t-Intersection sets in $AG(r, q^2)$ and two-character multisets in $\mathrm{PG}(3, q^2)$

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Abstract

In this article we construct new minimal intersection sets in $\mathrm{AG}(r, q^2)$ with respect to hyperplanes, of size q^{2r-1} and multiplicity t, where $t \in \{q^{2r-3} - q^{(3r-5)/2}, q^{2r-3} + q^{(3r-5)/2} - q^{(3r-3)/2}\}$, for r odd or $t \in \{q^{2r-3} - q^{(3r-4)/2}, q^{2r-3} - q^{r-2}\}$, for r even. As a byproduct, for any odd q we get a new family of two-character multisets in $\mathrm{PG}(3, q^2)$.

The essential idea is to investigate some point-sets in $\mathrm{AG}(r, q^2)$ satisfying the opposite of the algebraic conditions required in [1] for quasi–Hermitian varieties.

Keywords: Hermitian variety, quadric, two-character set.

1 Introduction

All non–degenerate Hermitian varieties of $\mathrm{PG}(r, q^2)$ are projectively equivalent; furthermore, they sport just two intersection numbers with hyperplanes, see [6]. Quasi–Hermitian varieties $\mathcal{V}$ of $\mathrm{PG}(r, q^2)$ are combinatorial objects which have the same size and the same intersection numbers with hyperplanes as a (non–degenerate) Hermitian variety $\mathcal{H}$; see [1] for details and some constructions. In the present paper we shall consider varieties $\mathcal{V}$ arising by taking algebraic conditions opposite to those of [1] and show that these are in turn interesting geometric objects with 3 intersection numbers. The topic is also of interest for applications, as the projective system induced by $\mathcal{V}$ will determine linear codes with few weights; see [7] for a description of this correspondence.

Fix a projective frame in $\mathrm{PG}(r, q^2)$ and assume the space to have homogeneous coordinates $(X_0, X_1, \ldots, X_r)$. Consider the affine plane $\mathrm{AG}(r, q^2)$ whose infinite

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hyperplane Π_∞ has equation $X_0 = 0$. Then, $\mathrm{AG}(r, q^2)$ has affine coordinates $(x_1, x_2, \ldots, x_r)$ where $x_i = X_i/X_0$ for $i \in \{1, \ldots, r\}$.

Consider now the non–degenerate Hermitian variety $\mathcal{H}$ with affine equation of the form

$$x_r^q - x_r = (b^q - b)(x_1^{q+1} + \ldots + x_{r-1}^{q+1}), \tag{1}$$

where $b \in GF(q^2) \setminus \mathrm{GF}(q)$. The set of the points at infinity of $\mathcal{H}$ is

$$\mathcal{F} = \{(0, x_1, \ldots, x_r) | x_1^{q+1} + \ldots + x_{r-1}^{q+1} = 0\}; \tag{2}$$

this can be regarded as a Hermitian cone of $\mathrm{PG}(r-1, q^2)$, projecting a Hermitian variety of $PG(r-2, q^2)$ from the point $P_\infty := (0, \ldots, 0, 1)$. In particular, observe that the hyperplane Π_∞ is tangent to $\mathcal{H}$ at P_∞.

For any $a \in \mathrm{GF}(q^2)^*$ and $b \in \mathrm{GF}(q^2) \setminus \mathrm{GF}(q)$, let $\mathcal{B} := \mathcal{B}(a, b)$ be the affine algebraic variety of equation

$$x_r^q - x_r + a^q(x_1^{2q} + \ldots + x_{r-1}^{2q}) - a(x_1^2 + \ldots + x_{r-1}^2) = (b^q - b)(x_1^{q+1} + \ldots + x_{r-1}^{q+1}). \tag{3}$$

It is shown in [1] that $\mathcal{B}(a, b)$, together with the points at infinity of $\mathcal{H}$, as given by (2), is a quasi–Hermitian variety $\mathcal{V}$ of $\mathrm{PG}(r, q^2)$ provided that either of the following algebraic conditions are satisfied: for q odd, r is odd and $4a^{q+1} + (b^q - b)^2 \neq 0$, or r is even and $4a^{q+1} + (b^q - b)^2$ is a non–square in $\mathrm{GF}(q)$; for q even, r is odd, or r is even and $\mathrm{Tr}\,(a^{q+1}/(b^q + b)^2) = 0$.

In this paper, as stated before, we shall study the variety $\mathcal{B}(a, b)$ when the opposite of the previous conditions holds. More precisely our main results are the following

Proposition 1.1. *Suppose q odd, $4a^{q+1} + (b^q - b)^2 = 0$ and r odd. Then $\mathcal{B}(a, b)$ is a set of q^{2r-1} points of $\mathrm{AG}(r, q^2)$ of characters:*

- *for $r \equiv 1 \pmod 4$ or $q \equiv 1 \pmod 4$*

$$q^{2r-3} - q^{(3r-5)/2}, q^{2r-3}, q^{2r-3} - q^{(3r-5)/2} + q^{3(r-1)/2}$$

- *for $r \equiv 3 \pmod 4$ and $q \equiv 3 \pmod 4$*

$$q^{2r-3} + q^{(3r-5)/2} - q^{3(r-1)/2}, q^{2r-3}, q^{2r-3} + q^{(3r-5)/2}.$$

- *for r even,*

$$q^{2r-3} - q^{(3r-4)/2}, q^{2r-3}, q^{2r-3} + q^{(3r-4)/2}.$$

Furthermore $\mathcal{B}(a, b)$ is always a minimal intersection set with respect to hyperplanes.

Theorem 1.2. *Suppose q odd and $4a^{q+1} + (b^q - b)^2 = 0$. In $PG(3, q^2)$ there exists a 2–character multiset $\overline{\mathcal{B}}(a, b)$ containing $\mathcal{B}(a, b)$ and characters either $q^3 - q^2$ and $2q^3 - q^2$ if $q \equiv 1 \pmod 4$, or q^2, and $q^3 + q^2$ if $q \equiv 3 \pmod 4$.*

These results are proved respectively in Section 3 and in Section 4.

Finally, in Section 5 we prove that in the remaining cases we again get minimal intersection sets of the same size but multiplicity $q^{2r-3} - q^{r-2}$.

2 Preliminaries

2.1 Intersection sets with respect to hyperplanes

A set of points $\mathcal{B}$ in a projective or an affine space is a *t-fold blocking set with respect to hyperplanes* if every hyperplane contains at least t points of $\mathcal{B}$. Such a set $\mathcal{B}$ is also known as a *t-intersection set*, or an *intersection set with multiplicity t*, or a *multiple intersection set*.

A point P of a t-intersection set $\mathcal{B}$ is said to be *essential* if $\mathcal{B} \setminus \{P\}$ is not a t-intersection set. When all points of $\mathcal{B}$ are essential then $\mathcal{B}$ is *minimal*. If the size of the intersection of $\mathcal{B}$ with an arbitrary hyperplane takes m values, say $v_1, \ldots, v_m$, then the non-negative integers $v_1, \ldots, v_m$ are called the *characters* of $\mathcal{B}$ and $\mathcal{B}$ is also an *m-character set*. We observe that if $\mathcal{B}$ is an m–character set consisting of n points and spanning the projective space where it is contained, then the linear code having as columns of its generator matrix the coordinates of the points of $\mathcal{B}$ has exactly m distinct nonzero weights and length n. The dimension k of this code is the vector dimension of the subspace spanned by $\mathcal{B}$.

Quasi-Hermitian varieties are examples of 2-character sets of $PG(r, q^2)$. In [1] a new infinite family of quasi–Hermitian varieties have been constructed by modifying some point-hyperplane incidences in $\mathrm{PG}(r, q^2)$. To this purpose, the authors kept the point set of $\mathrm{PG}(r, q^2)$ but replaced the hyperplanes with their images under a suitable quadratic transformation, obtaining a non–standard model Π of $\mathrm{PG}(r, q^2)$. This model arises as follows.

Fix a non-zero element $a \in \mathrm{GF}(q^2)$. For any choice $\mathbf{m} = (m_1, \ldots, m_{r-1}) \in GF(q^2)^{r-1}$ and $d \in \mathrm{GF}(q^2)$ let $\mathcal{Q}_a(\mathbf{m}, d)$ denote the quadric of equation

$$x_r = a(x_1^2 + \ldots + x_{r-1}^2) + m_1 x_1 + \ldots + m_{r-1} x_{r-1} + d. \tag{4}$$

Consider now the incidence structure $\Pi_a = (\mathcal{P}, \Sigma)$ whose points are the points of $\mathrm{AG}(r, q^2)$ and whose hyperplanes are the hyperplanes of $\mathrm{PG}(r, q^2)$ through the infinite point $P_\infty(0, 0, \ldots, 0, 1)$ together with the quadrics $\mathcal{Q}_a(\mathbf{m}, d)$ as $\mathbf{m}$ and d range as indicated above.

Lemma 2.1. *For every non-zero* $a \in \mathrm{GF}(q^2)$*, the incidence structure* $\Pi_a = (\mathcal{P}, \Sigma)$ *is an affine space isomorphic to* $\mathrm{AG}(r, q^2)$*.*

Completing Π_a with its points at infinity in the usual way gives a projective space isomorphic to $\mathrm{PG}(r, q^2)$. We shall make use of this non-standard model of $PG(r, q^2)$ in our work.

2.2 Multisets

A *multiset* in a r-dimensional projective space Π is a mapping $M : \Pi \to \mathbb{N}$ from points of Π into non-negative integers. The *points* of a multiset are the points P of Π with multiplicity $M(P) > 0$. Assume that the number of points of M, each of them counted with its multiplicity, is n. For any hyperplane π of Π, the non-negative integer $M(\pi) = \sum_{P \in \pi} M(P)$ is a *character* of the multiset M, whereas $n - M(\pi)$ is called a *weight* of M. If the set $\{M(\pi)\}_{\pi \in \Pi}$ consists of two non-negative integers only, then M is a 2*-character multiset.*

Suppose the points of M span a projective space $\mathrm{PG}(r, q)$. Then, it is possible to regard the coordinates of the points of M as the columns of a generator matrix of a code $\mathcal{C}$ of length n and dimension $r + 1$. In this case it is straightforward to see that the weights of M are indeed exactly the weights of $\mathcal{C}$. We observe that points with multiplicity greater than one correspond to repeated components in $\mathcal{C}$.

3 Proof of Proposition 1.1

From now on, we shall always silently assume $a \in \mathrm{GF}(q^2)^*$, $b \in \mathrm{GF}(q^2) \setminus \mathrm{GF}(q)$. Recall that for any quadric $\mathcal{Q}$, the radical $\mathrm{Rad}(\mathcal{Q})$ of $\mathcal{Q}$ is the subspace

$$\mathrm{Rad}(\mathcal{Q}) := \{x \in \mathcal{Q} : \forall y \in \mathcal{Q}, \langle x, y \rangle \subseteq \mathcal{Q}\},$$

where, as usual, by $\langle x, y \rangle$ we denote the line through x and y. It is well known that $\mathrm{Rad}(\mathcal{Q})$ is a subspace of $\mathrm{PG}(r, q^2)$.

Assume $\mathcal{B} := \mathcal{B}(a, b)$ to have Equation (3). It is straightforward to see that $\mathcal{B}(a, b)$ coincides with the affine part of the Hermitian variety $\mathcal{H}$ of equation (1) in the space Π_a; hence, any hyperplane π_{P_∞} of $\mathrm{PG}(r, q^2)$ passing through P_∞ meets $\mathcal{B}$ in $|\mathcal{H} \cap \pi_{P_\infty}| = q^{2r-3}$ points.

Now we are interested in the possible intersection sizes of $\mathcal{B}$ with a generic hyperplane

$$\pi : x_r = m_1 x_1 + \cdots + m_{r-1} x_{r-1} + d,$$

of $\mathrm{AG}(r, q^2)$ with coefficients $m_1, \ldots, m_r, d \in \mathrm{GF}(q^2)$. This is the same as to study the intersection of $\mathcal{H}$ with the quadrics $\mathcal{Q}_a(\mathbf{m}, d)$. Choose $\varepsilon \in \mathrm{GF}(q^2) \setminus \mathrm{GF}(q)$ such that $\varepsilon^q = -\varepsilon$; for any $z \in \mathrm{GF}(q^2)$ write $z = z^0 + \varepsilon z^1$ with $z^1, z^2 \in \mathrm{GF}(q)$. The number N of affine points which lie in $\mathcal{B} \cap \pi$ is the same as the number of points of the affine quadric $\mathcal{Q}$ of $\mathrm{AG}(2r-2, q)$ of equation

$$\sum_{i=1}^{r-1} \left((b^1 + a^1)\varepsilon^2(x_i^1)^2 + 2a^0 x_i^0 x_i^1 + (a^1 - b^1)(x_1^0)^2\right) + \sum_{i=1}^{r-1}(m_i^0 x_i^1 + m_i^1 x_i^0) + d^1 = 0. \tag{5}$$

Following the approach of [1], in order to compute N, we first count the number of points of the quadric at infinity $\mathcal{Q}_\infty := \mathcal{Q} \cap \Pi_\infty$ of $\mathcal{Q}$ and then we determine $N = |\mathcal{Q}| - |\mathcal{Q}_\infty|$. Observe that the quadric $\mathcal{Q}_\infty$ of $\mathrm{PG}(2r-3, q)$ has a matrix of the form

$$A_\infty = \begin{pmatrix} (a^1 - b^1) & a^0 & & & \\ a^0 & (b^1 + a^1)\varepsilon^2 & & & \\ & & \ddots & & \\ & & & (a^1 - b^1) & a^0 \\ & & & a^0 & (b^1 + a^1)\varepsilon^2 \end{pmatrix}.$$

Since $(a^0)^2 - \varepsilon^2[(a^1)^2 - (b^1)^2] = [a^{q+1} + (b^q - b)^2/4] = 0$, we have $\det A_\infty = 0$. This is possible if, and only if,

$$\det \begin{pmatrix} (a^1 - b^1) & a^0 \\ a^0 & (a^1 + b^1)\varepsilon^2 \end{pmatrix} = 0,$$

that is, each of the 2×2 blocks on the main diagonal of A_∞ has rank 1. Consequently, the rank of A_∞ is exactly $r - 1$.

If $a^1 = b^1$, then $a^0 = 0$, the matrix A_∞ is diagonal and the quadric $\mathcal{Q}_\infty$ is projectively equivalent to

$$(x_1^1)^2 + (x_2^1)^2 + \cdots + (x_{r-1}^1)^2 = 0.$$

Otherwise, take

$$M = \begin{pmatrix} 1 & 0 & & & \\ -a^0/(a^1 - b^1) & 1 & & & \\ & & \ddots & & \\ & & & 1 & 0 \\ & & & -a^0/(a^1 - b^1) & 1 \end{pmatrix};$$

a direct computation proves that

$$M^T A_\infty M = \begin{pmatrix} a^1 - b^1 & 0 & & & \\ 0 & 0 & & & \\ & & \ddots & & \\ & & & a^1 - b^1 & 0 \\ & & & 0 & 0 \end{pmatrix}.$$

Hence, $\mathcal{Q}_\infty$ is projectively equivalent to the quadric of rank $r-1$ with equation

$$(x_1^0)^2 + (x_2^0)^2 + \cdots + (x_{r-1}^0)^2 = 0.$$

For r odd we see that in both cases $\mathcal{Q}_\infty$ is either

- a cone with vertex $\mathrm{Rad}(\mathcal{Q}_\infty) \simeq \mathrm{PG}(r-2, q)$ and basis a hyperbolic quadric $Q^+(r-2, q)$ if $q \equiv 1 \pmod 4$ or $r \equiv 1 \pmod 4$, or
- a cone with vertex $\mathrm{Rad}(\mathcal{Q}_\infty) \simeq \mathrm{PG}(r-2, q)$ and basis an elliptic quadric $Q^-(r-2, q)$ if $q \equiv 3 \pmod 4$ and $r \equiv 3 \pmod 4$.

For r even, $\mathcal{Q}_\infty$ is a cone with vertex $\mathrm{Rad}(\mathcal{Q}_\infty) \simeq \mathrm{PG}(r-2, q)$ and basis a parabolic quadric $Q(r-2, q)$.

We now move to investigate the quadric $\mathcal{Q}$. Clearly, its rank is either $r-1$ or r. Observe that

- $\mathcal{Q}$ has rank $r-1$ if, and only if, there exist a linear function $f : \mathrm{GF}(q) \to \mathrm{GF}(q)$ such that for all $i = 1, \ldots, r-1$ we have $m_i^1 = f(m_i^0)$; also, the value of d_1 turns out to be uniquely determined. Thus, the number of distinct possibilities for the parameters is exactly q^r.

  Write now $\Pi_\infty = \Sigma \oplus \mathrm{Rad}(\mathcal{Q}_\infty)$. As Σ is disjoint from the radical of the quadratic form inducing $\mathcal{Q}_\infty$, we have that $\Sigma \cap \mathcal{Q}_\infty$ is a nondegenerate quadric (either hyperbolic, elliptic or parabolic according to the various conditions). Since $\mathcal{Q}$ has the same rank as $\mathcal{Q}_\infty$, we have $\dim \mathrm{Rad}(\mathcal{Q}) = \dim \mathrm{Rad}(\mathcal{Q}_\infty) + 1$. Observe that $\mathrm{Rad}(\mathcal{Q}) \cap \Pi_\infty \leq \mathrm{Rad}(\mathcal{Q}_\infty)$. Thus, $\mathrm{Rad}(\mathcal{Q}) \cap \Sigma = \{\mathbf{0}\}$ and Σ is also a direct complement of $\mathrm{Rad}(\mathcal{Q})$. It follows that $\mathcal{Q}$ is cone of vertex a $\mathrm{PG}(r-1, q)$ and basis a quadric of the same kind as the basis of $\mathcal{Q}_\infty$.
- $\mathcal{Q}$ has rank r in the remaining $q^{2r} - q^r$ possibilities. Here $\mathcal{Q}$ is a cone of vertex a $\mathrm{PG}(r-2, q)$ and basis a parabolic quadric $Q(r-1, q)$ for r odd or $\mathcal{Q}$ is a cone of vertex a $\mathrm{PG}(r-2, q)$ and basis a hyperbolic quadric $Q^+(r-1, q)$ or an elliptic quadric $Q^-(r-1, q)$ for r even.

We can now determine the complete list of sizes for r odd:

-
$$|\mathcal{Q}_\infty| = \frac{q^{2r-3}-1}{q-1} \pm q^{(3r-5)/2};$$

- in case $\mathrm{rank}(\mathcal{Q}) = r-1$, then
$$|\mathcal{Q}| = \frac{q^{2r-2}-1}{q-1} \pm q^{3(r-1)/2};$$

- in case $\mathrm{rank}(\mathcal{Q}) = r$,
$$|\mathcal{Q}| = \frac{q^{2r-2}-1}{q-1}$$

In particular, the possible values for $|\mathcal{Q}| - |\mathcal{Q}_\infty|$ are

$$q^{2r-3} + q^{3(r-1)/2} - q^{(3r-5)/2}, q^{2r-3} - q^{(3r-5)/2}$$

for $q \equiv 1 \pmod 4$ or $r \equiv 1 \pmod 4$ and

$$q^{2r-3} - q^{3(r-1)/2} + q^{(3r-5)/2}, q^{2r-3} + q^{(3r-5)/2}$$

for $q \equiv 3 \pmod 4$ and $r \equiv 3 \pmod 4$.

When r is even we get:

-
$$|\mathcal{Q}_\infty| = \frac{q^{2r-3}-1}{q-1};$$

- in case $\mathrm{rank}(\mathcal{Q}) = r-1$, then
$$|\mathcal{Q}| = \frac{q^{2r-2}-1}{q-1};$$

- in case $\mathrm{rank}(\mathcal{Q}) = r$,
$$|\mathcal{Q}| = \frac{q^{2r-2}-1}{q-1} \pm q^{(3r-4)/2}.$$

Thus, the possible list of cardinalities for $|\mathcal{Q}| - |\mathcal{Q}_\infty|$ is

$$q^{2r-3}, q^{2r-3} + q^{(3r-4)/2}, q^{2r-3} - q^{(3r-4)/2}.$$

Now we are going to show that $\mathcal{B}(a,b)$ is a minimal intersection set. First of all, we prove that for any $P \in \mathcal{B}(a,b)$ there exists a subspace $\Lambda_n(P) \simeq \mathrm{AG}(n,q^2)$, $1 \leq n \leq r-1$ through P such that $|\mathcal{B}(a,b) \cap \Lambda_n(P)| \leq q^{2n-1} - q^{n-1}$. The argument is by induction on n. Assume $n = 1$. Then, for any $P \in \mathcal{B}$ there exists at least one line ℓ through P such that $|\ell \cap \mathcal{B}| < q$, otherwise $\mathcal{B}$ would contain more than q^{2r-1} points. Suppose now that the result holds for $n = 1, \ldots, r-2$, take $P \in \mathcal{B}$ and suppose that any hyperplane π through P meets $\mathcal{B}$ in at least q^{2r-3} points. By induction, there exists a subspace $\pi' := \Lambda_{r-2}(P) \simeq \mathrm{AG}(r-2, q^2)$ through P meeting $\mathcal{B}$ in at most $q^{2r-5} - q^{r-3}$ points. By considering all hyperplanes containing π' we get $|\mathcal{B}| \geq (q^2+1)(q^{2r-3} - q^{2r-5} + q^{r-3}) + q^{2r-5} - q^{r-3} > q^{2r-1}$, a contradiction. Thus, through any $P \in \mathcal{B}(a,b)$ there exists a hyperplane meeting $\mathcal{B}(a,b)$ in $(q^{2r-3} - q^{(3r-5)/2})$ points for r odd or $(q^{2r-3} - q^{(3r-4)/2})$ for r even. This implies that $\mathcal{B}(a,b)$ is in all cases a minimal intersection set.

Corollary 3.1. *For q odd and $4a^{q+1} + (b^q - b)^2 = 0$, the number of hyperplanes N_j meeting $\mathcal{B}(a,b)$ in exactly j points are as follows:*

(a) *for r odd*

$$N_{q^{2r-3}+q^{(3r-5)/2}} = q^{2r} - q^r, \qquad N_{q^{2r-3}} = \frac{q^{2r}-1}{q^2-1} - 1,$$
$$N_{q^{2r-3}-q^{3(r-1)/2}+q^{(3r-5)/2}} = q^r.$$

(b) *for r even,*

$$N_{q^{2r-3}-q^{(3r-4)/2}} = \frac{1}{2}(q^{2r} - q^r) \qquad N_{q^{2r-3}} = q^r + \frac{q^{2r}-1}{q^2-1} - 1,$$
$$N_{q^{2r-3}+q^{3(r-4)/2}} = \frac{1}{2}(q^{2r} - q^r).$$

Proof. Case (a) is a direct consequence of the arguments of Theorem 1.1. In Case (b), when r is even, we need to count how often $\mathcal{Q}$ turns out to be elliptic rather than hyperbolic. For any choice of the parameters $m_1, \ldots, m_{r-1}, d$ there is exactly one quadric $\mathcal{Q}$ to consider. As $\mathcal{Q}_\infty$ is always a parabolic quadric, we can assume it to be fixed. Denote by $\sigma^0, \sigma^+, \sigma^-$ respectively the number of quadrics $\mathcal{Q}$ which are parabolic, elliptic or hyperbolic. Clearly σ_0 corresponds to the case in which $\mathrm{rank}(\mathcal{Q}) = \mathrm{rank}(\mathcal{Q}_\infty)$. We have

$$\sigma^+ + \sigma^0 + \sigma^- = q^{2r}, \qquad \sigma^0 = q^r.$$

Each point of $\mathcal{B}(a,b)$ lies on $\frac{q^{2r}-1}{q^2-1}$ hyperplanes; of these $\frac{q^{2r-2}-1}{q^2-1}$ pass through P_∞ (and they must be discounted). Thus, we get

$$q^{2r-2}|\mathcal{B}| = q^{4r-3} = \sigma^0 q^{2r-3} + \sigma^+(q^{2r-3} + q^{(3r-4)/2}) + \sigma^-(q^{2r-3} - q^{(3r-4)/2}) =$$
$$q^{2r-3}(\sigma^0 + \sigma^+ + \sigma^-) + q^{(3r-4)/2}(\sigma^+ - \sigma^-) = q^{4r-3} + (\sigma^+ - \sigma^-)q^{(3r-4)/2}.$$

Hence, $\sigma^+ = \sigma^- = \frac{1}{2}(q^{2r} - q^r)$. □

Remark 3.2. The quadric $\mathcal{Q}_a(\mathbf{m}, d)$ of Equation (4) shares its tangent hyperplane at P_∞ with the Hermitian variety (1).

The problem of the intersection of the Hermitian variety $\mathcal{H}$ with irreducible quadrics $\mathcal{Q}$ having the same tangent plane at a common point $P \in \mathcal{Q} \cap \mathcal{H}$ has been considered for $r = 3$ in [3, 4].

4 A family of two-character multisets in $PG(3, q^2)$

In [2, Theorem 4.1] it is shown that for $r = 2$, q odd and $4a^{q+1} + (b^q - b)^2 \neq 0$ or $r = 2$, q even and $\mathrm{Tr}\,(a^{q+1}/(b^q + b)^2) = 1$, the set $\mathcal{B}(a, b)$ can be completed to a 2–character multiset $\overline{\mathcal{B}}(a, b)$. An analogous result holds for $r = 3$. In this section we now prove Theorem 1.2.

Assume q odd and $4a^{q+1} + (b^q - b)^2 = 0$. From the proof of Proposition 1.1, the quadric $\mathcal{Q}_\infty$ is the union of two distinct planes for $q \equiv 1 \pmod 4$ or just a line for $q \equiv 3 \pmod 4$. Therefore, if $q \equiv 1 \pmod 4$ then either

$$N = q^3 + q^2 + q + 1 - (2q^2 + q + 1) = q^3 - q^2$$

or

$$N = 2q^3 + q^2 + q + 1 - (2q^2 + q + 1) = 2q^3 - q^2,$$

according as $\mathcal{Q}$ is either the join of a line to a conic or a pair of solids; hence, the list of intersection numbers of $\mathcal{B}(a, b)$ with affine hyperplanes is $q^3 - q^2$, q^3 and $2q^3 - q^2$.

If $q \equiv 3 \pmod 4$ we get either

$$N = q^3 + q^2 + q + 1 - q - 1 = q^3 + q^2,$$

or

$$N = q^2 + q + 1 - q - 1 = q^2,$$

according as $\mathcal{Q}$ is either the join of a line to a conic or a plane; therefore, in this case, the intersection numbers are q^2, q^3 and $q^3 + q^2$

Now consider the multiset $\overline{\mathcal{B}}(a, b)$ in $PG(3, q^2)$ arising from $\mathcal{B}(a, b)$ by assigning multiplicity bigger than 1 to just the point P_∞.

More in detail the points of the 2–character multiset $\overline{\mathcal{B}}(a, b)$ are exactly those of $\mathcal{B}(a, b) \cup \{P_\infty\}$ where each affine point of $\mathcal{B}(a, b)$ has multiplicity one, and P_∞ has either multiplicity $q^3 - q^2$ for $q \equiv 1 \pmod 4$, or multiplicity q^2 when $q \equiv 3 \pmod 4$. Our theorem follows.

Remark 4.1. Let $\mathcal{C}$ be the linear code associated to $\overline{\mathcal{B}}(a, b)$. In the first case $\mathcal{C}$ is a $[q^5 + q^3 - q^2, 4, q^5 - q^3]_{q^2}$ two-weight code, while in the second it has parameters $[q^5 + q^2, 4, q^5 - q^3]_{q^2}$. In either case the non–zero weights are q^5 and $q^5 - q^3$.

If A_i is the number of words in $\mathcal{C}$ of weight i then by Corollary 3.1 it follows that

$$A_{q^5} = (q^6 - q^3 + 1)(q^2 - 1); \qquad A_{q^5-q^3} = (q^4 + q^3 + q^2)(q^2 - 1).$$

5 Intersection sets with multiplicity $q^{2r-3} - q^{r-2}$

We keep the notation of the previous sections and examine the remaining cases. Even though the results we obtain are a direct consequence of the construction of [1], we provide some further technical details so that this paper can be considered self-contained.

Proposition 5.1. *Suppose r to be even and that either q is odd and $4a^{q+1}+(b^q-b)^2$ is a non–zero square in $\mathrm{GF}(q)$ or q is even and $\mathrm{Tr}\,(a^{q+1}/(b^q+b)^2) = 1$. Then, $\mathcal{B}(a,b)$ is a set of q^{2r-1} points of $\mathrm{AG}(r,q^2)$ with characters*

$$q^{2r-3} - q^{r-2}, q^{2r-3}, q^{2r-3} - q^{r-2} + q^{r-1}.$$

This is also a minimal intersection set with respect to hyperplanes.

Proof. We first discuss the nature of $\mathcal{Q}_\infty$. Observe that, under our assumptions, for q odd $(-1)^{r-1}\det A_\infty$ is always a square; hence, $\mathcal{Q}_\infty$ is a hyperbolic quadric.

For q even choose $\varepsilon \in \mathrm{GF}(q^2) \setminus \mathrm{GF}(q)$ such that $\varepsilon^2 + \varepsilon + \nu = 0$, for some $\nu \in \mathrm{GF}(q)\setminus\{1\}$ with $\mathrm{Tr}\,(\nu) = 1$. Then, $\varepsilon^{2q}+\varepsilon^q+\nu = 0$. Therefore, $(\varepsilon^q+\varepsilon)^2+(\varepsilon^q+\varepsilon) = 0$, whence $\varepsilon^q+\varepsilon+1 = 0$. With this choice of ε, the system given by (3) and (4) reads as

$$\begin{aligned}&(a^1+b^1)(x_1^0)^2 + [(a^0+a^1)+\nu(a^1+b^1)](x_1^1)^2 + b^1x_1^0x_1^1 + m_1^1x_1^0 + (m_1^0+m_1^1)x_1^1\\ &+\ldots+(a^1+b^1)(x_{r-1}^0)^2 + [(a^0+a^1)+\nu(a^1+b^1)](x_{r-1}^1)^2 + b^1x_{r-1}^0x_{r-1}^1\\ &+m_{r-1}^1x_{r-1}^0 + (m_{r-1}^0+m_{r-1}^1)x_{r-1}^1 + d^1 = 0.\end{aligned} \tag{6}$$

The discussion of the (possibly degenerate) quadric $\mathcal{Q}$ of Equation (6) may be carried out in close analogy to what has been done before.

Observe however that, as also pointed out in the remark before [5, Theorem 22.2.1], some caution is needed when quadrics and their classifications are studied in even characteristic. Indeed let A_∞ be the formal matrix associated to the quadric $\mathcal{Q}_\infty$ of equation

$$(a^1+b^1)(x_1^0)^2 + [(a^0+a^1)+\nu(a^1+b^1)](x_1^1)^2 + b^1x_1^0x_1^1 + \ldots$$

$$+(a^1+b^1)(x_{r-1}^0)^2 + [(a^0+a^1)+\nu(a^1+b^1)](x_{r-1}^1)^2 + b^1x_{r-1}^0x_{r-1}^1 = 0.$$

Its determinant is equal to

$$\det A_\infty = [4(a^1+b^1)(a^0+a^1+\nu(a^1+b^1)) + (b^1)^2]^{r-1}.$$

In order to encompass the case q even, $\det A_\infty$ needs to be regarded as a formal function in the polynomial ring $\mathrm{GF}(q)[z_0, z_1, z_2, z_3]$ evaluated in (a^0, a^1, b^0, b^1). This gives $\det A_\infty = b_1^2$. Here $b_1 \neq 0$, by our assumption $b^q \neq b$. From [5, Theorem 22.2.1 (i)], the quadric $\mathcal{Q}_\infty$ must be non-degenerate. Furthermore, by [5, Theorem 22.2.1 (ii)] and the successive Lemma 22.2.2 the nature of $\mathcal{Q}_\infty$ can be ascertained as follows. Let B the matrix obtained from A_∞ by omitting all the entries on its main diagonal, and define

$$\alpha = \frac{\det B - (-1)^{r-1} \det A_\infty}{4 \det B}$$

A straightforward computation shows that

$$\alpha = \frac{(b^1)^{2(r-1)} + (4(a^1 + b^1)(a^0 + a^1 + \nu(a^1 + b^1)) + (b^1)^2)^{r-1}}{4\,(b^1)^{2(r-1)}}.$$

Regard α also as a function in the polynomial ring $GF(q)[z_0, z_1, z_2, z_3]$ evaluated in (a^0, a^1, b^0, b^1). Hence we get

$$\alpha = \frac{(a^1 + b^1)(a^0 + a^1 + \nu(a^1 + b^1))}{(b^1)^2}.$$

Arguing as in [1, p. 439], we see that $\mathrm{Tr}_{\mathrm{GF}(q)|\mathrm{GF}(2)}(\alpha) = 0$ and, hence, $\mathcal{Q}_\infty$ is hyperbolic also for q even.

Now, in both cases q odd or q even we investigate the possible nature of $\mathcal{Q}$. Suppose $\mathcal{Q}$ to be non-singular; then

$$N = \frac{(q^{r-1}+1)(q^{r-1}-1)}{q-1} - \frac{(q^{r-1}+1)(q^{r-2}-1)}{q-1} = q^{r-2}(q^{r-1}+1).$$

If $\mathcal{Q}$ is singular, then

$$N = \frac{q(q^{r-1}+1)(q^{r-2}-1)}{q-1} - \frac{(q^{r-1}+1)(q^{r-2}-1)}{q-1} + 1 = q^{r-2}(q^{r-1}+1) - q^{r-1}.$$

This gives the possible intersection numbers.

Finally, in order to show that $\mathcal{B}(a,b)$ is a minimal $(q^{2r-3} - q^{r-2})$–fold blocking set we can use the same techniques as those adopted to prove that $\mathcal{B}(a,b)$ is a minimal blocking set in Section 3 for q odd and $4a^{q+1} + (b^q - b)^2 = 0$

$\square$

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