Unitals in PG(2, q^2) with a large 2-point stabiliser

L. Giuzzi [a,*], G. Korchmáros [b]

[a] Universita' degli studi di Brescia, Dipartimento di Matematica, Facolta' di Ingegneria, Via Valotti 9, 25133 Brescia, Italy

[b] Universita' degli Studi della Basilicata, Dipartimento di Matematica e Informatica, Campus di Macchia Romana, Via dell'Ateneo Lucano 10, I-85100 Potenza, Italy

ARTICLE INFO	ABSTRACT
Article history: Available online 11 April 2011 *Keywords:* Unital Point-stabiliser Projective plane	Let $\mathcal{U}$ be a unital embedded in the Desarguesian projective plane PG(2, q^2). Write M for the subgroup of PGL(3, q^2) which preserves $\mathcal{U}$. We show that $\mathcal{U}$ is classical if and only if $\mathcal{U}$ has two distinct points P, Q for which the stabiliser $G = M_{P,Q}$ has order $q^2 - 1$. © 2011 Elsevier B.V. All rights reserved.

1. Introduction

In the Desarguesian projective plane PG(2, q^2), a *unital* is defined to be a set of $q^3 + 1$ points containing either 1 or $q + 1$ points from each line of PG(2, q^2). Observe that each unital has a unique 1-secant at each of its points. The idea of a unital arises from the combinatorial properties of the non-degenerate unitary polarity π of PG(2, q^2). The set of absolute points of π is indeed a unital, called the *classical* or *Hermitian* unital. Therefore, the projective group preserving the classical unital is isomorphic to PGU(3, q) and acts on its points as PGU(3, q) in its natural 2-transitive permutation representation. Using the classification of subgroups of PGL(3, q^2), Hoffer [14] proved that a unital is classical if and only if it is preserved by a collineation group isomorphic to PSU(3, q^2). Hoffer's characterisation has been the starting point for several investigations of unitals in terms of the structure of their automorphism group, see [3,6,4,5,8,9,11,10,12,15,16]; see also the survey [2, Appendix B]. In PG(2, q^2) with q odd, Abatangelo [1] proved that a Buekenhout–Metz unital with a cyclic 2-point stabiliser of order $q^2 - 1$ is necessarily classical. In their talk at Combinatorics 2010, G. Donati e N. Durante have conjectured that Abatangelo's characterisation holds true for any unital in PG(2, q^2). In this note, we provide a proof of this conjecture.

Our notation and terminology are standard, see [2,13]. We shall assume $q > 2$, since all unitals in PG(2, 4) are classical.

2. Some technical lemmas

Let M be the subgroup of PGL(3, q^2) which preserves a unital $\mathcal{U}$ in PG(2, q^2). A *2-point stabiliser* of $\mathcal{U}$ is a subgroup of M which fixes two distinct points of $\mathcal{U}$.

Lemma 2.1. *Let $\mathcal{U}$ be a unital in* PG(2, q^2) *with a 2-point stabiliser G of order $q^2 - 1$. Then, G is cyclic, and there exists a projective frame in* PG(2, q^2) *such that G is generated by a projectivity with matrix representation*

$$\begin{pmatrix} \lambda & 0 & 0 \\ 0 & \mu & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

where λ is a primitive element of GF(q^2) *and μ is a primitive element of* GF(q).

* Corresponding author. Tel.: +39 030 3715739; fax: +39 030 3715745.

E-mail addresses: giuzzi@ing.unibs.it (L. Giuzzi), korchmaros@unibas.it (G. Korchmáros).

Proof. Let O, Y_∞ be two distinct points of $\mathcal{U}$ such that the stabiliser $G = M_{O,Y_\infty}$ has order $q^2 - 1$. Choose a projective frame in PG$(2, q^2)$ so that $O = (0, 0, 1)$, $Y_\infty = (0, 1, 0)$ and the 1-secants of $\mathcal{U}$ at those points are respectively $\ell_X : X_2 = 0$ and $\ell_\infty : X_3 = 0$. Write $X_\infty = (1, 0, 0)$ for the common point of ℓ_X and ℓ_∞. Observe that G fixes the vertices of the triangle $OX_\infty Y_\infty$. Therefore, G consists of projectivities with diagonal matrix representation. Now let $h \in G$ be a projectivity that fixes a further point $P \in \ell_X$ apart from O, X_∞. Then, h fixes ℓ_X point-wise; that is, h is a perspectivity with axis ℓ_X. Since h also fixes Y_∞, the centre of h must be Y_∞. Take any point $R \in \ell_X$ with $R \neq O$, X_∞. Obviously, h preserves the line $r = Y_\infty R$; hence, it also preserves $r \cap \mathcal{U}$. Since $r \cap \mathcal{U}$ comprises q points other than R, the subgroup H generated by h has a permutation representation of degree q in which no non-trivial permutation fixes a point. As $q = p^r$ for a prime p, this implies that p divides $|H|$. On the other hand, h is taken from a group of order $q^2 - 1$. Thus, h must be the trivial element in G. Therefore, G has a faithful action on ℓ_X as a 2-point stabiliser of PG$(1, q^2)$. This proves that G is cyclic. Furthermore, a generator g of G has a matrix representation

$$\begin{pmatrix} \lambda & 0 & 0 \\ 0 & \mu & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

with λ a primitive element of GF(q^2).

As G preserves the set $\Delta = \mathcal{U} \cap OY_\infty$, it also induces a permutation group $\bar{G}$ on Δ. Since any projectivity fixing three points of OY_∞ must fix OY_∞ point-wise, $\bar{G}$ is semiregular on Δ. Therefore, $|\bar{G}|$ divides $q - 1$. Now let F be the subgroup of G fixing Δ point-wise. Then, F is a perspectivity group with centre X_∞ and axis $\ell_Y : X_1 = 0$. Take any point $R \in \ell_Y$ such that the line $r = RX_\infty$ is a $(q + 1)$-secant of $\mathcal{U}$. Then, $r \cap \mathcal{U}$ is disjoint from ℓ_Y. Hence, F has a permutation representation on $r \cap \mathcal{U}$ in which no non-trivial permutation fixes a point. Thus, $|F|$ divides $q + 1$. Since $|G| = q^2 - 1$, we have $|\bar{G}| \leq q - 1$ and $|G| = |\bar{G}||F|$. This implies $|\bar{G}| = q - 1$ and $|F| = q + 1$. From the former condition, μ must be a primitive element of GF(q). $\quad\square$

Lemma 2.2. *In* PG$(2, q^2)$*, let* $\mathcal{H}_1$ *and* $\mathcal{H}_2$ *be two non-degenerate Hermitian curves which have the same tangent at a common point* P*. Denote by* $I(P, \mathcal{H}_1 \cap \mathcal{H}_2)$ *the intersection multiplicity of* $\mathcal{H}_1$ *and* $\mathcal{H}_2$ *at* P*. Then,*

$$I(P, \mathcal{H}_1 \cap \mathcal{H}_2) = q + 1. \tag{1}$$

Proof. Since up to projectivities there is a unique class of Hermitian curves in PG$(2, q^2)$, we may assume $\mathcal{H}_1$ to have equation $-X_1^{q+1} + X_2^q X_3 + X_2 X_3^q = 0$. Furthermore, as the projectivity group PGU$(3, q)$ preserving $\mathcal{H}_1$ acts transitively on the points of $\mathcal{H}_1$ in PG$(2, q^2)$, we may also suppose $P = (0, 0, 1)$. Within this setting, the tangent r of $\mathcal{H}_1$ at P coincides with the line $X_2 = 0$. As no term X_1^j with $0 < j \leq q$ occurs in the equation of $\mathcal{H}_1$, the intersection multiplicity $I(P, \mathcal{H}_1 \cap r)$ is equal to $q + 1$.

The equation of the other Hermitian curve $\mathcal{H}_2$ might be written as

$$F(X_1, X_2, X_3) = a_0 X_3^q X_2 + a_1 X_3^{q-1} G_1(X_1, X_2) + \cdots + a_q G_q(X_1, X_2) = 0,$$

where $a_0 \neq 0$ and deg $G_i(X_1, X_2) = i + 1$. Since the tangent of $\mathcal{H}_2$ at P has no other common point with $\mathcal{H}_2$, even over the algebraic closure of GF(q^2), no terms X_1^j with $0 < j \leq q$ can occur in the polynomials $G_i(X_1, X_2)$. In other words, $I(P, \mathcal{H}_2 \cap r) = q + 1$.

A primitive representation of the unique branch of $\mathcal{H}_1$ centred at P has components

$$x(t) = t, \qquad y(t) = ct^i + \cdots,$$

where i is a positive integer and $y(t) \in$ GF$(q^2)[[t]]$, that is, $y(t)$ stands for a formal power series with coefficients in GF(q^2). From $I(P, \mathcal{H}_1 \cap r) = q + 1$,

$$y(t)^q + y(t) - t^{q+1} = 0,$$

whence $y(t) = t^{q+1} + H(t)$, where $H(t)$ is a formal power series of order at least $q + 2$. That is, the exponent j in the leading term ct^j of $H(t)$ is larger than $q + 1$.

It is now possible to compute the intersection multiplicity $I(P, \mathcal{H}_1 \cap \mathcal{H}_2)$ using [13, Theorem 4.36]:

$$I(P, \mathcal{H}_1 \cap \mathcal{H}_2) = \mathrm{ord}_t F(t, y(t), 1) = \mathrm{ord}_t (a_0 t^{q+1} + G(t)),$$

with $G(t) \in$ GF$(q^2)[[t]]$ of order at least $q + 2$. From this, the assertion follows. $\quad\square$

Lemma 2.3. *In* PG$(2, q^2)$*, let* $\mathcal{H}$ *be a non-degenerate Hermitian curve and let* $\mathcal{C}$ *be a Hermitian cone whose centre does not lie on* $\mathcal{H}$*. Assume that there exist two points* $P_i \in \mathcal{H} \cap \mathcal{C}$*, with* $i = 1, 2$*, such that the tangent line of* $\mathcal{H}$ *at* P_i *is a linear component of* $\mathcal{C}$*. Then*

$$I(P_1, \mathcal{H} \cap \mathcal{C}) = q + 1. \tag{2}$$

Proof. We use the same setting as in the proof of Lemma 2.2 with $P = P_1$. Since the action of PGU(3, q) is 2-transitive on the points of $\mathcal{H}$, we may also suppose that $P_2 = (0, 1, 0)$. Then the centre of $\mathcal{C}$ is the point $X_\infty = (1, 0, 0)$, and $\mathcal{C}$ has equation $c^q X_2^q X_3 + c X_2 X_3^q = 0$ with $c \neq 0$. Therefore,

$$I(P, \mathcal{H} \cap \mathcal{C}) = \text{ord}_t(c^q y(t)^q + cy(t)) = \text{ord}_t(c^q t^{q+1} + K(t))$$

with $K(t) \in \text{GF}(q^2)[[t]]$ of order at least $q + 2$, whence the assertion follows. $\quad\square$

3. Main result

Theorem 3.1. *In* PG(2, q^2)*, let $\mathcal{U}$ be a unital and write M for the group of projectivities which preserves $\mathcal{U}$. If $\mathcal{U}$ has two distinct points P, Q such that the stabiliser $G = M_{P,Q}$ has order $q^2 - 1$, then $\mathcal{U}$ is classical.*

The main idea of the proof is to build up a projective plane of order q using, for the definition of points, non-trivial G-orbits in the affine plane AG(2, q^2) which arise from PG(2, q^2) by removing the line $\ell_\infty : X_3 = 0$ with all its points. For this purpose, take $\mathcal{U}$ and G as in Lemma 2.1, with $\mu = \lambda^{q+1}$, and define an incidence structure $\Pi = (\mathcal{P}, \mathcal{L})$ as follows.

1. Points are all non-trivial G-orbits in AG(2, q^2).
2. Lines are ℓ_Y, and the non-degenerate Hermitian curves of equation

$$\mathcal{H}_b : -X_1^{q+1} + bX_3 X_2^q + b^q X_3^q X_2 = 0, \tag{3}$$

with b ranging over $\text{GF}(q^2)^*$, together with the Hermitian cones of equation

$$\mathcal{C}_c : c^q X_2^q X_3 + c X_2 X_3^q = 0, \tag{4}$$

with c ranging over a representative system of cosets of $(\text{GF}(q), *)$ in $(\text{GF}(q^2), *)$.
3. Incidence is the natural inclusion.

Lemma 3.2. *The incidence structure $\Pi = (\mathcal{P}, \mathcal{L})$ is a projective plane of order q.*

Proof. In AG(2, q^2), the group G has $q^2 + q + 1$ non-trivial orbits, namely its q^2 orbits disjoint from ℓ_Y, each of length $q^2 - 1$, and its $q + 1$ orbits on ℓ_Y, these of length $q - 1$. Therefore, the total number of points in $\mathcal{P}$ is equal to $q^2 + q + 1$. By construction of Π, the number of lines in $\mathcal{L}$ is also $q^2 + q + 1$. Incidence is well defined as G preserves ℓ_Y and each Hermitian curve and cone representing lines of $\mathcal{L}$.

Now we count the points incident with a line in Π. Each G-orbit on ℓ_Y distinct from O and Y_∞ has length $q - 1$. Hence there are exactly $q + 1$ such G-orbits; in terms of Π, the line represented by ℓ_Y is incident with $q + 1$ points. A Hermitian curve $\mathcal{H}_b$ of Eq. (3) has q^3 points in AG(2, q^2) and meets ℓ_Y in a G-orbit, while it contains no points apart from O of line ℓ_X. As $q^3 - q = q(q^2 - 1)$, the line represented by $\mathcal{H}_b$ is incident with $q + 1$ points in $\mathcal{P}$. Finally, a Hermitian cone $\mathcal{C}_c$ of Eq. (4) has q^3 points in AG(2, q^2) and contains q points from ℓ_Y. One of these q points is O, the other $q - 1$ forming a non-trivial G-orbit. The remaining $q^3 - q$ points of $\mathcal{C}_c$ are partitioned into q distinct G-orbits. Hence, the line represented by $\mathcal{C}_c$ is also incident with $q + 1$ points. This shows that each line in Π is incident with exactly $q + 1$ points.

Therefore, it is enough to show that any two distinct lines of $\mathcal{L}$ have exactly one common point. Obviously this is true when one of these lines is represented by ℓ_Y. Furthermore, the point of $\mathcal{P}$ represented by ℓ_X is incident with each line of $\mathcal{L}$ represented by a Hermitian cone of Eq. (4). We are led to investigate the case where one of the lines of $\mathcal{L}$ is represented by a Hermitian curve $\mathcal{H}_b$ of Eq. (4), and the other line of $\mathcal{L}$ is represented by a Hermitian curve $\mathcal{H}$ which is either another Hermitian curve $\mathcal{H}_d$ of the same type of Eq. (3), or a Hermitian cone $\mathcal{C}_c$ of Eq. (4).

Clearly, both O and Y_∞ are common points of $\mathcal{H}_b$ and $\mathcal{H}$. From Kestenband's classification [17], see also [2, Theorem 6.7], $\mathcal{H}_b \cap \mathcal{H}$ cannot consist of exactly two points. Therefore, there exists another point, say $P \in \mathcal{H}_b \cap \mathcal{H}$. Since ℓ_X and ℓ_∞ are 1-secants of $\mathcal{H}_b$ at the points O and Y_∞, respectively, either P is on ℓ_Y or P lies outside the fundamental triangle. In the latter case, the G-orbit Δ_1 of P has size $q^2 - 1$ and represents a point in $\mathcal{P}$. Assume that $\mathcal{H}_b \cap \mathcal{H}$ contains a further point $Q \notin \Delta_1$ which does not belong to ℓ_Y and denote by Δ_2 its G-orbit. Then,

$$|\mathcal{H}_b \cap \mathcal{H}| \geq |\Delta_1| + |\Delta_2| = 2(q^2 - 1) + 2 = 2q^2.$$

However, from Bézout's theorem, see [13, Theorem 3.14],

$$|\mathcal{H}_b \cap \mathcal{H}| \leq (q + 1)^2.$$

Therefore, $Q \in \ell_Y$, and its G-orbit Δ_3 has length $q - 1$. Hence, $\mathcal{H}_b$ and $\mathcal{H}$ share $q + 1$ points on ℓ_Y. If $\mathcal{H} = \mathcal{H}_d$ is a Hermitian curve of Eq. (3), each of these $q + 1$ points is the tangency point of a common inflection tangent with multiplicity $q + 1$ of the Hermitian curves $\mathcal{H}_b$ and $\mathcal{H}$. Write $R_1, \ldots, R_{q+1}$ for these points. Then, by (1) the intersection multiplicity is $I(R_i, \mathcal{H}_b \cap \mathcal{H}_d) = q + 1$. This holds true also when $\mathcal{H}$ is a Hermitian cone $\mathcal{C}_c$ of Eq. (4); see Lemma 2.3. Therefore, in any case,

$$\sum_{i=1}^{q+1} I(R_i, \mathcal{H}_b \cap \mathcal{H}) = (q + 1)^2.$$

From Bézout's theorem, $\mathcal{H}_b \cap \mathcal{H} = \{R_1, \ldots, R_{q+1}\}$. Therefore, $\mathcal{H}_b \cap \mathcal{H} = \Delta_3 \cup \{O, Y_\infty\}$. This shows that if $Q \notin \ell_Y$, the lines represented by $\mathcal{H}_b$ and $\mathcal{H}$ have exactly one point in common. The above argument can also be adapted to prove this assertion in the case where $Q \in \ell_Y$. Therefore, any two distinct lines of $\mathcal{L}$ have exactly one common point. $\quad\square$

Proof of Theorem 3.1. Assume first $\mu = \lambda^{q+1}$. Construct a projective plane Π as in Lemma 3.2. Since $\mathcal{U} \setminus \{O, Y_\infty\}$ is the union of G-orbits, $\mathcal{U}$ represents a set Γ of $q + 1$ points in Π. From [7], $N \equiv 1(\mathrm{mod}\ p)$ where N is the number of common points of $\mathcal{U}$ with any Hermitian curve $\mathcal{H}_b$. In terms of Π, Γ contains some point from every line Λ in $\mathcal{L}$ represented by a Hermitian curve of Eq. (3). Actually, this holds true when the line Λ in $\mathcal{L}$ is represented by a Hermitian cone $\mathcal{C}$ of Eq. (4). To prove it, observe that $\mathcal{C}$ contains a line r distinct from both lines ℓ_X and ℓ_∞. Then $r \cap \mathcal{U}$ is not empty and contains neither O nor Y_∞. If P is a point in $r \cap \mathcal{U}$, then the G-orbit of P represents a common point of Γ and Λ. Since the line in $\mathcal{L}$ represented by ℓ_Y meets Γ, it turns out that Γ contains some point from every line in $\mathcal{L}$.

Therefore, Γ itself is a line in $\mathcal{L}$. Note that $\mathcal{U}$ contains no line. In terms of $PG(2, q^2)$, this yields that $\mathcal{U}$ coincides with a Hermitian curve of Eq. (3). In particular, $\mathcal{U}$ is a classical unital.

To investigate the case $\mu \neq \lambda^{q+1}$, we still work in the above plane Π. By a straightforward computation, the projectivity g given in Lemma 2.1 induces a non-trivial collineation on Π. Also, g preserves every Hermitian cone of Eq. (4) and the common line ℓ_X of these Hermitian cones. In terms of Π, $\bar{g}$ is a perspectivity with centre at the point represented by ℓ_X. Since g also preserves the line ℓ_Y, the axis of $\bar{g}$ is ℓ_Y, regarded as a line in Π. Therefore, every point of Π lying on ℓ_Y is fixed by g. Consequently, $\bar{g}^{q-1}$ is the identity collineation. As g has order $q^2 - 1$, this yields that g^{q+1} preserves every Hermitian curve of Eq. (3). Thus, $\mu^{q+1} = (\lambda^{q+1})^{q+1}$, whence $\mu = -\lambda^{q+1}$. In particular, $p \neq 2$.

Consider now the $q + 1$ non-trivial G-orbits in $\mathcal{U}$ with $G = \langle g \rangle$. For any point $P \in \Pi$, let n_P the number of the non-trivial G-orbits in $\mathcal{U}$ intersecting the set $\rho(P)$ representing P in $PG(2, q^2)$. Then $n_P = 1$ when $\rho(P)$ is the unique G-orbit in $\mathcal{U}$ which lies on ℓ_Y. Otherwise, $0 \leq n_P \leq 2$, with $n_P = 2$ if and only if $\rho(P)$ is not a G-orbit but the union of two H-orbits with $H = \langle g^2 \rangle$.

Let Γ be the multiset in Π consisting of all points with $n_P > 0$ and define the weight ν_P of P to be either 1 or 2, according as $n_P = 2$ or $n_P = 1$. Then, $\sum_{P \in \Gamma} \nu_P = 2q + 2$. We show that Γ is a 2-fold blocking multiset of Π. For this purpose, let $\mathcal{H}$ be either a Hermitian curve of Eq. (3) or a Hermitian cone of Eq. (4). Write m for the number of common points of $\mathcal{H}_b$ and $\mathcal{U}$, different from O and Y_∞; thus, the total number of common points is $N = m + 2$. As $N \equiv 1(\mathrm{mod}\ p)$, we have $m \geq 1$. Take $P \in \mathcal{H} \cap \mathcal{U}$. If $\nu_P = 2$, then the line representing $\mathcal{H}$ meets Γ in a point with weight 2. If $\nu_P = 1$, then the H-orbit of P has size $(q^2 - 1)/2$ and lies on both $\mathcal{H}$ and $\mathcal{U}$. Since $(q^2 - 1)/2 + 2 \not\equiv 1(\mathrm{mod}\ p)$, $\mathcal{H}$ and $\mathcal{U}$ must share a further point Q other than O and Y_∞. Therefore, the points P' and Q' of Π which represent the subsets containing P and Q are distinct. This shows that Γ meets the line represented by $\mathcal{H}$ in two distinct points. Therefore, Γ is a 2-fold blocking multiset.

Since Γ has at least one point with weight 2, this yields that Γ comprises all points of a line, each with weight 2. Hence, $\mathcal{U}$ coincides with either a Hermitian curve or a Hermitian cone. On the other hand, $\mathcal{U}$ is definitely not a Hermitian cone. As $\mu \neq \lambda^{q+1}$, $\mathcal{U}$ is neither a Hermitian curve; therefore, this case cannot actually occur. $\quad\square$

References

[1] L.M. Abatangelo, Una caratterizzazione gruppale delle curve Hermitiane, Matematiche 39 (1984) 101–110.
[2] S.G. Barwick, G.L. Ebert, Unitals in Projective Planes, in: Springer Monographs in Mathematics, 2008.
[3] L.M. Batten, Blocking sets with flag transitive collineation groups, Arch. Math. 56 (1991) 412–416.
[4] M. Biliotti, G. Korchmáros, Collineation groups preserving a unital of a projective plane of odd order, J. Algebra 122 (1989) 130–149.
[5] M. Biliotti, G. Korchmáros, Collineation groups preserving a unital of a projective plane of even order, Geom. Dedicata 31 (1989) 333–344.
[6] P. Biscarini, Hermitian arcs of PG(2, q^2) with a transitive collineation group on the set of ($q + 1$)-secants, Rend. Sem. Mat. Brescia 7 (1982) 111–124.
[7] A. Blokhuis, A. Brouwer, H. Wilbrink, Hermitian unitals are codewords, Discrete Math. 97 (1991) 63–68.
[8] A. Cossidente, G.L. Ebert, G. Korchmáros, A group-theoretic characterization of classical unitals, Arch. Math. 74 (2000) 1–5.
[9] A. Cossidente, G.L. Ebert, G. Korchmáros, Unitals in finite Desarguesian planes, J. Algebraic Combin. 14 (2001) 119–125.
[10] J. Doyen, Designs and automorphism groups, in: Surveys in Combinatorics, in: London Math. Soc. Lecture Note Ser., vol. 141, 1989, pp. 74–83.
[11] G.L. Ebert, K. Wantz, A group-theoretic characterization of Buekenhout–Metz unitals, J. Combin. Des. 4 (1996) 143–152.
[12] L. Giuzzi, A characterisation of classical unitals, J. Geom. 74 (2002) 86–89.
[13] J.W.P. Hirschfeld, G. Korchmáros, F. Torres, Algebraic Curves Over a Finite Field, Princeton Univ. Press, Princeton and Oxford, 2008, xx+696 pp.
[14] A.R. Hoffer, On unitary collineation groups, J. Algebra 22 (1972) 211–218.
[15] W.M. Kantor, On unitary polarities of finite projective planes, Canad. J. Math. 23 (1971) 1060–1077.
[16] W.M. Kantor, Homogeneous designs and geometric lattices, J. Combin. Theory Ser. A 38 (1985) 66–74.
[17] B.C. Kestenband, Unital intersections in finite projective planes, Geom. Dedicata 11 (1) (1981) 107–117.