New results on path-decompositions and their down-links

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Abstract

In [3] the concept of *down-link* from a (K_v, Γ)-design $\mathcal{B}$ to a (K_n, Γ')-design $\mathcal{B}'$ has been introduced. In the present paper the spectrum problems for $\Gamma' = P_4$ are studied. General results on the existence of path-decompositions and embeddings between path-decompositions playing a fundamental role for the construction of down-links are also presented.

Keywords: (K_v, Γ)-design; down-link; embedding.
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1 Introduction

Suppose $\Gamma \leq K$ to be a subgraph of K. A (K, Γ)-*design*, or Γ-*decomposition* of K, is a set of graphs isomorphic to Γ whose edges partition the edge set of K. Given a graph Γ, the problem of determining the existence of (K_v, Γ)-designs, also called Γ-*designs of order* v, where K_v is the complete graph on v vertices, has been extensively studied; see the surveys [4, 5]. In [3] we proposed the following definition.

Definition 1.1. *Given a* (K, Γ)*-design* $\mathcal{B}$ *and a* (K', Γ')*-design* $\mathcal{B}'$ *with* $\Gamma' \leq \Gamma$*, a* down-link *from* $\mathcal{B}$ *to* $\mathcal{B}'$ *is a function* $f : \mathcal{B} \to \mathcal{B}'$ *such that* $f(B) \leq B$*, for any* $B \in \mathcal{B}$*.*

When such a function f exists, we say that it is possible to *down-link* $\mathcal{B}$ to $\mathcal{B}'$.

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As seen in [3], down-links are closely related to metamorphoses [8], their generalizations [9] and embeddings [11]. In close analogy to embeddings, we introduced spectrum problems about down-links:

(I) For each admissible v, determine the set $\mathcal{L}_1\Gamma(v)$ of all integers n such that there exists *some* Γ-design of order v down-linked to a Γ'-design of order n.

(II) For each admissible v, determine the set $\mathcal{L}_2\Gamma(v)$ of all integers n such that *every* Γ-design of order v can be down-linked to a Γ'-design of order n.

In [3, Proposition 3.2], we proved that for any v such that there exists a (K_v, Γ)-design and any $\Gamma' \leq \Gamma$, the sets $\mathcal{L}_1\Gamma(v)$ and $\mathcal{L}_2\Gamma(v)$ are always non-empty. In the same paper the case $\Gamma' = P_3$ has been investigated in detail. Here we shall deal with the case $\Gamma' = P_4$. In order to get results about down-links to P_4-designs, we shall first study path-designs and their embeddings. More precisely, in Section 2 we determine sufficient conditions for the existence of P_4-decompositions of any graph Γ and P_k-decompositions of complete bipartite graphs. In Section 3, applying the results of Section 2, we are able to prove the existence of embeddings and down-links between path-designs. Section 4 is devoted to the cases of cycle systems and path-designs, with general theorems and directed constructions.

Throughout this paper the following standard notations will be used; see also [7]. For any graph Γ, write $V(\Gamma)$ for the set of its vertices and $E(\Gamma)$ for the set of its edges. If $\mathcal{B}$ is a collection of graphs, by $V(\mathcal{B})$ we will mean the set of the vertices of all its elements. By $t\Gamma$ we shall denote the disjoint union of t copies of graphs all isomorphic to Γ. As usual, $P_k = [a_1, \dots, a_k]$ is the path with $k-1$ edges and $C_k = (a_1, \dots, a_k)$, $k \geq 3$, is the cycle of length k. Also, $K_{m,n}$ is the complete bipartite graph with parts of size m and n. When we focus on the actual parts X and Y, $K_{X,Y}$ will be written.

2 Existence of some path-designs

In this section we present new results on the existence of path decompositions. Recall that a (K_n, P_k)-design exists if, and only if, $n(n-1) \equiv 0 \pmod{2(k-1)}$; see [13].

Proposition 2.1. *Let k be an even integer. For $x = k-2, k$ the complete bipartite graph $K_{k-1,x}$ admits a P_k-decomposition.*

Proof. Consider the bipartite graph $K_{A,I}$ where $A = \{a_1, \dots, a_{k-1}\}$ and $I = \{1, \dots, x\}$ with $x = k-2, k$.
Let $U^t = (1, \dots, 1)$ be an $\frac{x}{2}$-tuple. Set $P_1^t = (1, \dots, \frac{x}{2})$ and for $i = 1, \dots, \frac{x}{2}$,

$P_i^t = (i, i+1, \dots, \frac{x}{2}, 1, 2, \dots, i-1)$, $\overline{P}_i^t = P_i^t + \frac{x}{2}U$, $A_i = a_i U$, $\overline{A}_i = a_{i+\frac{k}{2}} U$.
If $k \equiv 0 \pmod 4$, consider the $\frac{x}{2} \times k$ matrices

$$M = (P_1\ A_1\ \overline{P}_1\ A_2\ P_2\ A_3\ \overline{P}_2 \dots P_i\ A_{2i-1}\ \overline{P}_i\ A_{2i} \dots P_{\frac{k}{4}}\ A_{\frac{k-2}{2}}\ \overline{P}_{\frac{k}{4}}\ A_{\frac{k}{2}})$$
$$\overline{M} = (\overline{P}_1\ \overline{A}_1\ P_1\ \overline{A}_2\ \overline{P}_2\ \overline{A}_3\ P_2 \dots \overline{P}_i\ \overline{A}_{2i-1}\ P_i\ \overline{A}_{2i} \dots \overline{P}_{\frac{k}{4}}\ \overline{A}_{\frac{k-2}{2}}\ P_{\frac{k}{4}}\ A_{\frac{k}{2}}).$$

If $k \equiv 2 \pmod 4$, consider the $\frac{x}{2} \times k$ matrices

$$M = (P_1\ A_1\ \overline{P}_1\ A_2\ P_2\ A_3\ \overline{P}_2 \dots P_i\ A_{2i-1}\ \overline{P}_i\ A_{2i} \dots P_{\frac{k+2}{4}}\ A_{\frac{k}{2}})$$
$$\overline{M} = (\overline{P}_1\ \overline{A}_1\ P_1\ \overline{A}_2\ \overline{P}_2\ \overline{A}_3\ P_2 \dots \overline{P}_i\ \overline{A}_{2i-1}\ P_i\ \overline{A}_{2i} \dots \overline{P}_{\frac{k+2}{4}}\ A_{\frac{k}{2}}).$$

In either case, the rows of M and $\overline{M}$, taken together, are the x paths of a P_k-decomposition of $K_{A,I}$. □

Theorem 2.2. *Let Γ be a graph with at least two vertices of degree $|V(\Gamma)|-1$. Then Γ admits a P_4-decomposition if, and only if, $|E(\Gamma)| \equiv 0 \pmod 3$. If $|E(\Gamma)| \equiv 1, 2 \pmod 3$, then Γ can be partitioned into a P_4-decomposition together with one or two (possibly connected) edges, respectively.*

Proof. The condition is obviously necessary. For sufficiency, let α and β be two vertices of degree $|V(\Gamma)|-1$. Delete α and β in Γ, as to obtain a graph G. Let G' be a maximal P_4-decomposable subgraph of G and remove from G the edges of G', determining a new graph G''. In general, G'' is not connected and its connected components are either isolated vertices or stars or cycles of length 3; call $\mathcal{I}$, $\mathcal{S}$ and $\mathcal{C}$ their (possibly empty) sets. Let Γ' be the graph obtained removing the edges of G' from Γ. Clearly, $|E(\Gamma)| \equiv 0 \pmod 3$ implies $|E(\Gamma')| \equiv 0 \pmod 3$; thus it remains to show that $E(\Gamma')$ is P_4-decomposable. Obviously α and β are of degree $|V(\Gamma)|-1$ also in Γ'. Let $A = \{\alpha, \beta\}$ and consider the following decomposition $\Gamma' = K_A \cup K_{A,\mathcal{I}} \cup (\mathcal{C} \cup K_{A,V(\mathcal{C})}) \cup (\mathcal{S} \cup K_{A,V(\mathcal{S})})$. We begin by providing, separately, P_4-decompositions of $K_{A,\mathcal{I}}$, $\mathcal{C} \cup K_{A,V(\mathcal{C})}$ and $\mathcal{S} \cup K_{A,V(\mathcal{S})}$.

i) It is easy to see that for any 3-subset of $\mathcal{I}$, say H_3, the graph K_{A,H_3} has a P_4-decomposition. Thus, depending on the congruence class modulo 3 of $|\mathcal{I}|$, $K_{A,\mathcal{I}}$ can be partitioned into a P_4-decomposition together with the following possible remnants.

(i_1) $\lvert\mathcal{I}\rvert \equiv 0 \pmod 3$	(i_2) $\lvert\mathcal{I}\rvert \equiv 1 \pmod 3$	(i_3) $\lvert\mathcal{I}\rvert \equiv 2 \pmod 3$
the set $\varnothing$	the path $[\alpha, h, \beta]$ with $h \in \mathcal{I}$	the cycle $(h_1, \alpha, h_2, \beta)$ with $h_1, h_2 \in \mathcal{I}$

Table 1: Case *i*.

ii) For any 3-cycle $C \in \mathcal{C}$, the graph $C \cup K_{A,V(C)}$ has a P_4-decomposition. Thus, $\mathcal{C} \cup K_{A,V(\mathcal{C})}$ also admits a P_4-decomposition.

iii) It is not difficult to see that, for any star $S_c \in \mathcal{S}$ of center c, the graph $S_c \cup K_{A,V(S_c)}$ has a partition into a P_4-decomposition together with either the path $[\alpha, c, \beta]$ or the graph $(\alpha, c, \beta, v) \cup [c, v]$, where v is any external vertex, depending on whether the number of vertices of S_c is odd or even. Let $\mathcal{S}_1$ (respectively $\mathcal{S}_2$) be the set of stars with an odd (even) number of vertices. For any three stars of $\mathcal{S}_1$ ($\mathcal{S}_2$) the remnants give P_4-decomposable graphs. So $\mathcal{S}_1 \cup K_{A,V(\mathcal{S}_1)}$, as well as $\mathcal{S}_2 \cup K_{A,V(\mathcal{S}_2)}$, can be partitioned into a P_4-decomposition together with the possible remnants outlined in Tables 2 and 3.

(iii_{11}) $\lvert\mathcal{S}_1\rvert \equiv 0 \pmod 3$	(iii_{12}) $\lvert\mathcal{S}_1\rvert \equiv 1 \pmod 3$	(iii_{13}) $\lvert\mathcal{S}_1\rvert \equiv 2 \pmod 3$
$\varnothing$	the path $[\alpha, c, \beta]$ where c is the center of a star	the cycle $(c_1, \alpha, c_2, \beta)$ where c_1, c_2 are centers of two stars

Table 2: Case iii_1: $\mathcal{S}_1 \cup K_{A,V(\mathcal{S}_1)}$.

(iii_{21}) $\lvert\mathcal{S}_2\rvert \equiv 0 \pmod 3$	(iii_{22}) $\lvert\mathcal{S}_2\rvert \equiv 1 \pmod 3$	(iii_{23}) $\lvert\mathcal{S}_2\rvert \equiv 2 \pmod 3$
$\varnothing$	the graph $(\alpha, c, \beta, v) \cup [c, v]$ where c is the center and v is an external vertex of a star	the graph $\bigcup_{i=1}^{2} (\alpha, c_i, \beta, v_i) \cup [c_i, v_i]$ where c_1, c_2 are centers and v_1, v_2 are external vertices of two stars

Table 3: Case iii_2: $\mathcal{S}_2 \cup K_{A,V(\mathcal{S}_2)}$.

The remnants from *i*), iii_1) and iii_2) together with the edge $[\alpha, \beta]$ can be combined in 27 different ways to obtain 27 connected graphs with t edges. It is a routine to check that we have exactly 9 cases with $t \equiv i \pmod 3$, for $i = 0, 1, 2$.

In Table 4 we will list in detail the 9 cases with $t \equiv 0 \pmod 3$ and, for each of them, in Table 5 we give the corresponding graph.

	i	iii_1	iii_2
a_1	$\varnothing$	$\varnothing$	iii_{22}
a_2	$\varnothing$	iii_{13}	iii_{23}
a_3	$\varnothing$	iii_{12}	$\varnothing$

	i	iii_1	iii_2
a_4	i_2	$\varnothing$	$\varnothing$
a_5	i_2	iii_{13}	iii_{22}
a_6	i_2	iii_{12}	iii_{23}

	i	iii_1	iii_2
a_7	i_3	$\varnothing$	iii_{23}
a_8	i_3	iii_{13}	$\varnothing$
a_9	i_3	iii_{12}	iii_{22}

Table 4: $t \equiv 0 \pmod 3$.

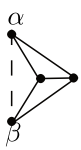

Figure 1: Case a_1.

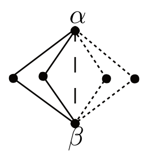

Figure 2: Case a_8.

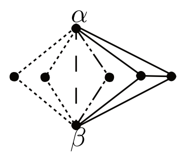

Figure 3: Cases a_5 and a_9.

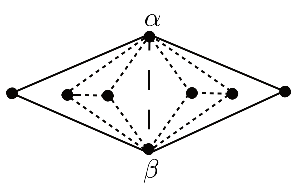

Figure 4: Cases a_2, a_6 and a_7.

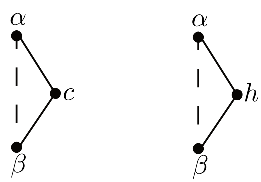

Figure 5: Cases a_3 and a_4.

Table 5: Graphs of the remnants plus edge $[\alpha, \beta]$.

It is easy to determine a P_4-decomposition of the graphs in Figures 1, 2, 3, 4. In cases a_3 and a_4 (Figure 5) a P_4-decomposition is clearly not possible, thus we proceed back tracking one step in the construction. How to deal with case a_3 is explained in Figure 6.

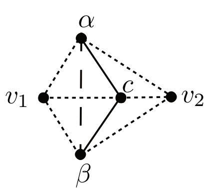

Figure 6: We recover the two P_4's from 2 radii of the star of center c.

In case a_4 we have to distinguish several subcases depending on the size of $\mathcal{I}$,$\mathcal{C}$ and $\mathcal{S}$. When $|\mathcal{I}| > 1$ see Figure 7. For $|\mathcal{I}| = 1$ and $|\mathcal{C}| \neq 0$, see Figure 8.

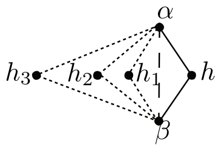

Figure 7: If $|\mathcal{I}| > 1$, we recover the two P_4's from 3 vertices of $\mathcal{I}$.

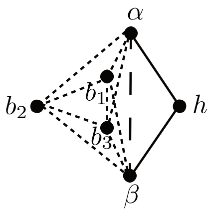

Figure 8: If $|\mathcal{I}| = 1$ and $|\mathcal{C}| \neq 0$ we recover the three P_4's from a C_3.

When $|\mathcal{I}| = 1$ and $|\mathcal{C}| = 0$ we have two possibilities. If there is one star of $\mathcal{S}$ with at least two edges, we proceed as explained in Figure 9.

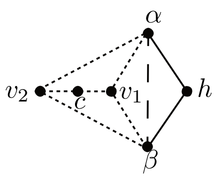

Figure 9: If $|\mathcal{I}| = 1$, $|\mathcal{C}| = 0$, and $\exists S_c \in \mathcal{S}$ with $P_3 \leq S_c$ we recover the two P_4's from 2 radii of S_c.

Otherwise, G'' consists of an isolated vertex h and a set $\mathcal{P}$ of disjoint P_2's. Since $|E(\Gamma')| \equiv 0 \pmod 3$, the size of $\mathcal{P}$ is also divisible by 3, let $|\mathcal{P}| = 3p$. It is easy to see that for any 3-subset of $\mathcal{P}$, say P^3, the graph K_{A,P^3} has a P_4-decomposition. After $p-1$ steps, the remnant is the graph in Figure 10, which likewise admits a P_4-decomposition. This concludes the case $t \equiv 0 \pmod 3$.

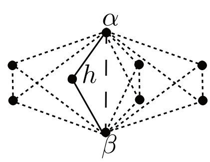

Figure 10: If $|\mathcal{I}| = 1$, $|\mathcal{C}| = 0$ and S is a disjoint union of P_2's we recover the 15 edges from the last but one step.

With similar arguments, when $t \equiv 1, 2 \pmod 3$ it is possible to find a P_4-decomposition of $E(\Gamma)$ leaving as remnants, respectively, one or two edges. □

3 Embeddings and down-links to P_4-designs

The results presented in the previous section are used to prove the existence of embeddings and down-links to path designs. In particular, we shall focus our attention on P_4-decompositions.

Theorem 3.1. *Any partial (K_v, P_4)-design can be embedded into a (K_n, P_4)-design for any admissible $n \geq v + 2$.*

Proof. Let $\mathcal{B}$ be a partial (K_v, P_4)-design. Let A be a set of vertices disjoint from $V(K_v)$ with $v + |A| \equiv 0, 1 \pmod 3$ and $|A| \geq 2$. Let Γ be the graph such that $V(\Gamma) = V(K_v) \cup A$ and $E(\Gamma) = E(K_{v+|A|}) \setminus E(\mathcal{B})$. Since $|A| \geq 2$, by Theorem 2.2 there exists a (Γ, P_4)-design $\mathcal{B}'$ and, clearly, $\mathcal{B} \cup \mathcal{B}'$ is a $(K_{v+|A|,P_4})$-design. □

Corollary 3.2. *For any (K_v, Γ)-design with $P_4 \leq \Gamma$*

$$\{n \geq v + 2 \mid n \equiv 0, 1 \pmod 3\} \subseteq \mathcal{L}_2\Gamma(v) \subseteq \mathcal{L}_1\Gamma(v).$$

Proof. Let $\mathcal{B}$ be a (K_v, Γ)-design with $P_4 \leq \Gamma$. Choose a P_4 in each block of $\mathcal{B}$ and call $\mathcal{P}$ the set of such P_4's. Obviously, $\mathcal{P}$ is a partial P_4-decomposition of K_v. Hence, by Theorem 3.1, $\mathcal{P}$ can be embedded into a (K_n, P_4)-design $\mathcal{B}'$ for any admissible $n \geq v + 2$. The construction also guarantees the existence of a down-link from $\mathcal{B}$ to $\mathcal{B}'$. □

Theorem 3.3. *For any even integer k, a P_k-design of order $n \equiv 0, 1 (\mathrm{mod}\, k-1)$ can be embedded into a P_k-design of any order $m > n + 1$ with $m \equiv 0, 1 (\mathrm{mod}\, k - 1)$.*

Proof. Let $\mathcal{B}$ be a (K_n, P_k)-design with $n \equiv 0, 1 \pmod{k - 1}$ and let $m = n + s \equiv 0, 1 (\mathrm{mod}\, k - 1)$. As $K_{n+s} = K_n \cup K_s \cup K_{n,s}$, for the existence of a (K_m, P_k)-design embedding $\mathcal{B}$ it is enough to find a P_k-decomposition of $K_s \cup K_{n,s}$. Since $n, n + s \equiv 0, 1 (\mathrm{mod}\, k - 1)$, one of the following cases occurs

- $n = \lambda(k-1), s = \mu(k-1) \Rightarrow K_s \bigcup K_{n,s} = K_s \bigcup \lambda\mu K_{k-1,k-1}$
- $n = \lambda(k-1), s = 1+\mu(k-1) \Rightarrow K_s \bigcup K_{n,s} = K_s \bigcup K_{\lambda(k-1),k+(\mu-1)(k-1)} =$
  $= K_s \bigcup \lambda K_{k-1,k} \bigcup \lambda(\mu-1) K_{k-1,k-1}$
- $n = 1+\lambda(k-1), s = \mu(k-1) \Rightarrow K_s \bigcup K_{n,s} = K_s \bigcup K_{k+(\lambda-1)(k-1),\mu(k-1)} =$
  $= K_s \bigcup \mu K_{k,k-1} \bigcup \mu(\lambda-1) K_{k-1,k-1}$
- $n = 1+\lambda(k-1), s = k-2+\mu(k-1) \Rightarrow K_s \bigcup K_{n,s} = K_s \bigcup K_{1+\lambda(k-1),s} =$
  $= K_s \bigcup K_{1,s} \bigcup K_{\lambda(k-1),s} = K_{s+1} \bigcup K_{\lambda(k-1),k-2+\mu(k-1)} =$
  $= K_{s+1} \bigcup \lambda K_{k-1,k-2} \bigcup \lambda\mu K_{k-1,k-1}$

So, to find a P_k-decomposition of $K_s \cup K_{n,s}$ it is sufficient to know P_k-decompositions of

- K_s and K_{s+1}, which exist by [13],
- $K_{k-1,k-1}$, whose existence is proved in [10],
- $K_{k-1,k}$ and $K_{k-1,k-2}$, whose existence follows from Proposition 2.1.

□

The following corollary is a straightforward consequence of Theorem 3.3.

Corollary 3.4. *If $n \in \mathcal{L}_i\Gamma(v)$, then*

$$\{m \geq n+2 \mid m \equiv 0,1 (\mathrm{mod}\, 3)\} \subseteq \mathcal{L}_i\Gamma(v).$$

Remark 3.5. *Set $\eta_i = \inf \mathcal{L}_i\Gamma(v)$. By Corollary 3.4, $\mathcal{L}_i\Gamma(v)$ contains all admissible values $m \geq \eta_i$ apart from (possibly) $\eta_i + 1$. Thus to exactly determine the spectra it is enough to compute η_i and ascertain if $\eta_i + 1 \in \mathcal{L}_i\Gamma(v)$.*

4 Cycle systems and path-designs

Here we shall provide some partial results on the existence of down-links from cycle systems and path-designs to P_4-designs.
We recall that a k-cycle system of order v, that is a (K_v, C_k)-design, exists if, and only if, $k \leq v$, v is odd and $v(v-1) \equiv 0 \pmod{2k}$; see [2], [12].

Theorem 4.1. *For any admissible v and any $k \geq 9$*

$$\left\{n \geq v - \left\lfloor \frac{k-9}{4} \right\rfloor \,\middle|\, n \equiv 0,1 \pmod 3\right\} \subseteq \mathcal{L}_2 C_k(v) \subseteq \mathcal{L}_1 C_k(v).$$

Proof. Let $k \geq 9$ and let $\mathcal{B}$ be a (K_v, C_k)-design. Write $t = \left\lfloor \frac{k-9}{4} \right\rfloor$. Take $t+2$ distinct vertices $x_1, x_2, \ldots, x_t, y_1, y_2 \in V(K_v)$. Observe that it is possible to extract from each block $C \in \mathcal{B}$ a P_4 whose vertices are different from $x_1, x_2, \ldots, x_t, y_1, y_2$, as we are forbidding at most $4(t+1)+2 = 4t+6 = k-3$ edges from any k-cycle. Use these P_4's for the down-link. Let S be the image of the down-link, considered as a subgraph of $K_{v-t} = K_v \smallsetminus \{x_1, \ldots, x_t\}$ and remove the edges of S from K_{v-t} to obtain a new graph R. It remains to show that R admits a P_4-decomposition. Observe that $|V(R)| = v-t$ and y_1, y_2 are two vertices of R of degree $v-t-1$. To apply Theorem 2.2 we have to distinguish some cases according to the congruence class modulo 3 of $v-t$.
If $v - t \equiv 0 (\mathrm{mod}\ 3)$, then $|E(R)| \equiv 0 (\mathrm{mod}\ 3)$ so the existence of a (R, P_4)-design is guaranteed by Theorem 2.2. Furthermore, if we add a vertex to K_{v-t} we can apply Theorem 2.2 also to $R' = R \cup K_{1,v-t}$ since $|E(R')| \equiv 0 (\mathrm{mod}\ 3)$. Hence there exist down-links from $\mathcal{B}$ to (K_{v-t}, P_4)-designs and

to (K_{v-t+1}, P_4)-designs.
If $v - t \equiv 1 (\text{mod } 3)$, then $|E(R)| \equiv 0 (\text{mod } 3)$, hence by Theorem 2.2, there exists a (R, P_4)-design. So we determine down-links from $\mathcal{B}$ to (K_{v-t}, P_4)-designs.
Finally, if $v-t \equiv 2 (\text{mod } 3)$, it is sufficient to add either $u = 1$ or $u = 2$ vertices to K_{v-t} and then apply Theorem 2.2 to $R'' = (K_{v-t} \cup K_u \cup K_{v-t,u}) \smallsetminus S$ in order to down-link $\mathcal{B}$ to (K_{v-t+1}, P_4)-designs or to (K_{v-t+2}, P_4)-designs, respectively. The statement follows from Remark 3.5. □

Arguing exactly as in the previous proof it is possible to prove the following result.

Theorem 4.2. *For any admissible v and any $k \geq 12$*

$$\left\{ n \geq v - \left\lfloor \frac{k-12}{4} \right\rfloor \,\middle|\, n \equiv 0, 1 \pmod 3 \right\} \subseteq \mathcal{L}_2 P_k(v) \subseteq \mathcal{L}_1 P_k(v).$$

4.1 Small cases

We shall now investigate in detail the spectrum problems for $\Gamma = C_4$ and $\Gamma = P_5$. In order to obtain our results, we shall extensively use the method of *gluing of down-links*, introduced in [3]. We briefly recall the main idea: a down-link from a (K_v, Γ)-design to a (K_n, Γ')-design can be constructed as union of down-links between partitions of the domain and the codomain. To give designs suitable for the down-link, we will use difference families; here we recall some preliminaries, for a survey see [1]. Let Γ be a graph. A set $\mathcal{F}$ of graphs isomorphic to Γ with vertices in $\mathbb{Z}_v$ is called a $(v, \Gamma, 1)$-*difference family* (DF, for short) if the list $\Delta\mathcal{F}$ of differences from $\mathcal{F}$, namely the list of all possible differences $x - y$, where (x, y) is an ordered pair of adjacent vertices of an element of $\mathcal{F}$, covers $\mathbb{Z}_v \smallsetminus \{0\}$ exactly once. In [6] it is proved that if $\mathcal{F} = \{B_1, \ldots, B_t\}$ is a $(v, \Gamma, 1)$-DF, then the collection of graphs $\mathcal{B} = \{B_i + g \mid B_i \in \mathcal{F},\ g \in \mathbb{Z}_v\}$ is a cyclic (K_v, Γ)-design.

Lemma 4.3. *For any $v \equiv 1, 9 \pmod{24}$, $v > 1$, there exists a down-link from a (K_v, C_4)-design to a (K_v, P_4)-design. For any $v \equiv 9, 17 \pmod{24}$ there exists a down-link from a (K_v, C_4)-design to a (K_{v+1}, P_4)-design.*

Proof. Take $v = s + 24t \geq 9$, with $s = 1, 9, 17$, and $V(K_v) = \mathbb{Z}_v$. Consider the set of 4-cycles

$$\mathcal{C} = \left\{ C^a = \left(0, a, \frac{v+1}{2}, \frac{v-1}{8} + a\right) \mid a = 1, 2, \ldots, \frac{v-1}{8} \right\}.$$

It is straightforward to check that

$$\Delta C^a = \pm \left\{ a, \frac{v+1}{2} - a, \frac{3v+5}{8} - a, \frac{v-1}{8} + a \right\}.$$

Hence $\Delta\mathcal{C} = \mathbb{Z}_v \setminus \{0\}$, so, by [6], the C^a are the $\frac{v-1}{8}$ base blocks of a cyclic (K_v, C_4)-design. The development of each base block gives v different 4-cycles, from each of which we extract the edge obtained by developing $[0, a]$. The obtained P_4's will be used to define a down-link in a natural way. The removed edges can be connected to complete the P_4-decomposition of K_v as follows: for each triple $\{[0, a+1], [0, a+2], [0, a+3]\}$, for $a \equiv 1 \pmod 3$ where $a \in \{1, 2, \ldots, \frac{v-1}{8}\}$, consider the three developments and connect the edges $\{[i+1, a+1+(i+1)], [i, a+2+i], [i, a+3+i]\}$ obtaining the paths $(i+1, a+i+2, i, a+i+3)$, with $i \in \mathbb{Z}_v$.
If $v \equiv 1 \pmod{24}$, we have the required P_4-decomposition.
If $v \equiv 9 \pmod{24}$, we have the required P_4-decomposition except for the development of $[0, 1]$. The v edges of such a development can be easily connected to give the v-cycle $C = (0, 1, \ldots, v-1)$, which obviously admits a P_4-decomposition. So, for $v \equiv 1, 9 \pmod{24}$, there exists a down-link from a (K_v, C_4)-design to a (K_v, P_4)-design. Under the assumption $v \equiv 9 \pmod{24}$, $n = v+1$ is also admissible. In this case, add the vertex α to $V(K_v)$ to obtain a K_{v+1} supporting the codomain of the down-link. Actually, the star $S_{[\alpha;V]}$ of center α and external vertices the elements of $V(K_v)$ has been added. Proceed as before till to the last but one step, namely do not decompose the v-cycle C obtained by developing $[0, 1]$. So it remains to determine a P_4-decomposition of the wheel $W = C \cup S_{[\alpha;V]}$. It is easy to see that W can be decomposed into $3 + 8t$ copies of the graph W' in Figure 11, which evidently admits a P_4-decomposition.

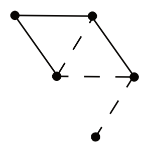

Figure 11: The graph W' as union of two P_4's.

If $v \equiv 17 \pmod{24}$, proceeding as before, we determine the required P_4-decomposition except for the two developments, say d_1 and d_2, of the edges $[0, 1]$ and $[0, \frac{v-1}{8}]$. Keeping in mind that we must also add a vertex, say α, to the codomain, we have to arrange the edges of d_1, d_2 and $S_{[\alpha;V]}$. It easy to see that we can obtain the P_4's as $[\alpha, 1+i, i, \frac{v-1}{8} + i]$, for $i \in \mathbb{Z}_v$.
So, for $v \equiv 9, 17 \pmod{24}$ there exists a down-link from a (K_v, C_4)-design to a (K_{v+1}, P_4)-design. $\square$

Theorem 4.4. *For any admissible $v > 1$,*

$$\mathcal{L}_1 C_4(v) = \{n \geq v \mid n \equiv 0,1 \pmod 3\}; \tag{1}$$

$$\{n \geq v+2 \mid n \equiv 0,1 \pmod 3\} \subseteq \mathcal{L}_2 C_4(v) \subseteq \{n \geq v \mid n \equiv 0,1 \pmod 3\}. \tag{2}$$

Proof. Let $\mathcal{B}$ and $\mathcal{B}'$ be, respectively, a (K_v, C_4)-design and a (K_n, P_4)-design. Suppose that $\mathcal{B}$ can be down-linked to $\mathcal{B}'$. Clearly, $n \geq v$. Hence $\mathcal{L}_2 C_4(v) \subseteq \mathcal{L}_1 C_4(v) \subseteq \{n \geq v \mid n \equiv 0,1 \pmod 3\}$.
To prove the reverse inclusion in (1) observe that a (K_v, C_4)-design exists if, and only if, $v \equiv 1 \pmod 8$ and a (K_n, P_4)-design exists if, and only if, $n \equiv 0,1 \pmod 3$. So it makes sense to look for a down-link from a (K_v, C_4)-design to a (K_v, P_4)-design only for $v \equiv 1,9 \pmod{24}$. Likewise, a down-link from a (K_v, C_4)-design to a (K_{v+1}, P_4)-design can exist only if $v \equiv 9,17 \pmod{24}$. The existence of such down-links is proved in Lemma 4.3. The statement of (1) follows from Remark 3.5. The other inclusion in (2) immediately follows from Corollary 3.2. □

Theorem 4.5. *For any admissible $v > 1$,*

$$\mathcal{L}_1 P_5(v) = \{n \geq v-1 \mid n \equiv 0,1 (\mathrm{mod}\, 3)\}; \tag{3}$$

$$\{n \geq v+2 \mid n \equiv 0,1 (\mathrm{mod}\, 3)\} \subseteq \mathcal{L}_2 P_5(v) \subseteq \{n \geq v \mid n \equiv 0,1 (\mathrm{mod}\, 3)\}. \tag{4}$$

Proof. The first inclusion in (4) follows from Corollary 3.2. In order to prove the second, it is sufficient to show that for any admissible v there exists a (K_v, P_5)-design $\mathcal{B}$ wherein no vertices can be deleted. In particular, this is the case if each vertex of K_v has degree 2 in at least one block of $\mathcal{B}$. First of all note that in a (K_v, P_5)-design there is at most one vertex with degree 1 in each block where it appears. Suppose that there actually exists a (K_v, P_5)-design $\overline{\mathcal{B}}$ with a vertex x as above. It is easy to see that in $\overline{\mathcal{B}}$ there is at least one block $P^1 = [x,a,b,c,d]$ such that the vertices a,b and c have degree two in at least another block. Let $P^2 = [x,d,e,f,g]$. By reassembling the edges of $P^1 \cup P^2$, it is possible to replace in $\overline{\mathcal{B}}$ these two paths with $P^3 = [d,x,a,b,c]$, $P^4 = [c,d,e,f,g]$ if $c \neq f,g$ or $P^5 = [a,x,d,c,g], P^6 = [a,b,c,e,d]$ if $c = f$ or $P^7 = [c,d,x,a,b], P^8 = [b,c,f,e,d]$ if $c = g$. Thus we have again a (K_v, P_5)-design. By the assumption on a,b,c all the vertices of this new design have degree two in at least one block.
Now we consider Relation (3). Let $\mathcal{B}$ and $\mathcal{B}'$ be respectively a (K_v, P_5)-design and a (K_n, P_4)-design. Suppose there exists a down-link $f : \mathcal{B} \to \mathcal{B}'$. Clearly, $n > v-2$. Hence, $\mathcal{L}_1 P_5(v) \subseteq \{n \geq v-1 \mid n \equiv 0,1 (\mathrm{mod}\, 3)\}$.
To show the reverse inclusion in (3) we prove the actual existence of designs providing down-links. Since a (K_v, P_5)-design exists if, and only if, $v \equiv 0,1 (\mathrm{mod}\, 8)$ and a (K_n, P_4)-design exists if, and only if, $n \equiv 0,1 (\mathrm{mod}\, 3)$, it makes sense to look for a down-link from a (K_v, P_5)-design to a (K_{v-1}, P_4)-design only if $v \equiv 1,8,16,17 (\mathrm{mod}\ 24)$. For the same reason, it makes sense

to construct a down-link from a (K_v, P_5)-design to a (K_v, P_4)-design only for $v \equiv 0, 1, 9, 16 (\mathrm{mod}\, 24)$. In view of Remark 3.5, in order to complete the proof, we have also to provide a down-link from a (K_v, P_5)-design to a (K_{v+1}, P_4)-design for every $v \equiv 0, 9 (\mathrm{mod}\, 24)$.
To determine the necessary down-links, we analyze a few basic cases and then apply the *gluing method*. To this end, we will use the following obvious relations in an appropriate way: $K_{a+b} = K_a \bigcup K_b \bigcup K_{a,b}$ and $K_{a+b,c} = K_{a,c} \cup K_{b,c}$. In particular,

$$\begin{aligned} K_{\ell+24t} &= K_\ell \cup K_{24t} \cup K_{\ell,24t}; \\ K_{24t} &= tK_{24} \cup \binom{t}{2} K_{24,24} = tK_{24} \cup 48\binom{t}{2} K_{3,4}; \\ K_{\ell=rs,24t} &= rK_{s,24t} = rtK_{s,24} = 6rtK_{s,4} = 8rtK_{s,3}. \end{aligned}$$

Let us now examine the possible cases.

- $(K_v, P_5) \rightarrow (K_{v-1}, P_4)$-design with $v = \ell + 24t > 1$, $\ell = 1, 8, 16, 17$.

P_5-design of order	basic components	$\rightarrow$	basic components	P_4-design of order
$1 + 24t$	(K_{25}, P_5), $(K_{3,4}, P_5)$		(K_{24}, P_4), $(K_{3,4}, P_4)$	$24t$
$8 + 24t$	(K_8, P_5), (K_{24}, P_5) $(K_{4,3}, P_5)$		(K_7, P_4), (K_{24}, P_4) $(K_{4,3}, P_4)$, $(K_{3,3}, P_4)$	$7 + 24t$
$16 + 24t$	(K_{16}, P_5), (K_{24}, P_5) $(K_{4,3}, P_5)$		(K_{15}, P_4), (K_{24}, P_4) $(K_{4,3}, P_4)$, $(K_{3,3}, P_4)$	$15 + 24t$
$17 + 24t$	(K_{17}, P_5), (K_{24}, P_5) $(K_{4,3}, P_5)$		(K_{16}, P_4), (K_{24}, P_4) $(K_{4,3}, P_4)$, $(K_{3,3}, P_4)$	$16 + 24t$

- $(K_v, P_5) \rightarrow (K_v, P_4)$-design with $v = \ell + 24t > 1$, $\ell = 0, 1, 9, 16$.

P_5-design of order	basic components	$\rightarrow$	basic components	P_4-design of order
$24t$	(K_{24}, P_5), $(K_{3,4}, P_5)$		(K_{24}, P_4), $(K_{3,4}, P_4)$	$24t$
$1 + 24t$	(K_9, P_5), (K_{16}, P_5) (K_{24}, P_5),$(K_{3,4}, P_5)$		(K_9, P_4), (K_{16}, P_4) (K_{24}, P_4),$(K_{3,4}, P_4)$	$1 + 24t$
$9 + 24t$	(K_9, P_5), (K_{24}, P_5) $(K_{3,4}, P_5)$		(K_9, P_4), (K_{24}, P_4) $(K_{3,4}, P_4)$	$9 + 24t$
$16 + 24t$	(K_{16}, P_5), (K_{24}, P_5) $(K_{3,4}, P_5)$		(K_{16}, P_4), (K_{24}, P_4) $(K_{3,4}, P_4)$	$16 + 24t$

- $(K_v, P_5) \rightarrow (K_{v+1}, P_4)$-design with $v = \ell + 24t > 1$, $\ell = 0, 9$.

P_5-design of order	basic components	$\rightarrow$	basic components	P_4-design of order
$24t$	(K_{24}, P_5), $(K_{3,4}, P_5)$		(K_{25}, P_4), $(K_{3,4}, P_4)$	$1 + 24t$
$9 + 24t$	(K_9, P_5), (K_{24}, P_5) $(K_{3,4}, P_5)$,$(K_{9,24}, P_5)$		(K_{10}, P_4), (K_{24}, P_4) $(K_{3,4}, P_4)$,$(K_{10,24}, P_4)$	$10 + 24t$

It is straightforward to show the existence of such basic down-links. For instance we provide a down-link ξ from a $(K_{9,24}, P_5)$-design to a $(K_{10,24}, P_4)$-design. Let $A = \{a, b, c, d, e, f, g, h, i\}$ and $B = \mathbb{Z}_{24}$, that is $K_{9,24} = K_{A,B}$. The following are the 54 paths of a P_5-decomposition of $K_{A,B}$:

$[\underline{6, a}, 12, b, 1]$	$[\underline{1, c}, 12, d, 6]$	$[\underline{6, e}, 18, f, 1]$	$[\underline{1, g}, 12, h, 0]$	$[\underline{12, i}, 0, a, 18]$
$[\underline{7, a}, 13, b, 2]$	$[\underline{2, c}, 13, d, 7]$	$[\underline{7, e}, 19, f, 2]$	$[\underline{2, g}, 13, h, 1]$	$[\underline{13, i}, 1, a, 19]$
$[\underline{8, a}, 14, b, 3]$	$[\underline{3, c}, 14, d, 8]$	$[\underline{8, e}, 20, f, 3]$	$[\underline{3, g}, 14, h, 2]$	$[\underline{14, i}, 2, a, 20]$
$[\underline{9, a}, 15, b, 4]$	$[\underline{4, c}, 15, d, 9]$	$[\underline{9, e}, 21, f, 4]$	$[\underline{4, g}, 15, h, 3]$	$[\underline{15, i}, 3, a, 21]$
$[\underline{10, a}, 16, b, 5]$	$[\underline{5, c}, 16, d, 10]$	$[\underline{10, e}, 22, f, 5]$	$[\underline{5, g}, 16, h, 4]$	$[\underline{16, i}, 4, a, 22]$
$[\underline{11, a}, 17, b, 0]$	$[\underline{0, c}, 17, d, 11]$	$[\underline{11, e}, 23, f, 0]$	$[\underline{0, g}, 17, h, 5]$	$[\underline{17, i}, 5, a, 23]$

$[\underline{18, b}, 6, c, 19]$	$[\underline{19, d}, 0, e, 12]$	$[\underline{12, f}, 6, g, 19]$	$[\underline{19, h}, 6, i, 18]$
$[\underline{19, b}, 7, c, 20]$	$[\underline{20, d}, 1, e, 13]$	$[\underline{13, f}, 7, g, 20]$	$[\underline{20, h}, 7, i, 19]$
$[\underline{20, b}, 8, c, 21]$	$[\underline{21, d}, 2, e, 14]$	$[\underline{14, f}, 8, g, 21]$	$[\underline{21, h}, 8, i, 20]$
$[\underline{21, b}, 9, c, 22]$	$[\underline{22, d}, 3, e, 15]$	$[\underline{15, f}, 9, g, 22]$	$[\underline{22, h}, 9, i, 21]$
$[\underline{22, b}, 10, c, 23]$	$[\underline{23, d}, 4, e, 16]$	$[\underline{16, f}, 10, g, 23]$	$[\underline{23, h}, 10, i, 22]$
$[\underline{23, b}, 11, c, 18]$	$[\underline{18, d}, 5, e, 17]$	$[\underline{17, f}, 11, g, 18]$	$[\underline{18, h}, 11, i, 23]$.

We obtain the image of any P_5 via ξ by removing the underlined edge. Now, to complete the codomain, we have to add a further vertex to A, say α, together with all the edges connecting α to the vertices of B. Thus, it remains to decompose the graph formed by the removed edges together with the star of center α and external vertices in B. Such a P_4-decomposition is listed below:

$[6, a, 9, \alpha]$	$[7, a, 10, \alpha]$	$[8, a, 11, \alpha]$	$[1, c, 4, \alpha]$	$[2, c, 5, \alpha]$	
$[3, c, 0, \alpha]$	$[9, e, 6, \alpha]$	$[10, e, 7, \alpha]$	$[11, e, 8, \alpha]$	$[4, g, 1, \alpha]$	
$[5, g, 2, \alpha]$	$[0, g, 3, \alpha]$	$[15, i, 12, \alpha]$	$[16, i, 13, \alpha]$	$[17, i, 14, \alpha]$	
$[22, d, 19, \alpha]$	$[23, d, 20, \alpha]$	$[21, d, 18, \alpha]$	$[12, f, 15, \alpha]$	$[13, f, 16, \alpha]$	
$[14, f, 17, \alpha]$	$[20, h, 21, \alpha]$	$[23, h, 22, \alpha]$	$[20, b, 23, \alpha]$	$[h, 18, b, 21]$	$[h, 19, b, 22]$.

□

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