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Codes and caps from orthogonal Grassmannians

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ABSTRACT

In this paper we investigate linear error correcting codes and projective caps related to the Grassmann embedding ε_k^{gr} of an orthogonal Grassmannian Δ_k. In particular, we determine some of the parameters of the codes arising from the projective system determined by $\varepsilon_k^{gr}(\Delta_k)$. We also study special sets of points of Δ_k which are met by any line of Δ_k in at most 2 points and we show that their image under the Grassmann embedding ε_k^{gr} is a projective cap.

1. Introduction

The overarching theme of this paper is the behaviour of the image of the Grassmann embedding ε_k^{gr} of an orthogonal Grassmannian Δ_k with $k \leqslant n$ with respect to linear subspaces of either maximal or minimal dimension. In the former case, we obtain the parameters of the linear error correcting codes arising from the projective system determined by the pointset $\varepsilon_k^{gr}(\Delta_k)$ and provide a bound on their minimum distance. In the latter, we consider and construct special sets of points of Δ_k that are met by each line of Δ_k in at most 2 points and show that the Grassmann embedding

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maps these sets in projective caps. Actually, an explicit construction of a family of such sets, met by any line in at most 1 point, is also provided, and a link with Hadamard matrices is presented.

The introduction is organised as follows: in Section 1.1 we shall provide a background on embeddings of orthogonal Grassmannians; Section 1.2 is devoted to codes arising from projective systems, while in Section 1.3 we summarise our main results and outline the structure of the paper.

1.1. Orthogonal Grassmannians and their embeddings

Let $V := V(2n+1,q)$ be a $(2n+1)$-dimensional vector space over a finite field $\mathbb{F}_q$ endowed with a non-singular quadratic form η of Witt index n. For $1 \leqslant k \leqslant n$, denote by $\mathcal{G}_k$ the k-Grassmannian of $\mathrm{PG}(V)$ and by Δ_k the k-polar Grassmannian associated to η, in short the latter will be called an *orthogonal Grassmannian*. We recall that $\mathcal{G}_k$ is the point-line geometry whose points are the k-dimensional subspaces of V and whose lines are sets of the form

$$\ell_{X,Y} := \{Z \mid X \subset Z \subset Y,\ \dim(Z) = k\},$$

where X and Y are any two subspaces of V with $\dim(X) = k-1$, $\dim(Y) = k+1$ and $X \subset Y$.

The orthogonal Grassmannian Δ_k is the proper subgeometry of $\mathcal{G}_k$ whose points are the k-subspaces of V totally singular for η. For $k < n$ the lines of Δ_k are exactly the lines $\ell_{X,Y}$ of $\mathcal{G}_k$ with Y totally singular; on the other hand, when $k = n$ the lines of Δ_n turn out to be the sets

$$\ell_X := \{Z \mid X \subset Z \subset X^\perp,\ \dim(Z) = n,\ Z \text{ totally singular}\}$$

with X a totally singular $(n-1)$-subspace of V and $X^\perp$ its orthogonal with respect to η. Note that the points of ℓ_X form a conic in the projective plane $\mathrm{PG}(X^\perp/X)$. Clearly, Δ_1 is just the orthogonal polar space of rank n associated to η; the geometry Δ_n can be regarded as its dual and is thus called the *orthogonal dual polar space* of rank n. Recall that the size of the pointset of Δ_k is $\prod_{i=0}^{k-1} \frac{q^{2(n-i)}-1}{q^{i+1}-1}$; see e.g. [23, Theorem 22.5.1].

Given a point-line geometry $\Gamma = (\mathcal{P}, \mathcal{L})$ we say that an injective map $e : \mathcal{P} \to \mathrm{PG}(V)$ is a *projective embedding* of Γ if the following conditions hold:

(1) $\langle e(\mathcal{P})\rangle = \mathrm{PG}(V)$;
(2) e maps any line of Γ onto a projective line.

Following [36], see also [10], when condition (2) is replaced by

(2′) e maps any line of Γ onto a non-singular conic of $\mathrm{PG}(V)$ and for all $l \in \mathcal{L}$, $\langle e(l)\rangle \cap e(\mathcal{P}) = e(l)$

we say that e is a *Veronese embedding* of Γ.

The dimension $\dim(e)$ of an embedding $e : \Gamma \to \mathrm{PG}(V)$, either projective or Veronese, is the dimension of the vector space V. When Σ is a proper subspace of $\mathrm{PG}(V)$ such that $e(\Gamma) \cap \Sigma = \emptyset$ and $\langle e(p_1), e(p_2)\rangle \cap \Sigma = \emptyset$ for any two distinct points p_1 and p_2 of Γ, then it is possible to define a new embedding e/Σ of Γ in the quotient space $\mathrm{PG}(V/\Sigma)$ called the *quotient of e over* Σ as $(e/\Sigma)(x) = \langle e(x), \Sigma\rangle/\Sigma$.

Let now $W_k := \bigwedge^k V$. The *Grassmann* or *Plücker embedding* $e_k^{gr} : \mathcal{G}_k \to \mathrm{PG}(W_k)$ maps the arbitrary k-subspace $\langle v_1, v_2, \ldots, v_k\rangle$ of V (hence a point of $\mathcal{G}_k$) to the point $\langle v_1 \wedge v_2 \wedge \cdots \wedge v_k\rangle$ of $\mathrm{PG}(W_k)$. Let $\varepsilon_k^{gr} := e_k^{gr}|_{\Delta_k}$ be the restriction of e_k^{gr} to Δ_k. For $k < n$, the mapping ε_k^{gr} is a projective embedding of Δ_k in the subspace $\mathrm{PG}(W_k^{gr}) := \langle \varepsilon_k^{gr}(\Delta_k)\rangle$ of $\mathrm{PG}(W_k)$ spanned by $\varepsilon_k^{gr}(\Delta_k)$. We call ε_k^{gr} the *Grassmann* embedding of Δ_k.

If $k = n$, then ε_n^{gr} is a Veronese embedding and maps the lines of Δ_n onto non-singular conics of $\mathrm{PG}(W_n)$. The dual polar space Δ_n affords also a projective embedding of dimension 2^n, namely the spin embedding ε_n^{spin}; for more details we refer the reader to either [11] or [7].

Let now ν_{2^n} be the usual quadric Veronese map $\nu_{2^n} : V(2^n, \mathbb{F}) \to V(\binom{2^n+1}{2}, \mathbb{F})$ given by

$$(x_1, \dots, x_{2^n}) \to \left(x_1^2, \dots, x_{2^n}^2, x_1x_2, \dots, x_1x_{2^n}, x_2x_3, \dots, x_2x_{2^n}, \dots, x_{2^n-1}x_{2^n}\right).$$

It is well known that ν_{2^n} defines a Veronese embedding of the point-line geometry $\mathrm{PG}(2^n-1, \mathbb{F})$ in $\mathrm{PG}(\binom{2^n+1}{2}-1, \mathbb{F})$, which will be also denoted by ν_{2^n}.

The composition $\varepsilon_n^{vs} := \nu_{2^n} \cdot \varepsilon_n^{spin}$ is a Veronese embedding of Δ_n in a subspace $\mathrm{PG}(W_n^{vs})$ of $\mathrm{PG}(\binom{2^n+1}{2}-1, \mathbb{F})$: it is called the *Veronese-spin embedding* of Δ_n.

We recall some results from [10] and [9] on the Grassmann and Veronese-spin embeddings of Δ_k, $k \leqslant n$. Observe that these results hold over arbitrary fields, even if in the present paper we shall be concerned just with the finite case.

Theorem 1. *Let $\mathbb{F}_q$ be a finite field with char$(\mathbb{F}_q) \neq 2$. Then,*

(1) $\dim(\varepsilon_k^{gr}) = \binom{2n+1}{k}$ *for any $n \geqslant 2$, $k \in \{1, \dots, n\}$.*
(2) $\varepsilon_n^{vs} \cong \varepsilon_n^{gr}$ *for any $n \geqslant 2$.*

When *char*$(\mathbb{F}_q) = 2$ there exist two subspaces $\mathcal{N}_1 \supset \mathcal{N}_2$ of $\mathrm{PG}(W_n^{vs})$, called *nucleus subspaces*, such that the following holds.

Theorem 2. *Let $\mathbb{F}_q$ be a finite field with char$(\mathbb{F}_q) = 2$. Then,*

(1) $\dim(\varepsilon_k^{gr}) = \binom{2n+1}{k} - \binom{2n+1}{k-2}$ *for any $k \in \{1, \dots, n\}$.*
(2) $\varepsilon_n^{vs}/\mathcal{N}_1 \cong \varepsilon_n^{spin}$ *for any $n \geqslant 2$.*
(3) $\varepsilon_n^{vs}/\mathcal{N}_2 \cong \varepsilon_n^{gr}$ *for any $n \geqslant 2$.*

1.2. Projective systems and codes

Error correcting codes are an essential component to any efficient communication system, as they can be used in order to guarantee arbitrarily low probability of mistake in the reception of messages without requiring noise-free operation; see [27]. An $[N, K, d]_q$ projective system Ω is a set of N points in $\mathrm{PG}(K-1, q)$ such that for any hyperplane Σ of $\mathrm{PG}(K-1, q)$,

$$|\Omega \setminus \Sigma| \geqslant d.$$

Existence of $[N, K, d]_q$ projective systems is equivalent to that of projective linear codes with the same parameters; see [22,6,15,38]. Indeed, given a projective system $\Omega = \{P_1, \dots, P_N\}$, fix a reference system $\mathfrak{B}$ in $\mathrm{PG}(K-1, q)$ and consider the matrix G whose columns are the coordinates of the points of Ω with respect to $\mathfrak{B}$. Then, G is the generator matrix of an $[N, K, d]$-code over $\mathbb{F}_q$, say $\mathcal{C} = \mathcal{C}(\Omega)$, uniquely defined up to code equivalence. Furthermore, as any word c of $\mathcal{C}(\Omega)$ is of the form $c = mG$ for some row vector $m \in \mathbb{F}_q^K$, it is straightforward to see that the number of zeroes in c is the same as the number of points of Ω lying on the hyperplane of equation $m \cdot x = 0$ where $m \cdot x = \sum_{i=1}^K m_i x_i$ and $m = (m_i)_1^K$, $x = (x_i)_1^K$. In particular, the minimum distance d of $\mathcal{C}$ is

$$d = \min_{\substack{\Sigma \leqslant \mathrm{PG}(K-1,q) \\ \dim \Sigma = K-2}} \big(|\Omega| - |\Omega \cap \Sigma|\big). \tag{1}$$

The link between incidence structures $\mathcal{S} = (\mathcal{P}, \mathcal{L})$ and codes is deep and it dates at least to [29]; we refer the interested reader to [3,8,34] for more details. Traditionally, two basic approaches have been proven to be most fruitful: either to regard the incidence matrix of $\mathcal{S}$ as a generator matrix for a binary code, see for instance [30,21], or to consider an embedding of $\mathcal{S}$ in a projective space and study either the code arising from the projective system thus determined or its dual; see e.g. [4,12,20] for codes related to the Segre embedding.

Codes based on projective Grassmannians belong to this latter class. They have been first introduced in [31] as generalisations of Reed–Muller codes of the first order and whenceforth extensively investigated; see also [32,28,18,19].

1.3. Organisation of the paper and main results

In Section 2 we study linear codes associated with the projective system $\varepsilon_k^{gr}(\Delta_k)$ determined by the embedding ε_k^{gr}.

We recall that a *partial spread* of a non-degenerate quadric is a set of pairwise disjoint generators; see also [14, Chapter 2]. A partial spread $\mathcal{S}$ is a *spread* if all the points of the quadric are covered by exactly one of its elements. We recall that for q odd the quadrics $Q(4n,q)$ do not admit spreads.

Main Result 1. *Let $\mathcal{C}_{k,n}$ be the code arising from the projective system $\varepsilon_k^{gr}(\Delta_k)$ for $1 \leqslant k < n$. Then, the parameters of $\mathcal{C}_{k,n}$ are*

$$N = \prod_{i=0}^{k-1} \frac{q^{2(n-i)}-1}{q^{i+1}-1}, \qquad K = \begin{cases} \binom{2n+1}{k} & \text{for } q \text{ odd}, \\ \binom{2n+1}{k} - \binom{2n+1}{k-2} & \text{for } q \text{ even}, \end{cases}$$
$$d \geqslant \psi_{n-k}(q)\big(q^{k(n-k)}-1\big)+1,$$

where $\psi_{n-k}(q)$ is the maximum size of a (partial) spread of the parabolic quadric $Q(2(n-k),q)$.

We observe that, in practice, we expect the bound on the minimum distance not to be sharp. As for codes arising from dual polar spaces of small rank, we have the following result where the minimum distance is exactly determined.

Main Result 2.

(i) *The code $\mathcal{C}_{2,2}$ arising from a dual polar space of rank 2 has parameters*

$$N = \big(q^2+1\big)(q+1), \qquad K = \begin{cases} 10 & \text{for } q \text{ odd}, \\ 9 & \text{for } q \text{ even}, \end{cases} \qquad d = q^2(q-1).$$

(ii) *The code $\mathcal{C}_{3,3}$ arising from a dual polar space of rank 3 has parameters*

$$N = \big(q^3+1\big)\big(q^2+1\big)(q+1), \qquad K = 35, \qquad d = q^2(q-1)\big(q^3-1\big) \quad \text{for } q \text{ odd}$$

and

$$N = \big(q^3+1\big)\big(q^2+1\big)(q+1), \qquad K = 28, \qquad d = q^5(q-1) \quad \text{for } q \text{ even}.$$

In Section 3, we introduce the notion of (m,v)-set of a partial linear space and the notion of polar m-cap of Δ_k. We prove that the image of a polar m-cap under the Grassmann embedding is a projective cap; see also [16] for caps contained in Grassmannians.

Main Result 3. *Suppose $1 \leqslant k \leqslant n$. Then,*

1. *the image $\varepsilon_k^{gr}(\mathfrak{X})$ of any polar m-cap $\mathfrak{X}$ of Δ_k is a projective cap of $\mathrm{PG}(W_k)$;*
2. *the image of $\varepsilon_n^{gr}(\Delta_n)$ is a projective cap.*

In Section 4 we give an explicit construction of some $(2^r, 1)$-sets contained in Δ_k with $r \leqslant \lfloor k/2 \rfloor$. This leads to the following theorem.

Main Result 4. *For any $r \leqslant \lfloor k/2 \rfloor$, the polar Grassmannian Δ_k contains a polar 2^r-cap $\mathfrak{X}$.*

Finally, in Section 5, we consider matrices H associated to the polar caps $\mathfrak{X}$ of Main Result 4 and prove that they are of Hadamard type. It is well known that these matrices lead to important codes; see [24, Chapter 3]. Then, it is shown that it is possible to introduce an order on the points of $\mathfrak{X}$ as to guarantee the matrix H to be in Sylvester form, thus obtaining a first order Reed–Muller code; see [24, p. 42].

2. Linear codes associated to Δ_k

2.1. General case

By Theorem 1, for q odd and $1 \leqslant k \leqslant n$, the Grassmann embedding ε_k^{gr} of Δ_k into $\mathrm{PG}(\bigwedge^k V)$ has dimension $\binom{2n+1}{k}$; by Theorem 2, for q even and $1 \leqslant k \leqslant n$, $\dim(\varepsilon_k^{gr}) = \binom{2n+1}{k} - \binom{2n+1}{k-2}$. As such, the image of ε_k^{gr} determines a projective code $\mathcal{C}_{k,n}^{gr} = \mathcal{C}(\varepsilon_k^{gr}(\Delta_k))$. Observe that $\mathcal{C}_{k,n}^{gr}$ can be obtained by the full Grassmann code, see [28], by deleting a suitable number of components; however, this does not lead to useful bounds on the minimum distance. The following lemma is a direct consequence of the definition of $\mathcal{C}_{k,n}^{gr}$.

Lemma 2.1. *The code $\mathcal{C}_{k,n}^{gr}$ is an $[N, K]$-linear code with*

$$N = \prod_{i=0}^{k-1} \frac{q^{2(n-i)} - 1}{q^{i+1} - 1}, \qquad K = \begin{cases} \binom{2n+1}{k} & \text{for } q \text{ odd}, \\ \binom{2n+1}{k} - \binom{2n+1}{k-2} & \text{for } q \text{ even}. \end{cases}$$

Given any m-dimensional subspace $X \leqslant V$ with $m > k$, in an analogous way as the one followed to define the k-Grassmannian $\mathcal{G}_k$ of $\mathrm{PG}(V)$ in Section 1, we introduce the k-Grassmannian $\mathcal{G}_k(X)$ of $\mathrm{PG}(X)$. More in detail, $\mathcal{G}_k(X)$ is the point-line geometry having as points the k-dimensional subspaces of X and as lines exactly the lines of $\mathcal{G}_k$ contained in $\mathcal{G}_k(X)$.

The following lemma is straightforward.

Lemma 2.2. *Suppose X to be a totally singular subspace with $\dim X = m$ and $k < m < n$; write $W_k(X) = \langle \varepsilon_k^{gr}(\mathcal{G}_k(X)) \rangle \leqslant W_k$. Then,*

$$\varepsilon_k^{gr}\big(\mathcal{G}_k(X)\big) = \varepsilon_k^{gr}(\Delta_k) \cap W_k(X) = e_k^{gr}\big(\mathcal{G}_k(X)\big).$$

Let X be a k-dimensional subspace of V contained in the non-degenerate parabolic quadric $Q(2n,q) \cong \Delta_1$. Define the *star $St(X)$ of X* as the set formed by the i-dimensional subspaces of $Q(2n,q)$, $k < i \leqslant n$, containing X. It is well known that $St(X)$ is isomorphic to a parabolic quadric $Q(2(n-k),q)$; see [37, Chapter 7].

Denote by $\psi_r(q)$ the maximum size of a (partial) spread of $Q(2r,q)$. Recall that for q even, $Q(2r,q)$ admits a spread; thus $\psi_r(q) = q^{r+1} + 1$. For q odd a general lower bound is $\psi_r(q) \geqslant q + 1$, even if improvements are possible in several cases; see [13], [14, Chapter 2].

Theorem 2.3. *If $k < n$, the minimum distance d of $\mathcal{C}_{k,n}^{gr}$ is at least*

$$s = \psi_{n-k}(q)\big(q^{k(n-k)} - 1\big) + 1.$$

Proof. It is enough to show that for any hyperplane Σ of $\mathrm{PG}(W_k)$ not containing $\varepsilon_k^{gr}(\Delta_k)$ there are at least s points in $\Phi = \varepsilon_k^{gr}(\Delta_k) \setminus \Sigma$ and then use (1). Recall that when q is odd, $\varepsilon_k^{gr}(\Delta_k)$ is not contained in any hyperplane.

Let E be a point of Δ_k such that $\varepsilon_k^{gr}(E) \in \Phi$; as such, E is a k-dimensional subspace contained in $Q(2n,q)$ and we can consider the star $St(E) \cong Q(2(n-k),q)$. Take Ψ as a partial spread of maximum size of $St(E)$. For any $X, X' \in \Psi$, since X and X' are disjoint in $St(E)$, we have $X \cap X' = E$.

Furthermore, for any $X \in \Psi$, by Lemma 2.2, $\varepsilon_k^{gr}(\mathcal{G}_k(X)) = \varepsilon_k^{gr}(\Delta_k) \cap W_k(X)$, where $W_k(X) = \langle \varepsilon_k^{gr}(\mathcal{G}_k(X))\rangle$. As X is an $(n-1)$-dimensional projective space, we have also that $\varepsilon_k^{gr}(\mathcal{G}_k(X))$ is isomorphic to the k-Grassmannian of an n-dimensional vector space. The hyperplane Σ meets the subspace $W_k(X)$ spanned by $\varepsilon_k^{gr}(\mathcal{G}_k(X))$ in a hyperplane Σ'. By [28, Theorem 4.1], wherein codes arising from projective Grassmannians are investigated and their minimal distance computed, we have $|W_k(X) \cap \Phi| \geqslant q^{k(n-k)}$. On the other hand, from

$$\varepsilon_k^{gr}\big(\mathcal{G}_k(X)\big) \cap \varepsilon_k^{gr}\big(\mathcal{G}_k(Y)\big) = \big\{\varepsilon_k^{gr}(E)\big\},$$

for any $X, Y \in \Psi$, it follows that $\varepsilon_k^{gr}(\Delta_k)$ has at least $\psi_{n-k}(q)(q^{k(n-k)} - 1) + 1$ points off Σ. This completes the proof. $\square$

Lemma 2.1 and Theorem 2.3 together provide Main Result 1.

In Section 2.2 we determine the minimum distance of $\mathcal{C}_{1,n}^{gr}$ for $k = 1$; Sections 2.3 and 2.4 are dedicated to the case of dual polar spaces of rank 2 and 3; in these latter cases the minimum distance is precisely computed.

2.2. Codes from polar spaces Δ_1

If $k = 1$, then Δ_k is just the orthogonal polar space and ε_1^{gr} is its natural embedding in $\mathrm{PG}(2n,q)$. Hence, the code $\mathcal{C}_{1,n}^{gr}$ is the code arising from the projective system of the points of a non-singular parabolic quadric $Q(2n,q)$ of $\mathrm{PG}(2n,q)$. To compute its minimum distance, in light of (1), it is enough to study the size of $Q(2n,q) \cap \Sigma$ where Σ is an arbitrary hyperplane of $\mathrm{PG}(2n,q)$. This intersection achieves its maximum at $(q^{2n-1}-1)/(q-1) + q^{n-1}$ when $Q(2n,q) \cap \Sigma$ is a non-singular hyperbolic quadric $Q^+(2n-1,q)$, see e.g. [23, Theorem 22.6.2]. Hence, the parameters of the code $\mathcal{C}_{1,n}^{gr}$ are

$$N = \big(q^{2n}-1\big)/(q-1); \qquad K = 2n+1; \qquad d = q^{2n-1} - q^{n-1}.$$

The full weight enumerator can now be easily computed, using, for instance, [23, Theorem 22.8.2].

2.3. Dual polar spaces of rank 2

2.3.1. Odd characteristic

Suppose that the characteristic of $\mathbb{F}_q$ is odd. By (2) in Theorem 1, the image $\varepsilon_2^{gr}(\Delta_2)$ of the dual polar space Δ_2 under the Grassmann embedding is isomorphic to the quadric Veronese variety $\mathcal{V}_2$ of $\mathrm{PG}(3,q)$, as embedded in $\mathrm{PG}(9,q)$. Length and dimension of the code $\mathcal{C}_{2,2}^{gr}$ directly follow from Theorem 1. By Eq. (1), the minimum distance of $\mathcal{C}_{2,2}^{gr}$ is $|\mathcal{V}_2| - m$, where

$$m := \max\big\{|\Sigma \cap \mathcal{V}_2|\colon\ \Sigma \text{ is a hyperplane of } \mathrm{PG}(9,q)\big\}.$$

It is well known, see e.g. [23, Theorem 25.1.3], that there is a bijection between the quadrics of $\mathrm{PG}(3,q)$ and the hyperplane sections of $\mathcal{V}_2$; thus, in order to determine m we just need to consider the maximum cardinality of a quadric Q in $\mathrm{PG}(3,q)$. This cardinality is $2q^2+q+1$, and corresponds to the case in which Q is the union of two distinct planes. Hence, we have the following theorem.

Theorem 2.4. *If q is odd, then the code $\mathcal{C}^{gr}_{2,2}$ is an $[N,K,d]_q$-linear code with the following parameters*

$$N=\left(q^2+1\right)(q+1), \qquad K=10, \qquad d=q^2(q-1).$$

The full spectrum of its weights is $\{q^3-q, q^3+q, q^3, q^3-q^2, q^3+q^2\}$.

Theorem 2.4 is part (i) of Main Result 2 for q odd.

2.3.2. Even characteristic

Assume that $\mathbb{F}_q$ has characteristic 2. By Theorem 2, let $\mathcal{N}_2$ be the nucleus subspace of $\mathrm{PG}(W_2^{vs})$ such that $\varepsilon_2^{gr} \cong \varepsilon_2^{vs}/\mathcal{N}_2$. It is possible to choose a basis $\mathfrak{B}$ of V so that η is given by $\eta(x_1,x_2,x_3,x_4,x_5)=x_1x_4+x_2x_5+x_3^2$; by [9], $\mathcal{N}_2$ can then be taken as the 1-dimensional subspace $\mathcal{N}_2=\langle(0,0,0,0,0,0,1,1,0,0)\rangle$. Clearly, the code $\mathcal{C}^{gr}_{2,2}$ has dimension $K=\dim(\varepsilon_2^{gr})=\dim(\varepsilon^{vs}/\mathcal{N}_2)=9$. To determine its minimum distance we use (1); in particular we need to compute $|\varepsilon_2^{gr}(\Delta_2)\cap\Sigma|$ with Σ an arbitrary hyperplane of the projective space defined by $\langle\varepsilon_2^{gr}(\Delta_2)\rangle$. Since $\langle\varepsilon_2^{gr}(\Delta_2)\rangle \cong \langle\varepsilon_2^{vs}(\Delta_2)/\mathcal{N}_2\rangle$, we have $\Sigma=\bar{\Sigma}/\mathcal{N}_2$ with $\bar{\Sigma}$ a hyperplane of $\langle\varepsilon_2^{vs}(\Delta_2)\rangle=\langle\mathcal{V}_2\rangle$ containing $\mathcal{N}_2$, where $\mathcal{V}_2$, as in Section 2.3.1, denotes the quadric Veronese variety of $\mathrm{PG}(3,q)$ in $\mathrm{PG}(9,q)$. As in the odd characteristic case, hyperplane sections of $\mathcal{V}_2$ bijectively correspond to quadrics of $\mathrm{PG}(3,q)$ and the maximum cardinality for a quadric Q of a 3-dimensional projective space is attained when Q is the union of two distinct planes, so it is $2q^2+q+1$. It is not hard to see that there actually exist degenerate quadrics Q of $\mathrm{PG}(3,q)$ which are union of two distinct planes and such that the corresponding hyperplane Σ_Q in $\langle\varepsilon_2^{vs}(\Delta_2)\rangle=\langle\mathcal{V}_2\rangle$ contains $\mathcal{N}_2$: for instance, one can take the quadric Q of equation $x_1x_2=0$. Hence, $\Sigma_Q/\mathcal{N}_2$ is a hyperplane of $\langle\varepsilon_2^{gr}(\Delta_2)\rangle \cong \langle\varepsilon_2^{vs}(\Delta_2)/\mathcal{N}_2\rangle$. As no line joining two points of $\varepsilon_2^{gr}(\Delta_2)$ passes through $\mathcal{N}_2$,

$$|\Sigma_Q\cap\mathcal{V}_2|=\left|\Sigma_Q/\mathcal{N}_2\cap\varepsilon_2^{gr}(\Delta_2)\right|=|Q|=2q^2+q+1.$$

So, $|\varepsilon_2^{gr}(\Delta_2)\cap\Pi|\leqslant 2q^2+q+1$ for every hyperplane Π of $\langle\varepsilon_2^{gr}(\Delta_2)\rangle$. This proves the following.

Theorem 2.5. *If q is even, then $\mathcal{C}^{gr}_{2,2}$ is a linear $[N,K,d]_q$-code with parameters*

$$N=\left(q^2+1\right)(q+1), \qquad K=9, \qquad d=q^2(q-1).$$

Theorem 2.5 is part (i) of Main Result 2 for q even.

2.4. Dual polar spaces of rank 3

2.4.1. Odd characteristic

Here $\mathbb{F}_q$ is assumed to have odd characteristic. By (2) in Theorem 1, the image of the Grassmann embedding $\varepsilon_3^{gr}\cong\varepsilon_3^{vs}$ spans a 34-dimensional projective space. Recall that the spin embedding ε_3^{spin} maps Δ_3 into the pointset Q_7^+ of a non-singular hyperbolic quadric of a 7-dimensional projective space; see e.g. [11] and [7]. Hence, $\varepsilon_3^{vs}(\Delta_3)=\nu_{2^3}(\varepsilon_3^{spin}(\Delta_3))=\nu_{2^3}(Q_7^+)$ is a hyperplane section of $\langle\nu_{2^3}(\mathrm{PG}(7,q))\rangle$. Using the correspondence induced by the quadratic Veronese embedding $\nu_{2^3}:\mathrm{PG}(7,q)\to\mathrm{PG}(35,q)$ between quadrics of $\mathrm{PG}(7,q)$ and hyperplane sections of the quadric Veronese variety $\mathcal{V}_2$ we see that the pointset $\varepsilon_3^{gr}(\Delta_3)\cong\nu_{2^3}(Q_7^+)$ is a hyperplane section of $\mathcal{V}_2$.

In order to determine the minimum distance d of the code $\mathcal{C}^{gr}_{3,3}$ we need now to compute

$$m=\max\left\{\left|\Sigma\cap\varepsilon_3^{gr}(\Delta_3)\right|:\ \Sigma \text{ is a hyperplane of } \mathrm{PG}(34,q)\right\}.$$

Note that $|\Sigma \cap \varepsilon_3^{gr}(\Delta_3)| = |\Sigma \cap \nu_{2^3}(\mathcal{Q}_7^+)|$ and $\Sigma = \bar{\Sigma} \cap \langle \varepsilon_3^{gr}(\Delta_3)\rangle$, where $\bar{\Sigma}$ is a hyperplane of $\langle \mathcal{V}_2\rangle \cong \mathrm{PG}(35,q)$ different from $\langle \nu_{2^3}(Q_7^+)\rangle = \langle \varepsilon_3^{gr}(\Delta_3)\rangle$. Because of the Veronese correspondence, $\bar{\Sigma} = \nu_{2^3}(Q)$ for some quadric Q of $\mathrm{PG}(7,q)$, distinct from Q_7^+. In particular,

$$\left|\varepsilon_3^{gr}(\Delta_3) \cap \Sigma\right| = \left|Q_7^+ \cap Q\right|.$$

Hence, in order to determine the minimum distance of the code, it suffices to compute the maximum cardinality m of $Q_7^+ \cap Q$ with Q_7^+ a given non-singular hyperbolic quadric of $\mathrm{PG}(7,q)$ and $Q \neq Q_7^+$ any other quadric of $\mathrm{PG}(7,q)$.

The study of the spectrum of the cardinalities of the intersection of any two quadrics has been performed in [17], in the context of functional codes of type $C_2(\mathcal{Q}^+)$, that is codes defined by quadratic functions on quadrics; see also [26, Remark 5.11]. In particular, in [17], the value of m is determined by careful analysis of all possible intersection patterns. Here we present an independent, different and shorter, argument leading to the same conclusion, based on elementary linear algebra. We point out that our technique could be extended to determine the full intersection spectrum of two quadrics.

Lemma 2.6. *Let $\mathcal{Q}^+$ be a given non-singular hyperbolic quadric of* $\mathrm{PG}(2n+1,q)$. *If $\mathcal{Q}$ is any other quadric of* $\mathrm{PG}(2n+1,q)$ *not containing any generator of $\mathcal{Q}^+$, then $|\mathcal{Q} \cap \mathcal{Q}^+| \leqslant (2q^n - q^{n-1} - 1)(q^n+1)/(q-1)$.*

Proof. The number of generators of $\mathcal{Q}^+$ is $\kappa(n) = 2(q+1)(q^2+1)\cdots(q^n+1)$. By the assumptions, any generator of $\mathcal{Q}^+$ meets $\mathcal{Q}$ in a quadric $\mathcal{Q}'$ of $\mathrm{PG}(n,q)$. It can be easily seen that $|\mathcal{Q}'|$ is maximal when $\mathcal{Q}'$ is the union of two distinct hyperplanes; hence, $|\mathcal{Q}'| \leqslant (2q^n - q^{n-1} - 1)/(q-1)$. Thus,

$$\left|\mathcal{Q}^+ \cap \mathcal{Q}\right| \leqslant \frac{(2q^n - q^{n-1} - 1)}{(q-1)} \cdot \frac{\kappa(n)}{\kappa(n-1)} = \frac{2q^{2n} - q^{2n-1} + q^n - q^{n-1} - 1}{q-1}. \qquad \square$$

Lemma 2.7. *Given a non-singular hyperbolic quadric $\mathcal{Q}^+$ in* $\mathrm{PG}(2n+1,q)$, *q odd, we have*

$$m = \max\left|\mathcal{Q}^+ \cap \mathcal{Q}\right| = \frac{2q^{2n} - q^{2n-1} + 2q^{n+1} - 3q^n + q^{n-1} - 1}{q-1},$$

as $\mathcal{Q} \neq \mathcal{Q}^+$ varies among all possible quadrics of $\mathrm{PG}(2n+1,q)$. *This number is attained only if the linear system generated by $\mathcal{Q}$ and $\mathcal{Q}^+$ contains a quadric splitting in the union of two distinct hyperplanes.*

Proof. Choose a reference system $\mathfrak{B}$ in $\mathrm{PG}(2n+1,q)$ wherein the quadric $\mathcal{Q}^+$ is represented by the matrix $C = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}$, with I and 0 respectively the $(n+1)\times(n+1)$-identity and null matrices.

If $\mathcal{Q}$ and $\mathcal{Q}^+$ were not to share any generator, then the bound provided by Lemma 2.6 on the size of their intersection would hold. Assume, instead, that $\mathcal{Q}$ and $\mathcal{Q}^+$ have at least one generator in common. We will determine the maximum intersection they can achieve; as this will be larger then the aforementioned bound, this will determine the actual maximum cardinality that is attainable. Under this hypothesis, we can suppose that $\mathcal{Q}$ is represented with respect to $\mathfrak{B}$ by a matrix of the form $S = \begin{pmatrix} 0 & M \\ M^T & B \end{pmatrix}$, with B an $(n+1)\times(n+1)$-symmetric matrix and M an arbitrary $(n+1)\times(n+1)$-matrix whose transpose is M^T.

Let $\binom{X}{Y}$ be the coordinates of a vector spanning a point of $\mathrm{PG}(2n+1,q)$ with X and Y column vectors of length $n+1$.

Then, $\langle \binom{X}{Y} \rangle \in \mathcal{Q} \cap \mathcal{Q}^+$ if and only if

$$\begin{pmatrix} X^T & Y^T \end{pmatrix} \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} \begin{pmatrix} X \\ Y \end{pmatrix} = 0 \quad \text{and} \quad \begin{pmatrix} X^T & Y^T \end{pmatrix} \begin{pmatrix} 0 & M \\ M^T & B \end{pmatrix} \begin{pmatrix} X \\ Y \end{pmatrix} = 0,$$

which is equivalent to

$$\begin{cases} X^T Y = 0, \\ 2X^T MY + Y^T BY = 0. \end{cases} \tag{2}$$

We need to determine M and B as to maximise the number of solutions of (2); in order to compute this number, we consider (2) as a family of linear systems in the unknown X, with Y regarded as a parameter. Four cases have to be investigated.

1. Take $Y = 0$. Then, any X is solution of (2); this accounts for $\frac{q^{n+1}-1}{q-1}$ points in the intersection.
2. When $Y \neq 0$ and Y is not an eigenvector of M, (2) is a system of two independent equations in $n+1$ unknowns. Hence, there are q^{n-1} solutions for X. If N is the total number of eigenvectors of M, the number of points in $\mathcal{Q} \cap \mathcal{Q}^+$ corresponding to this case is $\frac{q^{n-1}(q^{n+1}-(N+1))}{q-1}$.
3. If Y is an eigenvector of M and $Y^T BY \neq 0$, then (2) has no solutions in X.
4. Finally, suppose Y to be an eigenvector of M and $Y^T BY = 0$. Then, there are q^n values for X fulfilling (2). Denote by N_0 the number of eigenvectors Y of M such that $Y^T BY = 0$. Then, there are $\frac{q^n N_0}{q-1}$ distinct projective points in the intersection $\mathcal{Q} \cap \mathcal{Q}^+$ corresponding to this case.

The preceding argument shows

$$\begin{aligned} |\mathcal{Q} \cap \mathcal{Q}^+| &= \frac{q^{n-1}(q^{n+1} - N - 1)}{q-1} + \frac{q^n N_0}{q-1} + \frac{q^{n+1}-1}{q-1} \\ &= \frac{(qN_0 - N)q^{n-1}}{q-1} + \frac{(q^{n-1}+1)(q^{n+1}-1)}{q-1}. \end{aligned} \tag{3}$$

As $0 \leqslant N_0 \leqslant N \leqslant q^{n+1} - 1$, the maximum of (3) is attained for the same values as the maximum of $g(N_0, N) := (qN_0 - N)/(q-1)$, where N_0 and N vary among all allowable values. Clearly, when this quantity is maximal, it has the same order of magnitude as N_0. Several possibilities have to be considered:

(i) $N_0 = N = q^{n+1} - 1$; then, the matrix M has just one eigenspace of dimension $n+1$ and $B = 0$. From a geometric point of view this means $\mathcal{Q}^+ \equiv \mathcal{Q}$.

(ii) $N_0 = 2q^n - q^{n-1} - 1$ and $N = q^{n+1} - 1$; then,

$$g_1 := g\big(2q^n - q^{n-1} - 1, q^{n+1} - 1\big) = q^n - 1.$$

The matrix M has just one eigenspace $\mathcal{M}_{n+1}$ of dimension $n+1$ and $N_0/(q-1)$ is the maximum cardinality of a quadric of an n-dimensional projective space, corresponding to the union of two distinct hyperplanes.

(iii) $N_0 = N = q^n + q - 2$; then,

$$g_2 := g\big(q^n + q - 2, q^n + q - 2\big) = q^n + q - 2.$$

The matrix M has two distinct eigenspaces say $\mathcal{M}_n$ and $\mathcal{M}_1$, of dimension respectively n and 1 and eigenvalues λ_n and λ_1.

All other possible values of N_0, corresponding to the cardinality of quadrics in an $(n+1)$-dimensional vector space, are smaller than $2q^n - q^{n-1} - 1$. As $g_1 \leqslant g_2$, the choice of (iii) gives the maximum cardinality.

We now investigate the geometric configuration arising in Case (iii). Let $\mathbb{U} = (u_1, u_2, \dots, u_{n+1})$ be a basis of eigenvectors for M with $Mu_1 = \lambda_1 u_1$ and take D as a diagonalising matrix for M. So, the column D_i of D is the eigenvector u_i for $i = 1, \dots, n+1$ and $De_i = u_i$, with $\mathbb{E} = (e_1, e_2, \dots, e_{n+1})$ the canonical basis with respect to which M was originally written. We have $(M - \lambda_n I)De_1 = (\lambda_1 - \lambda_n)u_1$

and $(M-\lambda_n I)De_i=0$ for $i=2,\dots,n+1$. Hence, $(M-\lambda_n I)D$ is the null matrix except for the first column only. Thus,

$$D^T(M-\lambda_n I)D=\begin{pmatrix} s_0 & 0 & \dots & 0\\ s_1 & 0 & \dots & 0\\ \vdots & & & \vdots\\ s_n & 0 & \dots & 0\end{pmatrix}.$$

On the other hand, the matrix $B'=D^TBD$ represents a quadric in $\mathrm{PG}(n,q)$ containing both the point $\langle(1,0,0,\dots,0)\rangle$ and the hyperplane of equation $x_1=0$, where the coordinates are written with respect to $\mathbb{U}$. Thus,

$$D^TBD=\begin{pmatrix} 0 & r_1 & \dots & r_n\\ r_1 & 0 & \dots & 0\\ \vdots & & & \vdots\\ r_n & 0 & \dots & 0\end{pmatrix}.$$

It is now straightforward to see that

$$\operatorname{rank}\left(\begin{pmatrix} D^T & 0\\ 0 & D^T\end{pmatrix}\left(\begin{pmatrix} 0 & M\\ M^T & B\end{pmatrix}-\lambda_n\begin{pmatrix} 0 & I\\ I & 0\end{pmatrix}\right)\begin{pmatrix} D & 0\\ 0 & D\end{pmatrix}\right)=2.$$

In particular, as $\left(\begin{smallmatrix} D & 0\\ 0 & D\end{smallmatrix}\right)$ is invertible, also

$$\operatorname{rank}\left(\begin{pmatrix} 0 & M\\ M^T & B\end{pmatrix}-\lambda_n\begin{pmatrix} 0 & I\\ I & 0\end{pmatrix}\right)=2.$$

Hence, the quadric $\mathcal{Q}'=\mathcal{Q}-\lambda_n\mathcal{Q}^+$ is union of two distinct hyperplanes.

We remark that also in the case of (ii), the quadric $\mathcal{Q}-\lambda_{n+1}\mathcal{Q}^+$, where λ_{n+1} is the eigenvalue of M with multiplicity $n+1$, is union of two hyperplanes, as B has rank 2. $\square$

Theorem 2.8. *For q odd, $\mathcal{C}^{gr}_{3,3}$ is an $[N,K,d]_q$-linear code with the following parameters*

$$N=(q^3+1)(q^2+1)(q+1),\qquad K=35,\qquad d=q^2(q-1)(q^3-1).$$

Proof. By Lemma 2.7, for $n=3$ the maximum cardinality of the intersection of a hyperbolic quadric $\mathcal{Q}^+$ with any other quadric is $2q^5+q^4+3q^3+q+1$. The minimum distance follows from (1). $\square$

Theorem 2.8 is part (ii) of Main Result 2 for q odd.

2.4.2. Even characteristic

We now consider the case $\mathbb{F}_q=\mathbb{F}_{2^r}$. By (3) in Theorem 2, $\varepsilon_3^{gr}(\Delta_3)\cong(\varepsilon_3^{vs}/\mathcal{N}_2)(\Delta_3)$, where $\mathcal{N}_2$ is the nucleus subspace of $\langle\varepsilon_3^{vs}(\Delta_3)\rangle$. Note that, by definition of quotient embedding, any line joining two distinct points of $\varepsilon_3^{gr}(\Delta_3)$ is skew to $\mathcal{N}_2$.

As in the case of odd characteristic, the spin embedding ε_3^{spin} maps Δ_3 to the pointset of a non-singular hyperbolic quadric $\mathcal{Q}_7^+$ of a 7-dimensional projective space $\mathrm{PG}(7,q)$; see [7]. Hence, by [23, Theorem 25.1.3], $\mathrm{PG}(W_3^{vs})=\langle\varepsilon_3^{vs}(\Delta_3)\rangle=\langle\nu_{2^3}(\varepsilon_3^{spin}(\Delta_3))\rangle=\langle\nu_{2^3}(Q_7^+)\rangle$ is a hyperplane of the 35-dimensional projective space $\langle\nu_{2^3}(\mathrm{PG}(7,q))\rangle=\langle\mathcal{V}_2\rangle$, where $\mathcal{V}_2$ is, as usual, the quadric Veronese variety of $\mathrm{PG}(7,q)$.

It is always possible to choose a reference system of V wherein η is given by $\eta(x_1, x_2, x_3, x_4, x_5, x_6, x_7) = x_1x_5 + x_2x_6 + x_3x_7 + x_4^2$. Let $(x_{i,j})_{1\leqslant i\leqslant j\leqslant 8}$ be the coordinates of a vector x in $\langle \mathcal{V}_2\rangle$, written with respect to the basis $(e_i \otimes e_j)_{1\leqslant i\leqslant j\leqslant 8}$ of $\langle \mathcal{V}_2\rangle$, with $(e_i)_i^8$ a basis of the vector space defining the 7-dimensional projective space $\langle \varepsilon_3^{spin}(\Delta_3)\rangle$. Then, by [9], the equation of the hyperplane $\langle \varepsilon_3^{vs}(\Delta_3)\rangle$ in $\langle \mathcal{V}_2\rangle$ is $x_{1,8} + x_{2,7} + x_{3,6} + x_{4,5} = 0$, while $\mathcal{N}_2$ can be represented by the following system of 29 equations:

$$\begin{cases} x_{2,8} = x_{4,6}, & x_{1,1} = 0, & x_{1,5} = 0,\\ x_{2,3} = x_{1,4}, & x_{2,2} = 0, & x_{2,4} = 0,\\ x_{1,6} = x_{2,5}, & x_{3,3} = 0, & x_{2,6} = 0,\\ x_{1,7} = x_{3,5}, & x_{4,4} = 0, & x_{3,4} = 0,\\ x_{3,8} = x_{4,7}, & x_{5,5} = 0, & x_{3,7} = 0,\\ x_{5,8} = x_{6,7}, & x_{6,6} = 0, & x_{4,8} = 0,\\ x_{1,8} = x_{4,5}, & x_{7,7} = 0, & x_{5,6} = 0,\\ x_{2,7} = x_{4,5}, & x_{8,8} = 0, & x_{5,7} = 0,\\ x_{3,6} = x_{4,5}, & x_{1,2} = 0, & x_{6,8} = 0,\\ & x_{1,3} = 0, & x_{7,8} = 0. \end{cases}$$

Take now $\Sigma \neq W_3^{vs}$ as an arbitrary hyperplane of $\langle \nu_{2^3}(\mathrm{PG}(7, q))\rangle$ containing $\mathcal{N}_2$. Then, Σ has equation of the form

$$\sum_{1\leqslant i\leqslant j\leqslant 8} a_{i,j}x_{i,j} = 0,$$

with the coefficients $a_{i,j}$ fulfilling

$$\begin{cases} a_{1,4} = a_{2,3}, & a_{1,6} = a_{2,5},\\ a_{1,7} = a_{3,5}, & a_{2,8} = a_{4,6},\\ a_{3,8} = a_{4,7}, & a_{5,8} = a_{6,7},\\ a_{1,8} + a_{2,7} + a_{3,6} + a_{4,5} = 0. \end{cases}$$

By [23, Theorem 25.1.3], there is a quadric $\mathcal{Q}_\Sigma$ of the 7-dimensional projective space $\langle \varepsilon_3^{spin}(\Delta_3)\rangle = \langle \mathcal{Q}_7^+\rangle$ such that $\Sigma \cap \mathcal{V}_2 = \nu_{2^3}(\mathcal{Q}_\Sigma)$. Since $\mathcal{N}_2 \subset \Sigma$ and $\mathcal{N}_2$ is skew with respect to $\varepsilon_3^{gr}(\Delta_3)$,

$$\left|\mathcal{Q}_\Sigma \cap \mathcal{Q}_7^+\right| = \left|\Sigma/\mathcal{N}_2 \cap \varepsilon_3^{gr}(\Delta_3)\right|.$$

Observe that $\langle \varepsilon_3^{gr}(\Delta_3)\rangle \cong \mathrm{PG}(W_3^{vs}/\mathcal{N}_2)$ is a 27-dimensional projective space and $\Sigma/\mathcal{N}_2$ is an arbitrary hyperplane of $\langle \varepsilon_3^{gr}(\Delta_3)\rangle$. With the notation just introduced, we prove the following.

Lemma 2.9. *As $\mathcal{Q}_\Sigma$ varies among all the quadrics of* $\mathrm{PG}(7, q)$ *corresponding to hyperplanes Σ of* $\langle \nu_{2^3}(\mathrm{PG}(7, q))\rangle$ *containing $\mathcal{N}_2$,*

$$m = \max\left|\mathcal{Q}_\Sigma \cap \varepsilon_3^{spin}(\Delta_3)\right| = 2q^5 + q^4 + 2q^3 + q^2 + q + 1.$$

Proof. Suppose the pointset of $\varepsilon_3^{spin}(\Delta_3)$ to be that of the hyperbolic quadric $\mathcal{Q}_7^+$ of equation $x_1x_8 + x_2x_7 + x_3x_6 + x_4x_5 = 0$. As the bound of Lemma 2.6 holds also in even characteristic, we need to consider those hyperplanes $\Sigma = \nu_{2^3}(\mathcal{Q}_\Sigma)$ of $\langle \mathcal{V}_2 \rangle$ containing $\mathcal{N}_2$ and corresponding to quadrics $\mathcal{Q}_\Sigma$ of the 7-dimensional projective space $\langle \varepsilon_3^{spin}(\Delta_3) \rangle$ with at least one generator in common with $\mathcal{Q}_7^+$. In particular, we can assume Σ to have equation

$$\sum_{1 \leqslant i \leqslant j \leqslant 8} a_{i,j} x_{i,j} = 0,$$

where the coefficients $a_{i,j}$ satisfy

$$\begin{cases} a_{1,4} = a_{2,3}, \qquad a_{1,6} = a_{2,5}, \qquad a_{1,7} = a_{3,5}, \\ a_{2,8} = a_{4,6}, \qquad a_{3,8} = a_{4,7}, \\ a_{1,8} + a_{2,7} + a_{3,6} + a_{4,5} = 0, \\ a_{i,j} = 0 \quad \text{when } 5 \leqslant i \leqslant j. \end{cases}$$

Hence, the quadric $\mathcal{Q}_\Sigma$ has equation $\sum_{1 \leqslant i \leqslant j \leqslant 8} a_{i,j} x_i x_j = 0$, with the coefficients $a_{i,j}$ fulfilling the previous conditions. Thus, $\mathcal{N}_2$ is contained in the hyperplane $\Sigma = \nu_{2^3}(\mathcal{Q}_\Sigma)$, while $\mathcal{Q}_\Sigma \cap \mathcal{Q}_7^+$ contains the 3-dimensional projective space of equations $x_1 = x_2 = x_3 = x_4 = 0$.

Rewrite the equation of $\mathcal{Q}_\Sigma$ in a more compact form as

$$Y^T M^T X + \sum_{1 \leqslant i \leqslant j \leqslant 4} a_{i,j} x_i x_j = 0,$$

where

$$X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}, \qquad Y = \begin{pmatrix} x_5 \\ x_6 \\ x_7 \\ x_8 \end{pmatrix}, \qquad M = \begin{pmatrix} a_{1,5} & a_{1,6} & a_{1,7} & a_{1,8} \\ a_{1,6} & a_{2,6} & a_{2,7} & a_{2,8} \\ a_{1,7} & a_{3,6} & a_{3,7} & a_{3,8} \\ a_{4,5} & a_{2,8} & a_{3,8} & a_{4,8} \end{pmatrix}$$

with $a_{1,8} + a_{2,7} + a_{3,6} + a_{4,5} = 0$ and $a_{1,4} = a_{2,3}$.

We can also write the equation of $\mathcal{Q}_7^+$ as $Y^T J X = 0$ with $J = \begin{pmatrix} 0&0&0&1 \\ 0&0&1&0 \\ 0&1&0&0 \\ 1&0&0&0 \end{pmatrix}$.

Arguing as in the proof of Lemma 2.7, let $\langle \binom{X}{Y} \rangle$ be a point of $\mathrm{PG}(7, q)$. Then, $\langle \binom{X}{Y} \rangle \in \mathcal{Q}_\Sigma \cap \mathcal{Q}_7^+$ if, and only if,

$$\begin{cases} Y^T J X = 0, \\ Y^T M^T X + \sum_{1 \leqslant i \leqslant j \leqslant 4} a_{i,j} x_i x_j = 0, \end{cases} \tag{4}$$

where J, M are as previously defined.

Since $J^2 = I$, if we put $\bar{M} := J M^T$ and $\bar{Y}^T := Y^T J$, system (4) becomes as follows, where we have also included the conditions on the coefficients $a_{i,j}$:

$$\begin{cases} \bar{Y}^T X = 0, \\ \bar{Y}^T \bar{M} X + \sum_{1 \leqslant i \leqslant j \leqslant 4} a_{i,j} x_i x_j = 0, \\ trace(\bar{M}) = 0, \qquad a_{2,3} = a_{1,4}. \end{cases} \tag{5}$$

System (5) is the analogue of system (2) in Lemma 2.7 for $n = 3$, with the further restrictions $trace(\bar{M}) = 0$ and $a_{2,3} = a_{1,4}$. Hence, it is possible to perform the same analysis as before, in order to determine the number of its solutions. The maximum is achieved when $\bar{M}$ admits a unique eigenspace of dimension 4, as in Cases (i) and (ii) of Lemma 2.7. This means that $\bar{M}$ is similar to a diagonal matrix $diag(\lambda_1, \lambda_1, \lambda_1, \lambda_1)$, hence $trace(\bar{M}) = trace(diag(\lambda_1, \lambda_1, \lambda_1, \lambda_1)) = 4\lambda_1 = 0$ and the trace condition is satisfied.

Furthermore, if the coefficients $a_{i,j}$ in $\sum_{1\leqslant i\leqslant j\leqslant 4} a_{i,j}x_ix_j = 0$ are all 0, then $\mathcal{Q}_\Sigma = \mathcal{Q}_7^+$; this is the analogue of Case (i) of Lemma 2.7. Note that Case (iii) of Lemma 2.7 cannot happen, as if $\bar{M}$ were to admit two eigenspaces of dimensions respectively 1 and 3, then it would be similar to a diagonal matrix $diag(\lambda_1, \lambda_2, \lambda_2, \lambda_2)$, $\lambda_1 \neq \lambda_2$. However, $trace(\bar{M}) = trace(diag(\lambda_1, \lambda_2, \lambda_2, \lambda_2)) = \lambda_1 + 3\lambda_2 = 0$ gives $\lambda_1 = \lambda_2$ – a contradiction.

When the vectors satisfying the equation $\sum_{1\leqslant i\leqslant j\leqslant 4} a_{i,j}x_ix_j = 0$ represent points lying on two distinct planes of a 3-dimensional projective space, we have the analogue of Case (ii) of Lemma 2.7 and this achieves the maximum intersection size.

We have thus shown that the maximum value m for $|\mathcal{Q}_\Sigma \cap \mathcal{Q}^+|$ is attained for $\bar{M}$ similar to the diagonal matrix $diag(\lambda_1, \lambda_1, \lambda_1, \lambda_1)$ and $m = 2q^5 + q^4 + 2q^3 + q^2 + q + 1$. $\square$

Theorem 2.10. *For q even, the code $\mathcal{C}_{3,3}^{gr}$ is an $[N, K, d]_q$-linear code with*

$$N = \left(q^3 + 1\right)\left(q^2 + 1\right)(q + 1), \qquad K = 28, \qquad d = q^5(q - 1).$$

Theorem 2.10 is part (ii) of Main Result 2 for q even.

3. Projective and polar caps

In this section $\mathbb{F}$ is an arbitrary, possibly infinite, field. A *projective cap* of $\mathrm{PG}(n, \mathbb{F})$ is a set $\mathcal{C}$ of points of $\mathrm{PG}(n, \mathbb{F})$ which is met by no line of $\mathrm{PG}(n, \mathbb{F})$ in more than 2 points. A generalisation to an arbitrary point-line geometry $\Gamma = (\mathcal{P}, \mathcal{L})$ is as follows: an (m, v)*-set* $\mathcal{C} \subseteq \mathcal{P}$ is a set of m points which is met by any $\ell \in \mathcal{L}$ in *at most* v points.

Clearly, when Γ is a linear space, non-trivial (m, v)-sets can exist only for $v \geqslant 2$; however, when not all of the points of Γ are collinear, $(m, 1)$-sets are also interesting (consider, for instance, the case of ovoids in polar spaces).

In the present section we shall be dealing exclusively with $(m, 2)$-sets, henceforth called in brief m-caps. When $\Gamma = \mathrm{PG}(r, \mathbb{F})$, $\mathcal{G}_k$ or Δ_k we speak respectively of projective, Grassmann or polar m-caps.

In Theorem 3.4, it will be shown that the whole pointset of a dual polar space Δ_n is mapped by the Grassmann embedding into a projective cap, even if, clearly, the full pointset Δ_k for any $k \leqslant n$ could never be a polar cap of itself, as Δ_k, for $n > 1$, contains lines. For $k < n$, the Grassmann embedding ε_k^{gr} is a projective embedding, that is it maps lines of Δ_k onto projective lines; thus, $\varepsilon_k^{gr}(\Delta_k)$ cannot be a cap. However, in Theorem 3.2 and Corollary 3.3 we shall show that Grassmann and polar caps are mapped by ε_k^{gr} into projective caps; see also [16] and [5] for caps contained in classical varieties. This is significant as, when a geometry Γ is projectively embedded in a larger geometry, say Γ' and not all the points of Γ are collinear, then there might be m-caps of Γ which are not inherited by Γ'.

Theorem 3.1. *Let $1 \leqslant k \leqslant n$. If $\mathcal{C}$ is a polar m-cap of Δ_k, then $\mathcal{C}$ is a Grassmann m-cap of $\mathcal{G}_k$.*

Proof. Let P_1, P_2 and P_3 be three distinct points of $\mathcal{C}$. By way of contradiction, suppose P_1, P_2 and P_3 to be collinear in $\mathcal{G}_k$. So, P_1, P_2 and P_3 are three k-dimensional totally singular subspaces of V with $\dim(P_1 \cap P_2 \cap P_3) = k - 1$ and $\dim\langle P_1, P_2, P_3\rangle = k + 1$. Put $S := \langle P_1, P_2, P_3\rangle$.

If $\mathbb{F} = \mathbb{F}_2$, then $S = P_1 \cup P_2 \cup P_3$ is a singular subspace; hence, P_1, P_2 and P_3 are collinear in Δ_k. This contradicts the hypothesis.

If $\mathbb{F} \neq \mathbb{F}_2$, take $x \in S \setminus (P_1 \cup P_2 \cup P_3)$ and $y \in P_1 \setminus (P_1 \cap P_2 \cap P_3)$. The line $\langle x, y\rangle$ meets P_2 and P_3 in distinct points, say $y_2 \in P_2 \setminus P_3$ and $y_3 \in P_3 \setminus P_2$, as each P_i, $1 \leqslant i \leqslant 3$ is a hyperplane in S

and $x \notin (P_1 \cup P_2 \cup P_3)$. Then, the line $\langle x, y\rangle$ has three distinct singular points. Necessarily, $\langle x, y\rangle$ is a singular line; thus, x is a singular point and S is a totally singular subspace.

For $1 \leqslant k < n$, this means that P_1, P_2 and P_3 are collinear in Δ_k, contradicting the hypothesis on $\mathcal{C}$.

For $k = n$ we would have determined a totally singular subspace $S \leqslant V$ of dimension $n+1$. This is, again, impossible, as the maximal singular subspaces of V have dimension n. □

Theorem 3.2. *Let $1 \leqslant k \leqslant n$. If $\mathcal{C}$ is a Grassmann m-cap of $\mathcal{G}_k$, then $e_k^{gr}(\mathcal{C})$ is a projective cap of* $\mathrm{PG}(W_k)$.

Proof. Let P_1, P_2 and P_3 be three distinct points of $\mathcal{C}$. Put $\bar{P}_1 := e_k^{gr}(P_1)$, $\bar{P}_2 := e_k^{gr}(P_2)$ and $\bar{P}_3 := e_k^{gr}(P_3)$. By way of contradiction, suppose $\bar{P}_1$, $\bar{P}_2$ and $\bar{P}_3$ to be collinear in $\mathrm{PG}(W_k)$. The image $e_k^{gr}(\mathcal{G}_k)$ of the Plücker embedding e_k^{gr} of $\mathcal{G}_k$ is the intersection of (possibly degenerate) quadrics of $\mathrm{PG}(W_k)$. Since, by assumption, the projective line $\langle \bar{P}_1, \bar{P}_2\rangle$ meets $e_k^{gr}(\mathcal{G}_k)$ in three distinct points $\bar{P}_1$, $\bar{P}_2$ and $\bar{P}_3$, we have $\langle \bar{P}_1, \bar{P}_2\rangle \subseteq e_k^{gr}(\mathcal{G}_k)$, that is $\bar{P}_1$, $\bar{P}_2$ and $\bar{P}_3$ are on a line of $e_k^{gr}(\mathcal{G}_k)$. By [23, Theorem 24.2.5], P_1, P_2 and P_3 should be on a line of $\mathcal{G}_k$ and, thus, collinear in $\mathcal{G}_k$ – a contradiction. □

Corollary 3.3. *Let $1 \leqslant k \leqslant n$. If $\mathcal{C}$ is a polar m-cap of Δ_k, then $\varepsilon_k^{gr}(\mathcal{C})$ is a projective m-cap of* $\mathrm{PG}(W_k^{gr})$.

Theorem 3.4. *The image $\varepsilon_n^{gr}(\Delta_n)$ of the dual polar space Δ_n under the Grassmann embedding ε_n^{gr} is a projective cap of* $\mathrm{PG}(W_n^{gr})$.

Proof. We prove that $\varepsilon_n^{gr}(\Delta_n)$ does not contain any three collinear points. By way of contradiction, suppose $\varepsilon_n^{gr}(P_1)$, $\varepsilon_n^{gr}(P_2)$ and $\varepsilon_n^{gr}(P_3)$ to be three collinear points in $\mathrm{PG}(W_n^{gr})$ and put $\ell := \langle \varepsilon_n^{gr}(P_1), \varepsilon_n^{gr}(P_2)\rangle$. The image $e_n^{gr}(\mathcal{G}_n)$ of the projective Grassmannian $\mathcal{G}_n$ by the Plücker embedding e_n^{gr} is a variety obtained as the intersection of (possibly degenerate) quadrics of $\mathrm{PG}(W_n)$. Since ℓ is a projective line containing three points of $e_n^{gr}(\mathcal{G}_n)$, then $\ell \subset e_n^{gr}(\mathcal{G}_n)$. By [23, Theorem 24.2.5], its preimage $r = (e_n^{gr})^{-1}(\ell)$ is a line of $\mathcal{G}_n$. Hence, P_1, P_2 and P_3 are three distinct points of Δ_n lying on the line r of $\mathcal{G}_n$. This means that there are three distinct maximal subspaces p_1, p_2 and p_3 of V, totally singular with respect to η, intersecting in an $(n-1)$-dimensional subspace and spanning an $(n+1)$-dimensional subspace of V. This configuration is, clearly, impossible. □

Main Result 3 is a consequence of Corollary 3.3 and Theorem 3.4.

As recalled in Section 1.1, when $\mathbb{F} = \mathbb{F}_q$, the pointset of the dual polar space Δ_n is the set of all $(q^n+1)(q^{n-1}+1)\cdots(q+1)$ n-dimensional subspaces of V totally singular with respect to η. Thus, we get the following corollary.

Corollary 3.5. *Suppose $n \geqslant 2$ and $\mathbb{F} = \mathbb{F}_q$ a finite field. Then,*

(i) *For $q = p^h$, $p > 2$, the pointset $\varepsilon_n^{gr}(\Delta_n)$ is a cap of $\mathrm{PG}(\binom{2n+1}{n} - 1, q)$ of size $(q^n+1)(q^{n-1}+1)\cdots(q+1)$.*
(ii) *For $q = 2^h$, the pointset $\varepsilon_n^{gr}(\Delta_n)$ is a cap of $\mathrm{PG}(\binom{2n+1}{n} - \binom{2n+1}{n-2} - 1, q)$ of size $(q^n+1)(q^{n-1}+1)\cdots(q+1)$.*

Proof. By Theorem 3.4, $\varepsilon_n^{gr}(\Delta_n)$ is a cap of $\mathrm{PG}(W_n^{gr})$. Part (i) of the corollary follows from Part (1) of Theorem 1. Part (ii) follows from Part (1) of Theorem 2. □

We remark that Part (i) of Corollary 3.5 can also be proved using Part (2) of Theorem 1 together with the well-known result of [35] showing that the quadric Veronesean of $\mathrm{PG}(n, q)$ is a cap of $\mathrm{PG}(n(n+3)/2, q)$.

4. Construction of a polar cap of Δ_k

In this section $\mathbb{F}$ can be any, possibly infinite, field of odd characteristic. We shall determine a family of k-dimensional subspaces of V totally singular with respect to η providing a polar cap of Δ_k, for $k \leqslant n$. Observe that the caps we construct in this section all actually fulfil the stronger condition $v = 1$, that is no 2 of their points are on a line of Δ_k; furthermore, as all of the results of Section 3 for $v \leqslant 2$ apply, they determine caps of the ambient projective space by Theorem 3.3.

Up to a multiplicative non-zero constant, it is possible to choose without loss of generality a basis $\mathbb{B} = (e_1, e_2, \dots, e_{2n+1})$ for V in which the quadratic form η is given by

$$\eta(x_1, \dots, x_{2n+1}) = \sum_{i=1}^{n} x_i x_{n+i} + x_{2n+1}^2.$$

Denote by f_η the symmetric bilinear form obtained by polarising η and by $\perp$ the associated orthogonality relation. Given $I := \{1, \dots, 2n+1\}$, write $\binom{I}{k}$ for the set of all k-subsets of I.

For any set of indices $J = \{j_1, j_2, \dots, j_k\} \subset I$, $j_1 < j_2 < \dots < j_k$, define $e_J = e_{j_1} \wedge e_{j_2} \wedge \dots \wedge e_{j_k}$. The set $B_\wedge := (e_J)_{J \in \binom{I}{k}}$ is, clearly, a basis of W_k. For any i, with $1 \leqslant i \leqslant 2n+1$, let

$$i' := \begin{cases} i+n & \text{if } 1 \leqslant i \leqslant n, \\ i-n & \text{if } n < i \leqslant 2n, \\ 2n+1 & \text{if } i = 2n+1. \end{cases}$$

Observe that, with f_η defined as above and $1 \leqslant i \leqslant 2n$, we always get $f_\eta(e_i, e_{i'}) = 1$. Thus, the pair $\{e_i, e_{i'}\}$ is a *hyperbolic pair of vectors*; see [1, Chapter 3]. We shall say that $\{i, i'\}$ is a *hyperbolic pair of indices* if the corresponding set $\{e_i, e_{i'}\}$ is a hyperbolic pair of vectors.

Lemma 4.1. *Let $k \leqslant n$ and $r \leqslant \lfloor \frac{k}{2} \rfloor$. Suppose J to be a k-subset of I containing r hyperbolic pairs of indices. The following statements hold*:

(1) *If $2n+1 \notin J$, then there exists $\{m_1, m_2, \dots, m_r\} \subseteq \{1, 2, \dots, n\}$ such that $e_{m_i} \in \{e_j\}_{j \in J}^{\perp}$ for every $1 \leqslant i \leqslant r$.*

(2) *If $2n+1 \in J$, then there exists $\{m_1, m_2, \dots, m_r, \ell\} \subseteq \{1, 2, \dots, n\}$ such that $e_t \in \{e_j\}_{j \in J}^{\perp}$ for every $t \in \{m_1, \dots, m_r\} \cup \{\ell\}$.*

Proof. Write $J \cap \{1, 2, \dots, n\} = \{j_1, \dots, j_r, j_{r+1}, j_{r+2}, \dots, j_{r+s}\}$ and write $J \cap \{1', 2', \dots, n'\} = \{j'_1, j'_2, \dots, j'_r, j'_{r+s+1}, j'_{r+s+2}, \dots, j'_{k-r}\}$. Let $U = \{1, 2, \dots, n\} \setminus (J \cup J')$, where $J' = \{j' \colon\ j \in J\}$.

(1) If $2n+1 \notin J$, then $|U| = n - (r + k - 2r) = n - k + r \geqslant r$, since $n - k \geqslant 0$. Hence, there exists a subset $M_r = \{m_1, m_2, \dots, m_r\}$ of U of cardinality r. Clearly, $f_\eta(e_j, e_{m_i}) = 0$ for every $m_i \in M_r$ and every $j \in J$.

(2) Since $2n+1 \in J$, we have $|U| = n - (r + k - (2r+1)) = n - k + r + 1 \geqslant r + 1$, as $n - k \geqslant 0$. Hence, there exists a subset $\overline{M}_r = \{m_1, m_2, \dots, m_r, \ell\}$ of U of cardinality $r+1$. Clearly, $f_\eta(e_j, e_t) = 0$ for every $t \in \overline{M}_r$ and every $j \in J$. $\square$

4.1. First construction: $2n+1 \notin J$

Suppose $J = \{j_1, j_2, \dots, j_r, j'_1, j'_2, \dots, j'_r\} \cup \bar{J} \subset I$, where $\bar{J}$ does not contain any hyperbolic pair of indices, $|J| = k$ and $2n+1 \notin J$. By (1) in Lemma 4.1, there exists $M_r = \{m_1, m_2, \dots, m_r\} \subseteq \{1, 2, \dots, n\}$ such that $e_{m_i} \in \{e_j\}_{j \in J}^{\perp}$. We will construct a family of 2^r totally singular k-dimensional subspaces of V from these $m_i \in M_r$ as follows. Fix any bijection $\tau : \{j_1, j_2, \dots, j_r\} \to M_r$ and put

$$\begin{aligned} X_{\emptyset,\tau} := \langle & e_{j_1} + e_{\tau(j_1)}, e_{j_2} + e_{\tau(j_2)}, \dots, e_{j_r} + e_{\tau(m_r)}, \\ & e_{j'_1} - e_{\tau(j_1)'}, e_{j'_2} - e_{\tau(j_2)'}, \dots, e_{j'_r} - e_{\tau(j_r)'}, \{e_j\}_{j \in \bar{J}} \rangle. \end{aligned} \tag{6}$$

Table 1
Subspaces for $2n+1 \notin J$.

$$X_{\emptyset} := \langle e_{j_1}+e_{m_1}, e_{j_2}+e_{m_2}, \dots, e_{j_r}+e_{m_r}, e_{j'_1}-e_{m'_1}, e_{j'_2}-e_{m'_2}, \dots, e_{j'_r}-e_{m'_r}, \{e_j\}_{j\in\bar{J}}\rangle;$$

$$X_{m_1} := \langle e_{j_1}-e_{m'_1}, e_{j_2}+e_{m_2}, \dots, e_{j_r}+e_{m_r}, e_{j'_1}+e_{m_1}, e_{j'_2}-e_{m'_2}, \dots, e_{j'_r}-e_{m'_r}, \{e_j\}_{j\in\bar{J}}\rangle;$$

$$X_{m_2} := \langle e_{j_1}+e_{m_1}, e_{j_2}-e_{m'_2}, \dots, e_{j_r}+e_{m_r} e_{j'_1}-e_{m_1}, e_{j'_2}+e_{m_2}, \dots, e_{j'_r}-e_{m'_r}, \{e_j\}_{j\in\bar{J}}\rangle;$$

$$\dots$$

$$X_{m_1,\dots,m_r} := \langle e_{j_1}-e_{m'_1}, e_{j_2}-e_{m'_2}, \dots, e_{j_r}-e_{m'_r}, e_{j'_1}+e_{m_1}, e_{j'_2}+e_{m_2}, \dots, e_{j'_r}+e_{m_r}, \{e_j\}_{j\in\bar{J}}\rangle$$

For every non-empty subset S of M_r define $X_{S,\tau}$ to be the k-dimensional subspace of V spanned by the same vectors as $X_{\emptyset,\tau}$ in (6) except that when $\tau(j_i)\in S$, the vectors $e_{j_i}+e_{\tau(j_i)}$ and $e_{j'_i}-e_{\tau(j_i)'}$ are respectively replaced by $e_{j_i}-e_{\tau(j_i)'}$ and $e_{j'_i}+e_{\tau(j_i)}$. For simplicity in the following arguments, as well as in Section 4.1, we shall always assume $m_i=\tau(j_i)$ and write just X_S for $X_{S,\tau}$. For an example and an explicit description, see Table 1.

Theorem 4.2. *The set $\mathfrak{X}_k := \{X_S\}_{S\subseteq M_r}$ is a polar 2^r-cap of Δ_k.*

Proof. Clearly $|\mathfrak{X}_k| = 2^r$. We now prove $\mathfrak{X}_k \subset \Delta_k$ and that no two distinct elements of $\mathfrak{X}_k$ are collinear in Δ_k. By Lemma 4.1, it is straightforward to see that for any $S\subseteq M_r$, the subspace X_S is totally singular with respect to η. Let S and T be two arbitrary distinct subsets of M_r. Since $S\neq T$, there exists $u\in\{1,2,\dots,r\}$ such that $m_u\in S$ and $m_u\notin T$. So, $\langle e_{j_u}-e_{m'_u}, e_{j'_u}+e_{m_u}\rangle \nsubseteq X_S\cap X_T$. It follows that the distance $d(X_S,X_T) := k-\dim(X_S\cap X_T)$ between X_S and X_T, regarded as points of the collinearity graph of $\mathcal{G}_k$, is at least 2. As the collinearity graph of Δ_k is a subgraph of that of $\mathcal{G}_k$, this yields the result. $\square$

We observe that by Theorem 4.2, $\mathfrak{X}_k$ is also a $(2^r,1)$-set of $\mathcal{G}_k$.

For each $S\subseteq M_r$, denote by B_S the set formed by the first $2r$ generators of X_S, ordered as in Table 1, and by $\overline{X}_S = \langle B_S\rangle$ the subspace of X_S spanned by B_S.

Corollary 4.3. *The set $\overline{\mathfrak{X}}_{2r} = \{\overline{X}_S\}_{S\subseteq M_r}$ is a polar 2^r-cap of Δ_{2r}.*

Given an arbitrary $S\subseteq M_r$, the elements of B_S can be described as follows:

$$e_{j_1}+(-1)^{\chi_S(m_1)}e_{m_1+n\chi_S(m_1)}, \dots, e_{j_r}+(-1)^{\chi_S(m_r)}e_{m_r+n\chi_S(m_r)},$$
$$e_{j'_1}+(-1)^{\chi_S(m_1)+1}e_{m_1+n(1-\chi_S(m_1))}, \dots, e_{j'_r}+(-1)^{\chi_S(m_r)+1}e_{m_r+n(1-\chi_S(m_r))},$$

where χ_S is the characteristic function of S, that is $\chi_S(x)=1$ if $x\in S$ and $\chi_S(x)=0$ if $x\notin S$, and, as before, $x' := x+n$.

The Grassmann embedding ε_{2r}^{gr} applied to any of the singular subspaces $\overline{X}_S = \langle B_S\rangle$ determines a point $\varepsilon_{2r}^{gr}(\overline{X}_S) = \langle \bigwedge^{2r} B_S\rangle$ of $\mathrm{PG}(W_{2r}^{gr})$, with

$$\bigwedge^{2r} B_S = \left(e_{j_1}+(-1)^{\chi_S(m_1)}e_{m_1+n\chi_S(m_1)}\right)\wedge\cdots\wedge\left(e_{j_r}+(-1)^{\chi_S(m_r)}e_{m_r+n\chi_S(m_r)}\right)$$
$$\wedge\left(e_{j'_1}+(-1)^{\chi_S(m_1)+1}e_{m_1+n(1-\chi_S(m_1))}\right)\wedge\cdots\wedge\left(e_{j'_r}+(-1)^{\chi_S(m_r)+1}e_{m_r+n(1-\chi_S(m_r))}\right).$$

Hence, $\bigwedge^{2r} B_S$ is a sum of vectors of the form $\sigma_S(K)\cdot e_K$, where $\sigma_S(K)=\pm 1$ and $K\subseteq\{j_\ell, j'_\ell, m_\ell, m'_\ell\}_{\ell=1}^r$ has size $2r$ and contains at most r hyperbolic pairs of indices given by either $\{j_\ell, j'_\ell\}$ or $\{m_\ell, m'_\ell\}$.

It is possible to write $\bigwedge^{2r} B_S$ in a more convenient way by expanding the wedge products. To this end, let $T = \{t_1, \ldots, t_r\} \subseteq \{j_1, \ldots, j_r, m_1, \ldots, m_r\}$ with $t_\ell \in \{j_\ell, m_\ell\}$ for $1 \leqslant \ell \leqslant r$, $T' = \{t'_1, \ldots, t'_r\}$ and denote by $\mathcal{T}_r$ the family of all such sets T. The mapping sending every $T \in \mathcal{T}_r$ to $T \cap M_r$ is a bijection between $\mathcal{T}_r$ and the family of all the subsets of M_r. Hence, $|\mathcal{T}_r| = 2^r$.

Consider $U = \{u_1, \ldots, u_{2r}\} \subseteq \{j_1, \ldots, j_r, m_1, \ldots, m_r, j'_1, \ldots, j'_r, m'_1, \ldots, m'_r\}$ such that $|\{\{i, i'\} \subset U\}| < r$ and denote by $\mathcal{U}$ the family of all such sets.

In other words, every set $T \cup T'$ with $T \in \mathcal{T}_r$ is made up of precisely r hyperbolic pairs of indices, while any $U \in \mathcal{U}$ is made up of *at most* $r - 1$ hyperbolic pairs of indices. Then,

$$\bigwedge^{2r} B_S = \sum_{T \in \mathcal{T}_r} \sigma_S(T) e_{T,T'} + \sum_{U \in \mathcal{U}} \sigma_S(U) e_U, \tag{7}$$

where $e_{T,T'} := e_T \wedge e_{T'}$ and $\sigma_S(T)$, $\sigma_S(U)$ are shorthand notations for $\sigma_S(T \cup T')$ and $\sigma_S(U \cup U')$, respectively.

Put

$$\xi_S := \sum_{T \in \mathcal{T}_r} \sigma_S(T) e_{T,T'}. \tag{8}$$

In particular, $\xi_\emptyset := \sum_{T \in \mathcal{T}_r} \sigma_\emptyset(T) e_{T,T'}$, where, as it can be easily seen, $\sigma_\emptyset(T) = (-1)^{|T \cap M_r|}$.

By Corollaries 3.3 and 4.3, $\varepsilon^{gr}_{2r}(\overline{\mathfrak{X}}_{2r}) = \{\varepsilon^{gr}_{2r}(\overline{X}_S)\}_{S \subseteq M_r}$ is a projective cap of $\mathrm{PG}(W^{gr}_{2r})$. The function sending $\varepsilon^{gr}_{2r}(\overline{X}_S)$ to ξ_S is a bijection between $\varepsilon^{gr}_{2r}(\overline{\mathfrak{X}}_{2r})$ and the set $\{\xi_S\}_{S \subseteq M_r}$.

4.2. *Second construction:* $2n + 1 \in J$

We now move to Case (2) of Lemma 4.1. In close analogy to Section 4.1, we will introduce a family of 2^r totally singular k-dimensional subspaces of V. Most of the results previously proved hold unchanged when $2n + 1 \in J$.

Let $J = \{j_1, j_2, \ldots, j_r, j'_1, j'_2, \ldots, j'_r, 2n + 1\} \cup \bar{J} \subset I$, where $\bar{J}$ does not contain any hyperbolic pair of indices and $|J| = k$. By (2) in Lemma 4.1, there exists

$$\overline{M}_r = \{m_1, m_2, \ldots, m_r, \ell\} \subseteq \{1, 2, \ldots, n\}$$

such that $e_t \in \{e_j\}^{\perp}_{j \in J}$ for any $t \in \overline{M}_r$.

Put

$$\begin{aligned} \mathcal{X}_\emptyset := \langle & e_{j_1} + e_{m_1}, e_{j_2} + e_{m_2}, \ldots, e_{j_r} + e_{m_r}, e_{j'_1} - e_{m'_1}, e_{j'_2} - e_{m'_2}, \ldots, \\ & e_{j'_r} - e_{m'_r}, e_\ell + e_{2n+1} - e_{\ell'}, \{e_j\}_{j \in \bar{J}} \rangle. \end{aligned}$$

Clearly, $\mathcal{X}_\emptyset$ is totally singular.

As before, for every non-empty subset S of $M_r = \{m_1, m_2, \ldots, m_r\}$ define $\mathcal{X}_S$ to be the k-dimensional subspace of V spanned by the same vectors as $\mathcal{X}_\emptyset$, except that if $m_i \in S$, then $e_{j_i} + e_{m_i}$ and $e_{j'_i} - e_{m'_i}$ are respectively replaced by $e_{j_i} - e_{m'_i}$ and $e_{j'_i} + e_{m_i}$. For more details, see Table 2.

We thus determine 2^r totally singular k-dimensional subspaces of V each being at distance at least 2 from any other, when regarded as points in the collinearity graph of Δ_k. Hence, the following analogue of Theorem 4.2 holds.

Theorem 4.4. *The set* $\mathfrak{X}'_k = \{\mathcal{X}_S\}_{S \subseteq M_r}$ *is a polar* 2^r*-cap of* Δ_k.

Table 2
Subspaces for $2n+1 \in J$.

$$\mathcal{X}_{\emptyset} := \langle e_{j_1}+e_{m_1}, e_{j_2}+e_{m_2}, \dots, e_{j_r}+e_{m_r}, e_{j'_1}-e_{m'_1}, e_{j'_2}-e_{m'_2}, \dots, e_{j'_r}-e_{m'_r}, e_\ell+e_{2n+1}-e_{\ell'}, \{e_j\}_{j\in\bar{J}}\rangle;$$
$$\mathcal{X}_{m_1} := \langle e_{j_1}-e_{m'_1}, e_{j_2}+e_{m_2}, \dots, e_{j_r}+e_{m_r}, e_{j'_1}+e_{m_1}, e_{j'_2}-e_{m'_2}, \dots, e_{j'_r}-e_{m'_r}, e_\ell+e_{2n+1}-e_{\ell'}, \{e_j\}_{j\in\bar{J}}\rangle;$$
$$\mathcal{X}_{m_2} := \langle e_{j_1}+e_{m_1}, e_{j_2}-e_{m'_2}, \dots, e_{j_r}+e_{m_r}, e_{j'_1}-e_{m_1}, e_{j'_2}+e_{m_2}, \dots, e_{j'_r}-e_{m'_r}, e_\ell+e_{2n+1}-e_{\ell'}, \{e_j\}_{j\in\bar{J}}\rangle;$$
$$\dots$$
$$\mathcal{X}_{m_1,\dots,m_r} := \langle e_{j_1}-e_{m'_1}, e_{j_2}-e_{m'_2}, \dots, e_{j_r}-e_{m'_r}, e_{j'_1}+e_{m_1}, e_{j'_2}+e_{m_2}, \dots, e_{j'_r}+e_{m_r}, e_\ell+e_{2n+1}-e_{\ell'}, \{e_j\}_{j\in\bar{J}}\rangle$$

Arguing as in Section 4.1, let $\widehat{B}_S$ be the set consisting of the first $2r+1$ generators of $\mathcal{X}_S$ and $\overline{\mathcal{X}}_S = \langle \widehat{B}_S\rangle$ be the subspace of $\mathcal{X}_S$ spanned by $\widehat{B}_S$. In other words, $\widehat{B}_S := B_S \cup \{e_\ell + e_{2n+1} - e_{\ell'}\}$, with B_S defined as in Section 4.1.

The following corresponds to Corollary 4.3.

Corollary 4.5. *The set $\overline{\mathfrak{X}}'_{2r+1} = \{\overline{\mathcal{X}}_S\}_{S\subseteq M_r}$ is a polar 2^r-cap of Δ_{2r+1}.*

For any $S \subseteq M_r$, apply the Grassmann embedding ε^{gr}_{2r+1} to the singular subspaces $\overline{\mathcal{X}}_S = \langle \widehat{B}_S\rangle$. Hence, $\varepsilon^{gr}_{2r+1}(\overline{\mathcal{X}}_S) = \langle \bigwedge^{2r+1}\widehat{B}_S\rangle$ is the point of $\mathrm{PG}(W^{gr}_{2r+1})$ spanned by the vector $\bigwedge^{2r+1}\widehat{B}_S := \bigwedge^{2r} B_S \wedge (e_\ell + e_{2n+1} - e_{\ell'})$.

Expanding $\bigwedge^{2r+1}\widehat{B}_S$, we get an analogue of (7):

$$\bigwedge^{2r+1}\widehat{B}_S = \sum_{T\in\mathcal{T}_r}\sigma_S(T)e_{T,T',2n+1} + \sum_{\bar{U}\in\bar{\mathcal{U}}}\sigma_S(\bar{U})e_{\bar{U}}, \tag{9}$$

where $e_{T,T',2n+1} := e_T \wedge e_{T'} \wedge e_{2n+1}$, $\bar{U} \subseteq \bar{\mathcal{U}} = \mathcal{U} \cup \{l, l'\}$, $|\{(i,i') \subset \bar{U}\}| < r$, $|\bar{U}| = 2r+1$; the sets T, T', $\mathcal{T}_r$ and $\mathcal{U}$ are defined as in Section 4.1. The coefficients $\sigma_S(T)$ and $\sigma_S(\bar{U})$ are ± 1. Put

$$\bar{\xi}_S := \sum_{T\in\mathcal{T}_r}\sigma_S(T)e_{T,T',2n+1}. \tag{10}$$

By Corollaries 3.3 and 4.5, $\varepsilon^{gr}_{2r+1}(\overline{\mathfrak{X}}'_{2r+1}) = \{\varepsilon^{gr}_{2r+1}(\overline{\mathcal{X}}_S)\}_{S\subseteq M_r}$ is a projective cap of $\mathrm{PG}(W^{gr}_{2r+1})$. The function sending any element $\varepsilon^{gr}_{2r+1}(\overline{\mathcal{X}}_S)$ to the vector $\bar{\xi}_S$ is a bijection between $\varepsilon^{gr}_{2r+1}(\overline{\mathfrak{X}}'_{2r+1})$ and $\{\bar{\xi}_S\}_{S\subseteq M_r}$.

Observe that Main Result 4 is contained in Corollaries 4.3 and 4.5.

5. Hadamard matrices and codes from caps

Recall that a Hadamard matrix of order m is an $(m\times m)$-matrix H with entries ± 1 such that $HH^t = mI$, where I is the $(m\times m)$-identity matrix. Hadamard matrices have been widely investigated, as their existence, for $m > 2$, is equivalent to that of extendable symmetric 2-designs with parameters $(m-1, \frac{1}{2}m-1, \frac{1}{4}m-1)$; see [8], and also [25, Theorem 4.5]. It is well known that the point-hyperplane design of $PG(n,2)$ is a Hadamard $2-(2^{n+1}-1, 2^n-1, 2^{n-1}-1)$ design; any of the corresponding Hadamard matrices is called a *Sylvester matrix*; see [8, Example 1.31]. Indeed, the so-called recursive *Kronecker product construction*, see [25, Theorem 3.23], as

$$S_1 = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \qquad S_n = S_{n-1} \otimes \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

always gives a Sylvester matrix. In this section we shall show how it is possible to associate a Hadamard matrix of order 2^r to any polar cap $\overline{\mathfrak{X}}_{2r}$ of Δ_{2r} and $\overline{\mathfrak{X}}'_{2r+1}$ of Δ_{2r+1}. Recall that $\overline{\mathfrak{X}}_{2r}$ and $\overline{\mathfrak{X}}'_{2r+1}$ are introduced respectively in Corollaries 4.3 and 4.5 of Section 4. In particular, we shall make use of the vectors ξ_S and $\bar{\xi}_S$ therein computed respectively in Eqs. (8) and (10). We shall also introduce an order relation on the points of the cap itself, in order to prove that this matrix can be obtained by the recursive Sylvester construction. This will also provide a direct connection with first order Reed–Muller codes; for more details, see also [2].

At first, we need to take into account the two cases of Sections 4.1 and 4.2 separately. We adopt the same notation as is those sections.

For $2n+1 \notin J$, put $\mathcal{B}_S := \{\sigma_S(T)e_{T,T'}\}_{T\in\mathcal{T}_r}$ for $S \subseteq M_r$; see Eq. (8). Then, $\mathcal{B}_S$ is a basis of the linear space $L_{\mathcal{T}_r} := \langle e_{T,T'}\rangle_{T\in\mathcal{T}_r}$. In particular, $\mathcal{B}_\emptyset$ is a basis of $L_{\mathcal{T}_r}$ and $\xi_S = \sum_{T\in\mathcal{T}_r}\sigma_S(T)e_{T,T'} \in L_{\mathcal{T}_r}$. Thus, we can consider the coordinates $\{(\xi_S)_T\}_{T\in\mathcal{T}_r}$ of ξ_S with respect to $\mathcal{B}_\emptyset$. Clearly, $(\xi_S)_T = \sigma_S(T)\sigma_\emptyset(T)$. Observe that, while we have selected $\mathcal{B}_\emptyset$ as a basis, the result holds for any arbitrary fixed basis of the form $\mathcal{B}_S$.

If $2n+1 \in J$, let $\overline{\mathcal{B}}_S = \{\sigma_S(T)e_{T,T',2n+1}\}_{T\in\mathcal{T}_r}$; see Eq. (10). Then, $\overline{\mathcal{B}}_S$ is a basis of the linear space $\bar{L}_{\mathcal{T}_r} = \langle e_{T,T',2n+1}\rangle_{T\in\mathcal{T}_r}$. In particular, $\overline{\mathcal{B}}_\emptyset$ is a basis of $\bar{L}_{\mathcal{T}_r}$ and $\bar{\xi}_S \in \bar{L}_{\mathcal{T}_r}$; thus we consider the coordinates $\{(\bar{\xi}_S)_T\}_{T\in\mathcal{T}_r}$ of $\bar{\xi}_S$ with respect to the basis $\overline{\mathcal{B}}_\emptyset$ of $\bar{L}_{\mathcal{T}_r}$. Again, we have $(\bar{\xi}_S)_T = \sigma_S(T)\sigma_\emptyset(T)$.

Let $A_{\emptyset,r}$ be the $(2^r \times 2^r)$-matrix defined as follows. The rows are indexed by the subsets of $M_r = \{m_1,\dots,m_r\}$ and the columns by the members of $\mathcal{T}_r$. For $S \subseteq M_r$ and $T \in \mathcal{T}_r$ the T-entry of the row R_S corresponding to S is equal to $(\xi_S)_T = \sigma_S(T)\sigma_\emptyset(T)$ when $2n+1 \notin J$ and $(\bar{\xi}_S)_T = \sigma_S(T)\sigma_\emptyset(T)$ when $2n+1 \in J$. In particular, every entry of $A_{\emptyset,r}$ is either 1 or -1 and all entries in the row $R_\emptyset$ are equal to 1.

Lemma 5.1.

1. *When* $2n+1 \notin J$, $A_{\emptyset,r} = ((\xi_S)_T)_{\substack{S\subseteq M_r\\ T\in\mathcal{T}_r}}$ *with* $(\xi_S)_T = (-1)^{|S\cap T|}$.
2. *When* $2n+1 \in J$, $A_{\emptyset,r} = ((\bar{\xi}_S)_T)_{\substack{S\subseteq M_r\\ T\in\mathcal{T}_r}}$ *with* $(\bar{\xi}_S)_T = (-1)^{|S\cap T|}$.

Proof. Suppose $2n+1 \notin J$. The proof for the case $2n+1 \in J$ is entirely analogous.

Take $R = \{m_1,\dots,m_{\ell-1}\} \subseteq M_r$ and let $S = R \cup \{m_\ell\} \subseteq M_r$. Observe that ξ_S is obtained from ξ_R by replacing $e_{j_\ell} + e_{m_\ell}$ and $e_{j'_\ell} - e_{m'_\ell}$ by respectively $e_{j_\ell} - e_{m'_\ell}$ and $e_{j'_\ell} + e_{m_\ell}$ in $\bigwedge^{2r} B_R$. Clearly, if $m_\ell \notin T$, we have $(\xi_S)_T = (\xi_R)_T$, as ξ_R and ξ_S have exactly the same components with respect to all the vectors $e_{T,T'}$ which do not contain the term e_{m_ℓ}. On the other hand, when $m_\ell \in T$, the sign of the component of $e_{T,T'}$ must be swapped; thus, $(\xi_S)_T = -(\xi_R)_T$. As ξ_S can be obtained from the sequence

$$\xi_\emptyset \to \xi_{m_1} \to \xi_{m_1,m_2} \to \cdots \to \xi_R \to \xi_S$$

and $\xi_\emptyset = \mathbf{1}$, we have $(\xi_S)_T = (-1)^{|S\cap T|}$. This proves the lemma. □

Theorem 5.2. *The matrix* $A_{\emptyset,r}$ *is Hadamard.*

Proof. By Lemma 5.1, $(\xi_S)_T = (-1)^{\mathbf{S}\cdot\mathbf{T}}$, where $\mathbf{S}$ and $\mathbf{T}$ are the incidence vectors of S and $T \cap M_r$ with respect to M_r and $\cdot$ denotes the usual inner product. The result now is a consequence of [33, Lemma 4.7, p. 337]. □

In particular, $(\xi_S)_T = 1$ if, and only if, S and T share an even number of elements.

Corollary 5.3. *The design associated to the matrix* $A_{\emptyset,r}$ *is the point-hyperplane design of* $\mathrm{PG}(r,2)$; *in particular,* $A_{\emptyset,r}$ *is a Sylvester matrix.*

Proof. It is well known that for any hyperplane π of $\mathrm{PG}(r,2)$, there is a point $P_\pi \in \mathrm{PG}(r,2)$ such that

$$\pi = \{X \in \mathrm{PG}(r,2)\colon\ P_\pi \cdot X = 0\},$$

with $\cdot$ the usual inner product of $\mathrm{PG}(r,2)$ and $P_\pi = (p_1, p_2, \ldots, p_{r+1})$, $X = (x_1, x_2, \ldots, x_{r+1})$ binary vectors. In particular, $X \in \pi$ if and only if $P_\pi \cdot X = |\{i\colon\ p_i = x_i\}| \pmod 2 = 0$, that is to say if and only if the vectors P_π and X have an even number of 1's in common. By Lemma 5.1, it is now straightforward to see that the matrix $A'_{\emptyset,r}$ obtained from $A_{\emptyset,r}$ by deleting the all-1 row and column and replacing -1 with 0 contains the incidence vectors of the symmetric design of points and hyperplanes of a projective space $\mathrm{PG}(r,2)$. $\square$

Recall that equivalent Hadamard matrices give isomorphic Hadamard designs; the converse, however, is not true in general.

As anticipated, we now show how the rows and columns of $A_{\emptyset,r}$ or, equivalently, the points of the polar caps $\overline{X}_S$, might be ordered as to be able to describe it in terms of the Kronecker product construction.

Since both the rows and the columns of $A_{\emptyset,r}$ can be indexed by the subsets of M_r (for the columns we just consider $T \cap M_r$ with $T \in \mathcal{T}_r$) it is enough to introduce a suitable order $<_r$ on the set 2^{M_r} of all subsets of M_r. We proceed in a recursive way as follows:

- for $r = 1$, define $\emptyset <_1 \{m_1\}$;
- suppose we have ordered $2^{M_{r-1}}$, then for any $X, Y \subseteq M_r$, we say $X <_r Y$ if and only if
  1. $X <_{r-1} Y$ when $m_r \notin X \cup Y$;
  2. $m_r \notin X$ and $m_r \in Y$;
  3. $m_r \in X \cap Y$ and $(X \setminus \{m_r\}) <_{r-1} (Y \setminus \{m_r\})$.

Observe that $<_r$, when restricted to M_{r-1}, is the same as $<_{r-1}$. Thus, we shall drop the subscript from $<_r$, given that no ambiguity may arise.

As examples, for $r = 2$ we have

$$\emptyset < \{m_1\} < \{m_2\} < \{m_1, m_2\},$$

while, for $r = 3$,

$$\emptyset < \{m_1\} < \{m_2\} < \{m_1, m_2\} < \{m_3\} < \{m_1, m_3\} < \{m_2, m_3\} < \{m_1, m_2, m_3\}.$$

The minimum under $<$ is always $\emptyset$, and the maximum M_r. Using the order induced by $<$ on both the rows and the columns of $A_{\emptyset,r}$ we prove the following.

Theorem 5.4. *For any $r > 1$ we have $A_{\emptyset,r} = A_{\emptyset,r-1} \otimes A_{\emptyset,1}$.*

Proof. The matrix $A_{\emptyset,r}$ encodes the parity of the intersection of subsets of M_r; as we took the same order for columns and rows, $A_{\emptyset,r}$ is clearly symmetric. We now show that

$$A_{\emptyset,r} = \begin{pmatrix} A_{\emptyset,r-1} & A_{\emptyset,r-1} \\ A_{\emptyset,r-1} & -A_{\emptyset,r-1} \end{pmatrix} = A_{\emptyset,r-1} \otimes \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}.$$

Indeed, the elements indexing the first 2^{r-1} rows and columns of $A_{\emptyset,r}$ are all subsets of M_{r-1} in the order given by $<_{r-1}$. Thus, the minor they determine is indeed $A_{\emptyset,r-1}$. Observe now that if $m_r \in Y$ and $m_r \notin X$, then

$$(-1)^{|X \cap Y|} = (-1)^{|X \cap (Y \setminus \{m_r\})|}.$$

In particular, the entry in row $1 \leqslant x \leqslant 2^{r-1}$ and column $2^{r-1} < y \leqslant 2^r$ is the same as that in row x and column $y - 2^{r-1}$. It follows that the minor of $A_{\emptyset,r}$ comprising the first 2^{r-1} rows and the last 2^{r-1} columns is also $A_{\emptyset,r-1}$. By symmetry, this applies also to the minor consisting of the last 2^{r-1} rows and the first 2^{r-1} columns. Finally, consider an entry in row $2^{r-1} < x \leqslant 2^r$ and column $2^{r-1} < y \leqslant 2^r$. By definition of $<_r$, the sets X, Y indexing this entry are $X = X' \cup \{m_r\}$ and $Y = Y' \cup \{m_r\}$ where X' and Y' index the entry in row $x - 2^{r-1}$ and column $y - 2^{r-1}$. In particular, as $|X \cap Y| = |X' \cap Y'| + 1$,

$$(-1)^{|X\cap Y|} = -(-1)^{|X'\cap Y'|}.$$

It follows that this minor of $A_{\emptyset,r}$ is $-A_{\emptyset,r-1}$.

By Lemma 5.1,

$$A_{\emptyset,1} = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}.$$

The theorem now follows by recursion. □

By Theorem 5.4, the matrix $A_{\emptyset,r}$ is obtained by the Sylvester construction. As a corollary of Theorem 5.4, the codes associated to the caps constructed in Section 4 are Reed–Muller codes of the first order.

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