IOP PUBLISHING NONLINEARITY

Nonlinearity **21** (2008) 713–733 doi:10.1088/0951-7715/21/4/004

On the extensible viscoelastic beam

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Received 19 January 2008
Published 28 February 2008
Online at stacks.iop.org/Non/21/713

Recommended by E S Titi

Abstract

This work is focused on the equation

$$\partial_{tt}u + \partial_{xxxx}u + \int_0^\infty \mu(s)\partial_{xxxx}[u(t) - u(t-s)]\,\mathrm{d}s - \big(\beta + \|\partial_x u\|^2_{L^2(0,1)}\big)\partial_{xx}u = f$$

describing the motion of an extensible viscoelastic beam. Under suitable boundary conditions, the related dynamical system in the history space framework is shown to possess a global attractor of optimal regularity. The result is obtained by exploiting an appropriate decomposition of the solution semigroup, together with the existence of a Lyapunov functional.

Mathematics Subject Classification: 35B41, 35B65, 37B25, 45K05, 74K10

1. Introduction

1.1. The model equation

In this paper, we investigate the longtime behaviour of a nonlinear evolution problem describing the vibrations of an extensible viscoelastic beam. The full model is derived in the appendix, by combining the pioneering ideas of Woinowsky-Krieger [30] with the theory of Drozdov and Kolmanovskii [11]. Setting for simplicity $\alpha = \gamma = L = 1$ in (A.4) (see the appendix), we have, for $t > 0$,

$$\partial_{tt}u + \partial_{xxxx}u + \int_0^\infty \mu(s)\partial_{xxxx}[u(t) - u(t-s)]\,\mathrm{d}s - \big(\beta + \|\partial_x u\|^2_{L^2(0,1)}\big)\partial_{xx}u = f, \tag{1.1}$$

in the variable $u = u(x,t) : [0,1] \times \mathbb{R} \to \mathbb{R}$, accounting for the vertical deflection of the beam with respect to its reference configuration. Here, $f = f(x)$ is the lateral (static) load distribution, while the *memory kernel* μ is a nonnegative absolutely continuous function on $\mathbb{R}^+ = (0, \infty)$ (hence, differentiable almost everywhere) such that

$$\mu'(s) + \delta\mu(s) \leqslant 0, \qquad \int_0^\infty \mu(s)\,\mathrm{d}s = \kappa, \tag{1.2}$$

for some $\delta > 0$ and $\kappa > 0$. In particular, μ has an exponential decay of rate δ at infinity. The real constant β represents the axial force acting in the reference configuration. Namely, β is positive when the beam is stretched, negative when compressed. The *past history* of u (which need not fulfil the equation for negative times) is assumed to be known. Hence, the initial condition reads

$$u(x, t) = u_\star(x, t), \qquad (x, t) \in [0, 1] \times (-\infty, 0], \tag{1.3}$$

where $u_\star : [0, 1] \times (-\infty, 0] \to \mathbb{R}$ is a given function.

1.2. Boundary conditions

The choice of the boundary conditions for (1.1) deserves a brief discussion. In this work, we consider the case when both ends of the beam are hinged; namely, for every $t \in \mathbb{R}$, we assume

$$u(0, t) = u(1, t) = \partial_{xx}u(0, t) = \partial_{xx}u(1, t) = 0. \tag{1.4}$$

However, other types of boundary conditions with fixed ends are possible as well; for instance

$$u(0, t) = u(1, t) = \partial_x u(0, t) = \partial_x u(1, t) = 0, \tag{1.5}$$

when both ends are clamped, or

$$u(0, t) = u(1, t) = \partial_x u(0, t) = \partial_{xx} u(1, t) = 0, \tag{1.6}$$

when one end is clamped and the other one is hinged. In contrast, the so-called cantilever boundary condition, obtained by keeping one end clamped and leaving the other one free, is not consistent with the extensibility assumption, since, in that case, no geometric constraints compel the beam length to change.

In general, the differential operator ∂_{xxxx} acting on $L^2(0, 1)$ has different properties depending on the choice of the boundary conditions. Assuming (1.4), its domain is

$$\mathcal{D}(\partial_{xxxx}) = \{w \in H^4(0, 1) : w(0) = w(1) = w''(0) = w''(1) = 0\}.$$

This operator is strictly positive selfadjoint with compact inverse, and its discrete spectrum is given by $\lambda_n = n^4\pi^4$, $n \in \mathbb{N}$. Thus, $\lambda_1 = \pi^4$ is the smallest eigenvalue. Besides, the peculiar relation

$$(\partial_{xxxx})^{1/2} = -\partial_{xx} \tag{1.7}$$

holds true, with

$$\mathcal{D}(-\partial_{xx}) = H^2(0, 1) \cap H_0^1(0, 1).$$

The operator ∂_{xxxx} is still strictly positive selfadjoint with compact inverse if we consider (1.5) or (1.6), obviously, with different domains and different spectra. In particular, $\lambda_1 \simeq 5.06\,\pi^4$ in the first case, whereas $\lambda_1 \simeq 2.44\,\pi^4$ in the second one. As a consequence, in both situations the square root $(\partial_{xxxx})^{1/2}$ is well defined. Nevertheless, the *local* relation (1.7) is no longer true. This difference is crucial when recasting (1.1) into an abstract setting. Hence, if hinged ends are considered, the boundary value problem (1.1), (1.4) can be described by means of a single operator $A = \partial_{xxxx}$, which enters the equation at the powers 1 and 1/2. Conversely, if (1.5) or (1.6) is taken into account, one has to deal with two different operators, $A = \partial_{xxxx}$ and $B = -\partial_{xx}$, which do not commute. This fact is particularly relevant in the analysis of the *critical buckling load* ν_c, that is, the magnitude of the compressive axial force ($\beta < 0$) at which buckled stationary states appear. Indeed (cf [3]), such buckled stationary states occur when

$$-\beta < \nu_c = \min_{w \in \mathcal{D}(A)} \frac{\|w''\|^2}{\|w'\|^2},$$

where $\|\cdot\|$ is the standard norm on $L^2(0,1)$. Since (1.7) holds when (1.4) is assumed, this minimum value is directly related to λ_1; precisely,

$$\nu_c = \sqrt{\lambda_1} = \pi^2.$$

This is not the case under different boundary conditions. For instance, for (1.5), we have

$$\nu_c = 4\pi^2 > \sqrt{\lambda_1} \simeq 2.25\pi^2,$$

whereas, for (1.6),

$$\nu_c \simeq 2.04\pi^2 > \sqrt{\lambda_1} \simeq 1.56\pi^2.$$

Although different boundary conditions lead to different abstract settings, the techniques of this paper can be successfully adapted assuming (1.5) or (1.6) in place of (1.4), and the results proved here still hold provided that $\sqrt{\lambda_1}$ is replaced by ν_c.

1.3. The equation in the history space framework

In order to apply the powerful tools of the theory of strongly continuous semigroups, it is convenient to recast the original problem as a differential system in the *history space framework*. To this end, following Dafermos [8], we introduce the *relative displacement history*

$$\eta^t(x,s) = u(x,t) - u(x,t-s).$$

With this position, equation (1.1) turns into

$$\begin{aligned} &\partial_{tt}u + \partial_{xxxx}u + \int_0^\infty \mu(s)\partial_{xxxx}\eta(s)\,\mathrm{d}s - \big(\beta + \|\partial_x u\|^2_{L^2(0,1)}\big)\partial_{xx}u = f, \\ &\partial_t\eta = -\partial_s\eta + \partial_t u. \end{aligned} \tag{1.8}$$

Accordingly, initial condition (1.3) becomes

$$\begin{cases} u(x,0) = u_0(x), & x \in [0,1], \\ \partial_t u(x,0) = u_1(x), & x \in [0,1], \\ \eta^0(x,s) = \eta_0(x,s), & (x,s) \in [0,1] \times \mathbb{R}^+, \end{cases} \tag{1.9}$$

where we set

$$u_0(x) = u_\star(x,0), \qquad u_1(x) = \partial_t u_\star(x,0), \qquad \eta_0(x,s) = u_\star(x,0) - u_\star(x,-s).$$

As far as the boundary conditions are concerned, (1.4) translates into

$$\begin{aligned} &u(0,t) = u(1,t) = \partial_{xx}u(0,t) = \partial_{xx}u(1,t) = 0, \\ &\eta^t(0,s) = \eta^t(1,s) = \partial_{xx}\eta^t(0,s) = \partial_{xx}\eta^t(1,s) = 0, \\ &\eta^t(x,0) = \lim_{s\to 0}\eta^t(x,s) = 0, \end{aligned} \tag{1.10}$$

for every $t \geqslant 0$. Indeed, upon choosing a proper phase space accounting for the past history, and assuming (1.2), it can be easily shown that (1.8)–(1.10) are an equivalent formulation of the original problem (cf [20]).

1.4. Earlier contributions

In the fifties, Woinowsky-Krieger [30] and Hoff [22] proposed to modify the theory of the dynamic Euler–Bernoulli beam, assuming a nonlinear dependence of the axial strain on the deformation gradient. The resulting motion equation, namely, (1.1) with $\mu = 0$ and $f = 0$, has been considered for hinged ends in the papers [2,9], with particular reference to well-posedness results and to the analysis of the complex structure of equilibria. Adding an external viscous damping term $\gamma_0 \partial_t u$ ($\gamma_0 > 0$) to the original conservative model, one has the equation

$$\partial_{tt} u + \partial_{xxxx} u + \gamma_0 \partial_t u - \big(\beta + \|\partial_x u\|^2_{L^2(0,1)}\big)\partial_{xx} u = 0. \tag{1.11}$$

Stability properties of the unbuckled (trivial) and the buckled stationary states of (1.11) have been established in [3,10] and, more formally, in [28]. In particular, if $\beta > -\nu_c$, the exponential decay of solutions to the trivial equilibrium state has been shown. The global dynamics of solutions for a general β has been first tackled by Hale [21], who proved the existence of a global attractor for (1.11) subject to hinged ends, relying on the existence of a suitable Lyapunov functional. This result has been improved by Eden and Milani [12], both for hinged and clamped ends. In particular, they obtained some regularity of the attractor, provided that γ_0 is large enough. In both papers, however, the compactness of the ω-limit set is gained by means of the so-called method of α-contractions. Indeed, as remarked in [12, p 464], the authors were unable to exhibit a decomposition of the semigroup into the sum of a strictly contractive semigroup and a compact one. Similar results for related models can be found in [15,16].

A different class of problems arises when the original equation contains a stronger damping term accounting for the effects of the internal (structural) dissipation (see, for instance, [3,5,23,31]). In particular, for a beam of a viscoelastic material of the Kelvin–Voigt type, Ball [3] proposed a very general model, which we consider here in its reduced form

$$\partial_{tt} u + \partial_{xxxx} u + \gamma_0 \partial_{xxxxt} u - \big(\beta + \|\partial_x u\|^2_{L^2(0,1)}\big)\partial_{xx} u = 0. \tag{1.12}$$

In the same paper, he studied the asymptotic stability properties of the semigroup generated by the full model, including the presence of a small antidissipative term $-\gamma_1 \partial_t u$. The existence of a compact attractor and an exponential attractor for the full model subject to hinged or clamped ends has been proved in [12], but without regularity results. A similar problem was scrutinized in [4], where a nonlinear inextensible strongly damped beam is shown to have a flat inertial manifold. The longtime behaviour of (1.12), in the presence of an additional convolution term, has been considered also in [25], for the cantilever boundary condition. However, this work presents some serious flaws, imported from an unpublished (but nevertheless quoted by several authors) manuscript by Taboada and You.

1.5. Outline of the paper

Since the internal viscoelastic dissipation requires to take the whole past history into account, in equation (1.1) a structural damping of the memory type is considered. Indeed, system (1.8) (in the history space framework) provides a more accurate description of the behaviour of an extensible viscoelastic beam than (1.12), which can be viewed as a limiting equation when the system keeps a very short memory. Note that (1.12) can be formally obtained from (1.8) when the memory kernel is the Dirac mass. To the best of our knowledge, this is the first paper in the literature dealing with equation (1.1), where the damping is entirely carried out by the memory only. Due to the weak dissipation provided by the convolution term, equations with memory are usually more difficult to handle than the corresponding ones without memory. Nonetheless, we are here able to prove results on the global dynamics of (1.8) which are stronger than those previously obtained for (1.11) or (1.12) (cf [12]). In particular, for a general β, we prove the existence of a global attractor of *optimal regularity*, provided that the memory kernel μ has an exponential decay, without requiring any assumption on the strength of the dissipation term (lemma 5.1 suggests that the dissipation rate of the system is of the same order as $\kappa\partial_t u$). In any case, we point out that, with our techniques, the results obtained here can be (in fact, more easily) reproduced also for (1.11) and (1.12).

In section 2, we formulate an abstract version of the problem; the main results of the paper, concerning the existence of regular global attractors, are stated in section 3. Section 4 is focused on the existence of a Lyapunov functional. In section 5, we proceed with some preliminary estimates and the exponential stability result, while in section 6 we prove the asymptotic smoothing property of the semigroup generated by the abstract problem, via a suitable decomposition. The conclusions of the proofs of the main results are given in section 7. Finally, the appendix is devoted to a discussion of the physical model.

2. The dynamical system

We will consider an abstract version of problem (1.8)–(1.9). To this end, let $(H_0, \langle\cdot,\cdot\rangle, \|\cdot\|)$ be a real Hilbert space, and let $A : \mathcal{D}(A) \Subset H_0 \to H_0$ be a strictly positive selfadjoint operator. We denote by $\lambda_1 > 0$ the first eigenvalue of A. For $\ell \in \mathbb{R}$, we introduce the scale of Hilbert spaces

$$H_\ell = \mathcal{D}(A^{\ell/4}), \qquad \langle u, v\rangle_\ell = \langle A^{\ell/4}u, A^{\ell/4}v\rangle, \qquad \|u\|_\ell = \|A^{\ell/4}u\|.$$

In particular, $H_{\ell+1} \Subset H_\ell$ and

$$\sqrt{\lambda_1}\,\|u\|_\ell^2 \leqslant \|u\|_{\ell+1}^2.$$

Given μ satisfying (1.2), we consider the L^2-weighted spaces

$$\mathcal{M}_\ell = L^2_\mu(\mathbb{R}^+, H_{\ell+2}), \qquad \langle \eta, \xi\rangle_{\ell,\mu} = \int_0^\infty \mu(s)\langle \eta(s), \xi(s)\rangle_{\ell+2}\,\mathrm{d}s, \qquad \|\eta\|_{\ell,\mu}^2 = \langle \eta, \eta\rangle_{\ell,\mu},$$

along with the infinitesimal generator of the right-translation semigroup on $\mathcal{M}_0$, that is, the linear operator

$$T\eta = -D\eta, \qquad \mathcal{D}(T) = \{\eta \in \mathcal{M}_0 : D\eta \in \mathcal{M}_0,\ \eta(0) = 0\},$$

where D stands for the distributional derivative, and $\eta(0) = \lim_{s\to 0}\eta(s)$ in H_2. Besides, we denote by $\mathcal{M}_\ell^1$ the weighted Sobolev spaces

$$\mathcal{M}_\ell^1 = H^1_\mu(\mathbb{R}^+, H_{\ell+2}) = \{\eta \in \mathcal{M}_\ell : D\eta \in \mathcal{M}_\ell\},$$

normed by

$$\|\eta\|_{\mathcal{M}_\ell^1}^2 = \|\eta\|_{\ell,\mu}^2 + \|D\eta\|_{\ell,\mu}^2.$$

We will also encounter the functional

$$\mathcal{J}(\eta) = -\int_0^\infty \mu'(s)\|\eta(s)\|_2^2\,\mathrm{d}s,$$

which is finite provided that $\eta \in \mathcal{D}(T)$ (see [20]). From the assumptions on μ,

$$\|\eta\|_{0,\mu}^2 \leqslant \frac{1}{\delta}\mathcal{J}(\eta). \tag{2.1}$$

Finally, we define the product Hilbert spaces

$$\mathcal{H}_\ell = H_{\ell+2} \times H_\ell \times \mathcal{M}_\ell.$$

As remarked in [27], $\mathcal{H}_{\ell+1} \subset \mathcal{H}_\ell$, but $\mathcal{H}_{\ell+1} \not\Subset \mathcal{H}_\ell$.

The abstract problem. For μ satisfying (1.2), $\beta \in \mathbb{R}$ and $f \in H_0$, we investigate the evolution system on $\mathcal{H}_0$ in the unknowns $u(t) : [0, \infty) \to H_2$, $\partial_t u(t) : [0, \infty) \to H_0$ and $\eta^t : [0, \infty) \to \mathcal{M}_0$:

$$\begin{aligned} &\partial_{tt} u + Au + \int_0^\infty \mu(s) A\eta(s)\,\mathrm{d}s + \left(\beta + \|u\|_1^2\right) A^{1/2} u = f, \\ &\partial_t \eta = T\eta + \partial_t u, \end{aligned} \tag{2.2}$$

with initial conditions

$$(u(0), u_t(0), \eta^0) = (u_0, u_1, \eta_0) = z \in \mathcal{H}_0.$$

Remark 2.1. Problem (1.8)–(1.10) is just a particular case of the abstract system (2.2), obtained by setting $H_0 = L^2(0, 1)$ and $A = \partial_{xxxx}$ with boundary condition (1.4). Nonetheless, the abstract result applies to more general situations, including, for instance, viscoelastic plates.

System (2.2) generates a strongly continuous semigroup (or dynamical system) $S(t)$ on $\mathcal{H}_0$: for any initial data $z \in \mathcal{H}_0$, $S(t)z$ is the unique weak solution to (2.2), with related (twice the) energy given by

$$\mathcal{E}(t) = \|S(t)z\|^2_{\mathcal{H}_0} = \|u(t)\|_2^2 + \|\partial_t u(t)\|^2 + \|\eta^t\|^2_{0,\mu}.$$

Besides, $S(t)$ continuously depends on the initial data. We omit the proof of these facts, which can be demonstrated either by means of a Galerkin procedure (e.g. following the lines of [19, 27]) or with a standard fixed point method. In both cases, it is crucial to have uniform energy estimates on any finite time interval. Indeed, these estimates follow directly from the existence of a Lyapunov functional, as shown later in the work.

3. The global attractor

We now state the main results of the paper, on the existence and the optimal regularity of a global attractor for $S(t)$. The related proofs will be carried out in the following sections. We recall that the global attractor $\mathcal{A}$ is the unique compact subset of $\mathcal{H}_0$ which is at the same time

(i) attracting:

$$\lim_{t\to\infty} \delta(S(t)\mathcal{B}, \mathcal{A}) \to 0,$$

for every bounded set $\mathcal{B} \subset \mathcal{H}_0$;

(ii) fully invariant:

$$S(t)\mathcal{A} = \mathcal{A},$$

for every $t \geqslant 0$.

Here, δ stands for the Hausdorff semidistance in $\mathcal{H}_0$, defined as (for $\mathcal{B}_1, \mathcal{B}_2 \subset \mathcal{H}_0$)

$$\delta(\mathcal{B}_1, \mathcal{B}_2) = \sup_{z_1\in\mathcal{B}_1} \inf_{z_2\in\mathcal{B}_2} \|z_1 - z_2\|_{\mathcal{H}_0}.$$

We address the reader to the books [1, 21, 29] for a detailed presentation of the theory of attractors.

Theorem 3.1. *The semigroup $S(t)$ on $\mathcal{H}_0$ possesses a connected global attractor $\mathcal{A}$ bounded in $\mathcal{H}_2$, whose third component is included in $\mathcal{D}(T)$, bounded in $\mathcal{M}_0^1$ and pointwise bounded in H_4. Moreover, $\mathcal{A}$ coincides with the unstable manifold of the set $\mathcal{S}$ of the stationary points of $S(t)$, namely,*

$$\mathcal{A} = \left\{\tilde z(0) : \tilde z \text{ is a complete (bounded) trajectory of } S(t) \text{ and } \lim_{t\to\infty} \|\tilde z(-t) - \mathcal{S}\|_{\mathcal{H}_0} = 0\right\}.$$

Remark 3.2. Due to the regularity and the invariance of $\mathcal{A}$, we observe that $\tilde{z}(t) = S(t)z$, $t \geqslant 0$, is a strong solution to (2.2) whenever $z \in \mathcal{A}$.

The set $\mathcal{S}$ consists of the vectors of the form $(u, 0, 0)$, where u is a (weak) solution to the equation

$$Au + \big(\beta + \|u\|_1^2\big)A^{1/2}u = f.$$

It is then apparent that $\mathcal{S}$ is bounded in $\mathcal{H}_0$.

Corollary 3.3. *If $\mathcal{S}$ is finite, then*

$$\mathcal{A} = \Big\{\tilde{z}(0) : \lim_{t\to\infty} \|\tilde{z}(-t) - z_1\|_{\mathcal{H}_0} = \lim_{t\to\infty} \|\tilde{z}(t) - z_2\|_{\mathcal{H}_0} = 0\Big\},$$

for some $z_1, z_2 \in \mathcal{S}$.

Remark 3.4. The above corollary applies to the concrete problem (1.8)–(1.9). Taking for simplicity $f = 0$, the stationary solutions $(u, 0, 0)$ solve the boundary value problem on the interval $[0, 1]$

$$u'''' - \big(\beta + \|u'\|^2_{L^2(0,1)}\big)u'' = 0,$$
$$u(0) = u(1) = u''(0) = u''(1) = 0.$$

If $\beta \geqslant -\pi^2$, there is only the null solution. If $\beta < -\pi^2$, besides the null solution, there are also the buckled solutions (cf [10])

$$u_n^{\pm}(x) = A_n^{\pm} \sin(n\pi x), \qquad n \in \mathbb{N},$$

with

$$A_n^{\pm} = \pm\frac{1}{n\pi}\sqrt{-2\beta - 2n^2\pi^2}.$$

Hence, the total number of stationary solutions is $2n_\star + 1$, where

$$n_\star = \max\{n \in \mathbb{N} : \beta < -n^2\pi^2\}.$$

The attractor $\mathcal{A}$ is as regular as f permits.

Theorem 3.5. *If $f \in H_\ell$ for some $\ell > 0$, then $\mathcal{A}$ is bounded in $\mathcal{H}_{\ell+2}$, and its third component is bounded in $\mathcal{M}_\ell^1$ and pointwise bounded in $H_{\ell+4}$.*

For problem (1.8)–(1.9), theorem 3.5 and the usual Sobolev embeddings yield at once the following result.

Corollary 3.6. *Let $f \in C_0^\infty(0, 1)$. Then, the attractor $\mathcal{A}$ is a subset of*

$$C^\infty([0, 1]) \times C^\infty([0, 1]) \times C([0, \infty), C^\infty([0, 1])).$$

Remark 3.7. As a matter of fact, we can say more. For initial data on the attractor, taking the time derivative of (2.2), it is possible to show that the successive time derivatives of u and η are also regular. Besides, exploiting the explicit representation formula for η (see [27]), it is readily seen that the semigroup $S(t)$ enjoys the backward uniqueness property (on the whole phase space), so that $S(t)$ restricted on $\mathcal{A}$ is a strongly continuous group of operators. Therefore, if $z = (u_0, u_1, \eta_0) \in \mathcal{A}$, we can assert that

$$\eta_0(s) = u_0 - u(-s),$$

where $u(t)$ is the first component of the solution $S(t)z$ (which now exists for all $t \in \mathbb{R}$). In conclusion, the thesis of corollary 3.6 enhances to

$$\mathcal{A} \subset C^\infty([0, 1]) \times C^\infty([0, 1]) \times C^\infty([0, \infty), C^\infty([0, 1])).$$

If β is not too negative, in the absence of the forcing term f the long term dynamics is trivial.

Theorem 3.8. *If $f = 0$ and $\beta \geqslant -\sqrt{\lambda_1}$, then $\mathcal{A} = \{0\}$. In addition, if the strict inequality $\beta > -\sqrt{\lambda_1}$ holds, the convergence to the attractor is exponential; namely,*

$$\|S(t)z\|_{\mathcal{H}_0} \leqslant \mathcal{Q}(\|z\|_{\mathcal{H}_0})\mathrm{e}^{-\omega t},$$

for some $\omega > 0$ and some increasing positive function $\mathcal{Q}$.

Remark 3.9. If we assume to have, as a lateral load acting on the beam, a distributed linear elastic force restoring the straight position, namely,

$$f = f(u) = -ku, \qquad k > 0,$$

we obtain the exponential decay to the null solution, provided that the stiffness constant k is large enough, depending on the value of $\beta \in \mathbb{R}$ (see lemma 6.1). This result can be applied to solve the following feedback control problem: design a state function $f(\cdot, t) = F(u(t), \partial_t u(t), \eta^t)$ in order to achieve the robust exponential stabilization of the closed-loop system. Then, $f(\cdot, t) = -ku(t)$ provides the desired feedback control if k is sufficiently large.

Remark 3.10. With the techniques presented in this paper, analogous results can be proven for the much simpler model

$$\partial_{tt}u + Au + \gamma_0 A^q \partial_t u + \left(\beta + \|u\|_1^2\right)A^{1/2}u = f,$$

with $q \in [0, 1]$, which, in particular, subsumes (1.11)–(1.12). In this case, the semigroup acts on $H_2 \times H_0$, and the above theorems (clearly, limited to the first two components) hold, providing the existence of a global attractor of optimal regularity, without any requirement on the size of $\gamma_0 > 0$. This generalizes some previous results from [12, 21].

We conclude the section discussing some possible further developments.

Exponential attractors. As will be clear in the following, we will actually find regular sets which attract exponentially fast bounded subsets of $\mathcal{H}_0$. Applying some techniques devised in [13, 14], it is then possible to prove the existence of a regular compact set $\mathcal{C}$ of finite fractal dimension, called an exponential attractor or inertial set, which, contrary to $\mathcal{A}$, is only invariant for $S(t)$ (that is, $S(t)\mathcal{C} \subset \mathcal{C}$), but fulfils the stronger attraction property

$$\boldsymbol{\delta}(S(t)\mathcal{B}, \mathcal{C}) \leqslant \mathcal{Q}(\|\mathcal{B}\|_{\mathcal{H}_0})\mathrm{e}^{-\omega t}.$$

As a by-product, since such a set $\mathcal{C}$ necessarily contains the global attractor, we are able to assert that $\mathcal{A}$ has finite fractal dimension as well. This result and remark 3.9 solve the *spillover* problem with respect to the stabilization of the nonlinear beam system. Namely, we are able to design a linear state feedback control involving only finitely many modes which achieves the exponential stability of the full system for a general β, even when there might be some uncertainty in the values of the structural parameters (see [31], for instance).

Weakly dissipative kernels. Our results still hold under considerably weaker assumptions on μ. Indeed, following some ideas of [26], it is possible to replace the inequality

$$\mu'(s) + \delta\mu(s) \leqslant 0$$

with the milder condition

$$\mu' \leqslant 0, \qquad \mu(s + \sigma) \leqslant C\mu(\sigma)\mathrm{e}^{-\delta s},$$

for some $C \geqslant 1$, with the further requirement that the set where $\mu' = 0$ is not too big. This allows, for instance, to include kernels μ which have constant (hence, nondissipative) zones.

Singular limits. In the spirit of [6], one can consider, rather than (2.2), the family of systems

$$\partial_{tt}u + Au + \int_0^\infty \mu_\varepsilon(s)A\eta(s)\,\mathrm{d}s + \big(\beta + \|u\|_1^2\big)A^{1/2}u = f,$$
$$\partial_t\eta = T\eta + \partial_t u,$$

depending on a parameter $\varepsilon \in (0, 1]$, where

$$\mu_\varepsilon(s) = \frac{1}{\varepsilon^2}\mu\Big(\frac{s}{\varepsilon}\Big).$$

The limiting situation $\varepsilon = 0$ corresponds to the (strongly) damped beam equation

$$\partial_{tt}u + Au + \vartheta A\partial_t u + \big(\beta + \|u\|_1^2\big)A^{1/2}u = f,$$

where

$$\vartheta = \int_0^\infty s\mu(s)\,\mathrm{d}s.$$

Then, for every $\varepsilon \in [0, 1]$, we have a dynamical system $S_\varepsilon(t)$ acting on the phase space

$$\mathcal{H}_0^\varepsilon = \begin{cases} H_2 \times H_0 \times L^2_{\mu_\varepsilon}(\mathbb{R}^+, H_2) & \text{if } \varepsilon > 0,\\ H_2 \times H_0 & \text{if } \varepsilon = 0,\end{cases}$$

and it is possible to prove the existence of a family $\mathcal{C}_\varepsilon$ of exponential attractors for $S_\varepsilon(t)$ of (uniformly) bounded fractal dimension, which is stable in the singular limit $\varepsilon \to 0$ with respect to the symmetric Hausdorff distance.

4. The Lyapunov functional

The first step to prove the existence of the global attractor usually consists of finding a bounded absorbing set for $S(t)$. However, for the case under consideration, a direct proof of this fact seems to be out of reach, because of the extremely weak dissipation of the system, which is only due to the presence of the memory term. In order to overcome this obstacle, we need to pursue a different strategy. To this end, we begin to prove the existence of a Lyapunov functional. This is a function $\mathcal{L} \in C(\mathcal{H}_0, \mathbb{R})$ satisfying the following conditions:

(i) $\mathcal{L}(z) \to +\infty$ if and only if $\|z\|_{\mathcal{H}_0} \to +\infty$;
(ii) $\mathcal{L}(S(t)z)$ is nonincreasing for any $z \in \mathcal{H}_0$;
(iii) $\mathcal{L}(S(t)z) = \mathcal{L}(z)$ for all $t > 0$ implies that $z \in \mathcal{S}$.

In particular, the existence of a Lyapunov functional ensures that bounded sets have bounded orbits.

A word of warning. Throughout this paper, we will use several times the equality

$$\langle T\eta, \eta\rangle_{0,\mu} = -\tfrac{1}{2}\mathcal{J}(\eta),$$

which holds if $\eta \in \mathcal{D}(T)$ (see [20]). Thus, in the following, we will perform formal estimates, which can be rigorously justified in a suitable approximation scheme (see, e.g. [19]).

Proposition 4.1. *The function*

$$\mathcal{L}(u, v, \eta) = \|(u, v, \eta)\|^2_{\mathcal{H}_0} + \tfrac{1}{2}\big(\beta + \|u\|_1^2\big)^2 - 2\langle f, u\rangle$$

is a Lyapunov functional for $S(t)$.

Proof. The continuity of $\mathcal{L}$ and assertion (i) above are clear. Using (2.2), we obtain quite directly the inequality

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{L}(S(t)z) \leqslant -\delta\|\eta^t\|_{0,\mu}^2,$$

which proves the monotonicity of $\mathcal{L}$ along the trajectories departing from z. Finally, if $\mathcal{L}(S(t)z)$ is constant in time, we have that $\eta^t = 0$ for all t, which implies that $u(t)$ is constant. Hence, $z = S(t)z = (u_0, 0, 0)$ for all t, that is, $z \in \mathcal{S}$. □

Remark 4.2. When $f = 0$ and $\beta \geqslant -\sqrt{\lambda_1}$, it is easily seen that $\mathcal{S} = \{0\}$. Therefore, on account of the existence of a Lyapunov functional, the first assertion of theorem 3.8 is an immediate consequence of theorem 3.1.

The existence of a Lyapunov functional, along with the fact that $\mathcal{S}$ is a bounded set, allows us prove the existence of the attractor exploiting the following general result from [7] (cf also [21, 24]), tailored for our particular case.

Lemma 4.3. *Assume that, for every $R > 0$, there exist a positive function ψ_R vanishing at infinity and a compact set $\mathcal{K}_R \subset \mathcal{H}_0$ such that the semigroup $S(t)$ can be split into the sum $L(t) + K(t)$, where the one-parameter operators $L(t)$ and $K(t)$ fulfil*

$$\|L(t)z\|_{\mathcal{H}_0} \leqslant \psi_R(t) \qquad \text{and} \qquad K(t)z \in \mathcal{K}_R,$$

whenever $\|z\|_{\mathcal{H}_0} \leqslant R$ and $t \geqslant 0$. Then, $S(t)$ possesses a connected global attractor $\mathcal{A}$, which consists of the unstable manifold of the set $\mathcal{S}$. Moreover, $\mathcal{A} \subset \mathcal{K}_{R_0}$ for some $R_0 > 0$.

As a by-product, we deduce the existence of a bounded absorbing set $\mathcal{B}_0$. As shown in [7], $\mathcal{B}_0$ can be chosen to be the ball of $\mathcal{H}_0$ centred at zero of radius

$$R_0 = 1 + \sup\left\{\|z\|_{\mathcal{H}_0} : \mathcal{L}(z) \leqslant K\right\},$$

where

$$K = 1 + \sup_{z_0\in\mathcal{S}} \mathcal{L}(z_0).$$

Note that R_0 can be explicitly calculated in terms of the structural quantities of our system. Nonetheless, the drawback of this method relies on the fact that, given a bounded set $\mathcal{B} \subset \mathcal{H}_0$, we do not have a procedure to compute its entering time into the absorbing ball $\mathcal{B}_0$.

5. Preliminary estimates

With a view to applying lemma 4.3, we fix $R > 0$, and we consider initial data $z \in \mathcal{H}_0$ such that $\|z\|_{\mathcal{H}_0} \leqslant R$.

Notation. Till the end of the paper, C will denote a *generic* positive constant which depends (increasingly) only on R, unless otherwise specified, besides on the structural quantities of the system. The actual value of C may change, even within the same line of a given equation.

The existence of a Lyapunov functional ensures that

$$\mathcal{E}(t) \leqslant C. \tag{5.1}$$

In order to deal with the possible singularity of μ at zero, we choose $s_\star > 0$ such that

$$\int_0^{s_\star} \mu(s)\,\mathrm{d}s \leqslant \frac{\kappa}{4},$$

and we define

$$\rho(s) = \mu(s_\star)\chi_{(0,s_\star]}(s) + \mu(s)\chi_{(s_\star,\infty)}(s).$$

Then, we introduce the functional, devised in [26],

$$\Psi(t) = -\frac{4}{\kappa}\int_0^\infty \rho(s)\langle\partial_t u(t), \eta^t(s)\rangle\,\mathrm{d}s.$$

Note that, as $\rho(s) \leqslant \mu(s)$,

$$\Psi(t) \leqslant \frac{4}{\sqrt{\kappa\lambda_1}}\mathcal{E}(t).$$

Lemma 5.1. *For every $\gamma > 0$, the functional Ψ satisfies the differential inequality*

$$\frac{\mathrm{d}}{\mathrm{d}t}\Psi + 2\|\partial_t u\|^2 \leqslant \gamma\mathcal{E} + \gamma\|f\|^2 + \frac{C}{\gamma}\mathcal{J}(\eta).$$

Proof. The time derivative of Ψ is given by

$$\frac{\mathrm{d}}{\mathrm{d}t}\Psi = -\frac{4}{\kappa}\int_0^\infty \rho(s)\langle\partial_t u, \partial_t\eta(s)\rangle\,\mathrm{d}s - \frac{4}{\kappa}\int_0^\infty \rho(s)\langle\partial_{tt} u, \eta(s)\rangle\,\mathrm{d}s.$$

Arguing exactly as in [26, lemma 4.1], one can show that[3]

$$-\frac{4}{\kappa}\int_0^\infty \rho(s)\langle\partial_t u, \partial_t\eta(s)\rangle\,\mathrm{d}s \leqslant -2\|\partial_t u\|^2 + \frac{4\mu(s_\star)}{\kappa^2\lambda_1}\mathcal{J}(\eta).$$

Regarding the other term, using the first equation in (2.2), we obtain

$$-\frac{4}{\kappa}\int_0^\infty \rho(s)\langle\partial_{tt} u, \eta(s)\rangle\,\mathrm{d}s$$
$$= \frac{4}{\kappa}\int_0^\infty \rho(s)\langle\Theta, \eta(s)\rangle\,\mathrm{d}s + \frac{4}{\kappa}\int_0^\infty \rho(s)\int_0^\infty \mu(\sigma)\langle\eta(s), \eta(\sigma)\rangle_2\,\mathrm{d}\sigma\,\mathrm{d}s,$$

having set

$$\Theta = Au + (\beta + \|u\|_1^2)A^{1/2}u - f.$$

Recalling that $\rho(s) \leqslant \mu(s)$, and noting that, from (5.1),

$$\|\Theta\|_{-2} \leqslant C\|u\|_2 + \frac{1}{\sqrt{\lambda_1}}\|f\|,$$

by means of standard inequalities we are led to

$$\frac{4}{\kappa}\int_0^\infty \rho(s)\langle\Theta, \eta(s)\rangle\,\mathrm{d}s \leqslant \left(C\|u\|_2 + \frac{1}{\sqrt{\lambda_1}}\|f\|\right)\int_0^\infty \mu(s)\|\eta(s)\|_2\,\mathrm{d}s$$
$$\leqslant \gamma\|u\|_2^2 + \gamma\|f\|^2 + \frac{C}{\gamma}\|\eta\|_{0,\mu}^2.$$

Besides,

$$\frac{4}{\kappa}\int_0^\infty \rho(s)\int_0^\infty \mu(\sigma)\langle\eta(s), \eta(\sigma)\rangle_2\,\mathrm{d}\sigma\,\mathrm{d}s \leqslant \frac{4}{\kappa}\left(\int_0^\infty \mu(s)\|\eta(s)\|_2\,\mathrm{d}s\right)^2 \leqslant 4\|\eta\|_{0,\mu}^2.$$

In light of (2.1), collecting the above estimates, we reach the conclusion. □

The next step is to prove the existence of a dissipation integral, which will play a crucial role in the subsequent estimates.

[3] The cut-off at $s_\star$ is necessary, otherwise the rhs of the formula would contain the term $\mu(0)$, which can be infinite.

Lemma 5.2. *For every* $\nu > 0$ *small,*

$$\int_\tau^t \|\partial_t u(y)\| \, \mathrm{d}y \leqslant \nu(t-\tau) + \frac{C}{\nu^3},$$

for every $t \geqslant \tau \geqslant 0$.

Proof. For $\varepsilon \in (0, 1]$, we set

$$\Phi(t) = \mathcal{E}(t) + \beta\|u(t)\|_1^2 + \tfrac{1}{2}\|u(t)\|_1^4 - 2\langle f, u(t)\rangle + \varepsilon\langle \partial_t u(t), u(t)\rangle.$$

Exploiting (2.2), we have the equality

$$\frac{\mathrm{d}}{\mathrm{d}t}\Phi + \varepsilon\Phi + \mathcal{J}(\eta) + \frac{\varepsilon}{2}\|u\|_1^4 = \varepsilon\Big[2\|\partial_t u\|^2 + \|\eta\|_{0,\mu}^2 - \langle \eta, u\rangle_{0,\mu} + \varepsilon\langle \partial_t u, u\rangle - \langle f, u\rangle\Big]. \tag{5.2}$$

On account of (5.1), the rhs of (5.2) is easily seen to be less than εC. Hence,

$$\frac{\mathrm{d}}{\mathrm{d}t}\Phi + \varepsilon\Phi + \mathcal{J}(\eta) \leqslant \varepsilon C.$$

Then, we introduce the functional

$$\Gamma(t) = \Phi(t) + \sqrt{\varepsilon}\,\Psi(t).$$

Applying lemma 5.1 with $\gamma = \sqrt{\varepsilon}\,C$, and using (5.1), we end up with

$$\frac{\mathrm{d}}{\mathrm{d}t}\Gamma + 2\sqrt{\varepsilon}\,\|\partial_t u\|^2 \leqslant \varepsilon\Big[C + C\mathcal{E} + C\|f\|^2 - \Phi\Big] \leqslant \varepsilon C.$$

Integrating this inequality on $[\tau, t]$, we get

$$2\sqrt{\varepsilon}\int_\tau^t \|\partial_t u(y)\|^2 \, \mathrm{d}y \leqslant \varepsilon C(t-\tau) + \Gamma(\tau) - \Gamma(t),$$

and a further application of (5.1) entails

$$\int_\tau^t \|\partial_t u(y)\|^2 \, \mathrm{d}y \leqslant \sqrt{\varepsilon}\,C(t-\tau) + \frac{C}{\sqrt{\varepsilon}}.$$

Finally, by the Hölder and the Young inequalities,

$$\int_\tau^t \|\partial_t u(y)\| \, \mathrm{d}y \leqslant C\sqrt{t-\tau}\left(\sqrt{\varepsilon}\,(t-\tau) + \frac{1}{\sqrt{\varepsilon}}\right)^{1/2} \leqslant \varepsilon^{1/4}C(t-\tau) + \frac{C}{\varepsilon^{3/4}}.$$

Setting $\nu = \varepsilon^{1/4}C$, we are done. □

In a similar manner, we can provide the proof of the second assertion of theorem 3.8.

Corollary 5.3. *If* $f = 0$ *and* $\beta > -\sqrt{\lambda_1}$, *then*

$$\mathcal{E}(t) \leqslant C\mathrm{e}^{-\omega t},$$

for some $\omega > 0$.

Proof. We use the same functional Φ of the previous lemma. Since now $f = 0$ and $\beta > -\sqrt{\lambda_1}$, and noting that, from (5.1),

$$\|u\|_1^4 \leqslant C\|u\|_1^2 \leqslant C\mathcal{E},$$

we have

$$c\mathcal{E} \leqslant \Phi \leqslant C\mathcal{E},$$

for some $c = c(\beta)$, provided that ε is small enough. In this case, in light of (2.1), the rhs of (5.2) is bounded by

$$\tfrac{1}{2}\varepsilon c\mathcal{E} + 2\varepsilon\|\partial_t u\|^2 + \tfrac{1}{2}\mathcal{J}(\eta),$$

so that (5.2) turns into the inequality

$$\frac{\mathrm{d}}{\mathrm{d}t}\Phi + \frac{1}{2}\big[\varepsilon c\mathcal{E} + \mathcal{J}(\eta)\big] \leqslant 2\varepsilon\|\partial_t u\|^2.$$

At this point, we consider the functional

$$\Lambda(t) = \Phi(t) + \varepsilon\Psi(t).$$

On account of the bounds on Φ and Ψ, we readily have

$$\frac{c}{2}\mathcal{E} \leqslant \Lambda \leqslant C\mathcal{E},$$

for ε small enough. Applying lemma 5.1 with $f = 0$ and $\gamma = 2\varepsilon C$, and reducing once more ε (this time, in dependence of C), we finally obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\Lambda + \omega\Lambda \leqslant 0,$$

for some $\omega = \omega(R) > 0$. The Gronwall lemma together with the bounds on Λ entails

$$\mathcal{E}(t) \leqslant C\mathrm{e}^{-\omega t}.$$

Using the semigroup property of $S(t)$, we can easily remove the dependence on R in ω by means of a standard argument. Indeed, let $\|z\|_{\mathcal{H}_0} \leqslant R$. Then, for some $\tau = \tau(R) \geqslant 0$, we have that $\mathcal{E}(\tau)$ is less than (say) 1. Therefore, for $t > \tau$,

$$\mathcal{E}(t) \leqslant C\mathrm{e}^{\omega(1)\tau}\mathrm{e}^{-\omega(1)t} = C\mathrm{e}^{-\omega(1)t}.$$

On the other hand, $\mathcal{E}(t) \leqslant C$, for every $t \leqslant \tau$. □

6. The semigroup decomposition

By the interpolation inequality

$$\|u\|_1^2 \leqslant \|u\|\,\|u\|_2,$$

it is clear that, provided that $\alpha > 0$ is large enough,

$$\tfrac{1}{2}\|u\|_2^2 \leqslant \|u\|_2^2 + \beta\|u\|_1^2 + \alpha\|u\|^2 \leqslant m\|u\|_2^2, \tag{6.1}$$

for some $m = m(\beta, \alpha) \geqslant 1$.

Again, $R > 0$ is fixed and $\|z\|_{\mathcal{H}_0} \leqslant R$. Choosing $\alpha > 0$ such that (6.1) holds, we decompose the solution $S(t)z$ into the sum

$$S(t)z = L(t)z + K(t)z,$$

where

$$L(t)z = (v(t), \partial_t v(t), \xi^t) \qquad \text{and} \qquad K(t)z = (w(t), \partial_t w(t), \zeta^t)$$

solve the systems

$$\begin{cases} \partial_{tt} v + Av + \displaystyle\int_0^\infty \mu(s)A\xi(s)\,\mathrm{d}s + (\beta + \|u\|_1^2)A^{1/2}v + \alpha v = 0, \\ \partial_t\xi = T\xi + \partial_t v, \\ (v(0), \partial_t v(0), \xi^0) = z \end{cases} \tag{6.2}$$

and

$$\begin{aligned}
&\partial_{tt} w + Aw + \int_0^\infty \mu(s) A\zeta(s)\,\mathrm{d}s + (\beta + \|u\|_1^2)A^{1/2}w - \alpha v = f,\\
&\partial_t \zeta = T\zeta + \partial_t w,\\
&(w(0), \partial_t w(0), \zeta^0) = 0.
\end{aligned} \tag{6.3}$$

The next two lemmas show the asymptotic smoothing property of $S(t)$, for initial data bounded by R. We begin to prove the exponential decay of $L(t)z$.

Lemma 6.1. *There is* $\omega = \omega(R) > 0$ *such that*

$$\|L(t)z\|_{\mathcal{H}_0} \leqslant C\mathrm{e}^{-\omega t}.$$

Proof. Denoting

$$\mathcal{E}_0(t) = \|L(t)z\|_{\mathcal{H}_0}^2 = \|v(t)\|_2^2 + \|\partial_t v(t)\|^2 + \|\xi^t\|_{0,\mu}^2,$$

for $\varepsilon > 0$ to be determined, we set

$$\Phi_0(t) = \mathcal{E}_0(t) + \beta\|v(t)\|_1^2 + \alpha\|v(t)\|^2 + \|u(t)\|_1^2\|v(t)\|_1^2 + \varepsilon\langle \partial_t v(t), v(t)\rangle.$$

In light of (5.1) and (6.1), provided that ε is small enough, we have the bounds

$$\tfrac{1}{4}\mathcal{E}_0 \leqslant \Phi_0 \leqslant C\mathcal{E}_0,$$

Using (6.2), we compute the time derivative of Φ_0 as

$$\frac{\mathrm{d}}{\mathrm{d}t}\Phi_0 + \varepsilon\Phi_0 + \mathcal{J}(\xi) - 2\varepsilon\|\partial_t v\|^2 = 2\langle \partial_t u, A^{1/2}u\rangle\|v\|_1^2 + \varepsilon\|\xi\|_{0,\mu}^2 - \varepsilon\langle \xi, v\rangle_{0,\mu} + \varepsilon^2\langle \partial_t v, v\rangle.$$

Using (2.1) and (5.1), for ε small enough, we control the rhs by

$$C\|\partial_t u\|\mathcal{E}_0 + \frac{\varepsilon}{8}\mathcal{E}_0 + \frac{1}{2}\mathcal{J}(\xi).$$

Hence, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\Phi_0 + \frac{\varepsilon}{8}\mathcal{E}_0 + \frac{1}{2}\mathcal{J}(\xi) \leqslant 2\varepsilon\|\partial_t v\|^2 + C\|\partial_t u\|\mathcal{E}_0.$$

Then, we consider the further functional

$$\Psi_0(t) = -\frac{4}{\kappa}\int_0^\infty \rho(s)\langle \partial_t v(t), \xi^t(s)\rangle\,\mathrm{d}s.$$

Arguing exactly as in lemma 5.1 (now with $f = 0$ and $\gamma = 1/16$),

$$\frac{\mathrm{d}}{\mathrm{d}t}\Psi_0 + 2\|\partial_t v\|^2 \leqslant \frac{1}{16}\mathcal{E}_0 + C\mathcal{J}(\xi).$$

Thus, up to further reducing ε (in dependence of the value of C and, in turn, of R), the functional

$$\Lambda_0(t) = \Phi_0(t) + \varepsilon\Psi_0(t)$$

fulfils the differential inequality

$$\frac{\mathrm{d}}{\mathrm{d}t}\Lambda_0 + \frac{\varepsilon}{16}\mathcal{E}_0 \leqslant C\|\partial_t u\|\mathcal{E}_0.$$

Since for ε small the functional Λ_0 is equivalent to the energy $\mathcal{E}_0$, in view of lemma 5.2, we are in a position to apply a modified form of the Gronwall lemma (see, e.g. [18]), reported here below for the reader's convenience, which entails the desired conclusion. □

Lemma 6.2. *Let* $\varphi : [0, \infty) \to [0, \infty)$ *satisfy*

$$\varphi' + 2\varepsilon\varphi \leqslant g\varphi,$$

for some $\varepsilon > 0$ *and some positive function* g *such that*

$$\int_\tau^t g(y)\,\mathrm{d}y \leqslant c_0 + \varepsilon(t-\tau), \qquad \forall \tau \in [0, t],$$

with $c_0 \geqslant 0$. *Then, there exists* $c_1 \geqslant 0$ *such that*

$$\varphi(t) \leqslant c_1\varphi(0)\mathrm{e}^{-\varepsilon t}.$$

The next result provides the boundedness of $K(t)z$ in a more regular space.

Lemma 6.3. *The estimate*

$$\|K(t)z\|_{\mathcal{H}_2} \leqslant C$$

holds for every $t \geqslant 0$.

Proof. We denote

$$\mathcal{E}_1(t) = \|K(t)z\|^2_{\mathcal{H}_2} = \|w(t)\|_4^2 + \|\partial_t w(t)\|_2^2 + \|\zeta^t\|^2_{2,\mu}.$$

For $\varepsilon > 0$ small to be fixed later, we set

$$\Phi_1(t) = \mathcal{E}_1(t) + (\beta + \|u(t)\|_1^2)\|w(t)\|_3^2 + \varepsilon\langle\partial_t w(t), Aw(t)\rangle - 2\langle f, Aw(t)\rangle.$$

The interpolation inequality

$$\|w\|_3^2 \leqslant \|w\|_2\|w\|_4$$

and the fact that $\|w\|_2 \leqslant C$ (which follows by comparison from (5.1) and lemma 6.1) entail

$$\beta\|w\|_3^2 \geqslant -\tfrac{1}{2}\mathcal{E}_1 - C.$$

Therefore, in light of (5.1), provided that ε is small enough, we have the bounds

$$\tfrac{1}{2}\mathcal{E}_1 - C \leqslant \Phi_1 \leqslant C\mathcal{E}_1 + C.$$

Taking the time derivative of Φ_1, we find

$$\frac{\mathrm{d}}{\mathrm{d}t}\Phi_1 + \varepsilon\Phi_1 + \mathcal{J}(A^{1/2}\zeta) - 2\varepsilon\|\partial_t w\|_2^2 = 2\langle\partial_t u, A^{1/2}u\rangle\|w\|_3^2 + 2\alpha\langle A^{1/2}v, A^{1/2}\partial_t w\rangle$$
$$+ \varepsilon\big[\alpha\langle A^{1/2}v, A^{1/2}w\rangle + \|\zeta\|^2_{2,\mu} - \langle\zeta, w\rangle_{2,\mu} + \varepsilon\langle\partial_t w, Aw\rangle - \langle f, Aw\rangle\big].$$

Using (2.1), (5.1) and the above interpolation inequality, if ε is small enough, we control the rhs by

$$\frac{\varepsilon}{8}\mathcal{E}_1 + C\sqrt{\mathcal{E}_1} + \frac{1}{2}\mathcal{J}(A^{1/2}\zeta) + C \leqslant \frac{\varepsilon}{4}\mathcal{E}_1 + \frac{1}{2}\mathcal{J}(A^{1/2}\zeta) + C \leqslant \frac{\varepsilon}{2}\Phi_1 + \frac{1}{2}\mathcal{J}(A^{1/2}\zeta) + C.$$

Hence, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\Phi_1 + \frac{\varepsilon}{2}\Phi_1 + \frac{1}{2}\mathcal{J}(A^{1/2}\zeta) \leqslant 2\varepsilon\|\partial_t w\|^2 + C.$$

Then, we consider the functional

$$\Psi_1(t) = -\frac{4}{\kappa}\int_0^\infty \rho(s)\langle\partial_t w(t), \zeta^t(s)\rangle_2\,\mathrm{d}s.$$

As for the analogous cases seen before, we can prove that

$$\Psi_1(t) \leqslant \frac{4}{\sqrt{\kappa\lambda_1}}\mathcal{E}_1(t)$$

and

$$\frac{\mathrm{d}}{\mathrm{d}t}\Psi_1 + 2\|\partial_t w\|_2^2 \leqslant \varepsilon C\Phi_1 + \frac{1}{2\varepsilon}\mathcal{J}(A^{1/2}\zeta) + C.$$

Finally, we introduce

$$\Lambda_1(t) = \Phi_1(t) + \varepsilon\Psi_1(t),$$

which, for ε small enough, clearly fulfils the inequalities

$$\tfrac{1}{4}\mathcal{E}_1(t) - C \leqslant \Lambda_1(t) \leqslant C\mathcal{E}_1 + C.$$

Moreover,

$$\frac{\mathrm{d}}{\mathrm{d}t}\Lambda_1 + \varepsilon\Big(\frac{1}{2} - \varepsilon C\Big)\Lambda_1 \leqslant C.$$

Operating a further reduction of ε (depending on C), we conclude that

$$\frac{\mathrm{d}}{\mathrm{d}t}\Lambda_1 + \frac{\varepsilon}{4}\Lambda_1 \leqslant C.$$

Since $\Lambda_1(0) = 0$, from the Gronwall lemma and the controls satisfied by Λ_1, we obtain the desired estimate for $\mathcal{E}_1$. □

7. Proofs of the main results

We have now all the ingredients to complete the proofs of the main results.

Proof of theorem 3.1. The third component ζ^t of the solution $K(t)z$ to (6.3) admits the explicit representation (cf [27])

$$\zeta^t(s) = \begin{cases} w(t) - w(t-s) & \text{if } 0 < s \leqslant t, \\ w(t) & \text{if } s > t. \end{cases}$$

Accordingly,

$$D\zeta^t(s) = \begin{cases} \partial_t w(t-s) & \text{if } 0 < s \leqslant t, \\ 0 & \text{if } s > t. \end{cases}$$

Hence, lemma 6.3 and the summability of μ entail that

$$\|\zeta^t\|_{\mathcal{M}_0^1} + \sup_{s\in\mathbb{R}^+}\|\zeta^t(s)\|_4 \leqslant C, \qquad \forall t \geqslant 0.$$

Since R is fixed, and $C = C(R)$, we conclude that

$$K(t)z \in \mathcal{K}_R,$$

where

$$\mathcal{K}_R = \Big\{\bar{z} = (\bar{u}, \bar{v}, \bar{\eta}) : \|\bar{z}\|_{\mathcal{H}_2} + \|\bar{\eta}\|_{\mathcal{M}_0^1} + \sup_{s\in\mathbb{R}^+}\|\bar{\eta}(s)\|_4 \leqslant C,\ \ \bar{\eta}(0) = 0\Big\}.$$

According to [27, lemma 5.5], $\mathcal{K}_R$ is a compact subset of $\mathcal{H}_0$. Thus, in view of lemma 6.1, we can apply lemma 4.3, so obtaining in one step both the existence of the global attractor $\mathcal{A}$ and its regularity.

The proof of corollary 3.3 follows directly from the structure of $\mathcal{A}$, provided by theorem 3.1, and the existence of a Lyapunov functional. In particular, it must be

$$\mathcal{L}(z_1) > \mathcal{L}(z_2),$$

unless $z_1 = z_2$, in which case, the corresponding trajectory $z(t)$ is constant.

Proof of theorem 3.5. Since $\mathcal{A}$ is fully invariant for $S(t)$, it is enough to show that, given any $z \in \mathcal{A}$ (hence, in particular, $\|z\|_{\mathcal{H}_0} \leqslant C_0$ for some $C_0 \geqslant 0$), the estimate

$$\|K(t)z\|_{\mathcal{H}_{\ell+2}} \leqslant C_1 \tag{7.1}$$

holds true, for some $C_1 \geqslant 0$, and then repeat the very same passages of the former proof. Indeed, if $z \in \mathcal{A}$, for every $t \geqslant 0$ there is $z_t \in \mathcal{A}$ such that

$$z = S(t)z_t.$$

Moreover, by means of a standard bootstrap argument, it suffices to verify the validity of (7.1) assuming the boundedness of $\mathcal{A} = S(t)\mathcal{A}$ in $\mathcal{H}_\ell$. This can be shown by recasting word by word the proof of lemma 6.3, replacing the functional $\mathcal{E}_1$, Φ_1 and Ψ_1 with

$$\begin{aligned}
\mathcal{E}_\star &= \|w\|^2_{\ell+4} + \|\partial_t w\|^2_{\ell+2} + \|\zeta\|^2_{\ell+2,\mu},\\
\Phi_\star &= \mathcal{E}_\star + (\beta + \|u\|^2_1)\|w\|^2_{\ell+3} + \varepsilon\langle A^{\ell/4}\partial_t w, A^{(\ell+1)/4}w\rangle - 2\langle A^{\ell/4} f, A^{(\ell+1)/4}w\rangle,\\
\Psi_\star &= -\frac{4}{\kappa}\int_0^\infty \rho(s)\langle\partial_t w, \zeta(s)\rangle_{\ell+2}\,\mathrm{d}s.
\end{aligned}$$

The only slight difference with respect to the case $\ell = 0$ lies in the estimate of the term

$$2\alpha\langle A^{(\ell+2)/4}v, A^{(\ell+2)/4}\partial_t w\rangle + \alpha\varepsilon\langle A^{(\ell+2)/4}v, A^{(\ell+2)/4}w\rangle,$$

which, writing $v = u - w$, is controlled by

$$C\|\partial_t w\|_{\ell+2} + 2\alpha\|w\|_{\ell+2}\|\partial_t w\|_{\ell+2} + \varepsilon C\|w\|_{\ell+2} + \alpha\varepsilon\|w\|^2_{\ell+2}.$$

Then, one has to exploit the interpolation inequality

$$\|w\|_{\ell+2} \leqslant \|w\|_2^{2/(\ell+2)}\|w\|_{\ell+4}^{\ell/(\ell+4)} \leqslant C\|w\|_{\ell+4}^{\ell/(\ell+4)},$$

which is also needed to prove the corresponding controls between $\mathcal{E}_\star$ and $\Phi_\star$. The details are left to the reader.

Finally, the proof of theorem 3.8 follows directly from theorem 3.1 and corollary 5.3, in view of remark 4.2.

Appendix A. The physical model

This section is devoted to the derivation of model equation (1.1), in parallel to the well-established procedure devised in [28, 30] for the transverse motion of an extensible elastic homogeneous beam, but taking into account the energy loss due to the internal dissipation of the material, which translates into a linear viscoelastic response in bending. This is certainly more realistic than imposing any external damping in order to weaken vibrations. We consider the transverse motion of an extensible viscoelastic beam of constant cross section Ω and natural length L. According to the Euler–Bernoulli assumptions, we neglect rotational inertia and shear deformations. In the reference frame, at the natural (unloaded) rest configuration, the beam occupies the interval $[0, L]$ of the x-axis, while its motion occurs in the vertical plane x–z. As pointed out in [30], the extensibility of the beam comes into the picture when the ends are held at a fixed distance (which is not the case if one end is free). Here, we assume that the ends are fixed at the points 0 and $L + \Delta$, with $|\Delta| \ll L$, on the x-axis. Thus, if $\Delta \neq 0$, the natural (straight) configuration of the beam at equilibrium can be unstable, and buckling bifurcations may occur when $\Delta < 0$.

Calling $\boldsymbol{u}$ the displacement vector at the point (x, z), $u = u(x, t)$ the deflection and $w = w(x, t)$ the axial displacement of the middle line of the vibrating beam, we make the following assumptions.

(i) The deflection u, as well as its space derivative, is small compared with the beam thickness, so that only second order terms in the curvature are retained.
(ii) The Kirchhoff assumption is fulfilled, namely, any cross section remains perpendicular to the deformed longitudinal axis of the beam during the bending.
(iii) The axial velocity component $\partial_t w$ is negligible, so that the nonlinear terms in the curvature of the longitudinal axis can be replaced by their mean values.

As a consequence, it is straightforward to prove that the only nonzero component of the symmetric strain tensor

$$\boldsymbol{\epsilon} = \{\epsilon_{ij}\} = \tfrac{1}{2}(\nabla \boldsymbol{u} + \nabla \boldsymbol{u}^{\mathrm{T}})$$

is given by

$$\epsilon(x,z,t) = \epsilon_{11}(x,z,t) = \partial_x w(x,t) - z\partial_{xx}u(x,t) + \tfrac{1}{2}(\partial_x u(x,t))^2.$$

The classical scalar constitutive stress–strain relation for a homogeneous viscoelastic material reads

$$\sigma(x,z,t) = E\left[\epsilon(x,z,t) + \int_0^\infty g'(s)\epsilon(x,z,t-s)\,\mathrm{d}s\right],$$

where E is the constant Young's modulus and $g(s)$ is the relaxation measure. The bending moment $M(x,t)$ can be calculated according to the formula (see [11])

$$M(x,t) = -\int_\Omega z\sigma(x,z,t)\,\mathrm{d}\Omega.$$

Hence, substituting the expression for σ, we obtain

$$M(x,t) = EI\left[\partial_{xx}u(x,t) + \int_0^\infty g'(s)\partial_{xx}u(x,t-s)\,\mathrm{d}s\right],$$

where

$$I = \int_\Omega z^2\,\mathrm{d}\Omega$$

is the moment of inertia of the cross section. If a distributed lateral load $F(x,t)$ is applied to the beam, the balance equation in the z-direction can be written as

$$\partial_{xx}M(x,t) = \partial_x T(x,t) + F(x,t),$$

where the shearing stress T can be expressed in terms of the axial stress N by

$$T(x,t) = \int_\Omega N(x,z,t)\partial_x u(x,t)\,\mathrm{d}\Omega.$$

If we assume N to be constant, and we decompose the lateral load F into the sum of the inertia force $-\rho|\Omega|\partial_{tt}u(x,t)$, where $\rho > 0$ is the mass per unit of length, and an external load $\rho|\Omega| f(x)$, the deflection u satisfies the linear integrodifferential equation [11]

$$\rho\partial_{tt}u - \partial_x(N\partial_x u) + \frac{EI}{|\Omega|}\partial_{xx}\left[\partial_{xx}u + \int_0^\infty g'(s)\partial_{xx}u(t-s)\,\mathrm{d}s\right] = \rho f. \qquad \text{(A.1)}$$

In the usual theory, when one end is free to move in the axial direction, an extensionless deflection is obtained and N is assumed to vanish. Nevertheless, one often has to deal with both ends either clamped or hinged, in such a way that a tensile force, proportional to the elongation, is produced. In this case, we have to specify the nonconstant form of the axial stress N, in order to consider the extensibility of the beam. Therefore, the effect of the axial stress on the vibration process deserves further investigation. To this end, we assume the boundary conditions

$$w(0,t) = 0, \qquad w(L,t) = \Delta, \qquad u(0,t) = u(L,t) = \partial_{xx}u(0,t) = \partial_{xx}u(L,t) = 0$$

and the constitutive relation (see, for instance, [28])

$$N(x,z,t) = E\,\epsilon(x,z,t) = E\big(\partial_x w(x,t) - z\partial_{xx}u(x,t) + \tfrac{1}{2}(\partial_x u(x,t))^2\big).$$

Replacing N into the expression of T, and letting

$$\tilde{N}(x,t) = \frac{1}{|\Omega|}\int_\Omega N(x,z,t)\,\mathrm{d}\Omega = E\Big(\partial_x w(x,t) + \frac{1}{2}(\partial_x u(x,t))^2\Big),$$

we obtain the motion equation

$$\rho\partial_{tt}u - \partial_x(\tilde{N}\partial_x u) + \frac{EI}{|\Omega|}\partial_{xx}\left[\partial_{xx}u + \int_0^\infty g'(s)\partial_{xx}u(t-s)\,\mathrm{d}s\right] = \rho f, \qquad \text{(A.2)}$$

which is quasilinear, according to von Kármán theory, because of the cubic nonlinearity given by the term $\tilde{N}\partial_x u$. In addition, the presence of w in $\tilde{N}$ gives rise to the coupling with the balance equation in the x-direction. Now, as customary when the endpoints are not allowed to move, we assume that the axial inertial force is negligible, so that the balance in the x-direction reduces to (cf [28, 30])

$$\partial_x N(x,z,t) = 0\,.$$

As a consequence, we obtain $\tilde{N}(x,t) = N(t)$, and in (A.2) we can replace $\tilde{N}$ by its mean value on $[0, L]$. In particular, recalling that the axial velocity is negligible, namely, $w = w(x)$, we have

$$N(t) = N_0 + N_1(t),$$

where

$$N_0 = \frac{E}{L}\int_0^L \partial_x w(\xi)\,\mathrm{d}\xi = \frac{E}{L}[w(L) - w(0)] = \frac{E\Delta}{L}$$

is the axial tension in the reference configuration, while

$$N_1(t) = \frac{E}{2L}\int_0^L (\partial_x u(\xi,t))^2\,\mathrm{d}\xi$$

is the extra tension due to the deflection, which produces an elongation of the beam. In conclusion, setting

$$\alpha_0 = \frac{EI}{\rho|\Omega|}, \qquad \beta = \frac{E\Delta}{\rho L}, \qquad \gamma = \frac{E}{2\rho L}, \qquad \mu(s) = -\frac{EI}{\rho|\Omega|}g'(s),$$

the quasilinear motion equation (A.2) transforms into

$$\partial_{tt}u + \alpha_0\partial_{xxxx}u - \int_0^\infty \mu(s)\partial_{xxxx}u(t-s)\,\mathrm{d}s - \left[\beta + \gamma\int_0^L (\partial_x u(\xi,t))^2 d\xi\right]\partial_{xx}u = f. \qquad \text{(A.3)}$$

Here, $\gamma > 0$, whereas β can be either positive (traction) or negative (compression). In the (singular) limiting case $g(s) = g_0\delta(s)$, where $\delta(s)$ is the Dirac mass at zero, we have

$$\int_0^\infty g'(s)\partial_{xx}u(t-s)\,\mathrm{d}s = g_0\partial_{xx}\partial_t u,$$

which corresponds to the strong damping term of the model proposed by Ball [3]. According to the fading memory principle in thermodynamics, the memory kernel μ is a positive summable function. Besides, naming

$$\kappa = \int_0^\infty \mu(s)\,\mathrm{d}s > 0,$$

we assume that $\alpha_0 > \kappa$. Indeed, the quantity

$$\alpha = \alpha_0 - \kappa > 0$$

represents the relaxation modulus of the viscoelastic material, which is positive in solids. Hence, (A.3) can be rewritten in the more convenient form

$$\partial_{tt}u + \alpha\partial_{xxxx}u + \int_0^\infty \mu(s)\partial_{xxxx}[u(t) - u(t-s)]\,\mathrm{d}s - \big(\beta + \gamma\|\partial_x u\|^2_{L^2(0,L)}\big)\partial_{xx}u = f. \quad \text{(A.4)}$$

We finally remark that, in linear viscoelasticity, μ is usually assumed to be a nonnegative, absolutely continuous and nonincreasing function. This is useful for many purposes (e.g. for the definition of a quadratic energy) and is consistent with the real behaviour of a viscoelastic solid. We refer to the book [17] for a deeper analysis of the assumptions on μ. When the memory effects are neglected, we recover the model of [30]. If the transverse motion takes place in a viscous medium, whose resistance is proportional to the velocity, an additional (weak) damping term of the form $\partial_t u$ appears in the equation (see [3, 10, 12]).

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