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Existence of Maximal Elements of Semicontinuous Preorders

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Abstract

We discuss the existence of an upper semicontinuous multi-utility representation of a preorder $\precsim$ on a topological space (X, τ). We then prove that every weakly upper semicontinuous preorder $\precsim$ is extended by an upper semicontinuous preorder and use this fact in order to show that every weakly upper semicontinuous preorder on a compact topological space admits a maximal element.

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1 Introduction

In this paper we are concerned with the existence of maximal elements for weakly semicontinuous preorders on a compact topological space. A preorder

$\precsim$ on a topological space (X, τ) is said to be *weakly upper semicontinuous* (see e.g. Herden and Levin [10]) if for all $x, y \in X$ such that $x \prec y$ there exists an upper semicontinuous increasing real valued function f_{xy} such that $f_{xy}(x) < f_{xy}(y)$. Weakly (semi)continuous preorders were first introduced by Herden and Pallack [9] and then they were studied by Bosi and Herden[4, 5].

In order to prove the main result of this paper, according to which a weakly upper semicontinuous preorder on a compact topological space admits a maximal element, we use arguments concerning the *semicontinuous multi-utility representation of preorders*.

A preorder $\precsim$ on a topological space (X, τ) is said to have a *semicontinuous multi-utility representation* if a family $\mathcal{F}$ of upper semicontinuous real-valued increasing functions can be chosen in such a way that $x \precsim y$ if and only if $f(x) \leq f(y)$ for every $f \in \mathcal{F}$. The multi-utility representation of a preorder was fist introduced by Dubra, Maccheroni and Ok [7] in the context of expected utility and then it was studied by Evren and Ok [8] in a very rich paper concerning the topological context, following the finite case studied by Ok [11]. A more recent contribution has been provided by Bosi and Herden [6].

It is important to notice that weak upper semicontinuity is necessary for both the existence of a semicontinuous multi-utility representation and the existence of an upper semicontinuous *Richter-Peleg utility* (see Richter [13] and Peleg [12]). This observation motivates our interest in this kind of semicontinuity, which has been also recently considered by Herden and Levin [10] in connection with the existence of (semi)continuous utilities.

2 Notation and preliminaries

Let $\precsim$ be a preorder (i.e., a *reflexive* and *transitive* binary relation) on a set X. As usual, the strict part of $\precsim$ will be denoted by $\prec$. Then we set for every point $x \in X$ the following subsets of X:

$$l(x) = \{y \in X \mid y \prec x\}, \;\; r(x) = \{z \in X \mid x \prec z\},$$

$$d(x) = \{y \in X \mid y \precsim x\}, \;\; i(x) = \{z \in X \mid x \precsim z\}.$$

A subset D of X is said to be *decreasing* if $d(x) \subset D$ for all $x \in D$. By duality the concept of an *increasing* subset I of X is defined.

If $(X, \precsim)$ is a preordered set, then a function $u : (X, \precsim) \longrightarrow (\mathbb{R}, \leq)$ is said to be

(i) *increasing* if, for every $x, y \in X$, $[x \precsim y \Rightarrow u(x) \leq u(y)]$,

(ii) *order-preserving* if it is increasing and, for every $x, y \in X$, $[x \prec y \Rightarrow u(x) < u(y)]$.

In the economic literature, an order-preserving function is often referred to as a *Richter-Peleg utility function* (see e.g. Richter [13]). Denote by τ_{nat} the *natural topology* on the real line $\mathbb{R}$.

A preorder $\precsim$ on a topological space (X, τ) is said to be *weakly upper semicontinuous* if for every pair $(x, y) \in \prec$ there exists some open decreasing subset O of X such that $x \in O$ and $y \in X \backslash O$. One verifies immediately that a preorder $\precsim$ on (X, τ) is weakly upper semicontinuous if and only if for every pair $(x, y) \in \prec$ there exists an upper semicontinuous isotone (increasing) function $f : (X, \precsim, \tau) \longmapsto (\mathbb{R}, \leq, \tau_{nat})$ such that $f(x) < f(y)$. Indeed, let O be some open decreasing subset of X such that $x \in O$ and $y \in X \backslash O$. Then one may define the desired upper semicontinuous isotone function $f : (X, \precsim, \tau) \longmapsto (\mathbb{R}, \leq, \tau_{nat})$ by setting for all $z \in X$:

$$f(z) := \begin{cases} 0 & if\ z \in O \\ 1 & if\ z \in X \backslash 0 \end{cases}.$$

A preorder $\precsim$ on (X, τ) is said to satisfy *scmp* (*semicontinuous multi-utility representation property*) if a family $\mathcal{F}$ of upper semicontinuous real-valued increasing functions can be chosen in such a way that $x \precsim y$ if and only if $f(x) \leq f(y)$ for every $f \in \mathcal{F}$.

Further, following the terminology adopted by Ever and Ok [8], we say that a preorder $\precsim$ on a topolgical space (X, τ) is *upper semicontinuous* if $i(x) = \{z \in X \mid x \precsim z\}$ is a closed subset of X for every $x \in X$.

It is easily seen that an upper semicontinuous preorder is also weakly upper semicontinuous (see Herden and Levin [10]). Further, weak upper semicontinuity of a preorder is a necessary condition for the existence of an upper semicontinuous Richter-Peleg utility representation and also for the validity of scmp.

3 Semicontinuous multi-utilities and existence of maximal elements

Evren and Ok [8, Proposition 2] proved that every upper semicontinuous preorder $\precsim$ on a topological space (X, τ) satisfies scmp. This result can be slightly improved as follows.

Proposition 3.1 *Let $\precsim$ be a preorder on a topological space (X, τ). Then the following conditions are equivalent.*

(i) $\precsim$ *satisfies scmp.*

(ii) $\precsim$ *is upper semicontinuous.*

(iii) *For every* $x, y \in X$ *such that not*$(y \precsim x)$ *there exists an open decreasing subset* O_{xy} *of* X *such that* $x \in O_{xy}$, $y \notin O_{xy}$;

(iv) *For every* $x, y \in X$ *such that not*$(y \precsim x)$ *there exists an upper semicontinuous increasing function* $f_{xy} : (X, \precsim, \tau) \longrightarrow (\mathbb{R}, \leq, \tau_{nat})$ *such that* $f_{xy}(x) < f_{xy}(y)$.

(v) $\precsim$ *is weakly upper semicontinuous and for every* $x, y \in X$ *such that not*$(y \precsim x)$ *and not*$(x \precsim y)$ *there exist two upper semicontinuous increasing functions* f_{xy}, g_{xy} : $(X, \precsim, \tau) \longrightarrow (\mathbb{R}, \leq, \tau_{nat})$ *such that* $f_{xy}(x) < f_{xy}(y)$ *and* $g_{xy}(x) > g_{xy}(y)$.

Proof. (i) $\Rightarrow$ (ii). Assume that $\precsim$ satisfies scmp. If for two elements $x, z \in X$ we have that $z \in X \setminus i(x)$, then there exists an upper semicontinuous increasing function $f : (X, \precsim, \tau) \longrightarrow (\mathbb{R}, \leq, \tau_{nat})$ such that $f(z) < f(x)$ and therefore $f^{-1}(-\infty, f(x))$ is an open decreasing subset of X containing z such that $f^{-1}(-\infty, f(x)) \cap i(x) = \emptyset$. Therefore $i(x)$ is closed for every $x \in X$.
(ii) $\Rightarrow$ (iii). For two elements $x, y \in X$ such that *not*$(y \precsim x)$ we have that $O_{xy} = X \setminus i(y)$ is an open decreasing subset of X such that $x \in O_{xy}$, $y \notin O_{xy}$.
(iii) $\Rightarrow$ (iv). For two elements $x, y \in X$ such that *not*$(y \precsim x) \Leftrightarrow x \notin i(y)$ the desired upper semicontinuous increasing function $f_{xy} : (X, \precsim, \tau) \longrightarrow (\mathbb{R}, \leq, \tau_{nat})$ such that $f_{xy}(x) < f_{xy}(y)$ can be defined by setting for all $z \in X$:

$$f_{xy}(z) := \begin{cases} 0 & if\ z \in O_{xy} \\ 1 & if\ z \in X \backslash O_{xy} \end{cases},$$

where O_{xy} is an open decreasing subset of X with the indicated property.
(iv) $\Rightarrow$ (v). Immediate, since *not*$(y \precsim x)$ means that either $x \prec y$ or *not*$(y \precsim x)$ *and not*$(x \precsim y)$.
(v) $\Rightarrow$ (i). Immediate. □

Evren and Ok [8, Application after Proposition 2] observe that every upper smicontinuous preorder $\precsim$ on a topological space (X, τ) admits a *maximal element* (that is, there is an element $x \in X$ such that $x \prec y$ holds for no $y \in X$). They underlines the fact that this is a folk theorem that is well known unless they do not know who to attribute this result, which is very close to other famous theorems about the existence of maximal elements for binary relations on compact sets (see e.g., Bergstrom [3] and Alcantud [1]). The proof of this theorem is rather simple based on arguments that are perfectly analogous to those provided by Bergstrom in his classical theorem. Indeed, we can consider that, if $\precsim$ is an upper semicontinuous preorder on a topological space (X, τ), then

$$l^0(x) = X \backslash i(x) = \bigcup \{f^{-1}(-\infty, f(x)) : \ f \text{ is upper semicontinuous increasing}\}$$

is an open decreasing subset of X containing $l(x) = \{y \in X \mid y \prec x\}$ and not containing x.

We provide a generalization of this result by using the arguments of the previous section. Before presenting a lemma, we recall that a preorder $\precsim$ on a set X is said to be extended by a preorder $\lesssim$ on X if $\precsim \subset \lesssim$ and $\prec \subset <$.

Lemma 3.2 *Every weakly upper semicontinuous preorder $\precsim$ on a topological space (X, τ) is extended by an upper semicontinuous preorder $\lesssim$.*

Proof. Let $\precsim$ be a weakly upper semicontinuous preorder on (X, τ). Then we denote by $F(\precsim)$ the set of all upper semicontinuous increasing functions $f : (X, \precsim, \tau) \longrightarrow (\mathbb{R}, \leq, \tau_{nat})$ and set

$$\lesssim := \{(x, y) \in X \times X \mid \forall f \in F(\precsim) \ (f(x) \leq f(y))\}.$$

Clearly, $\lesssim$ is a weakly upper semicontinuous preorder on (X, τ) such that $\precsim \subset \lesssim$ and $\prec \subset <$. Hence, it suffices to verify that $\lesssim$ is upper semicontinuous. Let, therefore, (x, y) be a pair of $X \times X$ such that $not(y \lesssim x)$. Then there exists some function $f \in F(\precsim)$ such that $f(x) < f(y)$. The upper semicontinuity of f guarantees that $f^{-1}(-\infty, f(y))$ is an open decreasing subset of X such that $x \in f^{-1}(-\infty, f(y))$, $y \notin f^{-1}(-\infty, f(y))$. □

The following theorem is now an easy consequence.

Theorem 3.3 *Let $\precsim$ be a weakly upper semicontinuous preorder on a compact topological space (X, τ). Then there exists a maximal element for $\precsim$.*

Proof. By Lemma 3.2, the weakly upper semicontinuous preorder $\precsim$ on (X, τ) is extended by an upper semicontinuous preorder $\lesssim$, which admits a maximal element x since (X, τ) is compact. It is immediate to check that such a maximal element is also a maximal element for the preorder $\precsim$. □

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