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ORIGINAL ARTICLE

Mathematical utility theory and the representability of demand by continuous homogeneous functions

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Abstract The resort to utility-theoretical issues will permit us to propose a constructive procedure for deriving a homogeneous of degree one continuous function that gives raise to a primitive demand function under suitably mild conditions. This constitutes the first self-contained and elementary proof of a necessary and sufficient condition for an integrability problem to have a solution by continuous (subjective utility) functions.

Keywords Strong axiom of homothetic revelation · Revealed preference · Continuous homogeneous of degree one utility · Integrability of demand

JEL Classification D11

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1 Introduction

The integrability problem intends to ascertain when observed demand agrees with the postulates of the neoclassical model of consumer behavior (consumers' choices are derived from maximization of a utility, or perhaps only a preference, subject to a budget constraint). Two approaches provide different insights into that question: *revealed preference theory* (Uzawa 1971; Stigum 1973 or Mas-Colell 1978) and *integrability theory* (Hurwicz and Uzawa 1971). The partial answers that a number of well-known classical studies gave are highly ellaborated, technical, and lengthy when (semi)continuous utilities are involved, which caused a significant number of corrections and arguments in the specialized literature. Perhaps the only elementary approaches to the problem are Sondermann (1982), which inspires on Hurwicz and Richter (1971), as well as Alcantud and Rodríguez-Palmero (2002), that inspires on both works. Neither of them reaches full continuity of the subjective utilities that are used to explain the demand; nor have they provided necessary and sufficient conditions. The novelty there was the use of techniques that arise from mathematical utility theory. That possibility was fully exploited in Alcantud and Rodríguez-Palmero (2002), where a procedural proposal is explicited in order to benefit most from the appeal to arguments from utility theory. Such method transforms the integrability question into one of representability of the revealed preference by weak utilities, and then this associated binary relation is analyzed in order to check that it admits suitable utilities.

In this work that systematized approach will permit us to give a self-contained proof of a solution to the problem of representing a consumer's demand by means of an homogeneous function. Our procedure is *constructive* and uses standard tools. In order to reach our target, we first need to put forward some technical auxiliary results that concern the representability of homothetic preferences, and afterwards we will focus on the particular case of the preference revealed by a demand function. The final result will consist of a constructive procedure for deriving a *homogeneous of degree one and continuous* function that gives raise to a primitive demand function under suitably mild conditions.

In a much related line, Liu and Wong (2000) had approached the same problem recently. We shall compare our proposal to the proof that they gave of the main solution to their integrability problem. The milestone in achieving it is the appeal to the Compactness theorem of first-order predicate logic (cf. [Liu and Wong 2000, Lemma 1]). The use of such tool supposes a heavy handicap to the average researcher, which contrasts with the self-contained and simple technique developed herein.

We have organized this paper as follows. Section 2 focuses on the technical results that will be needed in order to solve the integrability issue under inspection. Among other useful situations, they concern the case where the underlying set is the positive orthant of a Euclidean space. Some relevant properties of the revealed homothetic preference, as well as the implications of the SHA, will be put forward in Section 3. Finally, Section 4 integrates all these different steps, furnishing a constructive solution to the representability questions we have announced above: Theorems 2 and 3 are the main objective of this research. Their relevance lies in their proofs: in our view, this is the first necessary and sufficient condition for an integrability problem to have a solution by continuous functions whose proof is elementary and self-contained. We summarize the consequences of our research in Section 5. An Appendix contains the proof of the main auxiliary result.

2 Homogeneous and continuous weak utilities

In this Section we focus on the possible existence of a homogeneous of degree one and continuous weak utility for an acyclic binary relation on a real cone in a topological real vector space. The non-negative orthant of $\mathbb{R}^n$ is an interesting example of a real cone in a Euclidean space. Our main tool will be Theorem 1 below, plus a technical ad hoc Corollary. The latter will produce a sufficient condition that will be key in our application to integrability theory. Before that, we shall explicit the definitions involved.

A binary relation $\succ$ on a set X is said to be *acyclic* if $x_1 \succ x_2 \succ \ldots \succ x_n$ implies $x_1 \neq x_n$ for all $x_1, \ldots, x_n \in X$. Given an acyclic binary relation $\succ$ on a set X, we shall denote by $\succ\!\succ$ the *transitive closure* of $\succ$ (i.e., $x \succ\!\succ y$ if and only if there exist $x_1, \ldots, x_n \in X$ such that $x = x_1 \succ x_2 \succ \ldots \succ x_n = y$), which is a *partial order* (i.e., it is irreflexive and transitive). A subset A of a set X endowed with an acyclic binary relation $\succ$ is said to be a *lower set* if $y \succ x$ and $y \in A$ imply $x \in A$. An acyclic binary relation $\succ$ on a topological space X is said to be *tc-upper semicontinuous* if $\{z \in X : x \succ\!\succ z\}$ is an open set for every $x \in X$, i.e., if its transitive closure is upper semicontinuous.

If A is any subset of a real vector space A, then define, for every real number t, $tA = \{tx : x \in A\}$. We recall that a real cone X in a topological real vector space E is a subset of E such that $tx \in X$ for every $x \in X$ and $t \in \mathbb{R}^{++}$. An acyclic binary relation on a real cone X in a topological real vector space E is said to be *homothetic* if $x \succ y$ implies $tx \succ ty$ for every $x, y \in X$ and $t \in \mathbb{R}^{++}$. A real-valued function $\varphi : X \longrightarrow \mathbb{R}$ on the real cone X is *homogeneous of degree one* if $\varphi(\lambda x) = \lambda\varphi(x)$ for every $x \in X$ and $\lambda > 0$.

A *weak utility* for an acyclic binary relation $\succ$ on a set X is a real-valued function u on X such that $u(x) > u(y)$ for all $x, y \in X$ with $x \succ y$.

THEOREM 1 Let $\succ$ *be an acyclic binary relation on a real cone X in a topological real vector space E. Then the following conditions are equivalent:*

(i) *There is a homogeneous of degree one, continuous weak utility u for* $\succ$.
(ii) *There is a countable family* $\{G_r\}_{r\in\mathbb{Q}}$ *of open lower subsets of X such that*

  (a) $\bigcup_{r\in\mathbb{Q}} G_r = X$, $\bigcap_{r\in\mathbb{Q}} G_r = \emptyset$;
  (b) $q < r$ *implies* $\overline{G_q} \subseteq G_r$ for all $q, r \in \mathbb{Q}$ ($\overline{C}$ *is the topological closure of* C);
  (c) $qG_r = G_{qr}$ for every $r \in \mathbb{Q}$ and $q \in \mathbb{Q}^{++}$;
  (d) *for every* $x, y \in X$ *such that* $x \succ\!\succ y$ *there exist* $r_1, r_2 \in \mathbb{Q}$ such that $r_1 > r_2$, $x \notin G_{r_1}$, $y \in G_{r_2}$.

Bosi and Zuanon (2000) already provided an axiomatization of the existence of a nonnegative, homogeneous of degree one and continuous order-preserving function for a not necessarily complete preorder. Theorem 1 is a quite natural generalization of their statement. Its proof is a not difficult variation on the same technique and thus we omit it.

REMARK 1 *If we seek upper semicontinuity only, then the conditions that characterize existence are (a),* (b$'$) $q < r$ *implies* $G_q \subseteq G_r$ *for all* $q, r \in \mathbb{Q}$*, (c) and (d).*

In subsequent Sections we will show that certain demand functions have an intrinsic structure that permits us to derive them from homogeneous of degree one and continuous functions. In order to ease this task we first provide a technical Corollary that will apply to that context. Its proof is given in the Appendix.[1]

COROLLARY 1 *Let* $\succ$ *be a tc-upper semicontinuous and homothetic acyclic binary relation on a real cone X in a topological real vector space E, and assume that the following conditions hold:*

(i) *for each non-minimal element* $x \in X$, *and for each* $y \in X$, *there is* $r > 0$ *rational such that* $rx \succ\!\succ y$;
(ii) *whenever* $x \succ\!\succ y$, *and for each* $\lambda \in (0,1)$, *then* $x \succ\!\succ \lambda y$.
Then there is a homogeneous of degree one, upper semicontinuous weak utility for $\succ$.
Suppose further
(iii) *there is* $\overline{x} \in X$ *that is not minimal for* $\succ$ *and* $G_1 = \{a \in X : \overline{x} \succ\!\succ a\}$ satisfies: for each $q > 1$ *rational,* $\overline{G_1} \subseteq qG_1$.
Then, there exists a homogeneous of degree one, continuous weak utility u for $\succ$.

3 Revealed preference and homotheticity

In this section we will perform an ad hoc study of some questions related to concepts involved in the search of a solution to the homogeneous representability problem. Formally this is a preliminary stage, since one must reach homothetic preferences that induce a given demand when a homogeneous of degree one function explains such demand function. In order to separate what depends on specific properties of the demand from common properties of more abstract models, we place ourselves in the general framework of choice structures first.

Therefore, let us fix a *choice structure* on a given set X, that is, a pair $(\mathcal{B}, c)$ where $\mathcal{B}$ is a collection of nonempty subsets of X and $c : \mathcal{B} \longrightarrow X$ is a correspondence such that $\emptyset \neq c(B) \subseteq B$ for all $B \in \mathcal{B}$. We say that the structure has univalued choices if $c(B)$ is a singleton for each $B \in \mathcal{B}$. Also, X will be a cone of a real vector space. This context permits to define the *homotheticity* of $\mathcal{R}$ binary relation on X. That concept means that $x \mathcal{R} y$ implies $(\lambda x) \mathcal{R} (\lambda y)$ for each $\lambda > 0$. *All choice structures will be univalued along this Section.*

[1]Besides, and provided that the primitive sets G_r are known, the proofs of the results above yield a precise functional form for the utility $u(x)$.

If $y \neq x$ and there is a $B \in \mathcal{B}$ such that $y \in B$ and $x = c(B)$ then we say that x is *directly revealed preferred to* y, and we write $x S y$ (cf. Samuelson (1938, 1950)). If there is a $B \in \mathcal{B}$ such that $y \in B$ and $x = c(B)$ then we say that x is (*weakly revealed preferred to*) y, and we write $x V y$ (cf. Clark 1988, Hurwicz and Richter 1971, p. 60). Let F be defined by: xFy if and only if there is $\lambda > 0$ with $\lambda x V \lambda y$. Then, F will be called the *homothetic closure* of the (weak) revealed preference V (the homothetic closure H of S is defined in Liu and Wong 2000, p. 291).

It is easy to check that $x H y$ if and only if ($x F y$ and $x \neq y$). By construction, H extends S and F extends V. Because V extends S -actually, $x S y$ if and only if ($x V y$ and $x \neq y$)-, F must extend H too.

For later use, we introduce:

$$WHA \; x F y \Rightarrow y S x \text{ false}$$

the weak axiom of revealed homothetic preference.

The choice structure $(\mathcal{B}, c)$ on X is *representable* if there exists a function $u : X \longrightarrow \mathbb{R}$ such that $c(B) = \{x \in B : u(x) \succcurlyeq u(y) \text{ for all } y \in B\}$, for any $B \in \mathcal{B}$.

We say that the choice structure $(\mathcal{B}, c)$ on X is *rationalizable* if there exists a *preference* (i.e., complete, transitive binary relation) $\succcurlyeq$ such that $c(B) = \{x \in B : x \succcurlyeq y \text{ for all } y \in B\}$, for any $B \in \mathcal{B}$. Obviously, representability implies rationalizability. Observe that univaluedness of c yields (a) xSy implies $x \succ y$, and therefore S is acyclic; and (b) $c(B) = \{x \in B : x \succ y \text{ for all } y \in B \setminus \{x\}\}$, for any $B \in \mathcal{B}$. Acyclicity of S constitutes Houthakker's *Strong Axiom of Revealed Preference* (SARP). Furthermore, this axiom implies Samuelson's *Weak Axiom of Revealed Preference* (WARP), which amounts to S being asymmetric. Moreover, it is well known that any of WARP or SARP implies that choices must be univalued, and that SARP is equivalent to rationalization (cf. Richter 1966, also Richter 1971, Corollary 1).

We next enunciate some fundamental properties that affect homothetic rationalizability. Given a univalued choice structure on a cone of a real vector space X that is rationalizable by a homothetic preference $\succcurlyeq$:

Property 1. It is clear that $\succcurlyeq$ must extend V, F and obviously the transitive closure F^* of F (which is transitive itself). The same can be said about S, H and the transitive closure H^* of H (which is transitive as well); the latter fact is mentioned in Remark 1 (a) of Liu and Wong (2000).

Property 2. The homothetic closure F of the (weak) revealed preference V rationalizes it. That is, for any fixed $B \in \mathcal{B}$ we show that $c(B) = \{x \in B : x F y \text{ for all } y \in B\}$. It is clear that $x = c(B)$ yields $x V y$ and then $x F y$ for all $y \in B$. Now let us take $x \in B$ with $x F y$ for every $y \in B$. Assume, by way of contradiction, $x \neq c(B)$; equivalently, there is $z \neq x$ such that $z = c(B)$. Then $z \succ x$ while $x \succcurlyeq z$ -because extends F and $x F z$-, a contradiction.

Property 3. The following SHA holds true[2]: $x H^* y \Rightarrow y S x$ false.

[2]Observe that the SHA axiom is defined in Liu and Wong (2000), as acyclicity of H, which is equivalent to the definition above. Property 3 is mentioned in Theorem 1 of Liu and Wong (2000), for a particular context.

This latter condition obtains because $x H^* y$ implies $x \succcurlyeq y$ (see Property 1) and this is not compatible with $y S x$ and thus with $y \succ x$ due to the univaluedness of c.

For certain demand functions on $\mathbb{R}^l_+$, Theorem 1 of Liu and Wong (2000) prove that SHA is sufficient for rationalizability by homothetic preferences too. The technique they use stems from Richter (1971) and Kannai (1992). It appeals to first-order predicate languages. That issue has been used by Liu and Wong to argue in favour of the empirical relevance of the H relation—cf. their Remark 1 (b)- which is observable as long as choices are. The same kind of argument follows after the latter properties with regard to either V or F.

It is trivial that any of SHA or WHA forces univalued choices. Moreover, SHA is stronger than WHA for, under SHA, any pair $x, y \in X$ such that $x F y$ must satisfy that $y S x$ is false. If $x = y$, this is obvious. Otherwise, because $x H y$ and then $x H^* y$, the SHA says $y S x$ false.

It is also true that WHA implies WARP because under WHA, S is asymmetric. Indeed, given a pair $x, y \in X$ such that $x S y$, since forcefully $x F y$ it follows that $y S x$ must be false.

Besides, SHA implies SARP because S is acyclic under SHA. Note that for any possible cycle $x_1, ..., x_n \in X$ such that $x_1 S .... S x_n S x_1$, because necessarily $x_1 H .... H x_n$ (i.e., $x_1 H^* x_n$) it follows that $x_n S x_1$ should be false, a contradiction.

Property 4. H rationalizes the structure, i.e., $c(B) = \{x \in B : x H y \textit{ for all } y \in B, y \neq x\}$. Let us fix $B \in \mathcal{B}$ and denote by x_B the only element in $c(B)$. It is plain that $c(B) = \{x \in B : x F y \textit{ for all } y \in B\} \subseteq \{x \in B : x H y \textit{ for all } y \in B, y \neq x\}$. Besides, given $x \in B$ such that $x H y$ whenever $y \in B \setminus \{x\}$, forcefully $x = x_B$. Otherwise $x H x_B H x$ and thus $x \sim x_B$ (because $\succcurlyeq$ extends H), contradicting the univaluedness of choice due to $x, x_B \in c(B) = \{x \in B : x \succcurlyeq y \textit{ for all } y \in B\}$. In short, $c(B) = \{x \in B : x H y \textit{ for all } y \in B, y \neq x\}$.

4 Homogeneous representability of demand

In this Section we apply the results we have explicited before in order to find necessary and sufficient conditions for certain demand functions to be explained as the result of optimizing a continuous, homogeneous of degree one function (subjective utility). As we have announced, the basis to proceed in this way will be Corollary 1. First we will engage in a study of the structure of our demand functions in order to be enabled to check that its assumptions apply.

The following context will be assumed in the remaining of this Section. X will denote a fixed consumption set (the non-negative orthant of $\mathbb{R}^n$ for some n). We adopt the usual notation for vector prices $p = (p^1, ..., p^n)$, income w and budget sets $B(p, w) = \{x \in X : p \cdot x \leqslant w\}$. $\mathcal{B}$ will be the collection of non-empty budget sets of X associated with the subset of price-income pairs $P \times M = \mathbb{R}^n_{++} \times (0, +\infty)$. Let h be a demand function on $\mathcal{B}$, that is, a function that selects exactly one element (denoted by $h(p, w)$ or by $h(B)$) for each $B = B(p, w) \in \mathcal{B}$.

Let us fix an arbitrary $B_0 = B(p_0, w_0)$, and denote by x_0 the only element in $h(B_0)$. Then, the following list of properties hold:

(1) For all $t \in (0,1)$: $x_0\, S\, (t x_0)$. Therefore, $q, r > 0$ and $r > q$ imply $(r x_0)\, H\, (q x_0)$.

Proof The first assertion is immediate: $t x_0 \in B(p_0, w_0)$ and $\{x_0\} = h(B_0)$. Concerning the second one: observe that $x_0\, S\, (\frac{q}{r}\, x_0)$, and thus $x_0\, H\, (\frac{q}{r}\, x_0)$ because H extends S. Since H is homothetic by construction, $(r x_0)\, H\, (q x_0)$.

(2) For all $x \in X$, there is $r \in \mathbb{Q}_{++} \setminus \{0\}$ with $(r\, x_0)\, H\, x$

Proof If $x = x_0$, the prior property proves the assertion: $x_0\, S\, \frac{x_0}{2}$, and now $(2 x_0)\, H\, x_0$ because H extends S and is homothetic.

If $x \neq x_0$, but $x \in B_0 = B(p_0, w_0)$, one has $x\, S\, x_0$ and so $x\, H\, x_0$.

Finally, if $x \neq x_0$ and also $x \in / B_0 = B(p_0, w_0)$, there is $r \in \mathbb{Q}$, $r \neq 0$, such that $p_0\, \frac{x}{r} < w_0$ and $\frac{x}{r} \neq x_0$. By construction, $x_0\, S\, \frac{x}{r}$, and now $(r x_0)\, H\, x$.

(3) Given $u : X \longrightarrow \mathbb{R}$ homogeneous of degree 1:

u generates h if and only if u is a weak utility for H

Proof This result follows easily; it parallels Lemma 1 in Alcantud and Rodríguez-Palmero (2002).

In order to apply the technique exposed in the previous sections, we are also interested in studying the behavior of all sets with a particular form. We keep $p_0 \gg 0$ arbitrary but fixed and for simplicity x_w will denote the only element in $h(p_0, w)$ for each positive income w. Let us denote $G_w = \{y \in X :\ x_w H^* y\}$, for each $w > 0$. It is plain that $B(p_0, w) \setminus \{x_w\} \subseteq G_w$. We have:

(4) $G_{rw} = r G_w$ for each $r > 0$

Proof We only need to prove $G_{rw} \subseteq r G_w$ for each $r > 0$. The converse inclusion is clearly immediate from that one (it requests that $G_w \subseteq \frac{1}{r} G_{rw}$ for each $r > 0$).

If $y \in G_{rw}$ then $x_{rw} H^* y$ by construction, therefore $\frac{1}{r} x_{rw} H^* \frac{1}{r} y$ by homotheticity of H^*. Because $\frac{1}{r} x_{rw} \in B(p_0, w)$, it follows that $x_w\, S\, \frac{1}{r} x_{rw}\, H^*\, \frac{1}{r} y$ and so $x_w\, H^*\, \frac{1}{r} y$. Thus, $\frac{1}{r} y \in G_w$ and $y \in r G_w$.

(5) For every $w > 0$ income, $y \in G_w$ implies $\lambda y \in G_w$ for each $\lambda \in (0,1)$

Proof We first check the assertion that $y \in G_1$ implies $\lambda y \in G_1$ for each $\lambda \in (0,1)$. Because $x_1 H^* y$ (x_1 is the only element in $h(p_0, 1)$), for some $x \in X$ forcefully $x_1 H^* x H y$ by definition of transitive closure (in case $x_1 H y$ we proceed with $x = x_1$). Then, $ax\, S\, ay$ for some $a > 0$ by construction, which means $ax\, S\, (a\lambda y)$ (because $\lambda(ay)$ is clearly available in the same demand situation for which ax is chosen, being ay affordable). In conclusion $x_1 H^* x H(\lambda y)$ and so $x_1 H^*(\lambda y)$, which means that $(\lambda y) \in G_1$ by definition.

Now for every $w > 0$ income, if $y \in G_w$ then $\lambda y \in G_w$ for each $\lambda \in (0,1)$. The reason is that $\frac{1}{w} y \in G_1$ because $y \in w G_1 = G_w$ by Property (4), and then $\frac{\lambda}{w} y \in G_1$ by the previous assertion, yielding $\lambda y \in w G_1 = G_w$.

(6) $w > w' > 0$ implies $G_{w'} \subseteq G_w$

Proof We know $G_{w'} = w' G_1 \subseteq w G_1 = G_w$.

The SHA is necessary for a demand function to be represented by a homogeneous of degree 1 function, being implied by homothetic rationalizability only. We want to investigate the implications of requesting that the demand behavior be consistent with that requirement. Thus, *assuming SHA henceforth*, we have the following further properties:

(7) $\lambda x_w \notin G_w$ for each $\lambda > 1$

Proof Should there exist $\lambda > 1$ for which $\lambda x_w \in G_w$, by (5) we would also have $x_w \in G_w$, that is, $x_w H^* x_w$. This contradicts acyclicity of H, i.e., SHA.

(8) h is exhaustive

Proof Assume that for the prices p_0 (which were fixed but arbitrary) it is true that there is an income w such that the demanded bundle x_w satisfies $p_0 x_w < w$.

We would then derive the existence of a $\lambda > 1$ for which $\lambda x_w \in B(p_0, w)$, and therefore $x_w S y$ by definition of S. But this means $\lambda x_w \in G_w$, against (7).

(9) $S = H$

Proof Observe that, according to Remark 8 in Liu and Wong (2000), because SHA holds then the demand function h satisfies $h(p, \lambda w) = \lambda h(p, w)$ for each $\lambda > 0$ and every $(p, w) \in P \times M$. This latter property easily yields homotheticity of S, that is to say, $S = H$.

(10) If $z \in G_w$ and $a \in X$ is such that $z > a$ (by this we mean: $z_i \geqslant a_i$ for each component i and $z_{i_0} > a_{i_0}$ for at least one component i_0) then $a \in G_w$.

Proof There must be $y \in X$ with $x_w H^* y H z$; should we have $x_w H z$ then we proceed with $y = x_w$. Because $S = H$, $y = h(\overline{p}, \overline{w})$ for some $(\overline{p}, \overline{w}) \in P \times M$ that satisfies $\overline{p} z \leqslant \overline{w}$. But now $\overline{p} a < \overline{w}$ because $P = \mathbb{R}^n_{++}$, therefore $y S a$.

Finally, an answer to the homogeneous representability problem will follow from:

(11) Suppose that

$$x S y \Rightarrow \exists B' = B(\overline{p}, \overline{w}) \in \mathcal{B} \text{ such that } \begin{cases} \overline{p} \cdot (tx + (1-t)y) \leqslant \overline{w} \text{ for some } t \in (0,1), \\ x \; S \; h(B') \end{cases}$$

Then H^* is upper semicontinuous.

Proof Observe that SARP was already assumed implicitely, for SHA holds by assumption. Now the transitive closure of S is upper semicontinuous; that fact is guaranteed by the proof of Proposition 2 in Alcantud and Rodríguez-Palmero (2002). By (9), the transitive closure of H, that is, H^*, is upper semicontinuous.

All the groundwork we have been performing heretofore culminates in the following result and the comments following it:

THEOREM 2 *For any demand function that satisfies the condition given in (11): SHA is equivalent to the existence of a continuous, homogeneous of degree 1 function that generates the demand.*

Proof Property 3 accounts for the necessary condition.

We now turn to sufficiency. Therefore, assume that SHA holds. Due to (3) we need an homogeneous of degree 1 continuous weak utility for either H or S because of (9). Corollary 1 assures the existence of the desired function. Note that S is acyclic by SHA and homothetic due to (9). Property (11) ensures that its transitive closure H^* is upper semicontinuous. The requirement (i) holds by (2). Property (5) accounts for $\{(ii)\}$: note that x not minimal means that $x = x_w = h(p, w)$ for some prices p and income w, and then $x H^* y$ means simply $y \in G_w$. So, we only need to check (iii), that is, $\overline{G_1} \subseteq qG_1$ for each $q > 1$ rational.

Fix $q > 1$ rational, and take $x \in \overline{G_1}$. There is $\varepsilon > 0$ such that every $z \in B(x, \varepsilon) \cap X$ satisfies $z > \frac{1}{q}x$ because either x is the null vector and the statement is trivial or otherwise $x > \frac{1}{q}x$. But there must be $z \in B(x, \varepsilon) \cap X \cap G_1$ because $x \in \overline{G_1}$, which forces $\frac{1}{q}x \in G_1$ by (10). Therefore, $x \in qG_1$.

In comparison to Theorem 3 in Liu and Wong (2000), and putting aside the much different techniques involved, Theorem 2 presumes an assumption weaker than full continuity in order to *check* that we can obtain continuous representability. The condition we have used in (11) was put forward in Alcantud and Rodríguez-Palmero (2002). The relationship of that requirement with others that appeared in related literature is explained there in detail. In particular, continuous demand functions display that behavior, a fact that yields the following sharpened consequence:

THEOREM 3 *Given h demand function on X: it is continuous and satisfies the SHA if and only if it can be represented by a continuous, homogeneous of degree 1 function.*

Proof We only need to justify that, because homogeneous of degree 1 utilities induce locally nonsatiated preferences on X, Proposition 3.AA.1 in Mas-Colell et al. (1995) applies and therefore the demand induced is continuous.

5 Conclusions

The extensive use of utility-theoretical techniques, as well as a deep analysis of the structure of the homogeneous representability problem, have permitted us to propose a self-contained *constructive* procedure for deriving a homogeneous of degree one continuous function giving raise to an original demand function under mild conditions. As far as we know *this is the first elementary and self-contained proof of a necessary and sufficient condition for an integrability problem to have a solution by continuous (subjective utility) functions*.

Our analysis produces several side remarks. Observe that unlike Liu and Wong we made no assumption on the openness or finiteness of the range of h in our main theorems. Also, our demand functions were not initially requested to be exhaustive, but we have observed in (8) that *assuming SHA forces exhaustiveness*, a fact that seems to have gone unnoticed so far. Moreover, we have taken a further step by observing that continuity of the demand was also necessary to have a solution. This was due to the fact that continuous utilities inducing locally nonsatiated preferences on X give raise to a continuous demand, provided that this demand is a function.

The constructive nature of our proof does not entail that applying our method in practice must be straightforward. Still, we can brief the reader on the functional form of a continuous, homogeneous of degree 1 function that induces h. This is obtained from the raw revealed preference S as follows. Through either S or its transitive clausure H^* we check SHA, and then define $G_1 = \{y \in X : x_w H^* y\}$ (associated with any fixed x_w). Now let

$$u(x) = inf\{r \in \mathbb{Q} : x \in G_r\}$$

where $G_r = rG_1$ for $r \in \mathbb{Q}$, $r > 0$ and $G_r = ø$ for $r \in \mathbb{Q}$, $r < 0$. Corollary 1 guarantees that this function is a continuous weak utility for $S = H$ and now (3) ensures that it induces the demand.

Appendix: proof of Corollary 1

Proof of Corollary 1: The case where all elements of X are minimal is trivial. Suppose therefore that there is $\bar{x} \in X$ that is not minimal respect to $\succ$. Define $G_1 = \{a \in X : \bar{x} \succ\!\succ a\}$, $G_{-1} = ø$. Take $G_q = qG_1$ for each $q > 0$ rational, and $G_q = ø$ for each $q \leqslant 0$ rational. These are open (by semicontinuity of $\succ\!\succ$) and lower (by homotheticity of $\succ$ and therefore of $\succ\!\succ$) subsets. Let us first check that the collection $\{G_q\}_{q \in \mathbb{Q}}$ satisfies conditions (a), (b'), (c) and (d) of Theorem 1 (see Remark 1) in order to ensure the existence of a homogeneous of degree one, upper semicontinuous weak utility u for $\succ$ under (i) and (ii).

In order to verify that condition (a) of Theorem 1 holds, fix any $x \in X$. Because there is $q > 0$ rational such that $q\bar{x} \succ\!\succ x$ by (i) and $\succ\!\succ$ is homothetic too, one has $\bar{x} \succ\!\succ \frac{x}{q}$, which means $\frac{x}{q} \in G_1$ or equivalently, $x \in qG_1 = G_q$. Hence, we have that $\bigcup_{r \in \mathbb{Q}} G_r = X$.

Furthermore, $\bigcap_{r \in \mathbb{Q}} G_r = ø$ by construction.

Now let us prove that condition (b') of Theorem 1 is verified (see Remark 1). Let us fix $q < r \in \mathbb{Q}$. Unless $q, r > 0$, assumption (b) holds trivially. Suppose therefore that $q, r > 0$ and take an arbitrary element in G_q, that is, an element with the form qa where $a \in G_1$. Because $qa = r(\frac{q}{r}a)$, in order to check $qa \in G_r = rG_1$ we only need to justify that $\frac{q}{r}a \in G_1$. This fact derives from $\bar{x} \succ\!\succ a$ plus (ii) above easily.

Assumption (c) of Theorem 1 holds by construction.

Finally, let us show that condition (d) of Theorem 1 holds. Consider any two elements $x, y \in X$ such that $x \succ\!\succ y$. Then either $x = z \succ y$ or there is $z \in X$ with $x \succ\!\succ z \succ y$. Clearly, we are done if we justify the existence of $r_1, r_2 \in \mathbb{Q}$ such that $r_1 > r_2$, $z \notin G_{r_1}$, $y \in G_{r_2}$, whatever the case holds. Let us define the set $A(z) = \{t > 0 : \bar{x} \succ\!\succ tz\}$. It is non-empty by ($i$). And it is bounded above: since z is not minimal there is $t_0 > 0$ such that $t_0 z \succ\!\succ \bar{x}$ by (i), and now acyclicity of $\succ$ prevents the existence of t arbitrarily large with $\bar{x} \succ\!\succ tz$ because of condition (ii). Denote $\lambda_0 = \sup A(z)$. Since $\succ$ is tc-upper semicontinuous we have that $\lambda_0 \notin A(z)$ due to the continuity of scalar multiplication. We have thus shown $\lambda_0 z \notin G_1$, i.e., $z \notin \frac{1}{\lambda_0} G_1$. Let us check that $\lambda_0 y \in G_1$, i.e., $y \in \frac{1}{\lambda_0} G_1$. Note that $\lambda_0 z \succ \lambda_0 y$ by homotheticity. Upper semicontinuity of $\succ\!\succ$ grants the existence of $\varepsilon_0 > 1$ with $\lambda_0 z \succ\!\succ (\lambda_0 \varepsilon_0) y$, therefore $\frac{\lambda_0}{\varepsilon_0} z \succ\!\succ \lambda_0 y$. By definition of λ_0, $\bar{x} \succ\!\succ \frac{\lambda_0}{\varepsilon_0} z \succ\!\succ \lambda_0 y$ and so $\lambda_0 y \in G_1$. Now fix any rational number $\overline{r_2}$ such that $0 < \lambda_0 < r_2$, $y \in \frac{1}{r_2} G_1$; such a number exists by upper semicontinuity of $\succ\!\succ$ and continuity of scalar multiplication, since we have that $\bar{x} \succ\!\succ_0 y$. Now let $\bar{r}_1$ be any rational number such that $\lambda_0 < \bar{r}_1 < \bar{r}_2$. Then $\bar{r}_1 z \notin G_1$ from the definition of λ_0. Finally, set $r_1 = \frac{1}{\bar{r}_1}$ and $r_2 = \frac{1}{\bar{r}_2}$. Hence, we have that $r_1 > r_2$, $z \notin G_{r_1} = r_1 G_1$, $y \in G_{r_2} = r_2 G_1$, and therefore condition (*d*) of Theorem 1 holds.

The final statement is immediate now, because in fact the stronger condition (*b*) is satisfied if we select the element in (*iii*) and proceed as before.

References

Alcantud JCR, Rodríguez-Palmero C (2002) A note on representability of consumer's behavior. Econ Theory 20:207–216

Bosi G, Zuanon M (2000) Homogeneous and continuous order-preserving functions for noncomplete preorders. Rendiconti per gli Studi Economici Quantitativi 16–24

Clark SA (1988) An extension theorem for rational choice functions. Rev Econ Stud LV 485–492

Hurwicz L, Richter MK (1971) Revealed preference without demand continuity assumptions. In: Chipman J et al (eds) Preferences, utility and demand. Harcourt, Brace, Jovanovich, New York

Hurwicz L, Uzawa H (1971) On the integrability of demand functions. In: Chipman J et al (eds) Preferences, utility and demand. Harcourt, Brace, Jovanovich, New York

Kannai Y (1992) Non-standard concave utility function. J Math Econ 21:51–58

Liu P-K, Wong K-C (2000) Revealed homothetic preference and technology. J Math Econ 34:287–314

Mas-Colell A (1978) On revealed preference analysis. Rev Econ Stud 45:121–131

Mas-Colell A, Whinston MD, Green JD (1995) Microeconomic theory. Oxford University Press

Richter MK (1966) Revealed preference theory. Econometrica 34:635–645

Richter MK (1971) Rational choice. In: Chipman J et al (eds) Preferences, utility and demand. Harcourt, Brace, Jovanovich, NewYork

Samuelson PA (1938) A note on the pure theory of consumer's behavior. Economica N.S. 5:61–71

Samuelson PA (1950) The problem of integrability in utility theory. Economica N.S. 17:355–385

Sondermann D (1982) Revealed preference: an elementary treatment. Econometrica 50:777–779

Stigum BP (1973) Revealed preference—a proof of Houthakker's theorem. Econometrica 41:411–423

Uzawa H (1971) Preference and rational choice in the theory of consumption. In: Chipman J et al (eds) Preferences, utility and demand. Harcourt, Brace, Jovanovich, New York