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Global Existence for Two Regularized MHD Models in Three Space-Dimension

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Abstract. The global existence of solutions for the 3D incompressible Euler equations is a major open problem. For the 3D *inviscid* MHD system, the global existence is an open problem as well. Our main concern in this paper is to understand which kind of regularization, of the form of α-regularization or partial viscous regularization, is capable to provide the global in time solvability for the 3D *inviscid* MHD system of equations. We consider two different regularized magnetohydrodynamic models for an incompressible fluid. In both cases, we provide a global existence result for the solution of the system.

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1. Introduction

The basic system of equations that one can consider in magnetohydrodynamics is obtained combining Maxwell's equations, which rule the magnetic field, with the Navier–Stokes equation, which governs the fluid motion; this system has form:

$$\boldsymbol{v}_t + (\boldsymbol{v}\cdot\nabla)\boldsymbol{v} - (\boldsymbol{B}\cdot\nabla)\boldsymbol{B} + \nabla\left(p + \frac{1}{2}|\boldsymbol{B}|^2\right) = \nu\Delta\boldsymbol{v}\,, \tag{1a}$$

$$\boldsymbol{B}_t + (\boldsymbol{v}\cdot\nabla)\boldsymbol{B} - (\boldsymbol{B}\cdot\nabla)\boldsymbol{v} = \mu\Delta\boldsymbol{B}\,, \tag{1b}$$

$$\nabla\cdot\boldsymbol{v} = \nabla\cdot\boldsymbol{B} = 0\,, \tag{1c}$$

$$(\boldsymbol{v},\boldsymbol{B})|_{t=0} = (\boldsymbol{v}_0,\boldsymbol{B}_0)\,, \qquad \boldsymbol{x}\in\mathbb{R}^n\,, \qquad n = 2,3\,, \tag{1d}$$

where the fluid velocity field $\boldsymbol{v}(\boldsymbol{x},t)$, the magnetic field $\boldsymbol{B}(\boldsymbol{x},t)$ and the pressure $p(\boldsymbol{x},t)$ are the unknowns, while $\nu \geqslant 0$ is the constant kinematic viscosity and $\mu \geqslant 0$ is the constant magnetic diffusivity. In this case, a homogeneous incompressible fluid is considered.

This problem has been deeply studied. If $\nu > 0$ and $\mu > 0$, then there exists a unique global solution in time when $n = 2$, while for $n = 3$ the problem is still open, as discussed in [13].

When $n = 2$, $\nu = 0$ and $\mu = 1$, local existence and small data global existence results have been established by Kozono [10] for bounded domains and by Casella–Secchi–Trebeschi [3] for unbounded domains.

When $n = 2$, $\nu = 1$ and $\mu = 0$, there is a regularity criterion for the solution in terms of $\boldsymbol{B}$ provided by Jiu–Niu [9], but the problem in its generality is still open.

As pointed out in [12] (see also the suggested bibliography), at the moment, there is no possibility to compute the turbulent behavior of fluids neither analytically nor via direct numerical simulation (this task is prohibitively expensive and disputable as well due to sensitivity of perturbation errors in the initial data). Hence, one can try to focus only on certain statistical features of the physical phenomenon through the employment of suitable models. This is sufficient in many practical applications.

Because of the success of Navier–Stokes-α models in producing solutions in excellent agreement with empirical data for a wide range of large Reynolds numbers and flow in infinite channels or pipes, it is natural to consider such a kind of regularization also for magnetohydrodynamic models.

In α models, a function (or several functions) is substituted in one or more of its occurrences with a regularized function; more precisely, the function $\boldsymbol{v}$ is substituted with $\boldsymbol{u}$, where

$$\boldsymbol{v} = (1 - \alpha^2 \Delta)\boldsymbol{u}\,, \qquad \alpha > 0\,.$$

This substitution is performed in nonlinear terms to make the nonlinearity milder, so that the solution becomes smoother.

Linshiz–Titi [12] have suggested several models. For instance, filtering only the velocity field, one can consider the following model:

$$\boldsymbol{v}_t + (\boldsymbol{u}\cdot\nabla)\boldsymbol{v} + \sum_{j=1}^{n} v_j \nabla u_j - (\boldsymbol{B}\cdot\nabla)\boldsymbol{B} + \nabla\left(p + \frac{1}{2}|\boldsymbol{B}|^2\right) = \nu\Delta\boldsymbol{v}\,, \tag{2a}$$

$$\boldsymbol{B}_t + (\boldsymbol{u}\cdot\nabla)\boldsymbol{B} - (\boldsymbol{B}\cdot\nabla)\boldsymbol{u} = \mu\Delta\boldsymbol{B}\,, \tag{2b}$$

$$\boldsymbol{v} = (1 - \alpha^2\Delta)\boldsymbol{u}\,, \qquad \alpha > 0\,, \tag{2c}$$

$$\nabla\cdot\boldsymbol{v} = \nabla\cdot\boldsymbol{u} = \nabla\cdot\boldsymbol{B} = 0\,, \tag{2d}$$

$$(\boldsymbol{v}, \boldsymbol{B})|_{t=0} = (\boldsymbol{v}_0, \boldsymbol{B}_0)\,. \tag{2e}$$

In this case, Linshiz–Titi [12] have shown a global existence result in a three-dimensional periodic box when $\nu > 0$ and $\mu > 0$, while Fan–Ozawa [8] have achieved the same result in the whole space $\mathbb{R}^2$ for both $(\nu = 1, \mu = 0)$ and $(\nu = 0, \mu = 1)$.

Another model is the so-called simplified Bardina model, which is studied by Cao–Lunasin–Titi in [2].

In [4], the following magnetohydrodynamic-α model, derived from Bardina model for incompressible fluids, is considered:

$$\boldsymbol{v}_t + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} - (\boldsymbol{B}\cdot\nabla)\boldsymbol{B} + \nabla p = \nu\Delta\boldsymbol{v} + \boldsymbol{f} \qquad \text{in } [0,T]\times\mathbb{R}^2\,, \tag{3a}$$
$$\boldsymbol{B}_t + (\boldsymbol{u}\cdot\nabla)\boldsymbol{B} - (\boldsymbol{B}\cdot\nabla)\boldsymbol{u} = \mu\Delta\boldsymbol{B} \qquad \text{in } [0,T]\times\mathbb{R}^2\,, \tag{3b}$$
$$\boldsymbol{v} = (1-\alpha^2\Delta)\boldsymbol{u}\,, \qquad \alpha > 0 \qquad \text{in } [0,T]\times\mathbb{R}^2\,, \tag{3c}$$
$$\nabla\cdot\boldsymbol{v} = \nabla\cdot\boldsymbol{u} = \nabla\cdot\boldsymbol{B} = 0 \qquad \text{in } [0,T]\times\mathbb{R}^2\,, \tag{3d}$$
$$(\boldsymbol{v},\boldsymbol{B})|_{t=0} = (\boldsymbol{v}_0,\boldsymbol{B}_0) \qquad \boldsymbol{x}\in\mathbb{R}^2\,. \tag{3e}$$

Once again, a global existence result is obtained in case $(\nu = 1, \mu = 0)$ and $\boldsymbol{f} \equiv \boldsymbol{0}$.

In [6], a double viscous version $(\mu\,, \nu > 0)$ of the previous model is considered. The case of a periodic box in space-dimension three is handled and global existence of solutions is achieved. Moreover, this result is complemented by the proof of the existence of a global attractor, whose fractal dimension is estimated from above. In [5], other MHDα models are considered.

It is well known that the global existence of solutions for the 3D incompressible Euler equations is a major open problem (see [1], [7]). For the 3D *inviscid* MHD system (i.e. (1) with $\nu = \mu = 0$), the global existence is an open problem as well (obviously, because the Euler equations correspond to the particular case $\boldsymbol{B} \equiv \boldsymbol{0}$).

Our main concern in this paper is to understand which kind of regularization, of the form of α-regularization or partial viscous regularization, is capable to provide the global in time solvability for the 3D *inviscid* MHD system of equations.

We will consider two different models.

First model. We begin by considering the case without viscosity nor diffusivity, but with regularizations both in the velocity $\boldsymbol{v}$ and the magnetic field $\boldsymbol{B}$:

$$\boldsymbol{v}_t + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} - (\boldsymbol{b}\cdot\nabla)\boldsymbol{b} + \nabla p = \boldsymbol{0} \qquad \text{in } [0,T]\times\Omega\,, \tag{4a}$$
$$\boldsymbol{B}_t + (\boldsymbol{u}\cdot\nabla)\boldsymbol{b} - (\boldsymbol{b}\cdot\nabla)\boldsymbol{u} = \boldsymbol{0} \qquad \text{in } [0,T]\times\Omega\,, \tag{4b}$$
$$\boldsymbol{v} = (1-\alpha^2\Delta)\boldsymbol{u}\,, \quad \boldsymbol{B} = (1-\beta^2\Delta)\boldsymbol{b}\,, \qquad \alpha,\beta > 0 \qquad \text{in } [0,T]\times\Omega\,, \tag{4c}$$
$$\nabla\cdot\boldsymbol{u} = \nabla\cdot\boldsymbol{b} = 0 \qquad \text{in } [0,T]\times\Omega\,, \tag{4d}$$
$$(\boldsymbol{v},\boldsymbol{B})|_{t=0} = (\boldsymbol{v}_0,\boldsymbol{B}_0) \qquad \boldsymbol{x}\in\Omega\,. \tag{4e}$$

Here, $\Omega = [0,L]^3 \subset \mathbb{R}^3$ and we assume periodic conditions on the initial data so that the corresponding solutions are space-periodic (this implies, in particular, that integrating by parts border terms disappear). Moreover, we assume that the initial data have zero mean, so that also the solutions have zero spatial mean (this simplifies some computations). Under these conditions, we have the following global existence results.

Theorem 1.1 (Weak Global Existence for the First Model). *Let us set*

$$\boldsymbol{u}_0 = (1-\alpha^2\Delta)^{-1}\boldsymbol{v}_0\,, \qquad \boldsymbol{b}_0 = (1-\beta^2\Delta)^{-1}\boldsymbol{B}_0\,,$$

and assume that $\boldsymbol{u}_0$, $\boldsymbol{b}_0 \in \mathrm{H}^1(\Omega)$ *and* $\nabla \cdot \boldsymbol{u}_0 = \nabla \cdot \boldsymbol{b}_0 = 0$*.*

Then, problem (4) *has a unique global solution* $(\boldsymbol{u}, \boldsymbol{b})$ *such that*

$$\boldsymbol{u}\,, \boldsymbol{b} \in \mathrm{L}^\infty(0, \infty; \mathrm{H}^1(\Omega))\,.$$

The couple $(\boldsymbol{u}, \boldsymbol{b})$ *is a* weak solution *of* (4) *in the sense of* (8) *(see Section* 2*).*

Theorem 1.2 (Strong Global Existence for the First Model). *Let us assume that the initial data satisfy* $\boldsymbol{v}_0$, $\boldsymbol{B}_0 \in \mathrm{L}^2(\Omega)$ *and* $\nabla \cdot \boldsymbol{v}_0 = \nabla \cdot \boldsymbol{B}_0 = 0$*.*

Then, problem (4) *has a unique global solution* $(\boldsymbol{v}, \boldsymbol{B})$ *such that, for each time* $T > 0$*, one has*

$$\boldsymbol{v}\,, \boldsymbol{B} \in \mathrm{L}^\infty(0, T; \mathrm{L}^2(\Omega))\,.$$

These results are shown in Section 2. Let us note that this model is particularly interesting since it preserves three physical quantities, that is to say the energy $E^{\alpha,\beta} = \frac{1}{2}\int_\Omega \big(\boldsymbol{v}(\boldsymbol{x}) \cdot \boldsymbol{u}(\boldsymbol{x}) + \boldsymbol{B}(\boldsymbol{x}) \cdot \boldsymbol{b}(\boldsymbol{x})\big)\,\mathrm{d}\boldsymbol{x}$, the cross helicity $H_{\mathrm{C}}^{\alpha,\alpha} = \frac{1}{2}\int_\Omega \big(\boldsymbol{u}(\boldsymbol{x}) \cdot \boldsymbol{b}(\boldsymbol{x}) + \alpha^2 \nabla \boldsymbol{u}(\boldsymbol{x}) \cdot \nabla \boldsymbol{b}(\boldsymbol{x})\big)\,\mathrm{d}\boldsymbol{x}$ (here we are assuming $\beta = \alpha$, which is absolutely reasonable) and the magnetic helicity $H_{\mathrm{M}}^{\alpha,\beta} = \frac{1}{2}\int_\Omega \big(\boldsymbol{a}(\boldsymbol{x}) \cdot \boldsymbol{b}(\boldsymbol{x}) + \beta^2 \nabla \boldsymbol{a}(\boldsymbol{x}) \cdot \nabla \boldsymbol{b}(\boldsymbol{x})\big)\,\mathrm{d}\boldsymbol{x}$, where $\boldsymbol{a}$ is a vector potential, so that $\boldsymbol{b} = \nabla \times \boldsymbol{a}$. Moreover, as $\alpha, \beta \to 0$, these quantities reduce to the corresponding conserved ideal quadratic invariants of the MHD equations.

Second model. Then, we consider the following model, with magnetic diffusivity $\mu > 0$ but no kinematic viscosity, and regularization only in the velocity $\boldsymbol{v}$, while $\boldsymbol{B}$ is the magnetic field:

$$\boldsymbol{v}_t + (\boldsymbol{u} \cdot \nabla)\boldsymbol{u} - (\boldsymbol{B} \cdot \nabla)\boldsymbol{B} + \nabla p = \mathbf{0} \qquad \text{in } [0,T] \times \Omega\,, \tag{5a}$$

$$\boldsymbol{B}_t + (\boldsymbol{u} \cdot \nabla)\boldsymbol{B} - (\boldsymbol{B} \cdot \nabla)\boldsymbol{u} = \mu \Delta \boldsymbol{B} \qquad \text{in } [0,T] \times \Omega\,, \tag{5b}$$

$$\boldsymbol{v} = (1 - \alpha^2 \Delta)\boldsymbol{u}\,, \qquad \alpha > 0 \qquad \text{in } [0,T] \times \Omega\,, \tag{5c}$$

$$\nabla \cdot \boldsymbol{u} = \nabla \cdot \boldsymbol{B} = 0 \qquad \text{in } [0,T] \times \Omega\,, \tag{5d}$$

$$(\boldsymbol{v}, \boldsymbol{B})|_{t=0} = (\boldsymbol{v}_0, \boldsymbol{B}_0) \qquad \boldsymbol{x} \in \Omega\,. \tag{5e}$$

The ideal version of system (5) conserves the energy and the magnetic helicity, but at the moment we are unable to find an invariant quantity corresponding to cross helicity.

Once again, we take $\Omega = [0, L]^3 \subset \mathbb{R}^3$ and assume periodic and zero mean initial data, so that the corresponding solutions are space-periodic and of zero spatial mean. Under the aforementioned conditions, we have the following result.

Theorem 1.3 (Strong Global Existence for the Second Model). *As to the initial data, we assume that they satisfy* $\boldsymbol{v}_0 \in \mathrm{L}^2(\Omega)$*,* $\boldsymbol{B}_0 \in \mathrm{H}^1(\Omega)$ *and* $\nabla \cdot \boldsymbol{v}_0 = \nabla \cdot \boldsymbol{B}_0 = 0$*.*

Then, problem (5) *has a unique global solution* $(\boldsymbol{v}, \boldsymbol{B})$ *such that, for each time* $T > 0$*, one has*

$$\boldsymbol{v} \in \mathrm{L}^\infty(0, T; \mathrm{L}^2(\Omega))\,, \qquad \boldsymbol{B} \in \mathrm{L}^\infty(0, T; \mathrm{H}^1(\Omega)) \cap \mathrm{L}^2(0, T; \mathrm{H}^2(\Omega))\,.$$

This result is shown in Section 3.

2. First Model

Here and in the next section, we will make use of the following identities, which hold provided $\nabla \cdot \boldsymbol{f} = 0$:

$$\int (\boldsymbol{f} \cdot \nabla)\boldsymbol{g} \cdot \boldsymbol{h} \,\mathrm{d}\boldsymbol{x} = -\int (\boldsymbol{f} \cdot \nabla)\boldsymbol{h} \cdot \boldsymbol{g} \,\mathrm{d}\boldsymbol{x} \tag{6}$$

$$\int (\boldsymbol{f} \cdot \nabla)\boldsymbol{g} \cdot \boldsymbol{g} \,\mathrm{d}\boldsymbol{x} = 0\,. \tag{7}$$

Moreover, to simplify notations, we set $\|\cdot\| = \|\cdot\|_{\mathrm{L}^2(\Omega)}$.

As to the local existence and uniqueness of a weak solution, let us note that we can restate system (4) in the form

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix}\boldsymbol{u}\\ \boldsymbol{b}\end{pmatrix} = F(\boldsymbol{u},\boldsymbol{b}) \doteq \begin{pmatrix}(1-\alpha^2\Delta)^{-1}[\mathscr{B}(\boldsymbol{b},\boldsymbol{b}) - \mathscr{B}(\boldsymbol{u},\boldsymbol{u})]\\ (1-\beta^2\Delta)^{-1}[\mathscr{B}(\boldsymbol{b},\boldsymbol{u}) - \mathscr{B}(\boldsymbol{u},\boldsymbol{b})]\end{pmatrix}\,, \tag{8}$$

where $\mathscr{B}(\boldsymbol{f},\boldsymbol{g}) = P[\boldsymbol{f} \cdot \nabla\boldsymbol{g}]$, P denoting the Helmholtz–Leray projection over the divergence free functions of L^2. Then $(\boldsymbol{u},\boldsymbol{b})$ will be a weak solution of (4) provided that it is a solution of (8).

We want to prove that the operator F is locally Lipschitz in H^1 equipped with the scalar product

$$\left\langle \begin{pmatrix}\boldsymbol{u}\\ \boldsymbol{b}\end{pmatrix}, \begin{pmatrix}\boldsymbol{w}\\ \boldsymbol{d}\end{pmatrix} \right\rangle = \Big((1-\alpha^2\Delta)^{1/2}\boldsymbol{u}, (1-\alpha^2\Delta)^{1/2}\boldsymbol{w}\Big)_{\mathrm{L}^2} + \Big((1-\beta^2\Delta)^{1/2}\boldsymbol{b}, (1-\beta^2\Delta)^{1/2}\boldsymbol{d}\Big)_{\mathrm{L}^2}\,.$$

We have

$$\begin{aligned} F_1 &\doteq \|\mathscr{B}(\boldsymbol{b}_1,\boldsymbol{b}_1) - \mathscr{B}(\boldsymbol{u}_1,\boldsymbol{u}_1) - \mathscr{B}(\boldsymbol{b}_2,\boldsymbol{b}_2) + \mathscr{B}(\boldsymbol{u}_2,\boldsymbol{u}_2)\|_{\mathrm{H}^{-1}}\\ &= \|\mathscr{B}(\boldsymbol{b}_1,\boldsymbol{b}_1-\boldsymbol{b}_2) + \mathscr{B}(\boldsymbol{b}_1-\boldsymbol{b}_2,\boldsymbol{b}_2) - \mathscr{B}(\boldsymbol{u}_1,\boldsymbol{u}_1-\boldsymbol{u}_2) - \mathscr{B}(\boldsymbol{u}_1-\boldsymbol{u}_2,\boldsymbol{u}_2)\|_{\mathrm{H}^{-1}} \end{aligned}$$

and also

$$\begin{aligned} \|\mathscr{B}(\boldsymbol{f},\boldsymbol{g})\|_{\mathrm{H}^{-1}} &\le \sup_{\|\nabla\boldsymbol{h}\|=1}\left|\int(\boldsymbol{f}\cdot\nabla\boldsymbol{g})\cdot\boldsymbol{h}\right| = \sup_{\|\nabla\boldsymbol{h}\|=1}\left|\int(\boldsymbol{f}\cdot\nabla\boldsymbol{h})\cdot\boldsymbol{g}\right|\\ &\le \|\nabla\boldsymbol{h}\|\,\|\boldsymbol{f}\|_{\mathrm{L}^6}\|\boldsymbol{g}\|_{\mathrm{L}^3} \le C\|\nabla\boldsymbol{f}\|\,\|\boldsymbol{g}\|^{1/2}\|\nabla\boldsymbol{g}\|^{1/2}\\ &\le C\|\nabla\boldsymbol{f}\|\,\|\nabla\boldsymbol{g}\|\,, \end{aligned}$$

having used the Hölder inequality,

$$\|\boldsymbol{f}\|_{\mathrm{L}^6} \le C\|\nabla\boldsymbol{f}\|\,, \qquad \|\boldsymbol{g}\|_{\mathrm{L}^3} \le C\|\boldsymbol{g}\|^{1/2}\|\nabla\boldsymbol{g}\|^{1/2}$$

and the Poincaré inequality. Thus we easily get

$$F_1 \le C(\|\nabla\boldsymbol{u}_1\| + \|\nabla\boldsymbol{u}_2\| + \|\nabla\boldsymbol{b}_1\| + \|\nabla\boldsymbol{b}_2\|)(\|\nabla(\boldsymbol{u}_1-\boldsymbol{u}_2)\| + \|\nabla(\boldsymbol{b}_1-\boldsymbol{b}_2)\|)\,.$$

Similarly,

$$\begin{aligned} F_2 &\doteq \|\mathscr{B}(\boldsymbol{b}_1,\boldsymbol{u}_1)-\mathscr{B}(\boldsymbol{u}_1,\boldsymbol{b}_1)-\mathscr{B}(\boldsymbol{b}_2,\boldsymbol{u}_2)+\mathscr{B}(\boldsymbol{u}_2,\boldsymbol{b}_2)\|_{\mathrm{H}^{-1}} \\ &= \|\mathscr{B}(\boldsymbol{b}_1,\boldsymbol{u}_1-\boldsymbol{u}_2)+\mathscr{B}(\boldsymbol{b}_1-\boldsymbol{b}_2,\boldsymbol{u}_2)-\mathscr{B}(\boldsymbol{u}_1,\boldsymbol{b}_1-\boldsymbol{b}_2)-\mathscr{B}(\boldsymbol{u}_1-\boldsymbol{u}_2,\boldsymbol{b}_2)\|_{\mathrm{H}^{-1}} \\ &\le C(\|\nabla\boldsymbol{u}_1\|+\|\nabla\boldsymbol{u}_2\|+\|\nabla\boldsymbol{b}_1\|+\|\nabla\boldsymbol{b}_2\|)(\|\nabla(\boldsymbol{u}_1-\boldsymbol{u}_2)\|+\|\nabla(\boldsymbol{b}_1-\boldsymbol{b}_2)\|)\,. \end{aligned}$$

Hence F is locally Lipschitz, using that $(1-\alpha^2\Delta)^{-1}$ is an isomorphism from H^{-1} onto H^1, and consequently we get the local existence and uniqueness of a weak solution through the Cauchy–Lipschitz theorem.

Second, in order to get an energy identity, we take the scalar product in H^1 (previously defined) of (8) with $(\boldsymbol{u},\boldsymbol{b})$. Using (7), (6) and integrating by parts when needed, we deduce the energy equality

$$\frac{\mathrm{d}}{\mathrm{d}t}\big(\|\boldsymbol{u}\|^2+\alpha^2\|\nabla\boldsymbol{u}\|^2+\|\boldsymbol{b}\|^2+\beta^2\|\nabla\boldsymbol{b}\|^2\big)=0\,,$$

or

$$\|\boldsymbol{u}\|^2+\alpha^2\|\nabla\boldsymbol{u}\|^2+\|\boldsymbol{b}\|^2+\beta^2\|\nabla\boldsymbol{b}\|^2=C_1\,, \tag{9}$$

where

$$C_1 \doteq \|\boldsymbol{u}_0\|^2+\alpha^2\|\nabla\boldsymbol{u}_0\|^2+\|\boldsymbol{b}_0\|^2+\beta^2\|\nabla\boldsymbol{b}_0\|^2\,.$$

Now, using the bound for the H^1 norm of the solution provided by the energy identity (9), we deduce that such a solution can be extended for all positive time (indeed, the time interval of local existence has a lower bound depending only on the initial data).

Hence, we have the global existence of a unique weak solution

$$\boldsymbol{u},\boldsymbol{b}\in \mathrm{L}^\infty(0,\infty;\mathrm{H}^1(\Omega))\,. \tag{10}$$

This concludes the proof of Theorem 1.1.

In order to prove Theorem 1.2 for strong solutions, we can proceed similarly. We only need an upper bound for higher derivatives. With this aim, we take the scalar product with $\boldsymbol{v}$ and $\boldsymbol{B}$, and integrate over Ω, getting

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\boldsymbol{v}\|^2+\int(\boldsymbol{u}\cdot\nabla)\boldsymbol{u}\cdot\boldsymbol{v}-\int(\boldsymbol{b}\cdot\nabla)\boldsymbol{b}\cdot\boldsymbol{v}=0\,, \tag{11}$$

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\boldsymbol{B}\|^2+\int(\boldsymbol{u}\cdot\nabla)\boldsymbol{b}\cdot\boldsymbol{B}-\int(\boldsymbol{b}\cdot\nabla)\boldsymbol{u}\cdot\boldsymbol{B}=0\,. \tag{12}$$

Using Gagliardo–Nirenberg inequality

$$\|\boldsymbol{u}\|_{\mathrm{L}^\infty}\leqslant C\|\Delta\boldsymbol{u}\|^{3/4}\|\boldsymbol{u}\|^{1/4}+C\|\boldsymbol{u}\| \tag{13}$$

and Poincaré inequality $\|\boldsymbol{u}\|\leqslant C\|\Delta\boldsymbol{u}\|$, we have

$$\|\boldsymbol{u}\|_{\mathrm{L}^\infty}\leqslant C\|\Delta\boldsymbol{u}\|^{3/4}\|\boldsymbol{u}\|^{1/4}\,,$$

and therefore

$$\left|\int(\boldsymbol{u}\cdot\nabla)\boldsymbol{u}\cdot\boldsymbol{v}\right| \leqslant \|\boldsymbol{u}\|_{\mathrm{L}^\infty}\|\nabla\boldsymbol{u}\|\,\|\boldsymbol{v}\| \leqslant C\|\Delta\boldsymbol{u}\|^{3/4}\|\boldsymbol{u}\|^{1/4}\|\nabla\boldsymbol{u}\|\,\|\boldsymbol{v}\| \leqslant C\|\boldsymbol{v}\|^{7/4}\|\boldsymbol{u}\|^{1/4}\|\nabla\boldsymbol{u}\|\,. \tag{14}$$

Proceeding similarly for the other terms in (11) and (12), we deduce

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(\|\boldsymbol{v}\|^2+\|\boldsymbol{B}\|^2) \leqslant & C\big(\|\boldsymbol{u}\|^{1/4}\|\nabla\boldsymbol{u}\|\,\|\boldsymbol{v}\|^{7/4}+\|\boldsymbol{b}\|^{1/4}\|\nabla\boldsymbol{b}\|\,\|\boldsymbol{B}\|^{3/4}\|\boldsymbol{v}\| \\ & +\|\boldsymbol{u}\|^{1/4}\|\nabla\boldsymbol{b}\|\,\|\boldsymbol{v}\|^{3/4}\|\boldsymbol{B}\|+\|\boldsymbol{b}\|^{1/4}\|\nabla\boldsymbol{u}\|\,\|\boldsymbol{B}\|^{7/4}\big) \\ \leqslant & C\big(\|\boldsymbol{v}\|^{7/4}+\|\boldsymbol{B}\|^{3/4}\|\boldsymbol{v}\|+\|\boldsymbol{v}\|^{3/4}\|\boldsymbol{B}\|+\|\boldsymbol{B}\|^{7/4}\big)\,,\end{aligned}$$

having used (10). Applying Young's inequality with exponents 7/3 and 7/4 to the middle terms, we get

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}(1+\|\boldsymbol{v}\|^2+\|\boldsymbol{B}\|^2) \leqslant C(\|\boldsymbol{v}\|^{7/4}+\|\boldsymbol{B}\|^{7/4}) &\leqslant C(\|\boldsymbol{v}\|^2+\|\boldsymbol{B}\|^2)^{7/8} \\ &\leqslant C(1+\|\boldsymbol{v}\|^2+\|\boldsymbol{B}\|^2)\,;\end{aligned}$$

the differential form of Gronwall lemma implies

$$1+\|\boldsymbol{v}(t)\|^2+\|\boldsymbol{B}(t)\|^2 \leqslant (1+\|\boldsymbol{v}_0\|^2+\|\boldsymbol{B}_0\|^2)\,\mathrm{e}^{Ct} \qquad \forall t>0\,,$$

and finally

$$\boldsymbol{v},\boldsymbol{B}\in\mathrm{L}^\infty(0,T;\mathrm{L}^2(\Omega)) \qquad \forall T>0$$

or

$$\boldsymbol{u},\boldsymbol{b}\in\mathrm{L}^\infty(0,T;\mathrm{H}^2(\Omega)) \qquad \forall T>0\,.$$

Remark 2.1. *Let us note that the same estimates hold also in the case* $\Omega=\mathbb{R}^3$*, with no need of periodicity hypotheses. The proof is indeed slightly simplified, since Gagliardo–Nirenberg estimate* (13) *is straightforwardly*

$$\|\boldsymbol{u}\|_{\mathrm{L}^\infty} \leqslant C\|\Delta\boldsymbol{u}\|^{3/4}\|\boldsymbol{u}\|^{1/4}\,.$$

Nevertheless, in this case one needs a different approach to prove local existence.

3. Second Model

First, let us note that local existence can be obtained by a contraction argument proceeding similarly as in [4] (indeed, in this case the technique can be simplified by considering the space $X_{m+2}=\mathrm{L}^\infty(0,T_0;\mathrm{H}^{m+2}(\Omega))$ instead of Θ_{m+2}); therefore, global existence is implied by the global in time a priori estimates provided below.

This approach (to get local and global existence) works for the whole space-domain $\Omega = \mathbb{R}^3$ as well.

We take the scalar product of equation (5a) with $\boldsymbol{u}$ and of equation (5b) with $\boldsymbol{B}$, integrate both equations over Ω and sum them up. Using (7), (6) and integrating by parts when needed, we deduce the energy equality

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\big(\|\boldsymbol{u}\|^2 + \alpha^2\|\nabla\boldsymbol{u}\|^2 + \|\boldsymbol{B}\|^2\big) + \mu\|\nabla\boldsymbol{B}\|^2 = 0\,,$$

or

$$\|\boldsymbol{u}(t)\|^2 + \alpha^2\|\nabla\boldsymbol{u}(t)\|^2 + \|\boldsymbol{B}(t)\|^2 + 2\mu\int_0^t \|\nabla\boldsymbol{B}(s)\|^2\,\mathrm{d}s = C_2\,, \tag{15}$$

where

$$C_2 \doteq \|\boldsymbol{u}_0\|^2 + \alpha^2\|\nabla\boldsymbol{u}_0\|^2 + \|\boldsymbol{B}_0\|^2\,.$$

Hence, provided $\boldsymbol{u}_0 \in \mathrm{H}^1(\Omega)$ and $\boldsymbol{B}_0 \in \mathrm{L}^2(\Omega)$, where $\boldsymbol{u}_0 = (1-\alpha^2\Delta)^{-1}\boldsymbol{v}_0$, we have

$$\boldsymbol{u} \in \mathrm{L}^\infty(0,\infty;\mathrm{H}^1(\Omega))\,, \qquad \boldsymbol{B} \in \mathrm{L}^\infty(0,\infty;\mathrm{L}^2(\Omega))\,. \tag{16}$$

Similarly, taking the scalar product of (5a) with $\boldsymbol{v}$ and integrating in space, we get

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\boldsymbol{v}\|^2 + \int(\boldsymbol{u}\cdot\nabla)\boldsymbol{u}\cdot\boldsymbol{v} - \int(\boldsymbol{B}\cdot\nabla)\boldsymbol{B}\cdot\boldsymbol{v} = 0 \tag{17}$$

while, testing (5b) by $(1-\alpha^2\Delta)\boldsymbol{B}$, we deduce

$$\begin{aligned}\int \boldsymbol{B}_t\cdot(1-\alpha^2\Delta)\boldsymbol{B} &+ \int(\boldsymbol{u}\cdot\nabla)\boldsymbol{B}\cdot(1-\alpha^2\Delta)\boldsymbol{B}\\ &- \int(\boldsymbol{B}\cdot\nabla)\boldsymbol{u}\cdot(1-\alpha^2\Delta)\boldsymbol{B} - \mu\int\Delta\boldsymbol{B}\cdot(1-\alpha^2\Delta)\boldsymbol{B} = 0\,.\end{aligned} \tag{18}$$

Integrating by parts, one easily compute

$$\int \boldsymbol{B}_t\cdot(1-\alpha^2\Delta)\boldsymbol{B} = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\big(\|\boldsymbol{B}\|^2 + \alpha^2\|\nabla\boldsymbol{B}\|^2\big)$$

and

$$-\mu\int\Delta\boldsymbol{B}\cdot(1-\alpha^2\Delta)\boldsymbol{B} = \mu\|\nabla\boldsymbol{B}\|^2 + \alpha^2\mu\|\Delta\boldsymbol{B}\|^2\,.$$

As to the second term, we assume sum from 1 to 3 for repeated indices and get

$$\begin{aligned}\left|\int(\boldsymbol{u}\cdot\nabla)\boldsymbol{B}\cdot(1-\alpha^2\Delta)\boldsymbol{B}\right| &= \alpha^2\left|\int(\boldsymbol{u}\cdot\nabla)\boldsymbol{B}\cdot(-\Delta\boldsymbol{B})\right|\\ &= \alpha^2\left|\int(\partial_h\boldsymbol{u}\cdot\nabla)\boldsymbol{B}\cdot\partial_h\boldsymbol{B} + \int(\boldsymbol{u}\cdot\nabla)\partial_h\boldsymbol{B}\cdot\partial_h\boldsymbol{B}\right|\\ &\leqslant \alpha^2\|\nabla\boldsymbol{u}\|_{\mathrm{L}^4}\|\nabla\boldsymbol{B}\|_{\mathrm{L}^4}\|\nabla\boldsymbol{B}\|\,,\end{aligned}$$

having used twice identity (7).

For the third integral, we use the identity

$$
\begin{aligned}
-\int(\boldsymbol{B}\cdot\nabla)\boldsymbol{u}\cdot(1-\alpha^2\Delta)\boldsymbol{B} &= -\int(1-\alpha^2\Delta)[(\boldsymbol{B}\cdot\nabla)\boldsymbol{u}]\cdot\boldsymbol{B}\\
&= -\int(\boldsymbol{B}\cdot\nabla)\boldsymbol{u}\cdot\boldsymbol{B}+\alpha^2\int B_j\,\partial_j\partial_i^2u_k\,B_k\\
&\quad+\alpha^2\int\partial_i^2B_j\,\partial_ju_k\,B_k+2\alpha^2\int\partial_iB_j\,\partial_{ij}^2u_k\,B_k\\
&= -\int(\boldsymbol{B}\cdot\nabla)\boldsymbol{v}\cdot\boldsymbol{B}-\alpha^2\int\partial_iB_j\,\partial_ju_k\,\partial_iB_k-\alpha^2\int\partial_iB_j\,\partial_iu_k\,\partial_jB_k
\end{aligned}
$$

to deduce the following estimate:

$$
\int(\boldsymbol{B}\cdot\nabla)\boldsymbol{u}\cdot(1-\alpha^2\Delta)\boldsymbol{B}\leqslant\int(\boldsymbol{B}\cdot\nabla)\boldsymbol{v}\cdot\boldsymbol{B}+2\alpha^2\|\nabla\boldsymbol{u}\|_{\mathrm{L}^4}\|\nabla\boldsymbol{B}\|_{\mathrm{L}^4}\|\nabla\boldsymbol{B}\|\,.
$$

Substituting these results in (18) and summing up with (17) yields

$$
\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\big(\|\boldsymbol{v}\|^2+\|\boldsymbol{B}\|^2+\alpha^2\|\nabla\boldsymbol{B}\|^2\big)+\mu\|\nabla\boldsymbol{B}\|^2+\alpha^2\mu\|\Delta\boldsymbol{B}\|^2\\
&\qquad\leqslant\int|(\boldsymbol{u}\cdot\nabla)\boldsymbol{u}\cdot\boldsymbol{v}|+3\alpha^2\|\nabla\boldsymbol{u}\|_{\mathrm{L}^4}\|\nabla\boldsymbol{B}\|_{\mathrm{L}^4}\|\nabla\boldsymbol{B}\|\,.
\end{aligned}
$$

Using Gagliardo–Nirenberg inequality

$$
\|\nabla\boldsymbol{u}\|_{\mathrm{L}^4}\leqslant C\|\Delta\boldsymbol{u}\|^{3/4}\|\nabla\boldsymbol{u}\|^{1/4}+C\|\boldsymbol{u}\|\leqslant C\|\Delta\boldsymbol{u}\|^{3/4}\|\nabla\boldsymbol{u}\|^{1/4}
$$

(thanks to Poincaré inequality) and the elliptic estimate $\|\Delta\boldsymbol{u}\|\leqslant C\|\boldsymbol{v}\|$, we obtain

$$
\begin{aligned}
&\|\nabla\boldsymbol{u}\|_{\mathrm{L}^4}\|\nabla\boldsymbol{B}\|_{\mathrm{L}^4}\|\nabla\boldsymbol{B}\|\\
&\qquad\leqslant C\|\boldsymbol{v}\|^{3/4}\|\nabla\boldsymbol{u}\|^{1/4}\|\Delta\boldsymbol{B}\|^{3/4}\|\nabla\boldsymbol{B}\|^{5/4}\\
&\qquad\leqslant\varepsilon(\|\Delta\boldsymbol{B}\|^{3/4})^{8/3}+C(\|\boldsymbol{v}\|^{3/4}\|\nabla\boldsymbol{u}\|^{1/4}\|\nabla\boldsymbol{B}\|^{5/4})^{8/5}\\
&\qquad=\varepsilon\|\Delta\boldsymbol{B}\|^2+C\|\nabla\boldsymbol{u}\|^{2/5}\|\nabla\boldsymbol{B}\|^{6/5}\|\boldsymbol{v}\|^{6/5}\|\nabla\boldsymbol{B}\|^{4/5}\,,
\end{aligned}
$$

where $\varepsilon>0$ is a small parameter. Combining with estimate (14) and exploiting

$$
\|\boldsymbol{u}\|\leqslant\|\boldsymbol{v}\|+\alpha^2\|\Delta\boldsymbol{u}\|\leqslant C\|\boldsymbol{v}\|\,,
$$

we deduce, for $\varepsilon=\alpha^2\mu/2$,

$$
\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\big(1+\|\boldsymbol{v}\|^2+\|\boldsymbol{B}\|^2+\alpha^2\|\nabla\boldsymbol{B}\|^2\big)+\mu\|\nabla\boldsymbol{B}\|^2+\frac{\alpha^2\mu}{2}\|\Delta\boldsymbol{B}\|^2\\
&\qquad\leqslant C\big(\|\nabla\boldsymbol{u}\|+\|\nabla\boldsymbol{u}\|^{2/5}\|\nabla\boldsymbol{B}\|^{6/5}\big)\big(1+\|\boldsymbol{v}\|^2+\|\boldsymbol{B}\|^2+\alpha^2\|\nabla\boldsymbol{B}\|^2\big)\,.
\end{aligned}
\tag{19}
$$

Setting $y(t) = 1 + \|\boldsymbol{v}(t)\|^2 + \|\boldsymbol{B}(t)\|^2 + \alpha^2\|\nabla\boldsymbol{B}(t)\|^2$, from Gronwall lemma we get

$$y(t) \leqslant y(0) \exp C \int_0^t \left(\|\nabla\boldsymbol{u}(\tau)\| + \|\nabla\boldsymbol{u}(\tau)\|^{2/5}\|\nabla\boldsymbol{B}(\tau)\|^{6/5}\right) \mathrm{d}\tau \,.$$

Now, recalling that $\|\nabla\boldsymbol{u}\| \in \mathrm{L}^\infty_t([0,\infty[)$ and

$$(\|\nabla\boldsymbol{B}\|^{6/5})^{5/3} = \|\nabla\boldsymbol{B}\|^2 \in \mathrm{L}^1_t([0,\infty[) \,,$$

as follows immediately from the energy identity (15), we have

$$\begin{aligned}&\int_0^t \left(\|\nabla\boldsymbol{u}(\tau)\| + \|\nabla\boldsymbol{u}(\tau)\|^{2/5}\|\nabla\boldsymbol{B}(\tau)\|^{6/5}\right) \mathrm{d}\tau \\ &\qquad \leqslant Ct + C\left(\int_0^t \mathrm{d}\tau\right)^{2/5} \left(\int_0^t \|\nabla\boldsymbol{B}\|^2 \,\mathrm{d}\tau\right)^{3/5} \\ &\qquad \leqslant C(t+1) \,.\end{aligned}$$

Therefore, we conclude

$$\boldsymbol{v} \in \mathrm{L}^\infty(0,T;\mathrm{L}^2(\Omega)) \,, \qquad \boldsymbol{B} \in \mathrm{L}^\infty(0,T;\mathrm{H}^1(\Omega)) \qquad \forall T > 0 \,.$$

Let us note in particular that

$$\boldsymbol{u} \in \mathrm{L}^\infty(0,T;\mathrm{H}^2(\Omega)) \qquad \forall T > 0 \,.$$

Moreover, integrating (19) in time shows that

$$\boldsymbol{B} \in \mathrm{L}^2(0,T;\mathrm{H}^2(\Omega)) \qquad \forall T > 0 \,.$$

This concludes the proof of Theorem 1.3.

After completing this work, the authors were informed of the paper by Larios–Titi [11] containing some results related to the first model.

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