CHAPTER 11

CLOAKING IN TERMS OF NONRADIATING CANCELLING CURRENTS

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A theory of nonradiating cancelling (NRC) currents is formulated in order to derive sufficient conditions for cloaking on the dyadic constitutive parameters of an anisotropic linear metamaterial of arbitrary shape illuminated by arbitrary sources. The link between the NRC current theory and the constitutive parameters of the cloak is established by applying the volumetric equivalence theorem. The constitutive parameters are eventually found as a function of two vector potentials satisfying simple boundary conditions. Two-dimensional and three-dimensional transformation optics cloaks are derived as particular cases of the general formulation.

1. Introduction

The "transformation optics" introduced by Pendry[1] is a powerful conceptual and practical tool to shape and address electromagnetic waves through the appropriate design of the constitutive parameters of media. This method was applied to design "electromagnetic cloaks", shells of anisotropic material capable of rendering any object within their interior cavities invisible to detection from outside the cloaks. The perfect cloak ensures that for any incident field the electromagnetic scattered field vanishes in the free space external to the cloaking shell, and the total field vanishes inside the free-space cavity of the shell. Thus, any object placed in the cavity does not perturb the electromagnetic field outside the cloak and to an external observer it appears as if the object and cloak were absent. Such invisibility cloaks require materials with inhomogeneous, anisotropic permittivities and permeabilities that cannot be found in nature (thus being referred to as metamaterials). An approximation to the ideal constitutive parameters for a circular cylinder was realized and experimentally characterized by Schurig et al.[2]

The formulation proposed by Pendry[1] is based on spatial coordinate transformations and corresponding transformations of Maxwell's equations that provide expressions for the required inhomogeneity and anisotropy of the permittivity and permeability of the cloaking metamaterial. A similar approximate (in the geometrical optics limit) method for cloaking was presented by Leonhardt[3], where the Helmholtz equation is transformed by conformal mapping to produce cloaking effects. Subsequently, Leonhardt and Philbin discussed the conformal mapping theory in the context of general relativity by analogy with the deviation of optical rays close to a gravitational mass[4]. More recently, cloaking has been reformulated as a boundary value problem with a single first-order Maxwell differential equation for linear anisotropic media[5]. This alternative formulation reveals the boundary values of the fields at the inner and outer surfaces of a cloak that yield zero scattered fields outside the cloak and zero total fields inside the cloak cavity. Moreover, in Ref. 5 the difference between the 2D case (cylindrical cloak) and the 3D case (spherical cloak) is discussed with reference to the behaviour of the fields and polarization densities at the inner surface of the cloak. A more extensive investigation of the singular behaviour of the fields and polarization densities has been carried out in Ref. 6. The formulation introduced in Ref. 5 has been further developed in Ref. 7 with an investigation of the group velocities and the engineering limitations of broadband cloaks.

Although the nonlinear transformation in Ref. 1 is physically appealing, different types of field-transformations can be adopted that do not resort to any space compressions. In Ref. 8, general affine transformations have been introduced with the purpose of finding different solutions of Maxwell's equations in anisotropic and bianisotropic media. Based on a simplification of the general transformation in Ref. 8, a linear transformation is used in Ref. 9 to categorize different types of bianisotropic metamaterials. Depending on the coefficients used in the linear transformation, artificial media are defined in Ref. 9 that can produce the fields in a prescribed fashion in the volume occupied by the medium.

In Ref. 10, the use of duality conditions on a linear transformation as that in Ref. 9 yields the definition of a new medium that possesses interesting invisibility and cloaking properties. The cloaking medium in Ref. 10 does not involve any space compression, and leads to field solutions with the Poynting vector that decreases in amplitude along the path without changing direction inside the medium. This way, the ray velocity matches the speed of light and thus the medium may not be subject to the narrow bandwidth limitation of transformation optics cloaks.

In this chapter, we present an alternative approach to cloaking, which is based on nonradiating cancelling (NRC) equivalent currents. This approach allows one to incorporate the existing two-dimensional (2D) and three-dimensional (3D) transformation optics formulations of cloaking in a general framework that provides new physical insight into the cloaking problem.

The chapter is structured as follows. In Section 2, the cloaking problem is introduced by means of a volumetric equivalence principle. A theory of NRC equivalent currents is formulated in Section 3 and applied in Section 4 to find a set of sufficient conditions for cloaking on the permittivity and permeability tensors of an anisotropic metamaterial. Section 5 shows how the 2D and 3D transformation optics cloaks can be derived as special cases of the previously introduced general formulation. Lastly, conclusions are drawn in Section 6.

2. Problem Formulation

2.1. Anisotropic Metamaterial Cloak

Consider an annular volume of anisotropic metamaterial immersed in free space, bounded by an external surface Σ and by an internal surface Σ' which represents the boundary of the cavity (Fig. 1a). Denote by V_{int} the cavity volume within Σ', by V_{ext} the external volume and by V the anisotropic metamaterial volume. The cavity region V_{int} is assumed to be filled with free space; however it can be filled with an arbitrary medium without changing the final result. In the following, we use bold characters for defining vectors and bold characters with double bars to denote dyadics; we suppress the time dependence $\exp(j\omega t)$.

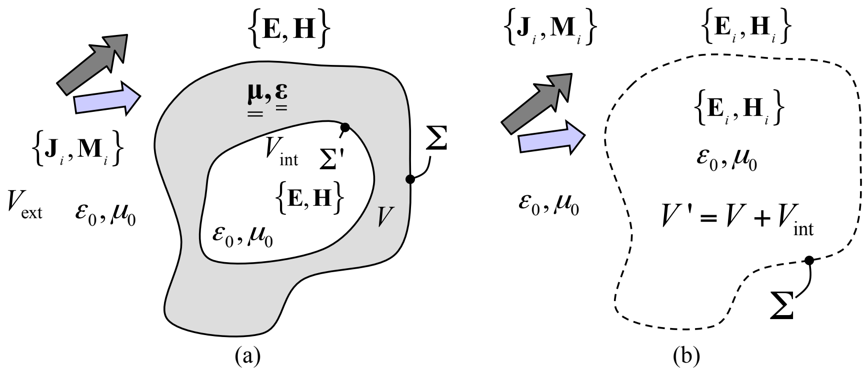

Fig. 1 Geometry for the problem: (a) anisotropic medium and definition of the internal fields, (b) Reference problem for the incident field.

The anisotropic metamaterial is illuminated by the incident field $\{\mathbf{E}_i, \mathbf{H}_i\}$ produced in absence of the metamaterial by the impressed electric and magnetic current densities $\{\mathbf{J}_i, \mathbf{M}_i\}$ located in the external region V_{ext} (Fig. 1b). The total electric and magnetic fields $\{\mathbf{E}, \mathbf{H}\}$ satisfy Maxwell's equations

$$\nabla \times \mathbf{E}(\mathbf{r}) = -j\omega\, \mathbf{B}(\mathbf{r}) - \mathbf{M}_i(\mathbf{r}) \tag{1}$$

$$\nabla \times \mathbf{H}(\mathbf{r}) = j\omega\, \mathbf{D}(\mathbf{r}) + \mathbf{J}_i(\mathbf{r}) \tag{2}$$

where the magnetic and electric induced fields (called the "inductions") $\mathbf{B}$ and $\mathbf{D}$ are related to the fields by the linear, inhomogeneous, anisotropic constitutive relations

$$\mathbf{B}(\mathbf{r}) = \underline{\underline{\boldsymbol{\mu}}}(\mathbf{r}) \cdot \mathbf{H}(\mathbf{r}) \tag{3}$$

$$\mathbf{D}(\mathbf{r}) = \underline{\underline{\boldsymbol{\varepsilon}}}(\mathbf{r}) \cdot \mathbf{E}(\mathbf{r}) \tag{4}$$

In the following, the functions $\mathbf{B}$ and $\mathbf{D}$ are intended to be continuously differentiable functions of $\mathbf{r}$ except possibly at the inner boundary Σ' where they may be discontinuous but do not contain delta functions. Possible delta function singularities in the inductions will be explicitly indicated as a separate term.

From here on we suppress and understand the space dependence of any function, unless otherwise indicated. The constitutive dyadics simplify outside V to those of the free-space

$$\underline{\underline{\boldsymbol{\mu}}} \rightarrow \mu_0 \underline{\underline{\mathbf{I}}} \; ; \; \underline{\underline{\boldsymbol{\varepsilon}}} \rightarrow \varepsilon_0 \underline{\underline{\mathbf{I}}} \quad \mathbf{r} \in V_{int}, V_{ext} \tag{5}$$

where $\underline{\underline{\mathbf{I}}}$ is the unit dyadic. The incident (unperturbed) fields $\{\mathbf{E}_i, \mathbf{H}_i\}$ obey the homogeneous Maxwell's equations in free space

$$\nabla \times \mathbf{E}_i = -j\omega\mu_0 \mathbf{H}_i - \mathbf{M}_i \tag{6}$$

$$\nabla \times \mathbf{H}_i = j\omega\varepsilon_0 \mathbf{E}_i + \mathbf{J}_i \tag{7}$$

The metamaterial shell cloaks the cavity if the *total external* field (field in V_{ext}) is equal to the incident field $\{\mathbf{E}_i, \mathbf{H}_i\}$ and the *total field in the cavity* vanishes, *that is*

$$\{\mathbf{E}, \mathbf{H}\}\big|_{\mathbf{r} \in V_{ext}} = \{\mathbf{E}_i, \mathbf{H}_i\} \tag{8}$$

$$\{\mathbf{E}, \mathbf{H}\}\big|_{\mathbf{r} \in V_{int}} = \{0, 0\} \tag{9}$$

In terms of the scattered fields $\{\mathbf{E}_s, \mathbf{H}_s\} \equiv \{\mathbf{E} - \mathbf{E}_i, \mathbf{H} - \mathbf{H}_i\}$, the cloaking conditions (8)-(9) become

$$\{\mathbf{E}_s, \mathbf{H}_s\}\big|_{\mathbf{r} \in V_{ext}} = \{0, 0\} \tag{10}$$

$$\{\mathbf{E}_s, \mathbf{H}_s\}\big|_{\mathbf{r} \in V_{int}} = \{-\mathbf{E}_i, -\mathbf{H}_i\} \tag{11}$$

namely, the scattered fields equal zero in the external volume and equal the negative of the incident fields inside the cavity.

Uniqueness theorems ensure that eqs. (8) and (9) are satisfied if Maxwell's equation are imposed in the three regions V_{ext}, V and V_{int}, along with proper boundary conditions on Σ and Σ'. In particular, on Σ the tangential components of the fields must be continuous, *that is*

$$\mathbf{E}\times\hat{\mathbf{n}}=\mathbf{E}_i\times\hat{\mathbf{n}}\ ;\qquad \mathbf{H}\times\hat{\mathbf{n}}=\mathbf{H}_i\times\hat{\mathbf{n}}\qquad \text{on } \Sigma \tag{12}$$

where $\hat{\mathbf{n}}$ is the external normal to Σ. On Σ' different sets of boundary conditions can be imposed to ensure zero total field inside the cavity (assuming a vanishingly small loss inside the cavity to eliminate source-free cavity resonances). In fact, uniqueness of the solution is ensured in a finite source-free volume if the tangential components of the electric field, *or* the tangential components of the magnetic field, *or* the normal components of both the electric and the magnetic fields are specified on the volume boundary [5,11]. Thus on Σ'^{+} we may apply alternatively one of the following conditions: the tangential component of electric field equal to zero, the tangential component of magnetic field equal to zero, or the normal components of both electric and magnetic field equal to zero, namely

$$(i)\qquad \mathbf{E}\times\hat{\mathbf{n}}'=0\ ;\qquad \text{on } \Sigma'^{+} \tag{13}$$

$$(ii)\qquad \mathbf{H}\times\hat{\mathbf{n}}'=0\ ;\qquad \text{on } \Sigma'^{+} \tag{14}$$

$$(iii)\qquad \mathbf{H}\cdot\hat{\mathbf{n}}'=0 \ \text{ and }\ \mathbf{E}\cdot\hat{\mathbf{n}}'=0\ ;\qquad \text{on } \Sigma'^{+} \tag{15}$$

where $\hat{\mathbf{n}}'$ is the external normal to Σ' and Σ'^{+} denotes the inner side of Σ' (see Fig. 2a). These conditions on the fields at the inner side of Σ' are transformed to conditions on the fields at the outer side of Σ' (*that is,* in the cloak-material region) if additional assumptions are made on surface currents and charges at Σ'. Assuming vanishing tangential magnetic surface current (or polarization) for (*i*), vanishing tangential electric surface current (or polarization) for (*ii*) and vanishing magnetic and electric surface charges (or polarization charges) for (*iii*), we obtain

$$(i)\qquad \mathbf{E}\times\hat{\mathbf{n}}'=0\ ;\qquad \text{on } \Sigma'^{-} \tag{16}$$

(*ii*) $\mathbf{H}\times\hat{\mathbf{n}}'=0$; on Σ'^{-} (17)

(*iii*) $\nabla\times\mathbf{E}\cdot\hat{\mathbf{n}}'=0$ and $\nabla\times\mathbf{H}\cdot\hat{\mathbf{n}}'=0$; on Σ'^{-} (18)

where (*iii*) can also be written as $\mathbf{B}\cdot\hat{\mathbf{n}}'=0$ and $\mathbf{D}\cdot\hat{\mathbf{n}}'=0$. It is assumed that for a given cloak geometry, an $\left(\underline{\underline{\boldsymbol{\varepsilon}}},\underline{\underline{\boldsymbol{\mu}}}\right)$ can be found that produces a solution to Maxwell's equations (1)-(2) with the constitutive relations (3)-(4) and the boundary conditions in (12) and (16)-(18). Moreover, different permittivity-permeability functions $\left(\underline{\underline{\boldsymbol{\varepsilon}}},\underline{\underline{\boldsymbol{\mu}}}\right)$ may produce the same zero scattered fields outside the cloak and yet different scattered fields within the cloak material.

2.2. Volumetric Equivalence Principle for Discontinuous Fields

From the standpoint of the "volumetric equivalence principle", the scattered fields $\left\{\mathbf{E}_s,\mathbf{H}_s\right\}$ are produced by equivalent "polarization" currents (with possible delta functions on the boundaries) radiating in free space. Therefore, the cloaking conditions in (10) and (11) mean that the equivalent polarization currents are *nonradiating* outside the metamaterial shell and *cancelling* the incident field inside the cavity of the shell.

We admit here the possibility that the fields are discontinuous only at the inner surface Σ', with *finite* different values on the two sides of the surface. As a consequence, eqs. (1)-(2) can be rewritten as [12]

$$\left\{\nabla\times\mathbf{E}\right\}+\nabla_{\Sigma'}\times\mathbf{E}=-j\omega\left\{\mathbf{B}\right\}-j\omega\mu_0\mathbf{M}^{\delta}-\mathbf{M}_i \quad (19)$$

$$\left\{\nabla\times\mathbf{H}\right\}+\nabla_{\Sigma'}\times\mathbf{H}=j\omega\left\{\mathbf{D}\right\}+j\omega\mathbf{P}^{\delta}+\mathbf{J}_i \quad (20)$$

where $\mathbf{M}^{\delta}$ and $\mathbf{P}^{\delta}$ are magnetic and electric delta-functions polarization densities that come from the discontinuities of **E** and **H**, respectively, $\left\{\mathbf{B}\right\}$ and $\left\{\mathbf{D}\right\}$ indicate the regular part of the magnetic and electric inductions, respectively, $\left\{\nabla\times\right\}$ denotes the curl operator without including the discontinuity surface (which leads to a regular function) and the operator $\nabla_{\Sigma'}\times$ denotes the surface curl operator defined by [12]

$$\begin{aligned}\nabla_{\Sigma'}\times\mathbf{E}\ \mathrm{B}\ \delta(n')\hat{\mathbf{n}}'\times\left[\mathbf{E}^{+}-\mathbf{E}^{-}\right]_{\Sigma'} &= -j\omega\mu_0\mathbf{M}^{\delta}\\ \nabla_{\Sigma'}\times\mathbf{H}\ \mathrm{B}\ \delta(n')\hat{\mathbf{n}}'\times\left[\mathbf{H}^{+}-\mathbf{H}^{-}\right]_{\Sigma'} &= j\omega\mathbf{P}^{\delta}\end{aligned}. \tag{21}$$

where the second equality of both the equations comes from the identification of the delta currents in (19)-(20) with the delta discontinuous part of the curl. In (21), $\hat{\mathbf{n}}'$ is the normal to Σ' directed toward the cavity, $\delta(n')$ is a Dirac delta function, whose argument is the distance to Σ' along the direction of $\hat{\mathbf{n}}'$ and $\mathbf{E}^{+}$ ($\mathbf{E}^{-}$) denotes the value of the function **E** on the positive (negative) side of Σ' (similar definition for **H**).
After subtracting (6)-(7) from (19)-(20) we obtain, outside the sources,

$$\nabla\times\left(\mathbf{E}-\mathbf{E}_i\right) = -j\omega\left\{\mathbf{B}\right\} + j\omega\mu_o\mathbf{H}_i - j\omega\mu_0\mathbf{M}^{\delta} \tag{22}$$

$$\nabla\times\left(\mathbf{H}-\mathbf{H}_i\right) = j\omega\left\{\mathbf{D}\right\} - j\omega\varepsilon_0\mathbf{E}_i + j\omega\mathbf{P}^{\delta} \tag{23}$$

Eqs. (22)-(23) can be rearranged as

$$\begin{aligned}\nabla\times\mathbf{E}_s &= -j\omega\mu_o\mathbf{H}_s - \mathbf{M}_{eq}\\ \nabla\times\mathbf{H}_s &= j\omega\varepsilon_0\mathbf{E}_s + \mathbf{J}_{eq}\end{aligned} \tag{24}$$

with the equivalent polarization currents $\mathbf{M}_{eq}, \mathbf{J}_{eq}$ defined by

$$\begin{aligned}\mathbf{M}_{eq} &= j\omega\left[\left(\underline{\underline{\boldsymbol{\mu}}} - \mu_o\underline{\underline{\mathbf{I}}}\right)\cdot\mathbf{H}\right]\chi_V + \delta(n')\left[\mathbf{E}^{+}-\mathbf{E}^{-}\right]_{\Sigma'}\times\hat{\mathbf{n}}'\\ \mathbf{J}_{eq} &= j\omega\left[\left(\underline{\underline{\boldsymbol{\varepsilon}}} - \varepsilon_0\underline{\underline{\mathbf{I}}}\right)\cdot\mathbf{E}\right]\chi_V + \delta(n')\hat{\mathbf{n}}'\times\left[\mathbf{H}^{+}-\mathbf{H}^{-}\right]_{\Sigma'}\end{aligned} \tag{25}$$

where χ_V is the characteristic function of V

$$\chi_V = \begin{cases}1 & \mathbf{r}\in V\\ 0 & \mathbf{r}\notin V\end{cases} \tag{26}$$

In (25) $\mathbf{E}^{+}$ and $\mathbf{H}^{+}$ are zero if we assume zero fields inside the cavity. The equivalent polarization electric and magnetic currents $\mathbf{J}_{eq}$ and $\mathbf{M}_{eq}$ are the sources of the scattered fields $\left\{\mathbf{E}_s, \mathbf{H}_s\right\}$ in free-space. These currents contain delta functions if the tangential components of the fields exhibit a jump across Σ'. Delta functions are not necessary if the jump occurs in the normal field components.

The perfect cloaking conditions in (10)-(11) are equivalent to the requirement that the currents $\left(\mathbf{J}_{eq},\mathbf{M}_{eq}\right)$ in (25) are "*nonradiating cancelling (NRC) currents*", namely that they *do not radiate* in V_{ext} and they *cancel* the incident field in V_{int}. The next section presents sufficient conditions for NRC currents.

3. Sufficient Conditions for NRC Currents

In this section we demonstrate two theorems on NRC currents, that will be referred to as Theorem A and Theorem B. Theorem A can be seen as the generalization to NRC currents of the theorems demonstrated by Devaney and Wolf [13] for nonradiating electric currents. Although it can be regarded as a particular case of Theorem B, it is demonstrated separately, since it is believed that this demonstration facilitates the understanding of Theorem B.

Theorem A: Let $\left\{\mathbf{F}^e,\mathbf{F}^h\right\}$ be two vector functions (called here "potentials") differentiable at any point of the volume V and $\{\hat{\mathbf{n}}, \hat{\mathbf{n}}'\}$ the normal unit vectors external to Σ and internal to Σ', respectively. Let the potentials satisfy the conditions

$$\mathbf{F}^e\times\hat{\mathbf{n}}=\mathbf{E}_i\times\hat{\mathbf{n}}\ ;\qquad \mathbf{F}^h\times\hat{\mathbf{n}}=\mathbf{H}_i\times\hat{\mathbf{n}}\qquad \text{on } \Sigma \tag{27}$$

$$\mathbf{F}^e\times\hat{\mathbf{n}}'=0\ ;\qquad \mathbf{F}^h\times\hat{\mathbf{n}}'=0\qquad \text{on } \Sigma' \tag{28}$$

where $\left\{\mathbf{E}_i,\mathbf{H}_i\right\}$ are arbitrary incident fields satisfying the free-space Maxwell's eqs. in $V'=V+V_{int}$. Then the currents

$$\begin{aligned}\mathbf{M}_{NRC}&=\left(-\nabla\times\mathbf{F}^e-j\omega\mu_o\mathbf{F}^h\right)\chi_V\\ \mathbf{J}_{NRC}&=\left(\nabla\times\mathbf{F}^h-j\omega\varepsilon_0\mathbf{F}^e\right)\chi_V\end{aligned} \tag{29}$$

are NRC currents (they do not radiate in V_{ext} and radiate minus the incident field in V_{int}); moreover, in V they produce a field equal to $\left\{\mathbf{E}_s,\mathbf{H}_s\right\}=\left\{\mathbf{F}^e-\mathbf{E}_i,\mathbf{F}^h-\mathbf{H}_i\right\}$. (Note that the curl in (29) is not intended to introduce any delta function on the surface when applied to the potentials because of the assumption of the regularity of $\mathbf{F}^{e,h}$).

Proof: At any point of the space one has

$$\begin{aligned}\nabla\times\left(\chi_V\mathbf{F}^{e,h}\right) &= \chi_V\nabla\times\mathbf{F}^{e,h}+\nabla\chi_V\times\mathbf{F}^{e,h}\\ &= \chi_V\nabla\times\mathbf{F}^{e,h}-\left(\delta(n)\hat{\mathbf{n}}+\delta(n')\hat{\mathbf{n}}'\right)\times\mathbf{F}^{e,h} =\\ &= \chi_V\nabla\times\mathbf{F}^{e,h}-\delta(n)\hat{\mathbf{n}}\times\left(\mathbf{E}_i,\mathbf{H}_i\right)\end{aligned} \tag{30}$$

where $\delta(n)$ is a Dirac delta function, whose argument is the distance along the direction of the unit normal $\hat{\mathbf{n}}$ to the surface Σ. In the latter equality we have used the fact that $\hat{\mathbf{n}}'\times\mathbf{F}^{e,h}$ vanishes on Σ'. The currents in (29) can be therefore rewritten as

$$\begin{aligned}\mathbf{M}_{NRC} &= -\nabla\times\left(\chi_V\mathbf{F}^{e}\right)-j\omega\mu_o\left(\chi_V\mathbf{F}^{h}\right)-\delta(n)\hat{\mathbf{n}}\times\mathbf{E}_i\\ \mathbf{J}_{NRC} &= \nabla\times\left(\chi_V\mathbf{F}^{h}\right)-j\omega\varepsilon_0\left(\chi_V\mathbf{F}^{e}\right)+\delta(n)\hat{\mathbf{n}}\times\mathbf{H}_i\end{aligned} \tag{31}$$

The fields $\left\{\mathbf{E}_s,\mathbf{H}_s\right\}$ radiated by these currents satisfy the equations

$$\begin{aligned}\nabla\times\left(\mathbf{E}_s-\chi_V\mathbf{F}^{e}\right) &= -j\omega\mu_o\left(\mathbf{H}_s-\chi_V\mathbf{F}^{h}\right)+\delta(n)\hat{\mathbf{n}}\times\mathbf{E}_i\\ \nabla\times\left(\mathbf{H}_s-\chi_{V'}\mathbf{F}^{h}\right) &= j\omega\varepsilon_0\left(\mathbf{E}_s-\chi_{V'}\mathbf{F}^{e}\right)+\delta(n)\hat{\mathbf{n}}\times\mathbf{H}_i\end{aligned} \tag{32}$$

This means that the fields $\mathbf{E}_S-\chi_V\mathbf{F}^{e}$ and $\mathbf{H}_S-\chi_V\mathbf{F}^{h}$ obey everywhere the Maxwell's equations in free-space with excitation given by the surface currents $\mathbf{M}_s=\mathbf{E}_i\times\hat{\mathbf{n}}\delta(n)$, $\mathbf{J}_s=\hat{\mathbf{n}}\times\mathbf{H}_i\delta(n)$ on Σ. These currents are equal to the equivalent currents provided by the Love formulation of the equivalence principle applied to the surface Σ in free space.[14] Thus, they radiate minus the incident field in $V+V_{int}$ and zero field in V_{ext}:

$$\begin{aligned}\left(\mathbf{E}_s-\chi_V\mathbf{F}^{e}\right) &= -\mathbf{E}_i\chi_{V+V_{int}}\\ \left(\mathbf{H}_s-\chi_V\mathbf{F}^{h}\right) &= -\mathbf{H}_i\chi_{V+V_{int}}\end{aligned} \tag{33}$$

which means that the fields radiated by the equivalent sources in (29) are

$$\begin{aligned}\mathbf{E}_s &= \chi_V\mathbf{F}^{e}-\mathbf{E}_i\chi_{V+V_{\mathrm{int}}}\\ \mathbf{H}_s &= \chi_V\mathbf{F}^{h}-\mathbf{H}_i\chi_{V+V_{\mathrm{int}}}\end{aligned} \tag{34}$$

and Theorem A is proved. W

Remark 1: If the incident field is zero, the currents become nonradiating in V_{int} and $V_{ext,}$ and Theorem A becomes the extension of the theorem of nonradiating currents by Devaney and Wolf [13] to the case of a combination of electric and magnetic sources.

Remark 2: The statement of Theorem A can be also rephrased by saying that the *total* field is $\left\{\mathbf{F}^e, \mathbf{F}^h\right\}$ in V and zero in V_{int} (*that is*, region V_{int} is impenetrable to any radiation from outside). Thus, having equivalent currents defined in (29) with boundary conditions in (27)-(28) is a sufficient condition to cloak the region of space V_{int} from an arbitrary incident field. The conditions in (28) can be mitigated if one allows nonvanishing surface currents on Σ', as stated in the next Theorem.

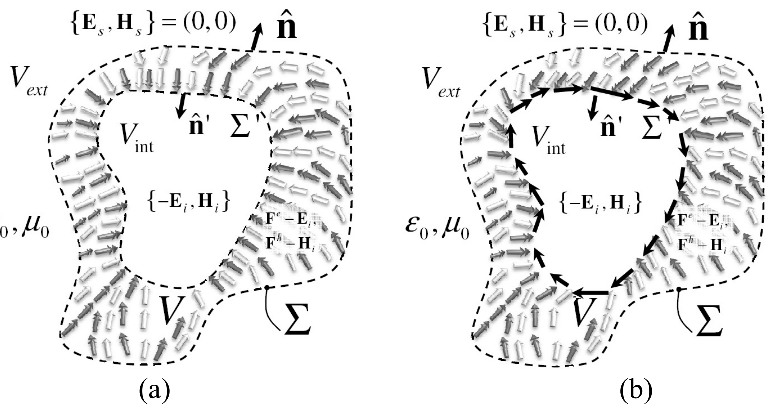

Fig. 2. Distribution of potentials $\mathbf{F}^e, \mathbf{F}^h$ inside V for the case of Theorem A (a) and Theorem B (b). The NRC currents are constructed via these potentials through (29) and (37) for (a) and (b), respectively. In (a), the tangential components of the potentials exhibit at Σ a jump equal to the value of the tangential components of the incident field, and zero tangential components at Σ'. In (b), the potentials have the same behaviour as in (a) at Σ and present a jump in the tangential components at Σ'; counter-radiating surface currents must be present at Σ' to cancel the radiation produced by the potential discontinuity.

Theorem B: Let $\left\{\mathbf{F}^e, \mathbf{F}^h\right\}$ be any two differentiable vector potentials defined at any point of the volume V that satisfy the conditions

$$\mathbf{F}^e \times \hat{\mathbf{n}} = \mathbf{E}_i \times \hat{\mathbf{n}} \;; \qquad \mathbf{F}^h \times \hat{\mathbf{n}} = \mathbf{H}_i \times \hat{\mathbf{n}} \qquad \text{on } \Sigma \tag{35}$$

$$\mathbf{F}^e \times \hat{\mathbf{n}}' = \mathbf{f}^e \;; \qquad \mathbf{F}^h \times \hat{\mathbf{n}}' = \mathbf{f}^h \qquad \text{on } \Sigma' \tag{36}$$

where $\mathbf{f}^e, \mathbf{f}^h$ are *arbitrary* continuous differentiable functions on Σ'. Then, the currents

$$\begin{aligned} \mathbf{M}_{NRC} &= \left(-\nabla\times\mathbf{F}^e - j\omega\mu_o\mathbf{F}^h\right)\chi_V - \mathbf{f}^e\delta\left(n'\right) \\ \mathbf{J}_{NRC} &= \left(\nabla\times\mathbf{F}^h - j\omega\varepsilon_0\mathbf{F}^e\right)\chi_V + \mathbf{f}^h\delta\left(n'\right) \end{aligned} \tag{37}$$

are NRC currents (do not radiate in V_{ext} and radiate minus the incident field in V_{int}); moreover, in V they radiate a field equal to $\left\{\mathbf{E}_s, \mathbf{H}_s\right\} = \left\{\mathbf{F}^e - \mathbf{E}_i, \mathbf{F}^h - \mathbf{H}_i\right\}$.

Proof: By applying the same steps as in the proof of Theorem A we obtain

$$\begin{aligned} \mathbf{M}_{NRC} &= \left(-\nabla\times\left(\chi_V\mathbf{F}^e\right) - j\omega\mu_o\mathbf{F}^h\right) - \hat{\mathbf{n}}\times\mathbf{E}_i\delta\left(n\right) - \hat{\mathbf{n}}'\times\mathbf{F}^e\delta\left(n'\right) - \mathbf{f}^e\delta\left(n'\right) \\ \mathbf{J}_{NRC} &= \left(\nabla\times\left(\chi_V\mathbf{F}^h\right) - j\omega\varepsilon_0\mathbf{F}^e\right) + \hat{\mathbf{n}}\times\mathbf{H}_i\delta\left(n\right) + \hat{\mathbf{n}}'\times\mathbf{F}^h\delta\left(n'\right) + \mathbf{f}^h\delta\left(n'\right) \end{aligned} \tag{38}$$

Since the last two terms cancel each other because of the boundary conditions, (38) coincides with (31); hence, following the same steps as for the proof of the previous theorem, we obtain

$$\begin{aligned} \mathbf{E}_s &= \chi_V\mathbf{F}^e - \mathbf{E}_i\chi_{V+V_{int}} \\ \mathbf{H}_s &= \chi_V\mathbf{F}^h - \mathbf{H}_i\chi_{V+V_{int}} \end{aligned} \tag{39}$$

and Theorem B is proved. W

Remark 3: Theorem B reduces to Theorem A for vanishing values of tangential field components on Σ'.

Remark 4: If the functions $\mathbf{f}^e, \mathbf{f}^h$ are divergence free along the surface, namely, $\nabla_s\cdot\mathbf{f}^{e,h} = 0$, we have

$$\nabla\times\mathbf{F}^{e,h}\cdot\hat{\mathbf{n}}' = 0 \text{ on } \Sigma' \tag{40}$$

since

$$\nabla_s\cdot\left(\mathbf{F}^{e,h}\times\hat{\mathbf{n}}'\right) = -\mathbf{F}^{e,h}\cdot\nabla_s\times\hat{\mathbf{n}}' + \nabla_s\times\mathbf{F}^{e,h}\cdot\hat{\mathbf{n}}' = \nabla\times\mathbf{F}^{e,h}\cdot\hat{\mathbf{n}}' \tag{41}$$

Remark 5: It is clear from eq. (38) that the jump in the tangential components of the potentials gives an extra contribution that has to be cancelled by surface polarization currents. Additional surface polarization currents are the price that has to be paid to relax the boundary conditions on the potentials at Σ'.

In the following section, Theorem B is applied to find sufficient conditions on the constitutive dyadics of an anisotropic metamaterial used for cloaking.

4. Sufficient Cloaking Conditions on the Constitutive Dyadic Parameters

The electric and magnetic currents coming from the application of the equivalence theorem to the fields of a cloak should now be identified with the NRC currents of Theorem B. This will lead to additional conditions on the potentials. The currents in (25), with $\mathbf{E}^+$ and $\mathbf{H}^+$ forced to zero inside the cavity become

$$\begin{aligned}\mathbf{M}_{eq} &= j\omega\left[\left(\underline{\underline{\boldsymbol{\mu}}} - \mu_o \underline{\underline{\mathbf{I}}}\right)\cdot \mathbf{H}\right]\chi_V - \delta(n')\left[\mathbf{E}^-\right]_{\Sigma'} \times \hat{\mathbf{n}}' \\ \mathbf{J}_{eq} &= j\omega\left[\left(\underline{\underline{\boldsymbol{\varepsilon}}} - \varepsilon_0 \underline{\underline{\mathbf{I}}}\right)\cdot \mathbf{E}\right]\chi_V - \delta(n')\hat{\mathbf{n}}'\times\left[\mathbf{H}^-\right]_{\Sigma'}\end{aligned} \tag{42}$$

They can be identified with the NRC currents of Theorem B such that

$$\begin{aligned}\mathbf{M}_{NRC} &= \left(-\nabla\times\mathbf{F}^e - j\omega\mu_o\mathbf{F}^h\right)\chi_V - \mathbf{f}^e\delta(n') \\ \mathbf{J}_{NRC} &= \left(\nabla\times\mathbf{F}^h - j\omega\varepsilon_0\mathbf{F}^e\right)\chi_V + \mathbf{f}^h\delta(n')\end{aligned} \tag{43}$$

with total internal fields coincident with the potentials (*that is*, with $\{\mathbf{E},\mathbf{H}\}\chi_V = \{\mathbf{E}_s + \mathbf{E}_i, \mathbf{H}_s + \mathbf{H}_i\}\chi_V = \{\mathbf{F}^e, \mathbf{F}^h\}\chi_V$) if one of the additional conditions (16)-(18) is imposed at Σ'. The equality $\left[\mathbf{M}_{eq}, \mathbf{J}_{eq}\right] = \left[\mathbf{M}_{NRC}, \mathbf{J}_{NRC}\right]$ applied to (42)-(43) leads through simple algebraic steps to the following theorem.

Theorem C: Sufficient conditions for cloaking on the permittivity and permeability tensors of a metamaterial are the fulfilment of the equations

$$\chi_V \nabla \times \mathbf{F}^e = -j\omega\, \underline{\underline{\boldsymbol{\mu}}} \cdot \mathbf{F}^h \chi_V$$
$$\chi_V \nabla \times \mathbf{F}^h = j\omega\, \underline{\underline{\boldsymbol{\varepsilon}}} \cdot \mathbf{F}^e \chi_V \tag{44}$$

where $\mathbf{F}^e, \mathbf{F}^h$ are *arbitrary* continuous potentials satisfying the boundary conditions.

$$\mathbf{F}^e \times \hat{\mathbf{n}} = \mathbf{E}_i \times \hat{\mathbf{n}}; \qquad \mathbf{F}^h \times \hat{\mathbf{n}} = \mathbf{H}_i \times \hat{\mathbf{n}} \qquad \text{on } \Sigma \tag{45}$$

and one of the alternative conditions

(*i*) $\mathbf{F}^e \times \hat{\mathbf{n}}' = 0$; on Σ'^- (46)

(*ii*) $\mathbf{F}^h \times \hat{\mathbf{n}}' = 0$; on Σ'^- (47)

(*iii*) $\nabla \times \mathbf{F}^h \cdot \hat{\mathbf{n}}' = 0$ and $\nabla \times \mathbf{F}^e \cdot \hat{\mathbf{n}}' = 0$; on Σ'^- (48)

Furthermore, $\mathbf{F}^e, \mathbf{F}^h$ represent the total electric and magnetic fields inside the cloak.

We emphasize that the conditions (46)-(48) are alternative sufficient conditions, *that is,* they do not need to be simultaneously verified. In particular, it is confirmed that the boundary conditions of $\hat{\mathbf{n}} \cdot \mathbf{B} = 0$ and $\hat{\mathbf{n}} \cdot \mathbf{D} = 0$ used in Ref. 5 at the inside inner surfaces of cloaks are sufficient conditions for solutions to both 2D and 3D cloaks to not scatter the incident fields outside the cloaks and to produce zero fields within the cavities of the cloaks. We also note that (44) tells us something more than the Maxwell's equation. Indeed, the vector potentials are quite arbitrary, except that they must be differentiable and they have to satisfy the boundary condition in (45) and one of the boundary conditions in (46)-(48). The operator $\nabla \times$ in (44) is intended to be applied to a continuous function, possibly defined as regular outside the volume V so as not to introduce delta functions on the surfaces.

The sufficient conditions (*i*)-(*iii*) prescribe different behaviours of the tangential surface polarization densities at the inner surface of the cloak. In fact, (*i*), (*ii*), and (*iii*) allow on Σ' the presence of electric surface currents $\mathbf{f}^h \delta(n')$, the presence of magnetic surface currents $\mathbf{f}^e \delta(n')$, and the presence of both electric and magnetic surface currents $\left[\mathbf{f}^h \delta(n'), \mathbf{f}^e \delta(n')\right]$, respectively.

Theorem C comes from Theorem B, which requires the conditions in (16)-(18), which in turn require vanishing tangential magnetic surface current, vanishing tangential electric surface current, and vanishing magnetic and electric surface charges, respectively. Thus, it must be assumed initially that the cloaks also satisfy these latter conditions. In the next section, it is confirmed that transformation optics cloaks do indeed satisfy these conditions.

Eq. (44) links the potentials with the constitutive tensors of the shell material. Since the potentials must satisfy conditions which depend on an incident field, in order to convert eqs. (44)-(48) into source-independent conditions on the constitutive parameters $\underline{\underline{\boldsymbol{\mu}}}, \underline{\underline{\boldsymbol{\varepsilon}}}$, the vector potentials shall necessarily be defined in terms of the incident field through a space dependent transformation that reduces to the identity in V_{ext}. An example is provided by the compression-type transformation of Transformation Optics[1] that will be considered in the following section.

5. Transformation Optics Cloaks

The present formulation includes the Transformation Optics (TO) cloaks as a special case of Theorem C. In a TO cloak, the incident field is "compressed" by a coordinate transformation into the cloak volume V and the constitutive parameters of the metamaterial that yield the same field compression are determined in terms of the Jacobian matrix of the coordinate transformation. In terms of ray theory, the trajectories of electromagnetic rays passing through the region of compressed space must conform to the local metric. Once the desired trajectories are determined through a conformal mapping applied to Cartesian straight trajectories in a "virtual free-space", the differential operators of the Maxwell's equations in the transformed space lead to space-dependent metric coefficients that can be reinterpreted in terms of constitutive relations of an anisotropic, inhomogeneous medium. In other words, the local metric of a deformed elemental volume is interpreted as a local change of components of the local permeability and permittivity dyadics. The local tensors produce a compression or an expansion of the local wavelength (with consequent change of local phase velocity) so as to equalize the phase delay from an input to an output face of a "compressed" elemental curvilinear volume to the corresponding phase delay of the "uncompressed" (Cartesian-coordinate) elemental volume. The final result is a medium that exhibits permittivity and permeability components smaller (greater) than the ones of free space in directions parallel (orthogonal) to the compression.

To reconstruct the above physically appealing picture within the framework of this formulation, let us introduce in the real space a Cartesian reference system with unit vectors $\hat{\mathbf{x}}_1, \hat{\mathbf{x}}_2, \hat{\mathbf{x}}_3$. The position vector is defined as $\mathbf{r} = x_1\hat{\mathbf{x}}_1 + x_2\hat{\mathbf{x}}_2 + x_3\hat{\mathbf{x}}_3$. Define a transformation

$$\mathbf{r}' = \mathbf{r}'(\mathbf{r}) \tag{49}$$

that maps the observation variable $\mathbf{r}$ of the real space into the observation variable $\mathbf{r}' = x_1'\hat{\mathbf{x}}_1' + x_2'\hat{\mathbf{x}}_2' + x_3'\hat{\mathbf{x}}_3'$ of a "virtual free-space", where (x_1', x_2', x_3') and $\hat{\mathbf{x}}_1', \hat{\mathbf{x}}_2', \hat{\mathbf{x}}_3'$ are orthogonal coordinates and unit vectors, respectively, of the virtual space. Eq. (49) can be written in components as

$$x_i' = x_i'(x_1, x_2, x_3) \quad i=1,2,3 \tag{50}$$

The inverse transformation is

$$x_i = x_i(x_1', x_2', x_3') \quad i=1,2,3 \tag{51}$$

For cloaking, the transformation in (50) must go smoothly to $\mathbf{r}' = \mathbf{r}$ for any $\mathbf{r}_\Sigma$ belonging to the external surface Σ; furthermore, the inverse transformation in (51) maps the origin of the virtual space into the inner surface Σ'. These conditions can be written as

$$\mathbf{r}'\left(\mathbf{r} \to \mathbf{r}_\Sigma^-\right) = \mathbf{r} \tag{52}$$

$$\mathbf{r}'\left(\mathbf{r} \to \mathbf{r}_{\Sigma'}^+\right) = \mathbf{0} \Leftrightarrow \mathbf{r}\left(\mathbf{r}' \to 0\right) = \mathbf{r}_{\Sigma'} \tag{53}$$

Assume also that the transformation becomes the identity outside V_{ext}, and 0 inside V_{int}, namely

$$\mathbf{r}'(\mathbf{r}) = \mathbf{r} \quad \mathbf{r} \in V_{ext}\ ; \quad \mathbf{r}'(\mathbf{r}) = 0 \quad \mathbf{r} \in V_{int} \tag{54}$$

We introduce as potentials the functions

$$\mathbf{F}^e(\mathbf{r}) = \underline{\underline{\mathbf{A}}}(\mathbf{r})\cdot\mathbf{E}_i\left[\mathbf{r}'(\mathbf{r})\right]$$
$$\mathbf{F}^h(\mathbf{r}) = \underline{\underline{\mathbf{A}}}(\mathbf{r})\cdot\mathbf{H}_i\left[\mathbf{r}'(\mathbf{r})\right] \tag{55}$$

where $\mathbf{E}_i(\mathbf{r}')$, $\mathbf{H}_i(\mathbf{r}')$ are the incident fields described in the virtual coordinate system. The boundary conditions (45) and (46)-(48) on the potentials turn into the following conditions on $\underline{\underline{\mathbf{A}}}(\mathbf{r})$

$$\hat{\mathbf{n}}\times\left(\underline{\underline{\mathbf{A}}}(\mathbf{r}_\Sigma)-\underline{\underline{\mathbf{I}}}\right)=0 \tag{56}$$

(*i*), (*ii*)
$$\hat{\mathbf{n}}'\times\underline{\underline{\mathbf{A}}}(\mathbf{r}_{\Sigma'})=0 \tag{57}$$

(*iii*)
$$\hat{\mathbf{n}}'\cdot\underline{\underline{\boldsymbol{\mu}}}\cdot\underline{\underline{\mathbf{A}}}(\mathbf{r}_{\Sigma'})=0 \;\; ; \;\; \hat{\mathbf{n}}'\cdot\underline{\underline{\boldsymbol{\varepsilon}}}\cdot\underline{\underline{\mathbf{A}}}(\mathbf{r}_{\Sigma'})=0 \tag{58}$$

where $\underline{\underline{\mathbf{I}}}$ is the identity matrix. The latter condition is the same applied in Ref. 5 for the special case of spherical and cylindrical cloaks. In the virtual space $\mathbf{r}'$, the incident fields satisfy the Maxwell equations

$$\nabla'\times\mathbf{H}_i(\mathbf{r}') = j\omega\varepsilon_0\,\mathbf{E}_i(\mathbf{r}')$$
$$\nabla'\times\mathbf{E}_i(\mathbf{r}') = -j\omega\mu_0\mathbf{H}_i(\mathbf{r}') \tag{59}$$

where $\nabla' = \hat{\mathbf{x}}_1'\partial/\partial x_1' + \hat{\mathbf{x}}_2'\partial/\partial x_2' + \hat{\mathbf{x}}_3'\partial/\partial x_3'$, namely, differentiation is performed in the coordinate of the virtual space. Inserting (55) into (44) and using (59) leads to

$$\nabla\times\left[\underline{\underline{\mathbf{A}}}(\mathbf{r})\cdot\mathbf{E}_i(\mathbf{r}')\right] = \frac{1}{\mu_0}\underline{\underline{\boldsymbol{\mu}}}(\mathbf{r})\cdot\underline{\underline{\mathbf{A}}}(\mathbf{r})\cdot\nabla'\times\mathbf{E}_i(\mathbf{r}') \tag{60}$$

$$\nabla\times\left[\underline{\underline{\mathbf{A}}}(\mathbf{r})\cdot\mathbf{H}_i(\mathbf{r}')\right] = \frac{1}{\varepsilon_0}\underline{\underline{\boldsymbol{\varepsilon}}}(\mathbf{r})\cdot\underline{\underline{\mathbf{A}}}(\mathbf{r})\cdot\nabla'\times\mathbf{H}_i(\mathbf{r}') \tag{61}$$

Assuming that a unique solution exists to (60) for $\mu_0^{-1}\underline{\underline{\boldsymbol{\mu}}}(\mathbf{r})$ for a given cloak geometry and transformation $\mathbf{r}'(\mathbf{r})$, then a unique solution exists to (61) for $\varepsilon_0^{-1}\underline{\underline{\boldsymbol{\varepsilon}}}(\mathbf{r})$, and it follows that

$$\frac{1}{\mu_0}\underline{\underline{\boldsymbol{\mu}}}(\mathbf{r}) = \frac{1}{\varepsilon_0}\underline{\underline{\boldsymbol{\varepsilon}}}(\mathbf{r}) = \underline{\underline{\boldsymbol{\alpha}}}(\mathbf{r}) \tag{62}$$

Thus, (60) and (61) may be rewritten as

$$\begin{aligned} &\underline{\underline{\mathbf{A}}}^{-1}(\mathbf{r})\cdot\underline{\underline{\boldsymbol{\alpha}}}^{-1}(\mathbf{r})\cdot\nabla\times\left[\underline{\underline{\mathbf{A}}}(\mathbf{r})\cdot\mathbf{E}_i(\mathbf{r}')\right]=\nabla'\times\mathbf{E}_i(\mathbf{r}')\\ &\underline{\underline{\mathbf{A}}}^{-1}(\mathbf{r})\cdot\underline{\underline{\boldsymbol{\alpha}}}^{-1}(\mathbf{r})\cdot\nabla\times\left[\underline{\underline{\mathbf{A}}}(\mathbf{r})\cdot\mathbf{H}_i(\mathbf{r}')\right]=\nabla'\times\mathbf{H}_i(\mathbf{r}') \end{aligned} \tag{63}$$

For any vector **a**, the expression of the curl in the real space can be transformed into the curl in the virtual space by [1,15,16]

$$\nabla\times\mathbf{a}=\det(\underline{\underline{\mathbf{M}}})\ \underline{\underline{\mathbf{M}}}^{-1}\cdot\nabla'\times\left([\underline{\underline{\mathbf{M}}}^{T}]^{-1}\cdot\mathbf{a}\right) \tag{64}$$

where $\underline{\underline{\mathbf{M}}}$ and $\underline{\underline{\mathbf{M}}}^{-1}$ are the Jacobian matrix of the transformation $\mathbf{r}'=\mathbf{r}'(\mathbf{r})$ and its inverse matrix, respectively (the latter is the Jacobian matrix of the inverse transformation)

$$\underline{\underline{\mathbf{M}}}=\left\{\frac{\partial x_i'}{\partial x_j}\right\}_{i,j=1,3}\ ;\ \underline{\underline{\mathbf{M}}}^{-1}=\left\{\frac{\partial x_i}{\partial x_j'}\right\}_{i,j=1,3} \tag{65}$$

The expression in (64) is still valid for a coordinate transformation between two general curvilinear orthogonal systems, provided that the proper scale factors are included in the Jacobian matrix (65) [16].
By using (64) in the first of eqs. (63) we obtain

$$\underline{\underline{\mathbf{A}}}^{-1}\cdot\underline{\underline{\boldsymbol{\alpha}}}^{-1}\cdot\det(\underline{\underline{\mathbf{M}}})\ \underline{\underline{\mathbf{M}}}^{-1}\cdot\nabla'\times\left[[\underline{\underline{\mathbf{M}}}^{T}]^{-1}\cdot\underline{\underline{\mathbf{A}}}\cdot\mathbf{E}_i(\mathbf{r}')\right]=\nabla'\times\mathbf{E}_i(\mathbf{r}') \tag{66}$$

where the dependence on **r** is suppressed. The above expression is satisfied for

$$\underline{\underline{\mathbf{A}}}\cdot\left(\underline{\underline{\mathbf{M}}}^{T}\right)^{-1}=\underline{\underline{\mathbf{A}}}^{-1}\cdot\underline{\underline{\boldsymbol{\alpha}}}^{-1}\cdot\det(\underline{\underline{\mathbf{M}}})\ \underline{\underline{\mathbf{M}}}^{-1}=\underline{\underline{\mathbf{I}}} \tag{67}$$

which may be rewritten as

$$\underline{\underline{\mathbf{A}}}=\underline{\underline{\mathbf{M}}}^{T};\quad \underline{\underline{\boldsymbol{\alpha}}}=\det(\underline{\underline{\mathbf{M}}})\underline{\underline{\mathbf{M}}}^{-1}\cdot\left(\underline{\underline{\mathbf{M}}}^{T}\right)^{-1} \tag{68}$$

The solutions for the fields and inductions within the cloak will be therefore given by

$$\begin{aligned} \mathbf{E}(\mathbf{r}) &= \underline{\underline{\mathbf{M}}}^T(\mathbf{r}) \cdot \mathbf{E}_i\left[\mathbf{r}'(\mathbf{r})\right] \\ \mathbf{H}(\mathbf{r}) &= \underline{\underline{\mathbf{M}}}^T(\mathbf{r}) \cdot \mathbf{H}_i\left[\mathbf{r}'(\mathbf{r})\right] \\ \mathbf{D}(\mathbf{r}) &= \varepsilon_0 \det(\underline{\underline{\mathbf{M}}})\underline{\underline{\mathbf{M}}}^{-1} \cdot \mathbf{E}_i\left[\mathbf{r}'(\mathbf{r})\right] \\ \mathbf{B}(\mathbf{r}) &= \mu_0 \det(\underline{\underline{\mathbf{M}}})\underline{\underline{\mathbf{M}}}^{-1} \cdot \mathbf{H}_i\left[\mathbf{r}'(\mathbf{r})\right] \end{aligned} \tag{69}$$

The boundary conditions (56)-(58) may be rewritten in terms of the Jacobian matrix as

$$\hat{\mathbf{n}} \times \underline{\underline{\mathbf{M}}}^T(\mathbf{r}_\Sigma) = \hat{\mathbf{n}} \times \underline{\underline{\mathbf{I}}} \tag{70}$$

(*i*), (*ii*)

$$\hat{\mathbf{n}}' \times \underline{\underline{\mathbf{M}}}^T(\mathbf{r}_{\Sigma'}) = 0 \tag{71}$$

(*iii*)

$$\hat{\mathbf{n}} \cdot \underline{\underline{\mathbf{M}}}^{-1} \det(\underline{\underline{\mathbf{M}}}) = 0 \tag{72}$$

In the following section, the fulfilment of these conditions is checked with reference to two- and three-dimensional cloaks of arbitrary shape. It will be found that (70) and (72) are satisfied in two-dimensional cloaks of arbitrary shape, while (71) is not in general satisfied. On the other hand, (70), (71) and (72) are simultaneously satisfied by three-dimensional cloaks of arbitrary shape.

5.1. Boundary Conditions Expressed in Terms of Covariant and Contravariant Vectors

The Jacobian matrix $\underline{\underline{\mathbf{M}}}$ and its inverse can be written in terms of the covariant ($\mathbf{g}_i$) and contravariant ($\mathbf{g}^i$) vectors of the coordinate transformation, which are defined by

$$\mathbf{g}_i = \frac{\partial \mathbf{r}}{\partial x_i'}, \quad \mathbf{g}^i = \nabla x_i' \tag{73}$$

Both covariant and contravariant vectors can be used as basis vectors for general curvilinear coordinate transformations. The covariant vectors are built along the coordinate axes, while the contravariant vectors are built to be perpendicular to the coordinate surfaces.

The Jacobian matrix and its inverse can be written in terms of covariant and contravariant vectors as follows

$$\underline{\underline{\mathbf{M}}} = \begin{bmatrix} \mathbf{g}^1 \\ \mathbf{g}^2 \\ \mathbf{g}^3 \end{bmatrix}; \quad \underline{\underline{\mathbf{M}}}^{-1} = \left[\mathbf{g}_1 \ , \quad \mathbf{g}_2 \ , \quad \mathbf{g}_3 \right] \tag{74}$$

The determinant of the Jacobian matrix can be written as

$$\det \underline{\underline{\mathbf{M}}} = \mathbf{g}^1 \cdot \left(\mathbf{g}^2 \times \mathbf{g}^3 \right) = \frac{1}{\mathbf{g}_1 \cdot \left(\mathbf{g}_2 \times \mathbf{g}_3 \right)} \tag{75}$$

It is now shown that the boundary conditions (52) and (53) on the coordinate transformation imply the fulfilment of condition (70) and at least one of the (71)-(72). In particular, eq. (52) implies

$$\underline{\underline{\mathbf{M}}}\left(\mathbf{r}_\Sigma \right) = \underline{\underline{\mathbf{I}}} \tag{76}$$

which directly yields the fulfilment of (70). The implications of condition (53) are investigated in the next section separately in the cases of two-dimensional and three-dimensional cloaks.

5.2. Three-Dimensional Cloaks

In the compression type transformation leading to three-dimensional cloaks, condition (53) means that a point in the virtual space is transformed into a finite surface in the real space. This implies

$$\lim_{\mathbf{r}' \to 0} \hat{\mathbf{n}}' \times \mathbf{g}^i = 0; \quad \lim_{\mathbf{r}' \to 0} \hat{\mathbf{n}}' \cdot \mathbf{g}^i = \lim_{\mathbf{r}' \to 0} \frac{\partial x_i'}{\partial n'} = b_i \tag{77}$$

$$\lim_{\mathbf{r} \to \mathbf{r}_{\Sigma'}^+} \hat{\mathbf{n}}' \times \mathbf{g}_i = \infty; \quad \lim_{\mathbf{r} \to \mathbf{r}_{\Sigma'}^+} \hat{\mathbf{n}}' \cdot \mathbf{g}_i = c_i \tag{78}$$

where b_i and c_i are finite constants different from zero. Eq. (78) shows that the components of the covariant vectors tangential to Σ' tend toward infinity when approaching Σ', while the value of the normal component remains finite. On the other hand, (77) implies that the contravariant vectors are all orthogonal to the surface with finite amplitude. This, in turn, implies the fulfilment of condition (71) since

$$\lim_{\mathbf{r} \to \mathbf{r}_{\Sigma'}^+} \hat{\mathbf{n}}' \times \underline{\underline{\mathbf{M}}}^T = \lim_{\mathbf{r} \to \mathbf{r}_{\Sigma'}^+} \hat{\mathbf{n}}' \times \left[\mathbf{g}^1 \quad \mathbf{g}^2 \quad \mathbf{g}^3 \right] = 0 \tag{79}$$

which is equivalent to

$$\hat{\mathbf{n}}'\times\mathbf{F}^{e}\left(\mathbf{r}_{\Sigma'}\right)=\hat{\mathbf{n}}'\times\mathbf{F}^{h}\left(\mathbf{r}_{\Sigma'}\right)=0 \tag{80}$$

Therefore, the tangential components of both the electric and magnetic fields vanish on Σ', and thus both (46) and (47) are satisfied. This also implies that there are no surface currents on Σ'. Hence, a three-dimensional cloak obtained through a compression-type coordinate transformation can always be cast in the framework of Theorem A.

On the other hand, since (77) implies that all the contravariant vectors are aligned with the unit vector $\hat{\mathbf{n}}'$ on Σ', and hence they are all parallel, it follows that

$$\lim_{\mathbf{r}\to\mathbf{r}_{\Sigma'}^{+}}\left[\det\underline{\underline{\mathbf{M}}}\right]=0 \tag{81}$$

This implies

$$\lim_{\mathbf{r}\to\mathbf{r}_{\Sigma'}^{+}}\det(\underline{\underline{\mathbf{M}}})\,\hat{\mathbf{n}}\cdot\underline{\underline{\mathbf{M}}}^{-1}=\lim_{\mathbf{r}\to\mathbf{r}_{\Sigma'}^{+}}\det(\underline{\underline{\mathbf{M}}})\,\hat{\mathbf{n}}\cdot\left[\mathbf{g}_1\quad\mathbf{g}_2\quad\mathbf{g}_3\right]=0 \tag{82}$$

and then yields the fulfilment of (72). Hence, also the normal components of the inductions vanish on Σ' inside the cloak, and condition (48) is also satisfied. We emphasize that, although only one of the three conditions (*i*)-(*iii*) in (46)-(48) is sufficient for cloaking (when combined with (45)), the TO 3D cloak fulfills all the three conditions for any shape. This is not true for 2D cloaks, as shown in the next subsection.

5.3. Two-Dimensional Cloaks

In the compression type transformation leading to 2D cloaks invariant along the z direction, condition (53) means that the line $x=y=0$ in the virtual space is transformed into a finite surface in the real space. This implies

$$\lim_{\mathbf{r}'\to 0}\hat{\mathbf{z}}\times\hat{\mathbf{n}}'\times\mathbf{g}^{i}=0\qquad\lim_{\mathbf{r}'\to 0}\hat{\mathbf{n}}'\cdot\mathbf{g}^{i}=\lim_{\mathbf{r}'\to 0}\frac{\partial x_i'}{\partial n'}=b_i\qquad\lim_{\mathbf{r}'\to 0}\hat{\mathbf{z}}\cdot\mathbf{g}^{i}=1 \tag{83}$$

$$\lim_{\mathbf{r}\to\mathbf{r}_{\Sigma'}^{+}}\hat{\mathbf{z}}\times\hat{\mathbf{n}}'\times\mathbf{g}_i=\infty\qquad\lim_{\mathbf{r}\to\mathbf{r}_{\Sigma'}^{+}}\hat{\mathbf{n}}'\cdot\mathbf{g}_i=c_i \tag{84}$$

where b_i and c_i are finite constants, in general different from 0. Eq. (84) shows that the components of the covariant vectors tangential to Σ' in the *xy* plane tend toward infinity when approaching Σ', while the value of the normal component remains finite. On the other hand, (83) implies that the contravariant vectors tend to have vanishing tangential components on Σ' in the *xy* plane, but finite tangential components along *z*. As a consequence, in this case we have

$$\lim_{\mathbf{r}\to\mathbf{r}_{\Sigma'}^+} \hat{\mathbf{z}}\times\hat{\mathbf{n}}'\times\underline{\underline{\mathbf{M}}}^T = \lim_{\mathbf{r}\to\mathbf{r}_{\Sigma'}^+} \hat{\mathbf{z}}\times\hat{\mathbf{n}}'\times\left[\mathbf{g}^1 \quad \mathbf{g}^2 \quad \mathbf{g}^3\right] = 0 \tag{85}$$

which implies the vanishing of the tangential components of the field in the *xy* plane

$$\hat{\mathbf{z}}\times\hat{\mathbf{n}}'\times\mathbf{E}\left(\mathbf{r}_{\Sigma'}\right) = \hat{\mathbf{z}}\times\hat{\mathbf{n}}'\times\mathbf{H}\left(\mathbf{r}_{\Sigma'}\right) = 0 \tag{86}$$

However, the same is not in general true for the *z*-components of the field, since we have

$$\lim_{\mathbf{r}\to\mathbf{r}_{\Sigma'}^+} \hat{\mathbf{z}}\cdot\underline{\underline{\mathbf{M}}}^T = \lim_{\mathbf{r}\to\mathbf{r}_{\Sigma'}^+} \hat{\mathbf{z}}\cdot\left[\mathbf{g}^1 \quad \mathbf{g}^2 \quad \mathbf{g}^3\right] \neq 0 \tag{87}$$

which implies

$$\hat{\mathbf{z}}\cdot\mathbf{E}\left(\mathbf{r}_{\Sigma'}\right)\neq 0;\quad \hat{\mathbf{z}}\cdot\mathbf{H}\left(\mathbf{r}_{\Sigma'}\right)\neq 0 \tag{88}$$

Hence, for TO 2D cloaks not all the tangential components of the fields vanish on Σ', (46) and (47) are not in general satisfied and both electric and magnetic surface currents may be present.[17] On the other hand, (83) implies that all the contravariant vectors are coplanar on Σ' and, as a consequence, it again follows that

$$\lim_{\mathbf{r}\to\mathbf{r}_{\Sigma'}^+}\left[\det\underline{\underline{\mathbf{M}}}\right] = 0 \tag{89}$$

From which

$$\lim_{\mathbf{r}\to\mathbf{r}_{\Sigma'}^+} \det(\underline{\underline{\mathbf{M}}})\hat{\mathbf{n}}\cdot\underline{\underline{\mathbf{M}}}^{-1} = \lim_{\mathbf{r}\to\mathbf{r}_{\Sigma'}^+} \det(\underline{\underline{\mathbf{M}}})\hat{\mathbf{n}}\cdot\left[\mathbf{g}_1 \quad \mathbf{g}_2 \quad \mathbf{g}_3\right]=0 \tag{90}$$

which leads to the fulfilment of (72) and then (48). Therefore, in two-dimensional cloaks the normal components of the inductions do vanish on Σ' while the tangential components of fields do not. This is indeed the same condition used in Ref. 5.

5.4. Field Behaviour Inside the Cloak in Terms of Covariant and Contravariant Vectors

The concept of covariant and contravariant vectors can be used to obtain a simple description of the field behaviour inside a generic TO cloak illuminated by a plane wave.

It is not restrictive to assume that the plane wave comes from the $-\hat{\mathbf{z}}$ direction with the electric field polarized along $\hat{\mathbf{y}}$ and the magnetic field polarized along $\hat{\mathbf{x}}$. Inside the cloak the fields are transformed according to eqs. (69)

$$\begin{aligned} \mathbf{E}(\mathbf{r}) &= \underline{\underline{\mathbf{M}}}^T(\mathbf{r}) \cdot \mathbf{E}_i\left[\mathbf{r}'(\mathbf{r})\right] \\ \mathbf{H}(\mathbf{r}) &= \underline{\underline{\mathbf{M}}}^T(\mathbf{r}) \cdot \mathbf{H}_i\left[\mathbf{r}'(\mathbf{r})\right] \end{aligned} \tag{91}$$

After writing the Jacobian matrix in terms of the contravariant vectors we obtain

$$\begin{aligned} \mathbf{E}(\mathbf{r}) &= \left[\mathbf{g}^1 \ , \quad \mathbf{g}^2 \ , \quad \mathbf{g}^3\right] \cdot \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} E_0 e^{-jk\mathbf{r}'(\mathbf{r})} = \mathbf{g}^2 E_0 e^{-jk\mathbf{r}'(\mathbf{r})} \\ \mathbf{H}(\mathbf{r}) &= \left[\mathbf{g}^1 \ , \quad \mathbf{g}^2 \ , \quad \mathbf{g}^3\right] \cdot \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \frac{E_0}{\zeta_0} e^{-jk\mathbf{r}'(\mathbf{r})} = \mathbf{g}^1 \frac{E_0}{\zeta_0} e^{-jk\mathbf{r}'(\mathbf{r})} \end{aligned} \tag{92}$$

where ζ_o is the free-space impedance. For the inductions, we have from (69)

$$\begin{aligned} \mathbf{D}(\mathbf{r}) &= \varepsilon_0 \det(\underline{\underline{\mathbf{M}}})\underline{\underline{\mathbf{M}}}^{-1} \cdot \mathbf{E}_i\left[\mathbf{r}'(\mathbf{r})\right] \\ \mathbf{B}(\mathbf{r}) &= \mu_0 \det(\underline{\underline{\mathbf{M}}})\underline{\underline{\mathbf{M}}}^{-1} \cdot \mathbf{H}_i\left[\mathbf{r}'(\mathbf{r})\right] \end{aligned} \tag{93}$$

that is

$$\mathbf{D}(\mathbf{r}) = \varepsilon_0 \det(\underline{\underline{\mathbf{M}}})\left[\mathbf{g}_1 \;,\quad \mathbf{g}_2 \;,\quad \mathbf{g}_3\right]\cdot\begin{bmatrix}0\\1\\0\end{bmatrix} E_0 e^{-jk\mathbf{r}'(\mathbf{r})} = \varepsilon_0 \det(\underline{\underline{\mathbf{M}}})\mathbf{g}_2 E_0 e^{-jk\mathbf{r}'(\mathbf{r})}$$

$$\mathbf{B}(\mathbf{r}) = \mu_0 \det(\underline{\underline{\mathbf{M}}})\left[\mathbf{g}_1 \;,\quad \mathbf{g}_2 \;,\quad \mathbf{g}_3\right]\begin{bmatrix}1\\0\\0\end{bmatrix} \frac{E_0}{\zeta_0} e^{-jk\mathbf{r}'(\mathbf{r})} = \mu_0 \det(\underline{\underline{\mathbf{M}}})\mathbf{g}_1 \frac{E_0}{\zeta_0} e^{-jk\mathbf{r}'(\mathbf{r})} \tag{94}$$

This means that at any point of the cloak the fields are aligned with the contravariant vectors and the inductions with the covariant vectors. The description in terms of covariant and contravariant vectors allows one to easily visualize the field structure in a TO cloak obtained through a generic coordinate transformation, and provides information on the field behaviour at the inner surface of the cloak, as shown in the previous section. The Poynting vector is aligned with a covariant vector, since

$$\mathbf{S}(\mathbf{r}) = \frac{1}{2}\mathbf{E}(\mathbf{r})\times\mathbf{H}^*(\mathbf{r}) = \frac{|E_0|^2}{\zeta_0}\mathbf{g}^2\times\mathbf{g}^1 = \frac{|E_0|^2}{\zeta_0}\det(\underline{\underline{\mathbf{M}}})\mathbf{g}_3 \tag{95}$$

This result was indeed expected, because in TO cloaks the ray-paths conform to the local metric in the compressed space.

6. Conclusions

The cloaking problem has been revisited in terms of nonradiating cancelling equivalent volumetric currents, that is, equivalent currents which do not radiate outside the external surface of the cloak and cancel the incident field inside the cloak cavity. This formulation leads to the determination of sufficient conditions for cloaking on the constitutive parameters of an arbitrarily shaped metamaterial shell, expressed in terms of two vector potentials satisfying simple conditions on the cloak boundary. The transformation optics cloak is recovered as a special case of the proposed formulation and the general boundary conditions satisfied by the fields at the inner surface of arbitrarily shaped three-dimensional and two-dimensional cloaks have been determined. In particular, it is found that all the components of the **E, H, D,** and **B** fields, except the normal components of **E** and **H,** must vanish at the inner surfaces of generic transformation optics three-dimensional cloaks, while in transformation optics two-dimensional cloaks there can also be nonvanishing tangential components at the inner surfaces, which are always associated with delta-function singularities in the tangential surface polarization densities. It is confirmed that the boundary conditions of $\hat{\mathbf{n}} \cdot \mathbf{B} = 0$ and $\hat{\mathbf{n}} \cdot \mathbf{D} = 0$ used in Ref. 5 at the inside inner surfaces of cloaks are sufficient conditions for solutions to both 2D and 3D cloaks to not scatter the incident fields outside the cloaks and to produce zero fields within the cavities of the cloaks.

Finally, the covariant and contravariant vectors of the coordinate transformation leading to transformational optics cloaks have been used to provide a simple and appealing picture of the field behavior inside a cloak illuminated by a plane wave.

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