At this point, to define the balancing function $\langle\Gamma_h\rangle(X) = x^{(d_h)}$, $h \in [0, p-1]$, we turn our attention to the proper design of a sequence $\mathcal{CS} = \{\Gamma_0, \Gamma_1, \ldots, \Gamma_{p-1}\}$, $p \in \mathbb{N}$, of mutually disjoint non-empty symmetric saturated subsets of $\mathbb{Z}_m^r$. The sequence $\mathcal{CS}$ must be defined so that the $d_h$'s (which are computed exclusively from the sequence $\mathcal{CS}$ using (19)) make (34) satisfied for some $\mathcal{F} = \{\varphi_i : i \in [1, k]\}$ to be defied appropriately. To this aim, let $m, r \in \mathbb{N}$ and

$$k \leq \frac{m^r - 1}{m - 1}.$$

Hence, let $\mathcal{P}$ be a partition of the set of check symbols $\mathbb{Z}_m^r$ into non-empty symmetric saturated subsets. Since $\mathcal{P}$ is a partition satisfying (17), it follows that

$$|\mathcal{P}| = \binom{r}{\lfloor R/2 \rfloor}_m, \qquad \sum_{\Gamma \in \mathcal{P}} |\Gamma| = |\mathbb{Z}_m^r| = m^r$$

and,

$$|\Gamma| \in \{(R+1) - 2x : x = 0, 1, \ldots, \lfloor R/2 \rfloor\},$$

for all $\Gamma \in \mathcal{P}$. Now, write

$$\mathcal{P} = \bigcup_{x=0}^{\lfloor R/2 \rfloor} \{\Gamma \in \mathcal{P} : |\Gamma| = (R+1) - 2x\} = \bigcup_{c=0}^{m-2} \left[ \bigcup_{x \in [(m-1)\mathbb{Z}+c]} \{\Gamma \in \mathcal{P} : |\Gamma| = (R+1) - 2x\} \right].$$

In this way, if we let

$$T_m(r, x) \stackrel{\text{def}}{=} \{\Gamma \in \mathcal{P} : |\Gamma| = (R+1) - 2x\}, \tag{40}$$

for all integer $x \in [0, \lfloor R/2 \rfloor]$; and

$$S_m(r, c) \stackrel{\text{def}}{=} \bigcup_{x \in [(m-1)\mathbb{Z}+c]} T_m(r, x) = \bigcup_{x \in [(m-1)\mathbb{Z}+c] \cap [0, \lfloor R/2 \rfloor]} T_m(r, x) = \tag{41}$$

$$T_m(r, c) \cup T_m(r, (m-1) + c) \cup \ldots \cup T_m(r, \lfloor R/2 \rfloor - [(\lfloor R/2 \rfloor - c) \bmod (m-1)]),$$

for all $c \in \mathbb{Z}_{m-1}$, then

$$\mathcal{P} = \bigcup_{x=0}^{\lfloor R/2 \rfloor} T_m(r, x) = \bigcup_{c=0}^{m-2} S_m(r, c).$$

Now, let

$$t_m(r, x) \stackrel{\text{def}}{=} |T_m(r, x)| = |\{\Gamma \in \mathcal{P} : |\Gamma| = (R+1) - 2x\}|,$$

for all integer $x \in [0, \lfloor R/2 \rfloor]$, and note that $|\Gamma| = (R+1) - 2x$ if, and only if $\min_{C \in \Gamma} w(C) = x$, for $x = 0, 1, \ldots, \lfloor R/2 \rfloor$. Since $\mathcal{P}$ is a partition, this implies that $t_m(r, 0) = \binom{r}{0}_m = 1$, and

$$t_m(r, 0) + t_m(r, 1) + \ldots + t_m(r, x) = |\,\{X \in \mathbb{Z}_m^r : w(X) = x\}\,| = \binom{r}{x}_m;$$

that is,

$$t_m(r, x) = \binom{r}{x}_m - [t_m(r, x-1) + t_m(r, x-2) + \ldots + t_m(r, 0)].$$

All this implies,

$$t_m(r,x) = \binom{r}{x}_m - \binom{r}{x-1}_m, \quad \text{with initial conditions } t_m(r,0) = \binom{r}{0}_m = 1. \tag{42}$$

In the Appendix the integer sequence $\{t_m(r,x) : \ x \in \mathbb{Z}\}$ is analyzed and some fundamental properties for the design of $\mathcal{CS}$ are proved. In particular, the following key theorem is proved.

***Theorem** 4:* The following relations hold.

$$\begin{array}{ll} |S_m(r,0)| = |S_m(r,1)| + 1 & \text{and} \\ |S_m(r,c)| = |S_m(r,m-c)|, & \text{for all } c \in [2, \lfloor m/2 \rfloor]. \end{array}$$

*Proof:* See the proof in the Appendix I. ■

Now, for notational convenience, if $m = 3$ then let $A_1 \stackrel{\text{def}}{=} S_m(r,0)$ and $B_1 \stackrel{\text{def}}{=} S_m(r,1)$. If instead $m \geq 4$ then let (note that the definition for $m = 3$ differs from the definition for $m \geq 4$),

$$\begin{array}{ll} A_1 \stackrel{\text{def}}{=} S_m(r,0), & B_1 \stackrel{\text{def}}{=} S_m(r,1), \\ A_2 \stackrel{\text{def}}{=} S_m(r,m-2), & B_2 \stackrel{\text{def}}{=} S_m(r,2), \\ A_3 \stackrel{\text{def}}{=} S_m(r,m-3), & B_3 \stackrel{\text{def}}{=} S_m(r,3), \\ A_4 \stackrel{\text{def}}{=} S_m(r,m-4), & B_4 \stackrel{\text{def}}{=} S_m(r,4), \\ \quad\vdots & \quad\vdots \\ A_c \stackrel{\text{def}}{=} S_m(r,m-c), & B_c \stackrel{\text{def}}{=} S_m(r,c), \\ A_{c+1} \stackrel{\text{def}}{=} S_m(r,m-c-1), & B_{c+1} \stackrel{\text{def}}{=} S_m(r,c+1), \\ \quad\vdots & \quad\vdots \\ A_{\lfloor m/2 \rfloor} \stackrel{\text{def}}{=} S_m(r,\lceil m/2 \rceil), & B_{\lfloor m/2 \rfloor} \stackrel{\text{def}}{=} S_m(r,\lfloor m/2 \rfloor); \end{array} \tag{43}$$

Now, from the partition $\mathcal{P}$, the sequence $\mathcal{CS} = \{\Gamma_0, \Gamma_1, \ldots, \Gamma_{p-1}\}$, with $p \leq |\mathcal{P}| = \binom{r}{\lfloor R/2 \rfloor}_m$ can be defined as follows. Let/pick

$$\begin{array}{l} \Gamma_0 \in A_1, \\ \Gamma_1 \in B_1 - \{\Gamma_0\}, \\ \Gamma_2 \in A_1 - \{\Gamma_0, \Gamma_1\}, \\ \Gamma_3 \in B_1 - \{\Gamma_0, \Gamma_1, \Gamma_2\}, \\ \quad\vdots \\ \Gamma_{|A_1 \cup B_1|-2} \in B_1 - \{\Gamma_0, \Gamma_1, \ldots, \Gamma_{|A_1 \cup B_1|-3}\}, \\ \Gamma_{|A_1 \cup B_1|-1} \in A_1 - \{\Gamma_0, \Gamma_1, \ldots, \Gamma_{|A_1 \cup B_1|-2}\}. \end{array} \tag{44}$$

Note that $|A_1| = |B_1| + 1$ because of Theorem 4, and so, the above sequence is well defined. Then, let/pick

$$\begin{array}{l} \Gamma_{|A_1 \cup B_1|} \in B_2, \\ \Gamma_{|A_1 \cup B_1|+1} \in A_2 - \{\Gamma_{|A_1 \cup B_1|}\}, \\ \Gamma_{|A_1 \cup B_1|+2} \in B_2 - \{\Gamma_{|A_1 \cup B_1|}, \Gamma_{|A_1 \cup B_1|+1}\}, \\ \Gamma_{|A_1 \cup B_1|+3} \in A_2 - \{\Gamma_{|A_1 \cup B_1|}, \Gamma_{|A_1 \cup B_1|+1}, \Gamma_{|A_1 \cup B_1|+2}\}, \\ \quad\vdots \\ \Gamma_{|A_1 \cup B_1|+|A_2 \cup B_2|-2} \in B_2 - \{\Gamma_{|A_1 \cup B_1|}, \Gamma_{|A_1 \cup B_1|+1}, \ldots, \Gamma_{|A_1 \cup B_1|+|A_2 \cup B_2|-3}\}, \\ \Gamma_{|A_1 \cup B_1|+|A_2 \cup B_2|-1} \in A_2 - \{\Gamma_{|A_1 \cup B_1|}, \Gamma_{|A_1 \cup B_1|+1}, \ldots, \Gamma_{|A_1 \cup B_1|+|A_2 \cup B_2|-2}\}. \end{array}$$

Note that $|A_2| = |B_2|$ because of Theorem 4, and so, the above sequence is well defined. In general, for $c = 2, 3, \ldots, \lfloor m/2 \rfloor$, let/pick

$$\begin{array}{l}
\Gamma_{\sum_{s=1}^{c-1}|A_s\cup B_s|} \in B_c, \\
\Gamma_{\sum_{s=1}^{c-1}|A_s\cup B_s|+1} \in A_c - \left\{\Gamma_{\sum_{s=1}^{c-1}|A_s\cup B_s|}\right\}, \\
\Gamma_{\sum_{s=1}^{c-1}|A_s\cup B_s|+2} \in B_c - \left\{\Gamma_{\sum_{s=1}^{c-1}|A_s\cup B_s|}, \Gamma_{\sum_{s=1}^{c-1}|A_s\cup B_s|+1}\right\}, \\
\Gamma_{\sum_{s=1}^{c-1}|A_s\cup B_s|+3} \in A_c - \left\{\Gamma_{\sum_{s=1}^{c-1}|A_s\cup B_s|}, \Gamma_{\sum_{s=1}^{c-1}|A_s\cup B_s|+1}, \Gamma_{\sum_{s=1}^{c-1}|A_s\cup B_s|+2}\right\}, \\
\qquad\qquad \vdots \\
\Gamma_{\sum_{s=1}^{c}|A_s\cup B_s|-2} \in B_c - \left\{\Gamma_{\sum_{s=1}^{c-1}|A_s\cup B_s|}, \Gamma_{\sum_{s=1}^{c-1}|A_s\cup B_s|+1}, \ldots, \Gamma_{\sum_{s=1}^{c}|A_s\cup B_s|-3}\right\}, \\
\Gamma_{\sum_{s=1}^{c}|A_s\cup B_s|-1} \in A_c - \left\{\Gamma_{\sum_{s=1}^{c-1}|A_s\cup B_s|}, \Gamma_{\sum_{s=1}^{c-1}|A_s\cup B_s|+1}, \ldots, \Gamma_{\sum_{s=1}^{c}|A_s\cup B_s|-2}\right\}.
\end{array} \tag{45}$$

Note that $|A_c| = |B_c|$ because of Theorem 4, and so, the above sequence is well defined. This process stops when the condition (29) of Theorem 2: $K + [(KR+K+R) \bmod 2] \leq \sum_{\Gamma_h \in \mathcal{CS}} |\Gamma_h|$, is satisfied. Note that the process eventually stops because we assumed $k \leq (m^r - 1)/(m-1)$, and so $K + [(KR+K+R) \bmod 2] \leq m^r = |\mathbb{Z}_m^r| = \sum_{\Gamma \in \mathcal{P}} |\Gamma|$. This means that the sequence $\mathcal{CS}$ just defined is well defined. The following theorem gives a constructive way to define a set of balancing functions $\mathcal{B}$ (and hence, a code design), for all $m \geq 2$.

***Theorem 5:*** Given $m, r, k \in \mathbb{N}$ such that $m \geq 2$ and $k \leq (m^r - 1)/(m-1)$, let the sequence $\mathcal{CS} = \{\Gamma_0, \Gamma_1, \ldots, \Gamma_{p-1}\}$ be defined by (45) and the integers $d_0, d_1, \ldots, d_{p-1} \in [1, k]$ be defined by (19). Then $0 = d_0 < d_1 < \ldots < d_{p-1}$. Furthermore, if we let

$$c \in [1, \lfloor m/2 \rfloor], \quad t \in [0, |A_c \cup B_c| - 1] \quad \text{and} \quad h = \sum_{s=1}^{c-1} |A_s \cup B_s| + t,$$

for all $h \in [0, p-1]$, then

$$d_h \in (m-1)\mathbb{Z} + (m-c), \tag{46}$$

$$0 = i(d_0) < i(d_1) < \ldots < i(d_{p-1}) \tag{47}$$

(and so, the information word digit positions $i(d_h)$'s are all distinct as in (33)) and

$$j(d_h) = m - c \in [\lceil m/2 \rceil, m-1]; \tag{48}$$

(and so, $j(d_h) \neq (m-1)/2$ if $m$ is odd) where $i(d)$ and $j(d)$ are defined in (22) and (23) respectively. So, let $\mathcal{F} = \{\varphi_i \colon\ i \in [1, k]\}$ be defined by letting,

$$\text{for all } h \in [0, p-1], \qquad \varphi_{i(d_h)} \stackrel{\text{def}}{=} f_m(j(d_h)) = f_m(m-c); \tag{49}$$

where $f_m(j)$ are the $j$-step invertible smooth complementation functions defined by (37) for all $j \in [\lceil m/2 \rceil, m-1]$. Then the set of functions $\mathcal{B} = \{\langle \Gamma_h \rangle \colon\ h \in [0, p-1]\}$ is a set of **one-to-one** balancing functions; where $\langle \Gamma_h \rangle (X) = X^{\left(\varphi_{i(d_h)}; d_h\right)} = X^{\left(\varphi_{i(d_h)}; i(d_h), j(d_h)\right)}$ is defined by (21).

*Proof:* See the proof in the Appendix II. ■

For example, when $m=4$, $r=3$, $R=(m-1)r=(4-1)3=9$ and $k=(m^r-1)/(m-1)=21$, the cardinality of partition $\mathcal{P}$ is $|\mathcal{P}|=\binom{3}{4}_4=12$. Such partition can be decomposed as follows

$$\mathcal{P}=\bigcup_{x=0}^{4} T_4(3,x)=\bigcup_{c=0}^{2}\left[\bigcup_{x\in[3\mathbb{Z}+c]\cap[0,4]} T_4(3,x)\right]=\bigcup_{c=0}^{2} S_4(3,c)$$

where $T_4(3,x)=\{\Gamma\in\mathcal{P}:\ |\Gamma|=10-2x\}$ (see (40)) and $S_4(3,c)=\bigcup_{x\in[3\mathbb{Z}+c]\cap[0,4]} T_4(3,x)$ (see (41)). Since (see (42))

$$\begin{aligned}
t_4(3,0)&=|T_4(3,0)|=|\{\Gamma\in\mathcal{P}:\ |\Gamma|=10\}|=1,\\
t_4(3,1)&=|T_4(3,1)|=|\{\Gamma\in\mathcal{P}:\ |\Gamma|=8\}|=2,\\
t_4(3,2)&=|T_4(3,2)|=|\{\Gamma\in\mathcal{P}:\ |\Gamma|=6\}|=3,\\
t_4(3,3)&=|T_4(3,3)|=|\{\Gamma\in\mathcal{P}:\ |\Gamma|=4\}|=4,\\
t_4(3,4)&=|T_4(3,4)|=|\{\Gamma\in\mathcal{P}:\ |\Gamma|=2\}|=2,
\end{aligned}$$

it follows that (see (43))

$$|A_1|=|S_4(3,0)|=\left|\bigcup_{x\in[3\mathbb{Z}]\cap[0,4]} T_4(3,x)\right|=\left|\bigcup_{x\in\{0,3\}} T_4(3,x)\right|=t_4(3,0)+t_4(3,3)=1+4=5$$

$$|B_1|=|S_4(3,1)|=\left|\bigcup_{x\in[3\mathbb{Z}+1]\cap[0,4]} T_4(3,x)\right|=\left|\bigcup_{x\in\{1,4\}} T_4(3,x)\right|=t_4(3,1)+t_4(3,4)=2+2=4$$

$$|B_2|=|A_2|=|S_4(3,2)|=\left|\bigcup_{x\in[3\mathbb{Z}+2]\cap[0,4]} T_4(3,x)\right|=\left|\bigcup_{x\in\{2\}} T_4(3,x)\right|=t_4(3,2)=3.$$

So, according to (44) and (45), we pick

$$\begin{aligned}
\Gamma_0 &\stackrel{\text{def}}{=} \{120,130,212,303\}\in A_1,\\
\Gamma_1 &\stackrel{\text{def}}{=} \{301,311\}\in B_1,\\
\Gamma_2 &\stackrel{\text{def}}{=} \{201,202,221,312\}\in A_1-\{\Gamma_0\},\\
\Gamma_3 &\stackrel{\text{def}}{=} \{310,320\}\in B_1-\{\Gamma_1\},\\
\Gamma_4 &\stackrel{\text{def}}{=} \{210,211,230,321\}\in A_1-\{\Gamma_0,\Gamma_2\},\\
\Gamma_5 &\stackrel{\text{def}}{=} \{010,011,012,022,032,123,223,323\}\in B_1-\{\Gamma_1,\Gamma_3\},\\
\Gamma_6 &\stackrel{\text{def}}{=} \{300,220,302,330\}\in A_1-\{\Gamma_0,\Gamma_2,\Gamma_4\},\\
\Gamma_7 &\stackrel{\text{def}}{=} \{100,020,021,031,113,132,232,332\}\in B_1-\{\Gamma_1,\Gamma_3,\Gamma_5\},\\
\Gamma_8 &\stackrel{\text{def}}{=} \{000,001,002,003,013,023,033,133,233,333\}\in A_1-\{\Gamma_0,\Gamma_2,\Gamma_4,\Gamma_6\},\\
\Gamma_9 &\stackrel{\text{def}}{=} \{101,030,103,122,213,313\}\in B_2=A_2,\\
\Gamma_{10} &\stackrel{\text{def}}{=} \{110,102,112,131,222,322\}\in A_2-\{\Gamma_9\}.\\
\Gamma_{11} &\stackrel{\text{def}}{=} \{200,111,121,203,231,331\}\in B_2-\{\Gamma_9,\Gamma_{10}\}.
\end{aligned}$$

Hence, from (19) and (49), a code design can be defined by the following balancing functions:

$$\begin{array}{ll}
\Big\langle \Gamma_0 \stackrel{\text{def}}{=} \{120, 130, 212, 303\} \Big\rangle(X) & \stackrel{\text{def}}{=} X^{\left(\varphi_{i(d_0)}; d_0\right)} = X^{(\varphi_0;0)} = X^{(f_4(3);0)} = X, \\
\Big\langle \Gamma_1 \stackrel{\text{def}}{=} \{301, 311\} \Big\rangle(X) & \stackrel{\text{def}}{=} X^{\left(\varphi_{i(d_1)}; d_1\right)} = X^{(\varphi_1;3)} = X^{(f_4(3);3)}, \\
\Big\langle \Gamma_2 \stackrel{\text{def}}{=} \{201, 202, 221, 312\} \Big\rangle(X) & \stackrel{\text{def}}{=} X^{\left(\varphi_{i(d_2)}; d_2\right)} = X^{(\varphi_2;6)} = X^{(f_4(3);6)}, \\
\Big\langle \Gamma_3 \stackrel{\text{def}}{=} \{310, 320\} \Big\rangle(X) & \stackrel{\text{def}}{=} X^{\left(\varphi_{i(d_3)}; d_3\right)} = X^{(\varphi_3;9)} = X^{(f_4(3);9)} \\
\Big\langle \Gamma_4 \stackrel{\text{def}}{=} \{210, 211, 230, 321\} \Big\rangle(X) & \stackrel{\text{def}}{=} X^{\left(\varphi_{i(d_4)}; d_4\right)} = X^{(\varphi_4;12)} = X^{(f_4(3);12)}, \\
\Big\langle \Gamma_5 \stackrel{\text{def}}{=} \{010, 011, 012, 022, 032, 123, 223, 323\} \Big\rangle(X) & \stackrel{\text{def}}{=} X^{\left(\varphi_{i(d_5)}; d_5\right)} = X^{(\varphi_6;18)} = X^{(f_4(3);18)}, \\
\Big\langle \Gamma_6 \stackrel{\text{def}}{=} \{300, 220, 302, 330\} \Big\rangle(X) & \stackrel{\text{def}}{=} X^{\left(\varphi_{i(d_6)}; d_6\right)} = X^{(\varphi_8;24)} = X^{(f_4(3);24)}, \\
\Big\langle \Gamma_7 \stackrel{\text{def}}{=} \{100, 020, 021, 031, 113, 132, 232, 332\} \Big\rangle(X) & \stackrel{\text{def}}{=} X^{\left(\varphi_{i(d_7)}; d_7\right)} = X^{(\varphi_{10};30)} = X^{(f_4(3);30)}, \\
\left\langle \Gamma_8 \stackrel{\text{def}}{=} \left\{ \begin{array}{l} 000, 001, 002, 003, 013, \\ 023, 033, 133, 233, 333 \end{array} \right\} \right\rangle(X) & \stackrel{\text{def}}{=} X^{\left(\varphi_{i(d_8)}; d_8\right)} = X^{(\varphi_{13};39)} = X^{(f_4(3);39)}, \\
\hline
\Big\langle \Gamma_9 \stackrel{\text{def}}{=} \{101, 030, 103, 122, 213, 313\} \Big\rangle(X) & \stackrel{\text{def}}{=} X^{\left(\varphi_{i(d_9)}; d_9\right)} = X^{(\varphi_{16};47)} = X^{(f_4(2);47)}, \\
\Big\langle \Gamma_{10} \stackrel{\text{def}}{=} \{110, 102, 112, 131, 222, 322\} \Big\rangle(X) & \stackrel{\text{def}}{=} X^{\left(\varphi_{i(d_{10})}; d_{10}\right)} = X^{(\varphi_{18};53)} = X^{(f_4(2);53)}, \\
\Big\langle \Gamma_{11} \stackrel{\text{def}}{=} \{200, 111, 121, 203, 231, 331\} \Big\rangle(X) & \stackrel{\text{def}}{=} X^{\left(\varphi_{i(d_{11})}; d_{11}\right)} = X^{(\varphi_{20};59)} = X^{(f_4(2);59)};
\end{array}$$

where the $m(=4)$-ary complementation functions are defined in (38) as

$$\varphi_0 = \varphi_1 = \varphi_2 = \varphi_3 = \varphi_4 = \varphi_6 = \varphi_8 = \varphi_{10} = \varphi_{13} = f_4(3) = \begin{pmatrix} 0 & 1 & 2 & \mathbf{3} \\ 1 & 1 & 1 & \mathbf{2} \\ 2 & 2 & 2 & \mathbf{1} \\ 3 & 2 & 1 & \mathbf{0} \end{pmatrix}$$

and

$$\varphi_{16} = \varphi_{18} = \varphi_{20} = f_4(2) = \begin{pmatrix} 0 & 1 & \mathbf{2} & 3 \\ 1 & 2 & \mathbf{3} & 2 \\ 2 & 1 & \mathbf{0} & 1 \\ 3 & 2 & \mathbf{1} & 0 \end{pmatrix}.$$

If $X = 332022210231211012100 \in \mathbb{Z}_4^{21}$ is an information word then the encoder computes

$$\begin{array}{llll}
\langle \Gamma_0 \rangle(X) = X & = X^{(0)} = X^{(0,3)} & = 332022210231211012100, & w(X^{(0)}) = 29, \\
\langle \Gamma_1 \rangle(X) & = X^{(3)} = X^{(1,3)} & = \underline{0}32022210231211012100, & w(X^{(3)}) = 26, \\
\langle \Gamma_2 \rangle(X) & = X^{(6)} = X^{(2,3)} & = 0\underline{0}2022210231211012100, & w(X^{(6)}) = 23, \\
\langle \Gamma_3 \rangle(X) & = X^{(9)} = X^{(3,3)} & = 00\underline{1}022210231211012100, & w(X^{(9)}) = 22, \\
\langle \Gamma_4 \rangle(X) & = X^{(12)} = X^{(4,3)} & = 001\underline{3}22210231211012100, & w(X^{(12)}) = 25, \\
\langle \Gamma_5 \rangle(X) & = X^{(18)} = X^{(6,3)} & = 00131\underline{1}210231211012100, & w(X^{(18)}) = 23, \\
\langle \Gamma_6 \rangle(X) & = X^{(24)} = X^{(8,3)} & = 0013111\underline{2}0231211012100, & w(X^{(24)}) = 23, \\
\langle \Gamma_7 \rangle(X) & = X^{(30)} = X^{(10,3)} & = 001311123\underline{1}31211012100, & w(X^{(30)}) = 25, \\
\langle \Gamma_8 \rangle(X) & = X^{(39)} = X^{(13,3)} & = 001311123102\underline{1}11012100, & w(X^{(39)}) = 22, \\
\hline
\langle \Gamma_9 \rangle(X) & = X^{(47)} = X^{(16,2)} & = 001311123102122\underline{2}12100, & w(X^{(47)}) = 26, \\
\langle \Gamma_{10} \rangle(X) & = X^{(53)} = X^{(18,2)} & = 00131112310212232\underline{0}100, & w(X^{(53)}) = 26, \\
\langle \Gamma_{11} \rangle(X) & = X^{(59)} = X^{(20,2)} & = 0013111231021223212\underline{2}0, & w(X^{(59)}) = 30, \leftarrow
\end{array}$$

Hence, an/the encoding of $X$ is

$$\mathcal{E}(X) = \langle \Gamma_{11} \rangle(X)231 = X^{(59)}231 = 001311123102122321220\,231.$$

In the sequence, the words above the line are obtained with the function $f_4(3)$, the ones below with the function $f_4(2)$. On receiving $YC = 0013111231021223212200\,231$, the decoder computes the balancing index $h_b(X) = 11$ from $C = 231 \in \Gamma_{11}$. Since $d_{11} = 59$ it computes $i = i(59) = \lceil 59/3 \rceil = 20$ and $j = j(59) = 58 \bmod 3 + 1 = 2$. Hence, it computes

$$\mathcal{D}(YC) = \langle \Gamma_{11} \rangle^{-1}(Y) =$$

$$Y^{)20,2(} = \overline{y_1y_2y_3y_4y_5y_6y_7y_8y_9y_{10}y_{11}y_{12}y_{13}y_{14}y_{15}y_{16}y_{17}y_{18}y_{19}}\varphi_{20}^{-1}(y_{20},2)y_{21} =$$

$$\overline{0013111231021223212}[f_4(2)]^{-1}(2,2)0 = \underline{3320222102312110121\mathbf{0}}0 = X$$

(note that the third column of $[f_4(2)]$ defines $[f_4(2)](0,2) = 2$, $[f_4(2)](1,2) = 3$, $[f_4(2)](2,2) = 0$ and $[f_4(2)](3,2) = 1$; hence, $[f_4(2)]^{-1}(2,2) = 0$). However, note that since any complementation function is obviously $(m-1)$-step invertible, the last two columns of $f_4(2)$ are permutation of the first column and we could have just chosen $f_4(2)$ only as complementation matrix. Obviously, this simplification may hold true only for $m = 2$, 3, 4 and 5. Already for $m \geq 6$ it seems we are forced to use two or more complementation functions as in the following other example. Consider $m = 6$, $r = 2$, $R = (m-1)r = (6-1)2 = 10$ and $k = (m^r - 1)/(m-1) = 7$. The cardinality of the partition $\mathcal{P}$ is $|\mathcal{P}| = \binom{2}{5}_m = 6$. Such partition can be decomposed as follows

$$\mathcal{P} = \bigcup_{x=0}^{5} T_6(2,x) = \bigcup_{c=0}^{4} \left[ \bigcup_{x \in [5\mathbf{Z}+c] \cap [0,5]} T_6(2,x) \right] = \bigcup_{c=0}^{4} S_6(2,c)$$

where $T_6(2,x) = \{\Gamma \in \mathcal{P} : \ |\Gamma| = 11 - 2x\}$ (see (40)) and $S_6(2,c) = \bigcup_{x \in [5\mathbf{Z}+c] \cap [0,5]} T_6(2,x)$ (see (41)). Since (see (42))

$$\begin{aligned}
t_6(2,0) &\stackrel{\text{def}}{=} |T_6(2,0)| = |\{\Gamma \in \mathcal{P} : \ |\Gamma| = 11\}| = 1, \\
t_6(2,1) &\stackrel{\text{def}}{=} |T_6(2,1)| = |\{\Gamma \in \mathcal{P} : \ |\Gamma| = 9\}| = 1, \\
t_6(2,2) &\stackrel{\text{def}}{=} |T_6(2,2)| = |\{\Gamma \in \mathcal{P} : \ |\Gamma| = 7\}| = 1, \\
t_6(2,3) &\stackrel{\text{def}}{=} |T_6(2,3)| = |\{\Gamma \in \mathcal{P} : \ |\Gamma| = 5\}| = 1, \\
t_6(2,4) &\stackrel{\text{def}}{=} |T_6(2,4)| = |\{\Gamma \in \mathcal{P} : \ |\Gamma| = 3\}| = 1, \\
t_6(2,5) &\stackrel{\text{def}}{=} |T_6(2,5)| = |\{\Gamma \in \mathcal{P} : \ |\Gamma| = 1\}| = 1,
\end{aligned}$$

it follows that (see (43))

$$|A_1| = |S_6(2,0)| = \left| \bigcup_{x \in [5\mathbf{Z}] \cap [0,5]} T_6(2,x) \right| = \left| \bigcup_{x \in \{0,5\}} T_6(2,x) \right| = t_6(2,0) + t_6(2,5) = 1 + 1 = 2,$$

$$|B_1| = |S_6(2,1)| = \left| \bigcup_{x \in [5\mathbf{Z}+1] \cap [0,5]} T_6(2,x) \right| = \left| \bigcup_{x \in \{1\}} T_6(2,x) \right| = t_6(2,1) = 1,$$

$$|B_2| = |S_6(2,2)| = \left| \bigcup_{x \in [5\mathbf{Z}+2] \cap [0,5]} T_6(2,x) \right| = \left| \bigcup_{x \in \{2\}} T_6(3,x) \right| = t_6(2,2) = 1,$$

$$|A_2| = |S_6(2,4)| = \left| \bigcup_{x \in [5\mathbf{Z}+4] \cap [0,5]} T_6(2,x) \right| = \left| \bigcup_{x \in \{4\}} T_6(3,x) \right| = t_6(2,4) = 1,$$

$$|A_3| = |B_3| = |S_6(2,3)| = \left|\bigcup_{x\in[5\mathbb{Z}+3]\cap[0,5]} T_6(2,x)\right| = \left|\bigcup_{x\in\{3\}} T_6(3,x)\right| = t_6(2,3) = 1.$$

So, according to (44) and (45), we pick

$$\begin{array}{l}
\Gamma_0 \stackrel{\text{def}}{=} \{32\} \in A_1,\\
\Gamma_1 \stackrel{\text{def}}{=} \{10,20,30,40,50,51,52,53,54\} \in B_1,\\
\Gamma_2 \stackrel{\text{def}}{=} \{00,01,02,03,04,05,15,25,35,45,55\} \in A_1 - \{\Gamma_0\},\\
\Gamma_3 \stackrel{\text{def}}{=} \{11,12,13,14,24,34,44\} \in B_2,\\
\Gamma_4 \stackrel{\text{def}}{=} \{22,23,33\} \in A_2,\\
\Gamma_5 \stackrel{\text{def}}{=} \{21,31,41,42,43\} \in B_3 = A_3.
\end{array}$$

In this way, from (19) and (49), a code design can be defined by the following balancing functions:

$$\begin{array}{ll}
\langle \Gamma_0 = \{32\}\rangle(X) & \stackrel{\text{def}}{=} X^{\left(\varphi_{i(d_0)};d_0\right)} = X^{(\varphi_0;0)} = X^{(f_6(5);0)},\\
\langle \Gamma_1 = \{10,20,30,40,50,51,52,53,54\}\rangle(X) & \stackrel{\text{def}}{=} X^{\left(\varphi_{i(d_1)};d_1\right)} = X^{(\varphi_1;5)} = X^{(f_6(5);5)},\\
\langle \Gamma_2 = \{00,01,02,03,04,05,15,25,35,45,55\}\rangle(X) & \stackrel{\text{def}}{=} X^{\left(\varphi_{i(d_2)};d_2\right)} = X^{(\varphi_3;15)} = X^{(f_6(5);15)},\\
\hline
\langle \Gamma_3 = \{11,12,13,14,24,34,44\}\rangle(X) & \stackrel{\text{def}}{=} X^{\left(\varphi_{i(d_3)};d_3\right)} = X^{(\varphi_5;24)} = X^{(f_6(4);24)},\\
\langle \Gamma_4 = \{22,23,33\}\rangle(X) & \stackrel{\text{def}}{=} X^{\left(\varphi_{i(d_4)};d_4\right)} = X^{(\varphi_6;29)} = X^{(f_6(4);29)},\\
\hline
\langle \Gamma_5 = \{21,31,41,42,43\}\rangle(X) & \stackrel{\text{def}}{=} X^{\left(\varphi_{i(d_5)};d_5\right)} = X^{(\varphi_7;33)} = X^{(f_6(3);33)};
\end{array}$$

where the $m(=4)$-ary complementation functions are defined in (39) as

$$\varphi_0 = \varphi_1 = \varphi_3 = f_6(5) = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & \mathbf{5}\\ 1 & 1 & 1 & 2 & 3 & \mathbf{4}\\ 2 & 2 & 2 & 2 & 2 & \mathbf{3}\\ 3 & 3 & 3 & 3 & 3 & \mathbf{2}\\ 4 & 4 & 4 & 3 & 2 & \mathbf{1}\\ 5 & 4 & 3 & 2 & 1 & \mathbf{0} \end{pmatrix}, \qquad \varphi_5 = \varphi_6 = f_6(4) = \begin{pmatrix} 0 & 1 & 2 & 3 & \mathbf{4} & 5\\ 1 & 2 & 3 & 4 & \mathbf{5} & 4\\ 2 & 2 & 2 & 2 & \mathbf{3} & 3\\ 3 & 3 & 3 & 3 & \mathbf{2} & 2\\ 4 & 3 & 2 & 1 & \mathbf{0} & 1\\ 5 & 4 & 3 & 2 & \mathbf{1} & 0 \end{pmatrix}$$

and

$$\varphi_7 = f_6(3) = \begin{pmatrix} 0 & 1 & 2 & \mathbf{3} & 4 & 5\\ 1 & 2 & 3 & \mathbf{4} & 4 & 4\\ 2 & 3 & 4 & \mathbf{5} & 4 & 3\\ 3 & 2 & 1 & \mathbf{0} & 1 & 2\\ 4 & 3 & 2 & \mathbf{1} & 1 & 1\\ 5 & 4 & 3 & \mathbf{2} & 1 & 0 \end{pmatrix}.$$

If $X = 1555122 \in \mathbb{Z}_6^7$ is an information word then,

$$\begin{array}{lll}
\langle \Gamma_0\rangle(X) = X = X^{(0)} = X^{(0,5)} = 1555122, & w(X^{(0)}) = 21, &\\
\langle \Gamma_1\rangle(X) \qquad = X^{(5)} = X^{(1,5)} = \underline{4}555122, & w(X^{(5)}) = 24, &\\
\langle \Gamma_2\rangle(X) \qquad = X^{(15)} = X^{(3,5)} = 40\underline{0}5122, & w(X^{(15)}) = 14, & \leftarrow\\
\hline
\langle \Gamma_3\rangle(X) \qquad = X^{(24)} = X^{(5,4)} = 4000\underline{5}22, & w(X^{(24)}) = 13, &\\
\langle \Gamma_4\rangle(X) \qquad = X^{(29)} = X^{(6,4)} = 40004\underline{3}2, & w(X^{(29)}) = 13, &\\
\hline
\langle \Gamma_5\rangle(X) \qquad = X^{(33)} = X^{(7,3)} = 400043\underline{5}, & w(X^{(33)}) = 16, & \leftarrow
\end{array}$$

Hence, an encoding of $X$ is $\mathcal{E}(X) = \langle \Gamma_2\rangle(X)35 = X^{(15)}35 = 4005122\,35$. In the sequence, the words above the first line are obtained with the function $f_6(5)$, the ones between the first and second line with the function $f_6(4)$, and the ones below the second line with the function $f_6(3)$. On receiving $YC = 4005122\,35$, the decoder computes the balancing index $h_b(X) = 2$ from $C = 35 \in \Gamma_2$. Since $d_2 = 15$ it computes $i = i(15) = \lceil 15/5 \rceil = 3$ and $j = j(15) = 14 \bmod 5 + 1 = 5$. Hence, it computes

$$\mathcal{D}(YC) = \langle \Gamma_2 \rangle^{-1}(Y) = Y^{)3,5(} = \overline{y_1 y_2} \varphi_3^{-1}(y_3, 5) y_4 y_5 y_6 y_7 =$$

$$\overline{40}[f_6(5)]^{-1}(0,5)5122 = \underline{15}\mathbf{5}5122 = X.$$

(note that the sixth column of $f_6(5)$ defines $[f_6(5)](0,5) = 5$, $[f_6(5)](1,5) = 4$, $[f_6(5)](2,5) = 3$, $[f_6(5)](3,5) = 2$, $[f_6(5)](4,5) = 1$ and $[f_6(5)](5,5) = 0$; hence, $[f_6(5)]^{-1}(0,5) = 5$). Since any complementation function is $(m-1)$-step invertible, also in this case we could have simplified the design and by using only two complementation functions by letting $\varphi_0 = \varphi_1 = \varphi_3 = \varphi_5 = \varphi_6 = f_6(4)$ and $\varphi_7 = f_6(3)$. However, when $m = 6$ or more it does not seem to be possible to give a code design with only one complementation function.

With regard to the complexity, also for this scheme lookup tables and/or circuits of size $O(p) = O(mk \log_m k)$ $m$-ary digits can be used to encode and decode the balancing index $h_b \in [0, p-1]$ to and from the check symbol $C_{h_b} \in \Gamma_{h_b} \in \mathcal{CS}$, respectively. To encode, we just need to do at most $p$ sequential steps to compute the $p$ balancing functions and each step requires $O(\log_m k)$ $m$-ary digits operations to compute a constant number of quantities (such as, the $m$-ary digits representing the integer $w(X^{(d_h)})$ from $w(X^{(d_{h-1})})$. Obviously, this can be done in $O(\log_m k)$ digit operations if we have a table look-up to compute the weight of any $m$-ary word of length $i(d_h) - i(d_{h-1}) = O(\log_m k)$). So, a total of $O(p \log_m k)$ $m$-ary digits operations are needed. Note that $p \leq \binom{r}{\lfloor R/2 \rfloor}_m \simeq \sqrt{6/[\pi(m+1)R]} \cdot m^r$ (see (29) in [7]) and, obviously, $m^{r-1} < K \leq m^r$ (from (29)), so $p = O(k/\sqrt{\log_m k})$. Hence, a total of $O(p \log_m k) = O(k\sqrt{\log_m k})$ $m$-ary digits operations are needed to encode (this is, considerably less than the simple scheme). While decoding, as in the simple scheme, a parallel circuit of size $O(mk \log_m k)$ can output from $h_b \in [0, p-1]$ a length $k$ vector to be added component-wise (according to the possibly at most $\lfloor m/2 \rfloor - 1$ complementation functions used in the design) to the received information part and obtain the original information word. So, a total of $O(1)$ $m$-ary digit operations are needed to decode.

## IV. **Transmitting extra information for the simple and refined coding schemes**

For the simple scheme, Weber and Immink [13] and Swart and Weber [8] proposed to transmit extra auxiliary data by exploiting the degree of freedom of selecting from more than one possible balanced encoding of a given information word. In fact, for the binary case, the authors in [13] showed that by choosing the balancing index of any given information word $X$, the encoder can convey some extra $\delta k = (1/2)\log_2 k - 0.916$ information bits on average. In this way, the minimum redundancy of the improved simple binary scheme becomes (note that (1) for $m = 2$ implies $r = n - k = \log_2 k + \Theta(\log \log k)$)

$$r' \stackrel{\text{def}}{=} r - \delta k = \frac{1}{2}\log_2 k + \Theta(\log \log k).$$

Here, to improve the redundancy (that is, to make $\delta k$ as large as possible), we not only propose to choose among the possibly many balancing indices of $X$, but also propose to add more balancing functions to the code design by using the unused check symbols. In this way, for any $m \geq 2$ and for both the simple and refined scheme, the new balancing functions are encoded by the unused check symbols and can simply be chosen to be the identity function. For example, consider the following code design obtained with the simple scheme for $m = 2$, $k = 6$ and $r = 4$.

$$\begin{aligned}
\langle 0011\rangle(X) &\stackrel{\text{def}}{=} X^{(0)} = x_1x_2x_3x_4x_5x_6 = X,\\
\langle 0101\rangle(X) &\stackrel{\text{def}}{=} X^{(1)} = \overline{x_1}x_2x_3x_4x_5x_6,\\
\langle 0110\rangle(X) &\stackrel{\text{def}}{=} X^{(2)} = \overline{x_1x_2}x_3x_4x_5x_6,\\
\langle 1001\rangle(X) &\stackrel{\text{def}}{=} X^{(3)} = \overline{x_1x_2x_3}x_4x_5x_6,\\
\langle 1010\rangle(X) &\stackrel{\text{def}}{=} X^{(4)} = \overline{x_1x_2x_3x_4}x_5x_6,\\
\langle 1100\rangle(X) &\stackrel{\text{def}}{=} X^{(5)} = \overline{x_1x_2x_3x_4x_5}x_6.
\end{aligned}$$

The above design can be improved by adding the extra 9 balancing functions $\langle C\rangle(X) \stackrel{\text{def}}{=} X$, for $C = 0000$, $0001$, $0010$, $0100$, $1000$, $0111$, $1011$, $1101$, $1110$ and $1111$. So, if $X = 100100 \in \mathbb{Z}_2^6$ then

$$\begin{array}{llll}
\langle 0011\rangle(X) = X^{(0)} & = 100100 = X, & w(X^{(0)}) = 2, & \\
\langle 0101\rangle(X) = X^{(1)} & = 000100, & w(X^{(1)}) = 1, & \\
\langle 0110\rangle(X) = X^{(2)} & = 010100, & w(X^{(2)}) = 2, & \\
\langle 1001\rangle(X) = X^{(3)} & = 011100, & w(X^{(3)}) = 3, & \leftarrow\\
\langle 1010\rangle(X) = X^{(4)} & = 011000, & w(X^{(4)}) = 2, & \\
\langle 1100\rangle(X) = X^{(5)} & = 011010, & w(X^{(5)}) = 3, & \leftarrow\\
\hline
\langle 0000\rangle(X) = X & = 100100, & w(X) = 2, & \\
\langle 0001\rangle(X) = X & = 100100, & w(X) = 2, & \\
\langle 0010\rangle(X) = X & = 100100, & w(X) = 2, & \\
\langle 0100\rangle(X) = X & = 100100, & w(X) = 2, & \\
\langle 1000\rangle(X) = X & = 100100, & w(X) = 2, & \\
\langle 0111\rangle(X) = X & = 100100, & w(X) = 2, & \leftarrow\\
\langle 1011\rangle(X) = X & = 100100, & w(X) = 2, & \leftarrow\\
\langle 1101\rangle(X) = X & = 100100, & w(X) = 2, & \leftarrow\\
\langle 1110\rangle(X) = X & = 100100, & w(X) = 2, & \leftarrow\\
\langle 1111\rangle(X) = X & = 100100, & w(X) = 2; &
\end{array}$$

and so $\mathcal{E}(X = 100100)$ can be chosen in the following $6$ $(> 2)$ different ways:

$$\begin{array}{l}
\mathcal{E}(X) = \langle 1001\rangle(X)1001 = X^{(3)}\ 1001 = 011100\,1001,\\
\mathcal{E}(X) = \langle 1100\rangle(X)1100 = X^{(5)}\ 1100 = 011010\,1100,\\
\hline
\mathcal{E}(X) = \langle 0111\rangle(X)0111 = X\ \ 0111 = 100100\,0111,\\
\mathcal{E}(X) = \langle 1011\rangle(X)1011 = X\ \ 1011 = 100100\,1011,\\
\mathcal{E}(X) = \langle 1101\rangle(X)1101 = X\ \ 1101 = 100100\,1101,\\
\mathcal{E}(X) = \langle 1110\rangle(X)1110 = X\ \ 1110 = 100100\,1110.
\end{array}$$

Table I and Table II respectively show the parameters of the simple and refined schemes with the improvements suggested above. In each table, the four columns refer to the value of $m = 2$, $3$, $4$ and $5$. For each $m$, the first and second subcolumns show the number $k$ of information digits and $r$ of check digits, respectively. The third subcolumn shows the quantity $k' \stackrel{\text{def}}{=} k + \delta k$;

where $\delta k$ is the amount of information coming from the balancing index choice and the unused check symbol contribution. The fourth subcolumn shows $\Delta \stackrel{\text{def}}{=} k_{opt} - k'$. The values in the third and fourth subcolumns which are above the double line "$\cdots = \cdots$" are exact values. In this case, $\Delta$ is computed as the difference between the maximum number $k_{opt} \stackrel{\text{def}}{=} \log_m \binom{n}{\lfloor N/2 \rfloor}_m$ of information digits that can be conveyed with a length $n = k + r$ $m$-ary balanced code. Also, $k' \stackrel{\text{def}}{=} k + \delta k$, with $\delta k$ computed by taking the average over all $m$-ary information words of length $k$. The values which are below the double line are approximated. In this last case, $k_{opt}$ is approximated with the formula $k_{opt} \approx n - (1/2)\log_m[n(m^2-1)] - (1/2)\log_m(\pi/6)$ (see (29) in [7]), whereas, $\delta k$ is computed by taking the average over 10 million samples. From the data in the table we conjecture $r' \stackrel{\text{def}}{=} n - k' = (1/2)\log_m k + \Theta(\log\log k)$, for all $m \in \mathbb{N}$, $m \geq 2$.

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## Appendix I

Here we prove the following theorem which directly gives the relations (used in the code design of Section III) in the statement of Theorem 4 once it is noted that the integer sequence $s_m(n,c)$ defined in (54) is such that $s_m(n,c) \stackrel{\text{def}}{=} |S_m(n,c)|$, with $S_m(n,c)$ defined in (41).

***Theorem 6:*** Given $m, n \in \mathbb{N}$ and $c \in \mathbb{Z}_{m-1}$, let $s_m(n,c)$ be defined in (54). Then the following relations hold.

$$\begin{aligned} s_m(n,0) &= s_m(n,1)+1 \quad \text{and} \\ s_m(n,c) &= s_m(n,m-c), \quad \text{for all } c \in [2, \lfloor m/2 \rfloor]. \end{aligned}$$

The above theorem follows from the combinatorial properties proved in Theorem 7 below of the $m$-nomial integer sequence. Some preliminaries are needed. Let $m \in \mathbb{N}$ be fixed. Given $n \in \mathbb{N}$, let the $m$-nomial integer sequence $\beta_m(n) \stackrel{\text{def}}{=} \left\{ \binom{n}{w}_m : w \in \mathbb{Z} \right\}$ be defined as

$$\binom{n}{w}_m \stackrel{\text{def}}{=} |\{X \in \mathbb{Z}_m^n : w(X) = w\}| = \sum_{y \in \mathbb{Z}_m} \binom{n-1}{w-y}_m, \tag{50}$$

where the initial conditions are $\binom{0}{0}_m \stackrel{\text{def}}{=} 1$ and $\binom{0}{w}_m \stackrel{\text{def}}{=} 0$. For example, when $m=5$,

$$\begin{aligned} \beta_5(0) &= \left\{ \ldots, 0, \tbinom{0}{0}_5 = 1, 0, \ldots \right\}, \\ \beta_5(1) &= \left\{ \ldots, 0, \tbinom{1}{0}_5 = 1,1,1,1,1 = \tbinom{1}{4}_5, 0, \ldots \right\}, \\ \beta_5(2) &= \left\{ \ldots, 0, \tbinom{2}{0}_5 = 1,2,3,4,5,4,3,2,1 = \tbinom{2}{8}_5, 0, \ldots \right\}, \\ \beta_5(3) &= \left\{ \ldots, 0, \tbinom{3}{0}_5 = 1,3,6,10,15,18,19,18,15,10,6,3,1 = \tbinom{3}{12}_5, 0, \ldots \right\}, \\ &\vdots \end{aligned}$$

Whereas, when $m=6$,

$$\begin{aligned} \beta_6(0) &= \left\{ \ldots, 0, \tbinom{0}{0}_6 = 1, 0, \ldots \right\}, \\ \beta_6(1) &= \left\{ \ldots, 0, \tbinom{1}{0}_6 = 1,1,1,1,1,1 = \tbinom{1}{5}_6, 0, \ldots \right\}, \\ \beta_6(2) &= \left\{ \ldots, 0, \tbinom{2}{0}_6 = 1,2,3,4,5,6,5,4,3,2,1 = \tbinom{2}{10}_6, 0, \ldots \right\}, \\ \beta_6(3) &= \left\{ \ldots, 0, \tbinom{3}{0}_6 = 1,3,6,10,15,21,25,27,27,25,21,15,10,6,3,1 = \tbinom{3}{15}_6, 0, \ldots \right\}, \\ &\vdots \end{aligned}$$

Now, given the $m$-nomial integer sequence, for all $m, n \in \mathbb{N}$, define the new integer sequence $\tau_m(n) \stackrel{\text{def}}{=} \{t_m(n,x) : x \in \mathbb{Z}\}$ as

$$t_m(n,x) \stackrel{\text{def}}{=} \binom{n}{x}_m - \binom{n}{x-1}_m. \tag{51}$$

For example, when $m=5$,

$$\begin{aligned} \tau_5(0) &= \{\ldots, 0, t_5(0,0) = 1, -1 = t_5(0,1), 0, \ldots\}, \\ \tau_5(1) &= \{\ldots, 0, t_5(1,0) = 1,0,0,0,0,-1 = t_5(1,5), 0, \ldots\}, \\ \tau_5(2) &= \{\ldots, 0, t_5(2,0) = 1,1,1,1,1,-1,-1,-1,-1,-1 = t_5(2,9), 0, \ldots\}, \\ \tau_5(3) &= \{\ldots, 0, t_5(3,0) = 1,2,3,4,5,3,1,-1,-3,-5,-4,-3,-2,-1 = t_5(3,13), 0, \ldots\}, \\ &\vdots \end{aligned}$$

Whereas, when $m=6$,

$$\begin{aligned} \tau_6(0) &= \{\ldots, 0, t_6(0,0) = 1, -1 = t_6(0,1), 0, \ldots\}, \\ \tau_6(1) &= \{\ldots, 0, t_6(1,0) = 1,0,0,0,0,0,-1 = t_6(1,6), 0, \ldots\}, \\ \tau_6(2) &= \{\ldots, 0, t_6(2,0) = 1,1,1,1,1,1,-1,-1,-1,-1,-1,-1 = t_6(2,11), 0, \ldots\}, \\ \tau_6(3) &= \{\ldots, 0, t_6(3,0) = 1,2,3,4,5,6,4,2,0, \\ &\qquad -2,-4,-6,-5,-4,-3,-2,-1 = t_6(3,16), 0, \ldots\}, \\ &\vdots \end{aligned}$$

The following theorem holds.

***Theorem** 7:* Given $m,n\in\mathbb{N}$, the following properties hold for the integer sequence $\tau_m(n)$ defined as in (51). As usual, let $N\stackrel{\text{def}}{=}(m-1)n$.

P1: For all $x\in\mathbb{Z}$, $t_m(n,(N+1)-x)=-t_m(n,x)$. Furthermore,

$$\begin{cases} t_m(n,x)=0 & \text{if } x\notin[0,N+1],\\ t_m(n,x)\geq 0 & \text{if } x\in[0,\lceil(N+1)/2\rceil),\\ t_m(n,(N+1)/2)=0 & \text{if } N \text{ is odd},\\ t_m(n,x)=-t_m(n,(N+1)-x)\leq 0 & \text{if } x\in(\lfloor(N+1)/2\rfloor,N+1]. \end{cases} \tag{52}$$

P2: The integer sequence $\tau_m(n)$ can also be defined by the recurrence realtion

$$t_m(n,x)\stackrel{\text{def}}{=}t_m(n-1,x)+t_m(n-1,x-1)+\ldots+t_m(n-1,x-(m-1)) \tag{53}$$

with initial conditions $t_m(0,0)\stackrel{\text{def}}{=}1$, $t_m(0,1)\stackrel{\text{def}}{=}-1$ and $t_m(0,x)\stackrel{\text{def}}{=}0 \iff x\neq 0,1$.

P3: Given $c\in\mathbb{Z}$, let $\tilde{s}_m(n,c)\stackrel{\text{def}}{=}\sum_{x\in(m-1)\mathbb{Z}+c}t_m(n,x)$. Then, for all $c\in\mathbb{Z}$,

$$\tilde{s}_m(n,c)=2\tilde{s}_m(n-1,c)+\tilde{s}_m(n-1,c-1)+\ldots+\tilde{s}_m(n-1,c-(m-2)).$$

P4: Given $c\in\mathbb{Z}$, let

$$s_m(n,c)\stackrel{\text{def}}{=}\sum_{x\in[(m-1)\mathbb{Z}+c]\cap\{x\in\mathbb{Z}:\,t_m(n,x)>0\}}t_m(n,x)= \tag{54}$$

$$\sum_{x\in[(m-1)\mathbb{Z}+c]\cap[0,\lceil(N+1)/2\rceil)}t_m(n,x)=\sum_{x\in[(m-1)\mathbb{Z}+c]\cap[0,\lfloor N/2\rfloor]}t_m(n,x).$$

Then, for all $c\in\mathbb{Z}_{m-1}$,

$$\tilde{s}_m(n,c)=s_m(n,c)-s_m(n,m-c)=\begin{cases} +1 & \text{if } c=0,\\ -1 & \text{if } c=1,\\ 0 & \text{if } c\in[2,m-2]. \end{cases} \tag{55}$$

*Proof:* The properties P1 and P2 follow directly from (50). The property P3 follows from (53) and $\tilde{s}_m(n-1,c)=\tilde{s}_m(n-1,c-(m-1))$, which holds for all $m,n\in\mathbb{N}$ and $c\in\mathbb{Z}$. With regard to P4, first note that

$$\{x\in\mathbb{Z}:\ t_m(n,x)>0\}\cup\{x\in\mathbb{Z}:\ t_m(n,x)=0\}=\{x\in\mathbb{Z}:\ t_m(n,x)\geq 0\}=$$

$$\{x\in\mathbb{Z}:\ x\in[0,\lceil(N+1)/2\rceil)\}=\{x\in\mathbb{Z}:\ x\in[0,\lfloor N/2\rfloor]\}$$

because of (52). Hence, all the equalities in (54) hold because

$$s_m(n,c)=\sum_{x\in[(m-1)\mathbb{Z}+c]\cap\{x\in\mathbb{Z}:\,t_m(n,x)\geq 0\}}t_m(n,x).$$

With regard to the leftmost equality in (55), the property P1 implies

$$\tilde{s}_m(n,c)=\sum_{x\in(m-1)\mathbb{Z}+c}t_m(n,x)= \tag{56}$$

$$\sum_{[x\in(m-1)\mathbb{Z}+c]\cap\{x\in\mathbb{Z}:\,t_m(n,x)\geq 0\}}t_m(n,x)-\sum_{[x\in(m-1)\mathbb{Z}+c]\cap\{x\in\mathbb{Z}:\,t_m(n,x)\leq 0\}}(-t_m(n,x))=$$

$$s_m(n,c) - \sum_{[x\in(m-1)\mathbb{Z}+c]\cap(\lfloor(N+1)/2\rfloor,N+1]} (-t_m(n,x)) =$$

$$s_m(n,c) - \sum_{[x\in(m-1)\mathbb{Z}+c]\cap(\lfloor(N+1)/2\rfloor,N+1]} t_m(n,(N+1)-x).$$

Now, since $N = (m-1)n \in (m-1)\mathbb{Z}$, if we let $y \stackrel{\text{def}}{=} (N+1)-x$ then $x \in (m-1)\mathbb{Z}+c \iff y \in (m-1)\mathbb{Z}+(m-c)$. Furthermore, $x \in (\lfloor(N+1)/2\rfloor, N+1] \iff y \in \left[0, (N+1) - \lfloor(N+1)/2\rfloor = \left\lceil \frac{N+1}{2} \right\rceil\right)$. Hence,

$$x \in [(m-1)\mathbb{Z}+c] \cap \left(\left\lfloor \frac{N+1}{2} \right\rfloor, N+1\right] \iff y \in [(m-1)\mathbb{Z}+(m-c)] \cap \left[0, \left\lceil \frac{N+1}{2} \right\rceil\right).$$

The above relation, (54) and (56) imply $\tilde{s}_m(n,c) = s_m(n,c) - s_m(n,m-c)$. Finally, using the property P3, the rightmost equality in (55) can be proved by induction on $n$. As a matter of fact, since $\tau_m(0) = \{\ldots, 0, t_m(0,0)=1, -1=t_m(0,1), 0, \ldots\}$, the rightmost equality of (55) is true for $n=0$. Now, the relation in P3 for $c=0$, the periodicity of $\tilde{s}_m(n,c)$ as a function of $c$ with period $m-1$, and the inductive hypothesis imply,

$$\tilde{s}_m(n+1,0) = 2\tilde{s}_m(n,0) + \tilde{s}_m(n,-1) + \tilde{s}_m(n,-2) + \ldots + \tilde{s}_m(n,-(m-2)) =$$

$$2\tilde{s}_m(n,0) + \tilde{s}_m(n,m-2) + \tilde{s}_m(n,-3) + \ldots + \tilde{s}_m(n,1) = +2+0+0+\ldots-1 = +1.$$

Also, for $c=1$, we have

$$\tilde{s}_m(n+1,1) = 2\tilde{s}_m(n,1) + \tilde{s}_m(n,0) + \tilde{s}_m(n,-1) + \ldots + \tilde{s}_m(n,1-(m-2)) =$$

$$2\tilde{s}_m(n,1) + \tilde{s}_m(n,0) + \tilde{s}_m(n,m-2) + \ldots + \tilde{s}_m(n,2) = -2+1+0+\ldots+0 = -1.$$

Whereas, for $c=2$, we have

$$\tilde{s}_m(n+1,2) = 2\tilde{s}_m(n,2) + \tilde{s}_m(n,1) + \tilde{s}_m(n,0) + \ldots + \tilde{s}_m(n,2-(m-2)) =$$

$$2\tilde{s}_m(n,2) + \tilde{s}_m(n,1) + \tilde{s}_m(n,0) + \ldots + \tilde{s}_m(n,3) = 0-1+1+\ldots+0 = 0;$$

and we analogously obtain the same conclusion for $c = 3, 4, \ldots, m-2$. ■

At this point, the property P4 of Theorem 7 implies Theorem 6.

## Appendix II

Here we give the proof of Theorem 5.

*Proof:* For all $h \in [0, p-1]$, the integer $d_h$ is defined by (19) and $0 = d_0 < d_1 < \ldots < d_{p-1}$ because none of the $\Gamma_h$'s is empty. Now, only the following cases are possible when $h \in [1, p-1]$.

C1: $c=1$, $h=t\in 2\mathbb{Z}$, $t \in [0, |A_1 \cup B_1| - 1]$. So, $\Gamma_h \in A_1 = S_m(r,0)$ and $\Gamma_{h-1} \in B_1 = S_m(r,1)$. In this case, $d_h - d_{h-1} \in (m-1)\mathbb{Z}$ (hence, $d_h - d_{h-1} \geq m-1$). In fact, from (41) and (40),

$$\left\lceil \frac{|\Gamma_h|}{2} \right\rceil \in \left\lceil \frac{R+1}{2} \right\rceil - [(m-1)\mathbb{Z}] \qquad \text{and} \qquad \left\lfloor \frac{|\Gamma_{h-1}|}{2} \right\rfloor \in \left\lfloor \frac{R+1}{2} \right\rfloor - [(m-1)\mathbb{Z}+1].$$

Hence, from (19) and $R = (m-1)r \in (m-1)\mathbb{Z}$,

$$d_h - d_{h-1} = \left\lfloor \frac{|\Gamma_{h-1}|}{2} \right\rfloor + \left\lceil \frac{|\Gamma_h|}{2} \right\rceil \in$$

$$\left\lfloor \frac{R+1}{2} \right\rfloor - [(m-1)\mathbb{Z} + 1] + \left\lceil \frac{R+1}{2} \right\rceil - [(m-1)\mathbb{Z}] =$$

$$R + 1 - 1 + (m-1)\mathbb{Z} = (m-1)\mathbb{Z}.$$

C2: $c = 1$, $h = t \in 2\mathbb{Z}+1$, $t \in [1, |A_1 \cup B_1| - 1]$. So, $\Gamma_h \in B_1 = S_m(r, 1)$ and $\Gamma_{h-1} \in A_1 = S_m(r, 0)$. In this case, $d_h - d_{h-1} \in (m-1)\mathbb{Z}$ (hence, $d_h - d_{h-1} \geq m-1$). In fact, from (41) and (40),

$$\left\lceil \frac{|\Gamma_h|}{2} \right\rceil \in \left\lceil \frac{R+1}{2} \right\rceil - [(m-1)\mathbb{Z} + 1] \qquad \text{and} \qquad \left\lfloor \frac{|\Gamma_{h-1}|}{2} \right\rfloor \in \left\lfloor \frac{R+1}{2} \right\rfloor - [(m-1)\mathbb{Z}].$$

As in C1, from (19) and $R \in (m-1)\mathbb{Z}$ we get $d_h - d_{h-1} = \lfloor |\Gamma_{h-1}|/2 \rfloor + \lceil |\Gamma_h|/2 \rceil \in (m-1)\mathbb{Z}$.

C3: $c = 2$, $h = |A_1 \cup B_1| \in 2\mathbb{Z}+1$, $t = 0$. So, $\Gamma_h \in B_2 = S_m(r, 2)$ and $\Gamma_{h-1} \in A_1 = S_m(r, 0)$. In this case, $d_h - d_{h-1} \in (m-1)\mathbb{Z} - 1 = (m-1)\mathbb{Z} + (m-2)$ (hence, $d_h - d_{h-1} \geq m-2$). In fact, from (41) and (40),

$$\left\lceil \frac{|\Gamma_h|}{2} \right\rceil \in \left\lceil \frac{R+1}{2} \right\rceil - [(m-1)\mathbb{Z} + 2] \qquad \text{and} \qquad \left\lfloor \frac{|\Gamma_{h-1}|}{2} \right\rfloor \in \left\lfloor \frac{R+1}{2} \right\rfloor - [(m-1)\mathbb{Z}].$$

So, from (19) and $R \in (m-1)\mathbb{Z}$ we get $d_h - d_{h-1} = \lfloor |\Gamma_{h-1}|/2 \rfloor + \lceil |\Gamma_h|/2 \rceil \in (m-1)\mathbb{Z} - 1$.

C4: $c \geq 2$, $h = \sum_{i=1}^{c-1} |A_i \cup B_i| + t \in 2\mathbb{Z}+1$, $t \in [1, |A_c \cup B_c| - 1]$. So, $\Gamma_h \in B_c = S_m(r, c)$ and $\Gamma_{h-1} \in A_c = S_m(r, m-c)$. In this case, $d_h - d_{h-1} \in (m-1)\mathbb{Z}$ (hence, $d_h - d_{h-1} \geq m-1$). In fact, from (41) and (40),

$$\left\lceil \frac{|\Gamma_h|}{2} \right\rceil \in \left\lceil \frac{R+1}{2} \right\rceil - [(m-1)\mathbb{Z} + c] \text{ and } \left\lfloor \frac{|\Gamma_{h-1}|}{2} \right\rfloor \in \left\lfloor \frac{R+1}{2} \right\rfloor - [(m-1)\mathbb{Z} + (m-c)].$$

So, from (19) and $R \in (m-1)\mathbb{Z}$ we get $d_h - d_{h-1} = \lfloor |\Gamma_{h-1}|/2 \rfloor + \lceil |\Gamma_h|/2 \rceil \in (m-1)\mathbb{Z}$.

C5: $c \geq 2$, $h = \sum_{i=1}^{c-1} |A_i \cup B_i| + t \in 2\mathbb{Z}$, $t \in [1, |A_c \cup B_c| - 1]$. So, $\Gamma_h \in A_c = S_m(r, m-c)$ and $\Gamma_{h-1} \in B_c = S_m(r, c)$. So, as above, $d_h - d_{h-1} \in (m-1)\mathbb{Z}$ (hence, $d_h - d_{h-1} \geq m-1$).

C6: $c \geq 2$, $h = \sum_{i=1}^{c} |A_i \cup B_i| \in 2\mathbb{Z}+1$, $t = 0$. So, $\Gamma_h \in B_{c+1} = S_m(r, c+1)$ and $\Gamma_{h-1} \in A_c = S_m(r, m-c)$. So, as above, $d_h - d_{h-1} \in (m-1)\mathbb{Z} - 1$ (hence, $d_h - d_{h-1} \geq m-2$).

From the definition (45) of $\mathcal{CS}$, all the above, (22), (23) and $d_0 = 0 \in (m-1)\mathbb{Z}$ inductively imply (46), (47) and (48). In particular, from (23), (46) and $c \in [1, \lfloor m/2 \rfloor]$, we get

$$j(d_h) = (d_h - 1) \bmod (m-1) + 1 = (m-c) - 1 + 1 = m - c \in [\lceil m/2 \rceil, m-1];$$

which is (48). At this point, since (47), it is possible to let $\varphi_{i(d_h)} \stackrel{\text{def}}{=} f_m(m-c)$ as in (49). In this way, $\mathcal{CS}$ and $\mathcal{F}$ match (that is, satisfy (34)) because the $f_m(m-c)$'s where defined to be $j$-step invertible smooth complementation functions for $j = m - c \in [\lceil m/2 \rceil, m-1]$ (and so, $j \neq (m-1)/2$ if $m$ is odd). Hence, $\mathcal{B}$ is a well defined set of **one-to-one** balancing function because of Theorem 3. ■

TABLE I

RESULTS OF SIMPLE SCHEME. THE FOUR COLUMNS ARE REFERRED TO THE VALUES OF $m = 2, 3, 4$ AND $5$. FOR EACH $m$, THE FIRST AND SECOND SUBCOLUMNS SHOW THE NUMBER OF INFORMATION DIGITS, $k$, AND THE NUMBER OF CHECK DIGITS, $r$, RESPECTIVELY. THE THIRD SUBCOLUMN SHOWS THE QUANTITY $k' \stackrel{\text{def}}{=} k + \delta k$, WHERE $\delta k$ IS THE INFORMATION GIVEN BY THE BALANCING INDEX CHOICE. THE FOURTH SUBCOLUMN SHOWS $\Delta \stackrel{\text{def}}{=} k_{opt} - k'$.

| $m=2$ | | | | $m=3$ | | | | $m=4$ | | | | $m=5$ | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $k$ | $r$ | $k'$ | $\Delta$ | $k$ | $r$ | $k'$ | $\Delta$ | $k$ | $r$ | $k'$ | $\Delta$ | $k$ | $r$ | $k'$ | $\Delta$ |
| 1 | 2 | 1.500 | 0.085 | 1 | 2 | 1.667 | 0.105 | 1 | 2 | 1.698 | 0.094 | 1 | 2 | 1.745 | 0.085 |
| 2 | 2 | 2.500 | 0.085 | 2 | 3 | 3.513 | 0.066 | 2 | 3 | 3.598 | 0.040 | 2 | 3 | 3.665 | 0.027 |
| 3 | 4 | 5.050 | 0.079 | 3 | 3 | 4.322 | 0.182 | 3 | 3 | 4.494 | 0.096 | 3 | 3 | 4.571 | 0.069 |
| 4 | 4 | 6.050 | 0.079 | 4 | 4 | 6.300 | 0.080 | 4 | 3 | 5.336 | 0.192 | 4 | 3 | 5.472 | 0.122 |
| 5 | 4 | 6.746 | 0.232 | 5 | 4 | 7.211 | 0.118 | 5 | 4 | 7.331 | 0.112 | 5 | 4 | 7.417 | 0.103 |
| 6 | 4 | 7.746 | 0.232 | 6 | 4 | 8.129 | 0.154 | 6 | 4 | 8.277 | 0.137 | 6 | 4 | 8.353 | 0.135 |
| 7 | 5 | 9.627 | 0.225 | 7 | 4 | 9.048 | 0.193 | 7 | 4 | 9.204 | 0.170 | 7 | 4 | 9.294 | 0.166 |
| 8 | 5 | 10.504 | 0.241 | 8 | 4 | 9.965 | 0.238 | 8 | 4 | 10.158 | 0.192 | 8 | 4 | 10.239 | 0.194 |
| 9 | 5 | 11.389 | 0.356 | 9 | 4 | 10.868 | 0.300 | 9 | 4 | 11.097 | 0.220 | 9 | 4 | 11.189 | 0.220 |
| 10 | 5 | 12.292 | 0.359 | 10 | 5 | 12.854 | 0.250 | 10 | 4 | 12.058 | 0.238 | 10 | 4 | 12.144 | 0.243 |
| 11 | 6 | 14.294 | 0.276 | 11 | 5 | 13.801 | 0.274 | 11 | 4 | 13.005 | 0.262 | 11 | 4 | 13.102 | 0.264 |
| 12 | 6 | 15.294 | 0.276 | 12 | 5 | 14.752 | 0.297 | 12 | 4 | 13.969 | 0.280 | 12 | 4 | 14.063 | 0.283 |
| 13 | 6 | 16.177 | 0.318 | 13 | 5 | 15.706 | 0.317 | 13 | 4 | 14.918 | 0.306 | 13 | 4 | 15.027 | 0.300 |
| 14 | 6 | 17.177 | 0.318 | 14 | 5 | 16.663 | 0.336 | 14 | 4 | 15.878 | 0.329 | 21 | 4 | 22.812 | 0.401 |
| 15 | 6 | 18.067 | 0.361 | 15 | 5 | 17.622 | 0.354 | 15 | 5 | 17.793 | 0.377 | 22 | 5 | 24.657 | 0.533 |
| 16 | 6 | 19.067 | 0.361 | 16 | 5 | 18.584 | 0.370 | 51 | 5 | 53.164 | 0.641 | 95 | 5 | 97.001 | 0.782 |
| 17 | 6 | 19.957 | 0.410 | 17 | 5 | 19.548 | 0.386 | 52 | 6 | 54.841 | 0.951 | 96 | 6 | 98.615 | 1.162 |
| 18 | 6 | 20.957 | 0.410 | 18 | 5 | 20.513 | 0.400 | 193 | 6 | 195.330 | 1.018 | 437 | 6 | 439.202 | 1.118 |
| 19 | 6 | 21.829 | 0.482 | 19 | 5 | 21.480 | 0.415 | 194 | 7 | 196.870 | 1.474 | 438 | 7 | 440.581 | 1.739 |
| 20 | 6 | 22.829 | 0.482 | 25 | 5 | 27.327 | 0.473 | 709 | 7 | 711.597 | 1.289 | 2033 | 7 | 2035.482 | 1.364 |
| 21 | 7 | 24.769 | 0.488 | 26 | 6 | 29.202 | 0.569 | 710 | 8 | 712.919 | 1.966 | 2034 | 8 | 2036.683 | 2.163 |
| 22 | 7 | 25.704 | 0.505 | 70 | 6 | 72.587 | 0.790 | 2697 | 8 | 2699.924 | 1.482 | 9541 | 8 | 9543.842 | 1.524 |
| 23 | 7 | 26.692 | 0.517 | 71 | 7 | 74.263 | 1.102 | 2698 | 9 | 2701.105 | 2.301 | 9542 | 9 | 9544.942 | 2.424 |
| 24 | 7 | 27.634 | 0.529 | 196 | 7 | 198.826 | 1.104 | – | – | – | – | – | – | – | – |
| 25 | 7 | 28.620 | 0.543 | 197 | 8 | 200.345 | 1.581 | – | – | – | – | – | – | – | – |
| 26 | 7 | 29.568 | 0.552 | 553 | 8 | 556.033 | 1.434 | – | – | – | – | – | – | – | – |
| 27 | 7 | 30.551 | 0.569 | 554 | 9 | 557.449 | 2.016 | – | – | – | – | – | – | – | – |
| 28 | 7 | 31.504 | 0.575 | 1569 | 9 | 1572.375 | 1.621 | – | – | – | – | – | – | – | – |
| 29 | 7 | 32.483 | 0.596 | 1570 | 10 | 1573.636 | 2.360 | – | – | – | – | – | – | – | – |
| 30 | 7 | 33.442 | 0.599 | 4476 | 10 | 4479.735 | 1.786 | – | – | – | – | – | – | – | – |
| 34 | 7 | 37.244 | 0.751 | 4477 | 11 | 4480.893 | 2.628 | – | – | – | – | – | – | – | – |
| 35 | 8 | 39.136 | 0.825 | – | – | – | – | – | – | – | – | – | – | – | – |
| 70 | 8 | 73.616 | 0.915 | – | – | – | – | – | – | – | – | – | – | – | – |
| 71 | 9 | 75.436 | 1.077 | – | – | – | – | – | – | – | – | – | – | – | – |
| 126 | 9 | 129.925 | 1.211 | – | – | – | – | – | – | – | – | – | – | – | – |
| 127 | 10 | 131.674 | 1.451 | – | – | – | – | – | – | – | – | – | – | – | – |
| 252 | 10 | 256.288 | 1.370 | – | – | – | – | – | – | – | – | – | – | – | – |
| 253 | 11 | 257.840 | 1.812 | – | – | – | – | – | – | – | – | – | – | – | – |
| 462 | 11 | 466.487 | 1.744 | – | – | – | – | – | – | – | – | – | – | – | – |
| 463 | 12 | 467.937 | 2.291 | – | – | – | – | – | – | – | – | – | – | – | – |
| 924 | 12 | 928.870 | 1.869 | – | – | – | – | – | – | – | – | – | – | – | – |
| 925 | 13 | 930.235 | 2.503 | – | – | – | – | – | – | – | – | – | – | – | – |
| 1716 | 13 | 1721.214 | 2.083 | – | – | – | – | – | – | – | – | – | – | – | – |
| 1717 | 14 | 1722.490 | 2.806 | – | – | – | – | – | – | – | – | – | – | – | – |
| 3432 | 14 | 3437.548 | 2.250 | – | – | – | – | – | – | – | – | – | – | – | – |
| 3433 | 15 | 3438.771 | 3.028 | – | – | – | – | – | – | – | – | – | – | – | – |
| 6434 | 15 | 6439.895 | 2.452 | – | – | – | – | – | – | – | – | – | – | – | – |
| 6435 | 16 | 6441.078 | 3.269 | – | – | – | – | – | – | – | – | – | – | – | – |

TABLE II

RESULTS OF REFINED SCHEME. THE FOUR COLUMNS ARE REFERRED TO THE VALUES OF $m = 2, 3, 4$ AND $5$. FOR EACH $m$, THE FIRST AND SECOND SUBCOLUMNS SHOW THE NUMBER OF INFORMATION DIGITS, $k$, AND THE NUMBER OF CHECK DIGITS, $r$, RESPECTIVELY. THE THIRD SUBCOLUMN SHOWS THE QUANTITY $k' \stackrel{\text{def}}{=} k + \delta k$, WHERE $\delta k$ IS THE INFORMATION GIVEN BY THE BALANCING INDEX CHOICE. THE FOURTH SUBCOLUMN SHOWS $\Delta \stackrel{\text{def}}{=} k_{opt} - k'$.

| $m=2$ | | | | $m=3$ | | | | $m=4$ | | | | $m=5$ | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $k$ | $r$ | $k'$ | $\Delta$ | $k$ | $r$ | $k'$ | $\Delta$ | $k$ | $r$ | $k'$ | $\Delta$ | $k$ | $r$ | $k'$ | $\Delta$ |
| 1 | 1 | 1.000 | 0.000 | 1 | 1 | 1.000 | 0.000 | 1 | 1 | 1.000 | 0.000 | 1 | 1 | 1.000 | 0.000 |
| 2 | 2 | 2.500 | 0.085 | 2 | 2 | 2.614 | 0.066 | 2 | 2 | 2.672 | 0.058 | 2 | 2 | 2.681 | 0.080 |
| 3 | 2 | 3.250 | 0.072 | 3 | 2 | 3.532 | 0.047 | 3 | 2 | 3.585 | 0.053 | 3 | 2 | 3.636 | 0.057 |
| 4 | 2 | 4.250 | 0.072 | 4 | 2 | 4.445 | 0.060 | 4 | 2 | 4.540 | 0.050 | 4 | 2 | 4.582 | 0.058 |
| 5 | 3 | 6.032 | 0.097 | 5 | 3 | 6.228 | 0.153 | 5 | 2 | 5.472 | 0.056 | 5 | 2 | 5.545 | 0.050 |
| 6 | 3 | 6.852 | 0.125 | 6 | 3 | 7.188 | 0.141 | 6 | 3 | 7.003 | 0.440 | 6 | 2 | 6.499 | 0.056 |
| 7 | 3 | 7.852 | 0.125 | 7 | 3 | 8.168 | 0.115 | 7 | 3 | 8.143 | 0.271 | 7 | 3 | 8.033 | 0.456 |
| 8 | 4 | 9.699 | 0.153 | 8 | 3 | 9.132 | 0.109 | 8 | 3 | 9.109 | 0.266 | 8 | 3 | 9.011 | 0.449 |
| 9 | 4 | 10.589 | 0.156 | 9 | 3 | 10.094 | 0.109 | 9 | 3 | 10.164 | 0.186 | 9 | 3 | 10.052 | 0.381 |
| 10 | 4 | 11.589 | 0.156 | 10 | 3 | 11.069 | 0.099 | 10 | 3 | 11.174 | 0.143 | 10 | 3 | 11.066 | 0.343 |
| 11 | 4 | 12.510 | 0.142 | 11 | 3 | 12.041 | 0.094 | 11 | 3 | 12.159 | 0.137 | 11 | 3 | 12.099 | 0.288 |
| 12 | 4 | 13.510 | 0.142 | 12 | 3 | 13.005 | 0.100 | 12 | 3 | 13.140 | 0.127 | 12 | 3 | 13.111 | 0.255 |
| 13 | 4 | 14.426 | 0.144 | 13 | 3 | 13.972 | 0.103 | 13 | 3 | 14.127 | 0.122 | 13 | 3 | 14.136 | 0.210 |
| 14 | 4 | 15.426 | 0.144 | 14 | 4 | 15.576 | 0.447 | 14 | 3 | 15.109 | 0.114 | 31 | 3 | 32.041 | 0.077 |
| 15 | 4 | 16.340 | 0.155 | 15 | 4 | 16.555 | 0.444 | 15 | 3 | 16.098 | 0.110 | 32 | 4 | 33.355 | 0.746 |
| 16 | 4 | 17.340 | 0.155 | 16 | 4 | 17.599 | 0.377 | 21 | 3 | 22.010 | 0.101 | 156 | 4 | 157.486 | 0.151 |
| 17 | 5 | 19.148 | 0.280 | 17 | 4 | 18.583 | 0.371 | 22 | 4 | 23.463 | 0.618 | 157 | 5 | 158.771 | 0.862 |
| 18 | 5 | 20.097 | 0.270 | 18 | 4 | 19.565 | 0.368 | 85 | 4 | 86.481 | 0.157 | 781 | 5 | 782.947 | 0.195 |
| 19 | 5 | 21.097 | 0.270 | 19 | 4 | 20.598 | 0.315 | 86 | 5 | 87.860 | 0.769 | 782 | 6 | 784.146 | 0.995 |
| 20 | 5 | 22.066 | 0.244 | 40 | 4 | 41.469 | 0.157 | 341 | 5 | 342.965 | 0.183 | 3906 | 6 | 3908.431 | 0.213 |
| 21 | 5 | 23.066 | 0.244 | 41 | 5 | 42.962 | 0.644 | 342 | 6 | 344.218 | 0.928 | 3907 | 7 | 3909.558 | 1.086 |
| 22 | 5 | 24.037 | 0.221 | 121 | 5 | 122.942 | 0.205 | 5461 | 7 | 5463.903 | 0.250 | – | – | – | – |
| 23 | 5 | 25.037 | 0.221 | 122 | 6 | 124.299 | 0.841 | 5462 | 8 | 5465.036 | 1.117 | – | – | – | – |
| 24 | 5 | 26.004 | 0.205 | 364 | 6 | 366.405 | 0.252 | – | – | – | – | – | – | – | – |
| 25 | 5 | 27.004 | 0.205 | 365 | 7 | 367.725 | 0.930 | – | – | – | – | – | – | – | – |
| 26 | 5 | 27.965 | 0.198 | 1093 | 7 | 1095.887 | 0.274 | – | – | – | – | – | – | – | – |
| 27 | 5 | 28.965 | 0.198 | 1094 | 8 | 1097.129 | 1.031 | – | – | – | – | – | – | – | – |
| 28 | 5 | 29.914 | 0.206 | 3280 | 8 | 3283.353 | 0.309 | – | – | – | – | – | – | – | – |
| 29 | 5 | 30.914 | 0.206 | 3281 | 9 | 3284.526 | 1.136 | – | – | – | – | – | – | – | – |
| 30 | 5 | 31.864 | 0.215 | – | – | – | – | – | – | – | – | – | – | – | – |
| 31 | 5 | 32.872 | 0.218 | – | – | – | – | – | – | – | – | – | – | – | – |
| 32 | 6 | 34.448 | 0.603 | – | – | – | – | – | – | – | – | – | – | – | – |
| 64 | 6 | 66.390 | 0.220 | – | – | – | – | – | – | – | – | – | – | – | – |
| 65 | 7 | 67.977 | 0.612 | – | – | – | – | – | – | – | – | – | – | – | – |
| 127 | 7 | 129.786 | 0.355 | – | – | – | – | – | – | – | – | – | – | – | – |
| 128 | 8 | 131.338 | 0.792 | – | – | – | – | – | – | – | – | – | – | – | – |
| 256 | 8 | 259.280 | 0.372 | – | – | – | – | – | – | – | – | – | – | – | – |
| 257 | 9 | 260.711 | 0.935 | – | – | – | – | – | – | – | – | – | – | – | – |
| 511 | 9 | 514.772 | 0.391 | – | – | – | – | – | – | – | – | – | – | – | – |
| 512 | 10 | 516.199 | 0.961 | – | – | – | – | – | – | – | – | – | – | – | – |
| 1024 | 10 | 1028.267 | 0.401 | – | – | – | – | – | – | – | – | – | – | – | – |
| 1025 | 11 | 1029.546 | 1.120 | – | – | – | – | – | – | – | – | – | – | – | – |
| 2047 | 11 | 2051.724 | 0.447 | – | – | – | – | – | – | – | – | – | – | – | – |
| 2048 | 12 | 2053.005 | 1.165 | – | – | – | – | – | – | – | – | – | – | – | – |
| 4096 | 12 | 4101.195 | 0.477 | – | – | – | – | – | – | – | – | – | – | – | – |
| 4097 | 13 | 4102.450 | 1.222 | – | – | – | – | – | – | – | – | – | – | – | – |
| 8191 | 13 | 8196.662 | 0.511 | – | – | – | – | – | – | – | – | – | – | – | – |
| 8192 | 14 | 8197.862 | 1.311 | – | – | – | – | – | – | – | – | – | – | – | – |