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Physica D 203 (2005) 55–79
PHYSICA D
www.elsevier.com/locate/physd

Navier–Stokes limit of Jeffreys type flows

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Received 17 February 2005; accepted 14 March 2005
Communicated by R. Temam

Abstract

We analyze a Jeffreys type model ruling the motion of a viscoelastic polymeric solution with linear memory in a two-dimensional domain with nonslip boundary conditions. For fixed values of the concentrations, we describe the asymptotic dynamics and we prove that, when the scaling parameter in the memory kernel (physically, the Weissenberg number of the flow) tends to zero, the model converges in an appropriate sense to the Navier–Stokes equations.

MSC: 35B25; 35B40; 35Q30; 45K05

Keywords: Jeffreys type models; Navier–Stokes equations; Singular limit; Global attractors; Robust exponential attractors

1. The equations

Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with smooth boundary $\partial\Omega$. For $t > 0$, we consider the following system of equations in the unknown variables $u = u(x, t) : \Omega \times [0, \infty) \to \mathbb{R}^2$, $\eta = \eta^t(x, s) : \Omega \times [0, \infty) \times (0, \infty) \to \mathbb{R}^2$ and $p = p(x, t) : \Omega \times [0, \infty) \to \mathbb{R}$:

$$\begin{cases} \partial_t u - \omega\Delta u - (1-\omega)\displaystyle\int_0^\infty k_\varepsilon(s)\Delta\eta(s)\,\mathrm{d}s + (u\cdot\nabla)\,u + \nabla p = f, \\ \partial_t \eta = -\partial_s \eta + u, \\ \operatorname{div} u = 0, \\ \operatorname{div} \eta = 0. \end{cases} \tag{1.1}$$

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doi:10.1016/j.physd.2005.03.007

Here, $\omega \in (0,1)$ is a fixed parameter, $f : \Omega \to \mathbb{R}^2$, whereas the so-called *memory kernel*

$$k_\varepsilon(s) = \frac{1}{\varepsilon^2} k\left(\frac{s}{\varepsilon}\right), \quad \varepsilon \in (0,1],$$

is the rescaling of a given smooth function $k : (0,\infty) \to [0,\infty)$ such that

$$\int_0^\infty k(s)\,\mathrm{d}s = \alpha < \infty,$$

and subject to the normalization condition

$$\int_0^\infty sk(s)\,\mathrm{d}s = 1.$$

The above equations describe the horizontal motion of a fluid of Jeffreys type in an infinite cylinder of cross section Ω (for simplicity, all the physical constants are set equal to 1). System (1.1) is supplemented with the *nonslip boundary conditions*, plus an additional condition on η (which, in fact, is again a boundary condition), namely

$$\begin{cases} u(t) = 0 & \text{on } \partial\Omega, \forall t \geq 0, \\ \eta^t(s) = 0 & \text{on } \partial\Omega, \forall t \geq 0,\ \forall s > 0, \\ \eta^t(0) = \lim\limits_{s\to 0} \eta^t(s) = 0 & \text{in } \Omega, \forall t \geq 0, \end{cases} \tag{1.2}$$

and the initial conditions

$$\begin{cases} u(0) = u_0 & \text{in } \Omega, \\ \eta^0 = \eta_0 & \text{in } \Omega \times (0,\infty). \end{cases} \tag{1.3}$$

Notice that $k_\varepsilon - (\alpha/\varepsilon)\delta_0 \to -\delta_0'$ in the sense of distributions as $\varepsilon \to 0$ (δ_0 being the Dirac mass at zero). Thus, from the formal equality $\partial_s \eta^t(0) = u(t)$, we recover the classical Navier–Stokes equations

$$\begin{cases} \partial_t u - \Delta u + (u \cdot \nabla)u + \nabla p = f, \\ \operatorname{div} u = 0, \end{cases} \tag{1.4}$$

with nonslip boundary condition. Of course, another way to obtain (1.4) from (1.1) is to take the limit $\omega \to 1$.

2. Introduction

Linear viscoelastic equations approximate the complete motion equations of real viscoelastic materials. They apply to the study of small deformations of the rest state, that is, when the history of deformation is sufficiently close to the rest history. Clearly, it is possible to conceive a deformation history with arbitrarily large strain rate values and, nevertheless, arbitrarily close to the rest history: a simple example is a periodic motion with very small amplitude but very large frequency. Although linear viscoelasticity is inconsistent with material frame indifference, many excellent works based on this approach have been written (see, for instance, [5,8,12,20]), leading sometimes to new ideas and techniques.

In this framework, we consider a linear incompressible viscoelastic fluid of Jeffreys type. Following [13], the stress-strain rate constitutive relation consists of a Newtonian contribution and a viscoelastic contribution. Precisely, at any fixed point $x \in \Omega$ and any time $t \geq 0$, the symmetric Cauchy stress tensor $\boldsymbol{T}$ is given by

$$\boldsymbol{T}(t) = -p(t)\boldsymbol{I} + 2\nu_{\mathrm{N}}\boldsymbol{D}(t) + 2\nu_{\mathrm{E}} \int_0^\infty \kappa(s)\boldsymbol{D}(t-s)\,\mathrm{d}s, \tag{2.1}$$

where p is the pressure,

$$\boldsymbol{D} = \tfrac{1}{2}[\nabla v + \nabla v^\top]$$

is the strain rate, v being the velocity of the fluid, subject to the incompressibility constraint

$$\mathrm{tr}\,\boldsymbol{D} = \mathrm{div}\, v = 0,$$

and κ is a smooth positive kernel satisfying the normalization condition

$$\int_0^\infty \kappa(s)\,\mathrm{d}s = 1.$$

The positive constants ν_{N} and ν_{E} represent the Newtonian and the elastic viscosity, respectively. The sum $\nu = \nu_{\mathrm{N}} + \nu_{\mathrm{E}}$ is the total viscosity of the model, also called *zero shear viscosity* (see [13, p. 542]). Introducing the dimensionless parameter $\omega = \nu_{\mathrm{N}}/\nu \in (0, 1)$, equation (2.1) reads

$$\boldsymbol{T}(t) = -p(t)\boldsymbol{I} + 2\nu\omega\boldsymbol{D}(t) + 2\nu(1-\omega) \int_0^\infty \kappa(s)\boldsymbol{D}(t-s)\,\mathrm{d}s. \tag{2.2}$$

In the limiting situation $\omega = 1$, we have Newtonian fluids, whereas $\omega = 0$ corresponds to elastic fluids. There are molecular arguments (see [3]) suggesting a constitutive equation of this type. Indeed, very dilute solutions of polymers might exhibit a stress depending both on the deformation history and on the instantaneous value of the strain rate, where the viscous part of the behavior is contributed by the solvent. Accordingly, the quantity $(1-\omega)$ is assumed to represent the concentration of the polymers, which behave as elastic fluids. When no polymers are present ($\omega = 1$), we are left with a pure Newtonian fluid of viscosity ν_{N} (the solvent). This model is also supported by some recent investigations. For instance, in [15] it is applied to the study of bubbly liquids, whereas in [14] it is used to describe, in the linear regime, the hydrodynamic behavior of a 3D lattice Boltzmann model with 32 discrete velocities.

The Jeffreys model is a tensorial generalization of a simple rheological element given by a dashpot and a Voight model in series. It is recovered as a particular case of (2.2), assuming that

$$\kappa(s) = \frac{1}{\lambda}\mathrm{e}^{-s/\lambda},$$

where λ is called *relaxation time*. In this case, the traceless part of the stress $\mathring{\boldsymbol{T}} = \boldsymbol{T} + p\boldsymbol{I}$ can be formally obtained as the solution of the following rate-type constitutive equation

$$\mathring{\boldsymbol{T}}(t) + \lambda\frac{\mathrm{d}}{\mathrm{d}t}\mathring{\boldsymbol{T}}(t) = 2\nu\left[\boldsymbol{D}(t) + \omega\lambda\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{D}(t)\right],$$

for any given past history of the strain rate $\boldsymbol{D}$. The quantity $\omega\lambda$ is called *retardation time*. A frame-indifferent generalization of the above equation has been suggested by Oldroyd [16], who introduced lower-convected (Oldroyd-A) and upper-convected (Oldroyd-B) time derivatives in place of the ordinary ones. It is worth noting that the Jeffreys model can be viewed as the unique linearization of both the Oldroyd models A and B (see [13, p. 39] for the proof). A more realistic case, called *generalized Jeffreys model of order n*, can be obtained from (2.2) taking the kernel

$$\kappa(s)=\sum_{j=1}^{n}\frac{1}{\lambda_j}e^{-s/\lambda_j}, \tag{2.3}$$

with $\lambda_1>\lambda_2>\cdots>\lambda_n>0$. Then, a straightforward calculation shows that $\mathring{\boldsymbol{T}}$ obeys the rate-type equation

$$\mathring{\boldsymbol{T}}(t)+\sum_{j=1}^{n}\alpha_j\frac{d^j}{dt^j}\mathring{\boldsymbol{T}}(t)=2\nu\left[\boldsymbol{D}(t)+\sum_{j=1}^{n}\beta_j\frac{d^j}{dt^j}\boldsymbol{D}(t)\right],$$

where the coefficients α_j and β_j, for $j=1,\dots,n-1$, are related to the *relaxation times* λ_j, and $\beta_n=\omega\alpha_n$. As pointed out in [2], this choice fits very well isothermal stress–relaxation experiments on polymeric materials. In some cases the following relation can be assumed

$$\lambda_j=\frac{\lambda_1}{j^2},\quad j=1,\dots,n. \tag{2.4}$$

However, the influence of the jth term in the kernel becomes negligible for large values of j, so that a definite value for n is not needed provided that it is assumed to be sufficiently large. It is well known that viscoelastic materials have an intrinsic (often called *elapsed*) time scale. Thus, a natural time Λ can be defined in terms of the memory kernel, which represents in some sense the length of the memory. Although no exact definition is available, following [2, p. 249] we assume here

$$\Lambda=\frac{\int_0^\infty s^2\kappa(s)\,ds}{2\int_0^\infty s\kappa(s)\,ds}.$$

This choice seems preferable on the basis of the molecular theories of polymers. Indeed, for kernels of the form (2.3), $\Lambda=\sum\lambda_j^2/\sum\lambda_j$. This value does not depend on n (if n is sufficiently large), and it is of the order of the largest relaxation time λ_1, at least when relation (2.4) holds.

In order to obtain the equations governing the evolution of the fluid flow, a careful description of the kinematics is needed. The motion χ of a simple fluid at time τ is described with respect to the actual configuration x at time t, namely

$$\chi^t(x,\tau)=x-q^t(x,\tau),\quad x\in\Omega,\ \tau\le t,$$

where q^t is the *relative displacement vector*. In particular, $\chi^t(x,t)=x$. The velocity v^t at time $\tau\le t$ of the fluid particle which occupies the position x at time t is defined as

$$v^t(x,\tau)=\partial_\tau\chi^t(x,\tau)=-\partial_\tau q^t(x,\tau).$$

Often, the superscript t is ignored, as we did in the first part of this section. Since the same fluid particle occupied the position $\chi^t(x,\tau)$ at time $\tau\le t$, introducing the *Eulerian* velocity $u(\chi,\tau)$ of the fluid at position χ and time τ, we have

$$u(\chi^t(x,\tau),\tau)=v^t(x,\tau).$$

Accordingly, the acceleration is given by

$$\partial_\tau v^t(x,\tau) = \partial_\tau u(\chi^t(x,\tau),\tau) + (u(\chi^t(x,\tau),\tau)\cdot\nabla_\chi)u(\chi^t(x,\tau),\tau).$$

In particular, when $\tau = t$, we have $v^t(x,t) = u(x,t)$, so that

$$\partial_\tau v^t(x,\tau)_{|\tau=t} = \partial_t u(x,t) + (u(x,t)\cdot\nabla)u(x,t).$$

Accounting for linear viscoelasticity (cf. [2]), we consider

$$\boldsymbol{E}^t(x,\tau) = -\tfrac{1}{2}[\nabla q^t(x,\tau) + \nabla q^t(x,\tau)^\top],$$

subject to the linearized incompressibility constraint

$$\operatorname{tr}\boldsymbol{E}^t = -\operatorname{div} q^t = 0.$$

A formal integration by parts transforms the convolution integral appearing in (2.2) into

$$\int_0^\infty \kappa(s)\boldsymbol{D}^t(t-s)\,\mathrm{d}s = -\int_0^\infty k(s)\boldsymbol{E}^t(t-s)\,\mathrm{d}s,$$

where we put $k(s) = -\kappa'(s)$. This new kernel satisfies the normalization condition

$$\int_0^\infty s k(s)\,\mathrm{d}s = 1.$$

Hence, (2.2) takes the form

$$\boldsymbol{T}(t) = -p(t)\boldsymbol{I} + 2\nu\omega\boldsymbol{D}(t) - 2\nu(1-\omega)\int_0^\infty k(s)\boldsymbol{E}^t(t-s)\,\mathrm{d}s. \tag{2.5}$$

Assuming a constant density ρ, the motion equations of a Jeffreys type incompressible fluid are derived substituting (2.5) into the usual balance equations, so obtaining the differential system

$$\begin{cases} \rho[u_t + (u\cdot\nabla)u] - \nu\omega\Delta u - \nu(1-\omega)\displaystyle\int_0^\infty k(s)\Delta q^t(t-s)\,\mathrm{d}s + \nabla p = f, \\ \operatorname{div} u = \operatorname{div} q = 0, \end{cases}$$

where f is the density of driving force per unit volume and q^t satisfies the integral equation

$$q^t(x,\tau) = \int_\tau^t u(x - q^t(x,y), y)\,\mathrm{d}y, \quad \tau \le t.$$

Since we are dealing with small perturbations of the rest state, we can make the approximation

$$q^t(x,\tau) = \int_\tau^t [u(x,y) - (q^t(x,y)\cdot\nabla)u(x,y)]\,\mathrm{d}y, \quad \tau \le t. \tag{2.6}$$

In order to rewrite the system in dimensionless variables, let D and U be the characteristic length and velocity of the fluid flow, respectively. The ratio D/U is called *kinematic time* of the flow. Thus, considering the dimensionless space and time variables $\xi = x/D$, $\vartheta = Ut/D$, $\theta = U\tau/D$ and $\sigma = s/\Lambda$, and introducing the *Reynolds number* $Re = \rho UD/\nu$ and the *Weissenberg number* $We = \Lambda U/D$, we are led to

$$\begin{cases} Re[\hat{u}_\vartheta + (\hat{u}\cdot\nabla_\xi)\hat{u}] - \omega\Delta_\xi\hat{u} - \dfrac{1-\omega}{We}\displaystyle\int_0^\infty \hat{k}(\sigma)\Delta_\xi\hat{q}^\vartheta(\vartheta - We\sigma)\,\mathrm{d}\sigma + \nabla_\xi\hat{p} = \hat{f}, \\ \operatorname{div}_\xi\hat{u} = \operatorname{div}_\xi\hat{q} = 0, \end{cases}$$

having set

$$\hat{u}(\xi,\vartheta) = \frac{1}{U}u(x,t), \quad \hat{q}^\vartheta(\xi,\theta) = \frac{1}{D}q^t(x,\tau), \quad \hat{p}(\xi,\vartheta) = \frac{D}{\nu U}p(x,t), \quad \hat{f}(\xi) = \frac{D^2}{\nu U}f(x), \quad \hat{k}(\sigma) = \Lambda^2 k(s).$$

For simplicity, we put $Re = 1$ and $We = \varepsilon$. Removing the *hats*, and renaming the space and time variables, we obtain

$$\begin{cases} u_t + (u\cdot\nabla)u - \omega\Delta u - \dfrac{1-\omega}{\varepsilon}\displaystyle\int_0^\infty k(s)\Delta q^t(t-\varepsilon s)\,\mathrm{d}s + \nabla p = f, \\ \operatorname{div} u = \operatorname{div} q = 0. \end{cases} \tag{2.7}$$

By recursion, the integral equation (2.6) yields the following power expansion of $q^t(t-\varepsilon s)$ with respect to ε

$$q^t(x, t-\varepsilon s) = \varepsilon\int_0^s u(x,t-\varepsilon y)\,\mathrm{d}y - \varepsilon^2\int_0^s\left(\int_0^y u(x,t-\varepsilon w)\,\mathrm{d}w\cdot\nabla\right)u(x,t-\varepsilon y)\,\mathrm{d}y + \varepsilon^3\cdots$$

It is then convenient to introduce an additional variable (cf. [5]), namely, the *integrated past history* η of u, defined as

$$\eta^t(x,s) = \int_0^s u(x,t-y)\,\mathrm{d}y, \qquad s \geq 0.$$

It is readily seen that $\eta^t(\varepsilon s)$ is the first order approximation of $q^t(t-\varepsilon s)$ in ε. Besides, η fulfills the system

$$\begin{cases} \partial_t\eta^t(s) = -\partial_s\eta^t(s) + u(t), \\ \operatorname{div}\eta = 0, \\ \eta^t(0) = 0. \end{cases}$$

Since we are focused on the analysis for small values of ε (in fact, the limit $\varepsilon \to 0$), we are allowed to replace q with η in (2.7). This, after a change of variable in the integral, leads to (1.1). It is however worth mentioning that some authors directly obtain (1.1), by simply approximating $q^t(t-s)$ with $\eta^t(s)$ from the very beginning, that is, in the definition of the tensor $\boldsymbol{E}$ (see, e.g., [1,17,21]).

The aim of the present work is to establish convergence results, as the scale parameter ε tends to 0, of system (1.1) to the classical Navier–Stokes equations (1.4), formally corresponding to $\varepsilon = 0$. This will allow us to say that, if the measure $sk_\varepsilon(s)\,\mathrm{d}s$ is close to the Dirac mass, then the contribution of the memory is negligible, and (1.4) is in fact a good approximation of the original system. Indeed, we prove that single trajectories of (1.1) converge to the corresponding ones of (1.4) on finite-time intervals as $\varepsilon \to 0$ (see the following Remark 8.3). But, in particular, we are interested to the longterm behavior. Hence, we want to “measure" the distance of the two systems as $t \to \infty$. The correct way to do that is to see how the final objects describing the asymptotic dynamics (i.e., the attractors) differ in term of ε. To carry out this program, we first construct for every $\varepsilon \in [0,1]$ a dynamical system $S_\varepsilon(t)$ in a proper phase-space, and we study its asymptotic properties, proving the existence of global attractors $\mathcal{A}_\varepsilon$ and exponential attractors $\mathcal{E}_\varepsilon$, uniform with respect to ε (at least, if ε is small enough, or if ω is sufficiently close to 1). Then, loosely speaking, we show that $\mathcal{A}_\varepsilon \to \mathcal{A}_0$ and $\mathcal{E}_\varepsilon \to \mathcal{E}_0$ as $\varepsilon \to 0$, so proving the desired stability property. This is made possible relying on some techniques recently devised in [4], and successfully applied to the reaction-diffusion equation with memory. We point out that, assuming nonslip boundary conditions, the presence of the nonlinear term $(u \cdot \nabla)u$ introduces some nontrivial difficulties, that need to be treated with a new approach. Finally, we provide a physical interpretation of the obtained results.

We remark that, throughout the work, we assume that $\omega \in (0, 1)$ is a *fixed* value, whereas ε will be allowed to move in the interval $[0, 1]$. Nonetheless, it is also of some interest to see how the result depends on the selected value ω. Therefore, we will explicitly write down the dependencies on ω. For instance, in the limit case $\omega = 0$ we do not have the semigroup of solutions. Hence, as it is to be expected, we will find bounds for the solutions that blow up as $\omega \to 0$.

3. The mathematical setting

For the mathematical setting of problems (1.1)–(1.3), we consider the Hilbert space

$$H^0 = \{u \in [L^2(\Omega)]^2 : \operatorname{div} u = 0, u \cdot n_{|\partial\Omega} = 0\},$$

n being the outward normal to $\partial\Omega$. Naming $P : [L^2(\Omega)]^2 \to H^0$ the orthogonal projection onto H^0, the operator $A = -P\Delta$ is well known to be positive, with the first eigenvalue $\lambda_A > 0$, and with compact inverse on H^0. Thus, for any $r \in \mathbb{R}$, we introduce the scale of compactly nested Hilbert spaces $H^r = \operatorname{dom}(A^{r/2})$, endowed with the norms and the inner products

$$\|u\|_{H^r} = \|A^{r/2}u\| \quad \text{and} \quad \langle u, v\rangle_{H^r} = \langle A^{r/2}u, A^{r/2}v\rangle,$$

where $\|\cdot\|$ and $\langle\cdot,\cdot\rangle$ denote the norm and the inner product in $[L^2(\Omega)]^2$, respectively. Two particular instances are

$$H^1 = \{u \in [H_0^1(\Omega)]^2 : \operatorname{div} u = 0\} \quad \text{and} \quad H^2 = H^1 \cap [H^2(\Omega)]^2.$$

Also, $H^{-r} = (H^r)^*$ (dual space). For further use, we recall the Ladyzhenskaya inequality in dimension two

$$\|u\|_{L^4}^2 \le c\|u\|\|u\|_{H^1}, \quad \forall u \in H^1.$$

Defining the trilinear form

$$b(u, v, w) = \langle (u \cdot \nabla)v, w\rangle, \quad u, w \in H^0, v \in H^1,$$

and using the Green formula together with the boundary condition, under suitable regularity assumptions on u, η and f, the term involving p disappears, and we can rewrite the first two equations of (1.1) as the equality in H^{-1}

$$\partial_t u + \omega A u + (1-\omega)\int_0^\infty k_\varepsilon(s) A\eta(s)\,\mathrm{d}s + B(u, u) = f,$$

where the bilinear form $B : H^1 \times H^1 \to H^{-1}$ is given by

$$\langle B(u, v), w\rangle = b(u, v, w), \quad \forall u, v, w \in H^1.$$

Here, $\langle \cdot, \cdot\rangle$ stands for the duality pairing between H^1 and H^{-1}. We shall exploit some important properties of b in the sequel, such as $b(u, v, v) = 0$ and the inequality

$$b(u, v, w) \le c_b \|u\|^{1/2} \|A^{1/2} u\|^{1/2} \|A^{1/2} v\|^{1/2} \|Av\|^{1/2} \|w\|, \tag{3.1}$$

that holds for some $c_b > 0$ and all $u \in H^1$, $v \in H^2$ and $w \in H^0$. For more details concerning the mathematical setting of the Navier–Stokes equations and related results we address the reader to the books [22–24].

Next, we turn our attention to the memory kernel. It is convenient to define

$$\mu(s) = (1 - \omega)k(s),$$

and, accordingly,

$$\mu_\varepsilon(s) = (1 - \omega)k_\varepsilon(s).$$

Then, we assume $\mu \in C^1(\mathbb{R}^+) \cap L^1(\mathbb{R}^+)$, with $\mathbb{R}^+ = (0, \infty)$, such that $\mu \ge 0$ and the dissipation condition

$$\mu'(s) + \delta\mu(s) \le 0 \tag{3.2}$$

is verified for some $\delta > 0$ and all $s \in \mathbb{R}^+$. Notice that

$$\int_0^\infty \mu_\varepsilon(s)\,\mathrm{d}s = \frac{\alpha}{\varepsilon}(1 - \omega). \tag{3.3}$$

For $r \in \mathbb{R}$, we introduce the Lebesgue spaces with respect to the measure $\mu_\varepsilon(s)\,\mathrm{d}s$

$$\mathcal{M}^r_\varepsilon = L^2_{\mu_\varepsilon}(\mathbb{R}^+; H^{r+1}),$$

endowed with the usual inner product. Then, we consider the operator T_ε on $\mathcal{M}^0_\varepsilon$ with domain

$$\mathrm{dom}(T_\varepsilon) = \{\eta \in \mathcal{M}^0_\varepsilon : \partial_s \eta \in \mathcal{M}^0_\varepsilon, \eta(0) = 0\},$$

defined as

$$T_\varepsilon \eta = -\partial_s \eta, \quad \eta \in \mathrm{dom}(T_\varepsilon),$$

∂_s being the distributional derivative with respect to s. Due to (3.2), there holds

$$\langle T_\varepsilon \eta, \eta\rangle_{\mathcal{M}^0_\varepsilon} \le -\frac{\delta}{2\varepsilon} \|\eta\|^2_{\mathcal{M}^0_\varepsilon}, \quad \forall \eta \in \mathrm{dom}(T_\varepsilon). \tag{3.4}$$

In view of the boundary conditions, the third equation of (1.1) reads

$$\partial_t \eta = T_\varepsilon \eta + u.$$

If $u \in L^1(0, T; H^1)$, the (mild) solution in $\mathcal{M}^0_\varepsilon$ of the above equation on the time-interval $[0, T]$ has the explicit representation formula

$$\eta^t(s) = \begin{cases} \displaystyle\int_0^s u(t-y)\,\mathrm{d}y, & 0 < s \le t, \\ \eta^0(s-t) + \displaystyle\int_0^t u(t-y)\,\mathrm{d}y, & s > t. \end{cases}$$

More details on the memory equation can be found in [10]. Compared to the compact embedding $H^2 \Subset H^1$, the embedding $\mathcal{M}^1_\varepsilon \subset \mathcal{M}^0_\varepsilon$ is, in general, not compact. In order to find a space compactly embedded in $\mathcal{M}^0_\varepsilon$, we proceed as follows (see [4]). Given $\eta \in \mathcal{M}^0_\varepsilon$, we introduce the *tail function*

$$\mathbb{T}^\varepsilon_\eta(x) = \varepsilon \int_{(0,\frac{1}{x})\cup(x,\infty)} \mu_\varepsilon(s)\|A^{1/2}\eta(s)\|^2\,\mathrm{d}s, \quad x \ge 1.$$

Then, we define the Banach space

$$\mathcal{L}^1_\varepsilon = \{\eta \in \mathcal{M}^1_\varepsilon : \eta \in \mathrm{dom}(T_\varepsilon), \sup_{x\ge 1} x\mathbb{T}^\varepsilon_\eta(x) < \infty\}$$

endowed with the norm

$$\|\eta\|^2_{\mathcal{L}^1_\varepsilon} = \|\eta\|^2_{\mathcal{M}^1_\varepsilon} + \varepsilon\|T_\varepsilon\eta\|^2_{\mathcal{M}^0_\varepsilon} + \sup_{x\ge 1} x\mathbb{T}^\varepsilon_\eta(x).$$

By [18, Lemma 5.5], $\mathcal{L}^1_\varepsilon \Subset \mathcal{M}^0_\varepsilon$.

Along this paper, we denote by c a generic positive constant depending only on the structural data of the problem; further dependencies will be declared on occurrence. All the quantities appearing in the sequel (in particular, c) are understood to be independent of ε and (unless otherwise specified) of ω. Besides, we will tacitly use the Hölder and the Young inequalities, as well as the standard Sobolev embeddings. Finally, given a Banach space X, the symbol $B_X(R)$ stands for the closed ball in X centered at zero of radius R.

We conclude this section reporting a generalized version of the Gronwall lemma, that will be needed later.

Lemma 3.1. *Let Φ be an absolutely continuous function on $[0, t_\infty)$ (with $t_\infty > 1$; possibly, $t_\infty = \infty$) that fulfills, for almost every $t \in (0, t_\infty)$, the differential inequality*

$$\frac{\mathrm{d}}{\mathrm{d}t}\Phi(t) \le f(t)\Phi(t) + g(t) + \beta.$$

Here, $\beta \ge 0$; $\int_\tau^t f(y)\,\mathrm{d}y \le c - \sigma(t-\tau)$, for $\sigma > 0$ and $c \in \mathbb{R}$; and g is a positive function such that $\int_t^{t+1} g(y)\,\mathrm{d}y \le \gamma < \infty$, for all $t < t_\infty - 1$. Then

$$\Phi(t) \le \mathrm{e}^c\mathrm{e}^{-\sigma t}\Phi(0) + \frac{2\gamma\mathrm{e}^c\mathrm{e}^\sigma}{\mathrm{e}^\sigma - 1} + \frac{\beta\mathrm{e}^c}{\sigma}, \quad \forall t \in [0, t_\infty).$$

4. The dynamical system

In light of the preceding sections, we construct the dynamical system associated with problems (1.1)–(1.3). Let $\varepsilon \in [0,1]$. For $j = 0, 1/2, 1$, we introduce the Hilbert spaces

$$\mathcal{H}^j_\varepsilon = \begin{cases} H^j \times \mathcal{M}^j_\varepsilon, & \text{if}\, \varepsilon > 0, \\ H^j, & \text{if}\, \varepsilon = 0, \end{cases}$$

and the Banach space

$$\mathcal{Z}^1_\varepsilon = \begin{cases} H^1 \times \mathcal{L}^1_\varepsilon, & \text{if}\ \varepsilon > 0, \\ H^1, & \text{if}\, \varepsilon = 0, \end{cases}$$

endowed with the standard norms. Notice that $\mathcal{Z}^1_\varepsilon \Subset \mathcal{H}^0_\varepsilon$. When $\varepsilon = 0$, we agree to interpret the pair (u, η) just as u. For further convenience, we also introduce the *lifting map* $\mathbb{L}_\varepsilon : \mathcal{H}^0_0 \to \mathcal{H}^0_\varepsilon$, and the *projection maps* $\mathbb{P} : \mathcal{H}^0_\varepsilon \to \mathcal{H}^0_0$ and $\mathbb{Q}_\varepsilon : \mathcal{H}^0_\varepsilon \to \mathcal{M}^0_\varepsilon$, given by

$$\mathbb{L}_\varepsilon u = (u, 0), \quad \mathbb{P}(u, \eta) = u, \quad \mathbb{Q}_\varepsilon(u, \eta) = \eta.$$

As anticipated in Section 2, we will work with a fixed $\omega \in (0, 1)$, and we will be interested in analyzing the behavior of the equations in dependence of ε (more specifically, as $\varepsilon \to 0$).

Assuming

$$f \in H^0 \quad \text{independent of time},$$

system (1.1)–(1.3) translates into the following

Problem $\mathbf{P}_\varepsilon$. Given $(u_0, \eta_0) \in \mathcal{H}^0_\varepsilon$, find $(u, \eta) \in C([0, \infty), \mathcal{H}^0_\varepsilon)$ solution to

$$\begin{cases} \partial_t u + \omega A u + \displaystyle\int_0^\infty \mu_\varepsilon(s) A\eta(s)\, \mathrm{d}s + B(u, u) = f, \\ \partial_t \eta = T_\varepsilon \eta + u, \end{cases}$$

for $t > 0$, satisfying the initial conditions $u(0) = u_0$ and $\eta^0 = \eta_0$.

The adopted notation allows us to include within the above formulation also the limiting Problem P_0, corresponding to the classical Navier–Stokes equations (1.4), provided that we interpret the term $\int_0^\infty \mu_\varepsilon(s) A\eta(s)\, \mathrm{d}s$ for $\varepsilon = 0$ as $(1 - \omega)Au(t)$. All the results given in the sequel will be proved for $\varepsilon > 0$. The corresponding proofs for $\varepsilon = 0$ are already known. In fact, they can be recovered with little effort making the above position.

There holds

Theorem 4.1. *For every $\varepsilon \in [0, 1]$, Problem P_ε defines a strongly continuous semigroup (or dynamical system) $S_\varepsilon(t)$ on the phase-space $\mathcal{H}^0_\varepsilon$.*

The proof of this result, based on a standard Galerkin approximation scheme, is omitted (but see, for instance, [9]). We detail for further scopes the continuous dependence estimate: for any $z_1, z_2 \in \mathcal{H}^0_\varepsilon$ and any $t \geq 0$

$$\|S_\varepsilon(t)z_1 - S_\varepsilon(t)z_2\|^2_{\mathcal{H}^0_\varepsilon} + \omega\|\mathbb{P}S_\varepsilon(t)z_1 - \mathbb{P}S_\varepsilon(t)z_2\|^2_{L^2(0,t;H^1)} \leq \mathrm{e}^{c(1+t)/\omega^4}\|z_1 - z_2\|^2_{\mathcal{H}^0_\varepsilon}, \tag{4.1}$$

for some $c > 0$ depending (increasingly) only on the $\mathcal{H}^0_\varepsilon$-norms of z_1 and z_2.

Remark 4.2. We point out that we do not have a well-posedness result when $\omega = 0$, unless $\varepsilon = 0$ either, in which case we recover the Navier–Stokes equations.

Remark 4.3. Theorem 4.1, as well as the results that will follow, hold the same if we add to f a nonlinear term $\phi(u)$, provided that $\phi \in C^1(\mathbb{R}^2, \mathbb{R}^2)$, the (2×2)-matrix $[\omega\lambda\mathbb{I} - \phi'(x)]$ is positive whenever $|x|$ is large enough, for some $\lambda > \lambda_A$, and

$$|\phi'(x)| \leq c(1 + |x|^2), \quad \forall x \in \mathbb{R}^2.$$

Indeed, in the estimates of higher-order, this term can be controlled by means of the Agmon inequality in dimension two for the operator A (see, e.g. [23]), that is,

$$\|u\|^2_{L^\infty} \leq c\|u\|\|Au\|, \quad \forall u \in H^2.$$

5. Global attractors

We begin to investigate the dissipative features of $S_\varepsilon(t)$. The first result is

Proposition 5.1. *There exist $\kappa_0 > 0$ and $C_0 \geq 0$ such that*

$$\|S_\varepsilon(t)z\|_{\mathcal{H}^0_\varepsilon} \leq \|z\|_{\mathcal{H}^0_\varepsilon}\mathrm{e}^{-\omega\kappa_0 t} + \frac{C_0}{\omega},$$

for any $t \geq 0$ and any $z \in \mathcal{H}^0_\varepsilon$.

Proof. Let $(u(t), \eta^t) = S_\varepsilon(t)z$ be the solution to Problem P_ε corresponding to the initial data $z \in \mathcal{H}^0_\varepsilon$. Multiplying the first equation of Problem P_ε by u in H^0 and the second one by η in $\mathcal{M}^0_\varepsilon$, on account of (3.4), we obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(\|u\|^2 + \|\eta\|^2_{\mathcal{M}^0_\varepsilon}) + \omega\|A^{1/2}u\|^2 + \frac{\delta}{2\varepsilon}\|\eta\|^2_{\mathcal{M}^0_\varepsilon} \leq \langle f, u\rangle,$$

due to the equality $b(u, u, u) = 0$. Thus, by standard computations,

$$\frac{\mathrm{d}}{\mathrm{d}t}(\|u\|^2 + \|\eta\|^2_{\mathcal{M}^0_\varepsilon}) + \frac{\omega\lambda_A}{2}\|u\|^2 + \frac{\delta}{\varepsilon}\|\eta\|^2_{\mathcal{M}^0_\varepsilon} + \omega\|A^{1/2}u\|^2 \leq \frac{2}{\omega\lambda_A}\|f\|^2.$$

Setting $\kappa_0 = \min\{\lambda_A/4, \delta/2\}$, we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\|S_\varepsilon(t)z\|^2_{\mathcal{H}^0_\varepsilon} + 2\omega\kappa_0\|S_\varepsilon(t)z\|^2_{\mathcal{H}^0_\varepsilon} + \omega\|A^{1/2}u(t)\|^2 \leq \frac{c}{\omega},$$

and the result follows from the Gronwall lemma. □

As a straightforward consequence, we have

Corollary 5.2. *There exists* $R_0 = R_0(\omega) > 0$, *with* $R_0 \approx 1/\omega$, *such that the* (*bounded*) set $\mathcal{B}^0_\varepsilon = B_{\mathcal{H}^0_\varepsilon}(R_0)$ *is absorbing for* $S_\varepsilon(t)$ *on* $\mathcal{H}^0_\varepsilon$, *uniformly in* ε.

Moreover, integrating the last differential inequality in the above proof, we easily obtain

Corollary 5.3. *There exists* $C_1 \ge 0$ *such that*

$$\int_\tau^t \|A^{1/2}\mathbb{P}S_\varepsilon(y)z\|^2\,\mathrm{d}y \le \frac{R^2}{\omega} + \frac{C_1}{\omega^3}(1+t-\tau),$$

for all $t > \tau \ge 0$ *and all* $z \in B_{\mathcal{H}^0_\varepsilon}(R)$.

We are now ready to prove

Theorem 5.4. *For any* $\varepsilon \in [0, 1]$, *the dynamical system* $S_\varepsilon(t)$ on $\mathcal{H}^0_\varepsilon$ *possesses a* (*unique*) *connected global attractor* $\mathcal{A}_\varepsilon \subset \mathcal{H}^0_\varepsilon$.

Proof. For any initial data $z = (u_0, \eta_0)$ belonging to the absorbing set $\mathcal{B}^0_\varepsilon$, we decompose $S_\varepsilon(t)z$ into the sum

$$S_\varepsilon(t)z = L_\varepsilon(t)z + K_\varepsilon(t)z,$$

where $L_\varepsilon(t)z = (v(t), \xi^t)$ and $K_\varepsilon(t)z = (w(t), \zeta^t)$ solve the problems

$$\begin{cases} \partial_t v + \omega A v + \int_0^\infty \mu_\varepsilon(s) A\xi(s)\,\mathrm{d}s + B(u, v) = 0, \\ \partial_t \xi = T_\varepsilon \xi + v, \\ (v(0), \xi^0) = (u_0, \eta_0), \end{cases} \tag{5.1}$$

and

$$\begin{cases} \partial_t w + \omega A w + \int_0^\infty \mu_\varepsilon(s) A\zeta(s)\,\mathrm{d}s + B(u, w) = f, \\ \partial_t \zeta = T_\varepsilon \zeta + w, \\ (w(0), \zeta^0) = (0, 0). \end{cases} \tag{5.2}$$

By the previous results, we know that

$$\|S_\varepsilon(t)z\|_{\mathcal{H}^0_\varepsilon} \le \frac{c}{\omega} \quad \text{and} \quad \int_\tau^t \|A^{1/2}u(y)\|^2\,\mathrm{d}y \le \frac{c}{\omega^3}(1+t-\tau). \tag{5.3}$$

Here and in the sequel, c is independent of the particular $z \in \mathcal{B}^0_\varepsilon$. Arguing like in the proof of Proposition 5.1, it is easily seen that

$$\|L_\varepsilon(t)z\|_{\mathcal{H}^0_\varepsilon} \le R_0 \mathrm{e}^{-\omega\kappa_0 t}. \tag{5.4}$$

Next, we focus our attention on $K_\varepsilon(t)$. Let us set

$$E(t) = \tfrac{1}{2}\|K_\varepsilon(t)z\|^2_{\mathcal{H}^1_\varepsilon} = \tfrac{1}{2}(\|A^{1/2}w(t)\|^2 + \|\zeta^t\|^2_{\mathcal{M}^1_\varepsilon}).$$

Multiplying the first equation of (5.2) by Aw in H^0 and the second by ζ in $\mathcal{M}^1_\varepsilon$, we obtain

$$\frac{\mathrm{d}E}{\mathrm{d}t} + \omega\|Aw\|^2 + \langle T_\varepsilon A\zeta, A\zeta\rangle_{\mathcal{M}^0_\varepsilon} = -b(u, w, Aw) + \langle f, Aw\rangle. \tag{5.5}$$

By (3.1), (3.4) and (5.3), we deduce

$$\begin{aligned}\frac{\mathrm{d}E}{\mathrm{d}t} + \omega\|Aw\|^2 + \frac{\delta}{2\varepsilon}\|\zeta\|^2_{\mathcal{M}^1_\varepsilon} &\le c_b\|u\|^{1/2}\|A^{1/2}u\|^{1/2}\|A^{1/2}w\|^{1/2}\|Aw\|^{3/2} + \|f\|\|Aw\| \\ &\le \frac{\omega}{2}\|Aw\|^2 + \frac{c}{\omega^5}\|A^{1/2}u\|^2\|A^{1/2}w\|^2 + \frac{c}{\omega},\end{aligned}$$

thus we end up with the differential inequality

$$\frac{\mathrm{d}E}{\mathrm{d}t} + \frac{\omega}{2}\|Aw\|^2 \le \frac{c}{\omega^5}\|A^{1/2}u\|^2 E + \frac{c}{\omega}. \tag{5.6}$$

In light of (5.3), the Gronwall lemma furnishes

$$\|K_\varepsilon(t)z\|_{\mathcal{H}^1_\varepsilon} \le \mathrm{e}^{c(1+t)/\omega^8}, \quad \forall t \ge 0. \tag{5.7}$$

Using now a general argument (see [4, Section 3.1]), we obtain an analogous bound for the $\mathcal{Z}^1_\varepsilon$-norm of $K_\varepsilon(t)z$. Hence, for every fixed $T > 0$ we have found a compact set $K_T \subset \mathcal{H}^0_\varepsilon$ such that

$$\bigcup_{z\in\mathcal{B}^0_\varepsilon} K_\varepsilon(t)z \subset K_T, \quad \forall t \in [0, T].$$

This fact, together with the exponential decay estimate (5.4), imply the existence of the global attractor, by means of standard methods of the theory of dynamical systems (see, for instance, [11]). □

Remark 5.5. For the sake of simplicity, we chose at the beginning to set the Reynolds number $Re = 1$. However, observe that if ω/Re is large enough, we also obtain a regularity result for the attractor. Indeed, if it is so, in the proof of Theorem 5.4 we can apply Lemma 3.1 in place of the usual Gronwall lemma, so getting a bound independent of t. This in turn yields the existence of an attracting set bounded in $\mathcal{Z}^1_\varepsilon$.

6. Regular exponentially attracting sets

Our next task is to prove that the evolution system under consideration enjoys a stronger dissipativity property. Namely, there exists a regular (exponentially) attracting set which, in addition, absorbs itself under the action of $S_\varepsilon(t)$. As a byproduct, we obtain a regularity result for the global attractor. This occurs either if ω is sufficiently close to 1, or if ε is small enough (in dependence of ω). More precisely, the main result of this section is

Theorem 6.1. *There exist $\varepsilon_0 = \varepsilon_0(\omega) \in (0, 1]$ and $R_1 = R_1(\omega) > 0$ such that, denoting $\mathcal{I}_\omega = [0, \varepsilon_0]$ and $\mathcal{B}^1_\varepsilon = B_{\mathcal{Z}^1_\varepsilon}(R_1)$, the inequality*

$$\mathrm{dist}_{\mathcal{H}^0_\varepsilon}(S_\varepsilon(t)\mathcal{B}^0_\varepsilon, \mathcal{B}^1_\varepsilon) \le R_0\mathrm{e}^{-\omega\kappa_0 t} \tag{6.1}$$

holds for every $\varepsilon \in \mathcal{I}_\omega$ and any $t \ge 0$, where dist *is the usual Hausdorff semidistance. In particular, ε_0 is an increasing function of ω, and there exists $\omega_0 < 1$ such that $\varepsilon_0(\omega) = 1$ for every $\omega \ge \omega_0$. Besides, $\varepsilon_0(\omega) \approx \omega^8$. Finally, there* *exists* $t_1 > 0$ *such that*

$$S_\varepsilon(t)\mathcal{B}^1_\varepsilon \subset \mathcal{B}^1_\varepsilon, \quad \forall t \geq t_1. \tag{6.2}$$

The quantities $R_0 > 0$ and $\kappa_0 > 0$ appearing in the above formula are the radius of the absorbing ball $\mathcal{B}^0_\varepsilon$ and the exponential decay rate of system (5.1), respectively (cf. Section 5). Recall that $R_0 \approx 1/\omega$. Concerning R_1, as it will be clear from the proof, the dependence on ω is of the form e^{c/ω^8}. Thus, the above picture is perfectly coherent with the physical interpretation. Indeed, when ω is sufficiently close to 1 the contribution of the memory is negligible, and the system is close to the classical Navier–Stokes equations, which are well known to possess a regular absorbing set (see, e.g., [24]). Conversely, if ω is small we (formally) recover the classical situation letting $\varepsilon \to 0$. In this latter case, however, the bounds will blow up as $\omega \to 0$.

Before going to the proof (which is postponed to the next section), let us see some interesting consequences of Theorem 6.1. First, since the global attractor is the smallest attracting set, there holds

Corollary 6.2. *For every* $\varepsilon \in \mathcal{I}_\omega$, *the global attractor* $\mathcal{A}_\varepsilon$ *is a bounded subset of* $\mathcal{Z}^1_\varepsilon$, *with a (uniform as* $\varepsilon \in \mathcal{I}_\omega$*) bound of the form* e^{c/ω^8}. *Moreover,* $\mathcal{A}_\varepsilon$ *has finite fractal dimension, with an upper bound for the dimension independent of* $\varepsilon \in \mathcal{I}_\omega$.

The last assertion is actually an immediate consequence of Theorem 8.1 (see Section 8). Due to this extra regularity, we are also able to show that the *backwards uniqueness property* holds on the global attractor, that is

Proposition 6.3. *For every* $\varepsilon \in \mathcal{I}_\omega$ *the semigroup* $S_\varepsilon(t)$ *uniquely extends to a strongly continuous group of operators on* $\mathcal{A}_\varepsilon$.

Finally, the family of global attractors $\{\mathcal{A}_\varepsilon\}$ is upper semicontinuous at $\varepsilon = 0$, with respect to the Hausdorff semidistance in $\mathcal{H}^0_\varepsilon$.

Proposition 6.4. *There holds*

$$\lim_{\varepsilon \to 0}[\mathrm{dist}_{\mathcal{H}^0_\varepsilon}(\mathcal{A}_\varepsilon, \mathbb{L}_\varepsilon \mathcal{A}_0)] = 0.$$

Equivalently,

$$\lim_{\varepsilon \to 0}[\mathrm{dist}_{H^0}(\mathbb{P}\mathcal{A}_\varepsilon, \mathcal{A}_0) + \sup_{z \in \mathcal{A}_\varepsilon} \|\mathbb{Q}_\varepsilon z\|_{\mathcal{M}^0_\varepsilon}] = 0.$$

The proofs of the above two propositions do not differ too much from the proofs of the analogous results for the reaction-diffusion equation with memory (see [4]), and are therefore omitted.

7. Proof of Theorem 6.1

It seems convenient to break the proof into some lemmas. Throughout this section, we will keep the notation of the proof of Theorem 5.4.

Lemma 7.1. *There exists a (small) constant* $\lambda > 0$ *such that, setting* $\varepsilon_1 = \lambda\omega^8(1-\omega)$,

$$\sup_{z \in \mathcal{B}^0_\varepsilon} \|K_\varepsilon(t)z\|_{\mathcal{H}^1_\varepsilon} \leq \mathrm{e}^{c/\omega^8}, \quad \forall t \geq 0,$$

for any $\varepsilon \in [0, \varepsilon_1]$ *and some* $c > 0$.

Proof. We repeat the proof of Theorem 5.4 up to formula (5.5). Recall that, by (5.3) and (5.4), there hold

$$\sup_{z\in\mathcal{B}_\varepsilon^0}[\|u(t)\|+\|w(t)\|]\leq\frac{c_0}{\omega},\quad \forall t\geq 0, \tag{7.1}$$

and

$$\sup_{z\in\mathcal{B}_\varepsilon^0}\int_\tau^t\|A^{1/2}u(y)\|^2\,\mathrm{d}y\leq\frac{c_0^2}{\omega^3}(1+t-\tau),\quad \forall t>\tau\geq 0, \tag{7.2}$$

for some $c_0>0$ (independent of ε and ω). A further multiplication of the first equation of (5.2) by $A\zeta^t(s)$ in H^0 and an integration in $\mu_\varepsilon(s)\,\mathrm{d}s$ give

$$\begin{aligned}-\int_0^\infty\mu_\varepsilon(s)\langle\partial_t w, A\zeta(s)\rangle\,\mathrm{d}s&=\omega\int_0^\infty\mu_\varepsilon(s)\langle Aw, A\zeta(s)\rangle\,\mathrm{d}s+\left\|\int_0^\infty\mu_\varepsilon(s)A\zeta(s)\,\mathrm{d}s\right\|^2\\&\quad+\int_0^\infty\mu_\varepsilon(s)b(u,w,A\zeta(s))\,\mathrm{d}s-\int_0^\infty\mu_\varepsilon(s)\langle f, A\zeta(s)\rangle\,\mathrm{d}s.\end{aligned}$$

Introducing the functional

$$L(t)=-\int_0^\infty\mu_\varepsilon(s)\langle w(t), A\zeta^t(s)\rangle\,\mathrm{d}s,$$

exploiting (3.3) and the representation formula

$$\zeta_s^t(s)=\begin{cases}w(t-s), & 0<s\leq t,\\ 0, & s>t,\end{cases}$$

we deduce the equality

$$-\int_0^\infty\mu_\varepsilon(s)\langle\partial_t w, A\zeta(s)\rangle\,\mathrm{d}s=\frac{\mathrm{d}L}{\mathrm{d}t}+\frac{\alpha}{\varepsilon}(1-\omega)\|A^{1/2}w\|^2-\int_0^t\mu_\varepsilon(s)\langle Aw, w(t-s)\rangle\,\mathrm{d}s,$$

which in turn yields

$$\begin{aligned}&\frac{\mathrm{d}L}{\mathrm{d}t}+\frac{\alpha}{\varepsilon}(1-\omega)\|A^{1/2}w\|^2\\&\quad=\int_0^t\mu_\varepsilon(s)\langle Aw, w(t-s)\rangle\,\mathrm{d}s+\omega\int_0^\infty\mu_\varepsilon(s)\langle Aw, A\zeta(s)\rangle\,\mathrm{d}s\\&\qquad+\left\|\int_0^\infty\mu_\varepsilon(s)A\zeta(s)\,\mathrm{d}s\right\|^2+\int_0^\infty\mu_\varepsilon(s)b(u,w,A\zeta(s))\,\mathrm{d}s-\int_0^\infty\mu_\varepsilon(s)\langle f, A\zeta(s)\rangle\,\mathrm{d}s.\end{aligned}\tag{7.3}$$

For $K>0$ to be specified later, we consider the functional

$$\mathcal{E}(t)=E(t)+\frac{K\varepsilon}{\omega^8(1-\omega)}L(t).$$

In view of (7.1), there holds

$$\frac{K\varepsilon}{\omega^8(1-\omega)}L(t) \le \frac{K\sqrt{\alpha\varepsilon}}{\omega^8\sqrt{1-\omega}}\|w\|\|\zeta\|_{\mathcal{M}^1_\varepsilon} \le \frac{K^2\alpha c_0^2\varepsilon}{\omega^{18}(1-\omega)} + \frac{1}{4}\|\zeta\|^2_{\mathcal{M}^1_\varepsilon},$$

which provides the comparison

$$\frac{1}{2}E - \frac{c\varepsilon}{\omega^{18}(1-\omega)} \le \mathcal{E} \le 2E + \frac{c\varepsilon}{\omega^{18}(1-\omega)}. \tag{7.4}$$

Addition of (5.5) and (7.3) times $K\varepsilon/\omega^8(1-\omega)$, on account of (3.4), entails

$$\begin{aligned}
&\frac{d\mathcal{E}}{dt} + \omega\|Aw\|^2 + \frac{K\alpha}{\omega^8}\|A^{1/2}w\|^2 + \frac{\delta}{2\varepsilon}\|\zeta\|^2_{\mathcal{M}^1_\varepsilon}\\
&\quad \le \langle f, Aw\rangle - \frac{K\varepsilon}{\omega^8(1-\omega)}\int_0^\infty \mu_\varepsilon(s)\langle f, A\zeta(s)\rangle\, ds + \frac{K\varepsilon}{\omega^8(1-\omega)}\left\|\int_0^\infty \mu_\varepsilon(s)A\zeta(s)\, ds\right\|^2\\
&\qquad + \frac{K\varepsilon}{\omega^7(1-\omega)}\int_0^\infty \mu_\varepsilon(s)\langle Aw, A\zeta(s)\rangle\, ds + \frac{K\varepsilon}{\omega^8(1-\omega)}\int_0^t \mu_\varepsilon(s)\langle Aw, w(t-s)\rangle\, ds\\
&\qquad - b(u,w,Aw) + \frac{K\varepsilon}{\omega^8(1-\omega)}\int_0^\infty \mu_\varepsilon(s) b(u,w,A\zeta(s))\, ds.
\end{aligned}$$

We now proceed to control the terms of the right-hand side of the above inequality. We have

$$\begin{aligned}
\langle f, Aw\rangle - \frac{K\varepsilon}{\omega^8(1-\omega)}\int_0^\infty \mu_\varepsilon(s)\langle f, A\zeta(s)\rangle\, ds &\le \frac{\omega}{5}\|Aw\|^2 + \frac{K^2\alpha\varepsilon}{\omega^{15}(1-\omega)}\|\zeta\|^2_{\mathcal{M}^1_\varepsilon} + \frac{3}{2\omega}\|f\|^2,\\
\frac{K\varepsilon}{\omega^8(1-\omega)}\left\|\int_0^\infty \mu_\varepsilon(s)A\zeta(s)\, ds\right\|^2 &\le \frac{K\alpha}{\omega^8}\|\zeta\|^2_{\mathcal{M}^1_\varepsilon},\\
\frac{K\varepsilon}{\omega^7(1-\omega)}\int_0^\infty \mu_\varepsilon(s)\langle Aw, A\zeta(s)\rangle\, ds &\le \frac{\omega}{5}\|Aw\|^2 + \frac{5K^2\alpha\varepsilon}{4\omega^{15}(1-\omega)}\|\zeta\|^2_{\mathcal{M}^1_\varepsilon},\\
\frac{K\varepsilon}{\omega^8(1-\omega)}\int_0^t \mu_\varepsilon(s)\langle Aw, w(t-s)\rangle\, ds &\le \frac{\omega}{5}\|Aw\|^2 + \frac{5K^2\alpha^2c_0^2}{4\omega^{19}}.
\end{aligned}$$

Thanks to (3.1) and (7.1),

$$-b(u,w,Aw) \le \frac{\omega}{5}\|Aw\|^2 + \frac{125c_b^4c_0^2}{\omega^5}\|A^{1/2}u\|^2\|A^{1/2}w\|^2,$$

and, by the same token,

$$\frac{K\varepsilon}{\omega^8(1-\omega)}\int_0^\infty \mu_\varepsilon(s) b(u,w,A\zeta(s))\, ds \le \frac{\omega}{5}\|Aw\|^2 + \frac{5c_b^4c_0^2}{64\omega^3}\|A^{1/2}u\|^2\|A^{1/2}w\|^2 + \frac{K^2\alpha\varepsilon}{\omega^{16}(1-\omega)}\|\zeta\|^2_{\mathcal{M}^1_\varepsilon}.$$

At this point, we set the value of K to be

$$K = \frac{8005c_b^4c_0^4}{8\alpha}.$$

Collecting the above inequalities, we find

$$\frac{\mathrm{d}\mathcal{E}}{\mathrm{d}t}+\frac{K\alpha}{\omega^8}\|A^{1/2}w\|^2+\Big[\frac{\delta}{2\varepsilon}-\frac{K\alpha}{\omega^8}\Big(1+\frac{13K\varepsilon}{4\omega^8(1-\omega)}\Big)\Big]\|\zeta\|^2_{\mathcal{M}^1_\varepsilon}\leq\frac{K\alpha}{8c_0^2\omega^5}\|A^{1/2}u\|^2\|A^{1/2}w\|^2+\frac{c}{\omega^{19}},$$

where $c>0$ depends only on the structural data of the problem and is independent of ε and ω. Properly choosing a (small) positive constant λ, independent of ε and ω, and setting $\varepsilon_1=\lambda\omega^8(1-\omega)$, the coefficient of $\|\zeta\|^2_{\mathcal{M}^1_\varepsilon}$ satisfies

$$\frac{\delta}{2\varepsilon}-\frac{K\alpha}{\omega^8}\Big(1+\frac{13K\varepsilon}{4\omega^8(1-\omega)}\Big)\geq\frac{K\alpha}{\omega^8},\quad\forall\varepsilon\in(0,\varepsilon_1].$$

This yields the differential inequality

$$\frac{\mathrm{d}\mathcal{E}}{\mathrm{d}t}+\frac{2K\alpha}{\omega^8}E\leq\frac{K\alpha}{4c_0^2\omega^5}\|A^{1/2}u\|^2E+\frac{c}{\omega^{19}},$$

which, by (7.4), transforms into

$$\frac{\mathrm{d}\mathcal{E}}{\mathrm{d}t}+\frac{K\alpha}{\omega^8}\mathcal{E}\leq\frac{K\alpha}{2c_0^2\omega^5}\|A^{1/2}u\|^2\mathcal{E}+\frac{c}{\omega^{15}}\|A^{1/2}u\|^2+\frac{c}{\omega^{19}}.$$

Since, by the choice of K,

$$\frac{K\alpha}{2c_0^2\omega^5}\int_\tau^t\|A^{1/2}u(y)\|^2\,\mathrm{d}y\leq\frac{K\alpha}{2\omega^8}(1+t-\tau),\quad\forall t>\tau\geq0,$$

Lemma 3.1 gives

$$\mathcal{E}(t)\leq e^{c/\omega^8},\quad\forall t\geq0.$$

Using again (7.4), the desired inequality follows. □

Lemma 7.2. *There exists $\omega_0\in(0,1)$ such that, if $\omega\in[\omega_0,1)$, there holds*

$$\sup_{z\in\mathcal{B}^0_\varepsilon}\|K_\varepsilon(t)z\|_{\mathcal{H}^1_\varepsilon}\leq c,\quad\forall t\geq0,$$

for any $\varepsilon\in[\varepsilon_1,1]$ and some constant $c>0$.

Proof. To simplify the argument, let us work with ω greater than or equal to a fixed positive value, say, $\omega\geq1/2$. Consequently, in the considered range, $(1-\omega)/\varepsilon\leq\Lambda$, having set $\Lambda=256/\lambda$. Notice first that, by virtue of (5.7),

$$\|A^{1/2}w(t)\|^2+\|\zeta^t\|^2_{\mathcal{M}^1_\varepsilon}\leq c,\quad\forall t\in[0,1].\tag{7.5}$$

Thus, integrating (5.6), and taking into account Corollary 5.3, we obtain the further bound

$$\int_0^1\|Aw(y)\|^2\,\mathrm{d}y\leq c.\tag{7.6}$$

We now multiply the first equation of (5.2) by Aw in H^0. This leads to

$$\frac{\mathrm{d}}{\mathrm{d}t}\|A^{1/2}w\|^2 + 2\omega\|Aw\|^2 = -2\int_0^\infty \mu_\varepsilon(s)\langle A\zeta(s), Aw\rangle\,\mathrm{d}s - 2b(u, w, Aw) + 2\langle f, Aw\rangle.$$

Since

$$-2\int_0^\infty \mu_\varepsilon(s)\langle A\zeta(s), Aw\rangle\,\mathrm{d}s \le \frac{1}{6}\|Aw\|^2 + 6\alpha\Lambda\|\zeta\|^2_{\mathcal{M}^1_\varepsilon},$$

and estimating as before the remaining two terms of the right-hand side of the above equality, we end up with

$$\frac{\mathrm{d}}{\mathrm{d}t}\|A^{1/2}w\|^2 + \frac{1}{2}\|Aw\|^2 - 6\alpha\Lambda\|\zeta\|^2_{\mathcal{M}^1_\varepsilon} \le c + c\|A^{1/2}u\|^2\|A^{1/2}w\|^2.$$

In view of (7.5), we select a constant $C > 0$, independent of ε, ω and of the initial data z, such that

$$\mathcal{F}(t) = -6\alpha\Lambda\|\zeta\|^2_{\mathcal{M}^1_\varepsilon} + C > 0, \quad \forall t \in [0, 1].$$

Accordingly, we rewrite the above inequality as

$$\frac{\mathrm{d}}{\mathrm{d}t}\|A^{1/2}w\|^2 + \frac{1}{2}\|Aw\|^2 + \mathcal{F} \le c + c\|A^{1/2}u\|^2\|A^{1/2}w\|^2. \tag{7.7}$$

Relying on the representation formula for ζ, it is immediate to check that the function $\mathcal{F}$ is (Hölder) continuous on $[0, \infty)$. Therefore,

$$t_\infty = \sup\{t \ge 0 : \mathcal{F}(\tau) \ge 0, \forall \tau \in [0, t]\} > 1.$$

Of course, t_∞ may depend on ω, ε and z. Recalling that (cf. Corollary 5.3, which clearly holds for v, and thus for w as well)

$$\sup_{t\ge 0}\int_t^{t+1}[\|A^{1/2}u(y)\|^2 + \|A^{1/2}w(y)\|^2]\,\mathrm{d}y \le c,$$

we are in a position to apply the uniform Gronwall lemma [24, Lemma III.1.1]to (7.7), which, together with (7.5), entail

$$\|A^{1/2}w(t)\| \le c, \quad \forall t \in [0, t_\infty).$$

Then, integrating (7.7) and exploiting the bound (7.6), we also get the integral estimate

$$\sup_{t\in[0,t_\infty-1)}\int_t^{t+1}\|Aw(y)\|^2\,\mathrm{d}y \le c.$$

It is important to observe that the constant c appearing in the last two inequalities is *independent* of the value t_∞ (as well as of ω, ε, z). At this point, multiplying the second equation of (5.2) by ζ in $\mathcal{M}^1_\varepsilon$, we get the differential inequality

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\zeta\|^2_{\mathcal{M}^1_\varepsilon} + \frac{\delta}{2\varepsilon}\|\zeta\|^2_{\mathcal{M}^1_\varepsilon} \le \frac{2\alpha}{\delta}(1-\omega)\|Aw\|^2.$$

Hence, applying Lemma 3.1, we conclude that

$$\|\zeta^t\|_{\mathcal{M}^1_\varepsilon} \le c(1-\omega), \quad \forall t \in [0, t_\infty).$$

To finish the proof, we are left to show that $t_\infty = \infty$ for every initial data z. Indeed, the above bound also furnishes the estimate

$$\mathcal{F}(t) \ge -c(1-\omega) + C, \quad \forall t \in [0, t_\infty).$$

It is then clear that, up to possibly redefining ω_0 to be closer to 1, $\mathcal{F}(t) \ge C/2$ for all $t \in [0, t_\infty)$, which, due to the continuity of $\mathcal{F}$, forces the equality $t_\infty = \infty$. □

Defining $\varepsilon_0(\omega)$ as

$$\varepsilon_0(\omega) = \begin{cases} \lambda\omega^8(1-\omega), & \text{if } \omega < \omega_0, \\ 1, & \text{if } \omega \ge \omega_0, \end{cases}$$

and collecting Lemma 7.1 and 7.2, we have proved

Lemma 7.3. *For any $\varepsilon \in \mathcal{I}_\omega$ there holds*

$$\sup_{z \in \mathcal{B}^0_\varepsilon} \|K_\varepsilon(t)z\|_{\mathcal{H}^1_\varepsilon} \le \mathrm{e}^{c/\omega^8}, \quad \forall t \ge 0.$$

As we already pointed out, the above lemma does not suffice to entail the required compactness. However, using *verbatim* the results of [4, Section 3.1], we find an analogous estimate for the remaining part of the norm of $K_\varepsilon(t)z$ in $\mathcal{Z}^1_\varepsilon$. Namely,

Lemma 7.4. *For any $\varepsilon \in \mathcal{I}_\omega$ there holds*

$$\varepsilon\|T_\varepsilon\zeta^t\|^2_{\mathcal{M}^0_\varepsilon} + \sup_{x \ge 1} \mathbb{T}^\varepsilon_{\zeta^t}(x) \le \mathrm{e}^{c/\omega^8}, \quad \forall t \ge 0.$$

Finally, collecting (5.4), Lemma 7.3 and Lemma 7.4, we have proved formula (6.1).

The steps to prove formula (6.2) are basically the same, and are left to the interested reader. Possibly, one has to increase the quantities R_1 and ω_0 and decrease ε_0, obtained from the proof of (6.1) (nonetheless, the dependence of R_1 and ε_0 on ω does not change). Clearly, the redefined set $\mathcal{B}^1_\varepsilon$ is *a fortiori* exponentially attracting. Incidentally, $t_1 \approx \omega^8$.

Remark 7.5. Since $\mathcal{B}^1_\varepsilon$ absorbs itself after a finite time t_1, it is straightforward to see that there exists $Q_1 = Q_1(\omega) > 0$, with $Q_1 \approx \mathrm{e}^{c/\omega^8}$, such that, for every $\varepsilon \in \mathcal{I}_\omega$ and every $t \ge 0$,

$$\sup_{z \in \mathcal{B}^1_\varepsilon} \left[\|S_\varepsilon(t)z\|_{\mathcal{Z}^1_\varepsilon} + \int_t^{t+1} \|A\mathbb{P}S_\varepsilon(y)z\|^2 \,\mathrm{d}y\right] \le Q_1. \tag{7.8}$$

8. A robust family of exponential attractors

The main result of the paper is the existence of a family of exponential attractors $\mathcal{E}_\varepsilon$, for all ε small enough, which is *robust* (that is, continuous in a proper sense) with respect to the singular limit $\varepsilon \to 0$. Clearly, besides robustness, the novelty here is the existence of $\mathcal{E}_\varepsilon$ for $\varepsilon > 0$. Indeed, the existence of $\mathcal{E}_0$, corresponding to the classical case, is well-known since many years (cf. [6] and references therein).

Theorem 8.1. *For every $\varepsilon \in \mathcal{I}_\omega$ there exists a set $\mathcal{E}_\varepsilon$, compact in $\mathcal{H}^0_\varepsilon$ and bounded in $\mathcal{Z}^1_\varepsilon$, which satisfies the following conditions.*

(i) $S_\varepsilon(t)\mathcal{E}_\varepsilon \subset \mathcal{E}_\varepsilon$, *for every* $t \geq 0$.
(ii) *There exist* $\kappa > 0$ *and a positive increasing function* M (*both independent of* $\varepsilon \in \mathcal{I}_\omega$) *such that, for every bounded set* $\mathcal{B} \subset B_{\mathcal{H}^0_\varepsilon}(R)$, *there holds*

$$\mathrm{dist}_{\mathcal{H}^0_\varepsilon}(S_\varepsilon(t)\mathcal{B}, \mathcal{E}_\varepsilon) \leq M(R)e^{-\kappa t}, \quad \forall t \geq 0.$$

(iii) *The fractal dimension of* $\mathcal{E}_\varepsilon$ *in* $\mathcal{H}^0_\varepsilon$ *is uniformly bounded with respect to* $\varepsilon \in \mathcal{I}_\omega$.
(iv) *There exist* $\Theta \geq 0$ *and* $\vartheta \in (0, 1/8]$ *such that*

$$\mathrm{dist}^{\mathrm{sym}}_{\mathcal{H}^0_\varepsilon}(\mathcal{E}_\varepsilon, \mathbb{L}_\varepsilon\mathcal{E}_0) \leq \Theta\varepsilon^\vartheta.$$

Here, $\mathrm{dist}^{\mathrm{sym}}$ denotes the usual symmetric Hausdorff distance. The quantities κ, M, Θ and ϑ depend on the particular value of $\omega \in (0, 1)$.

In order to prove the theorem, let us introduce the spaces

$$\mathcal{H}'_\varepsilon = \begin{cases} H^{1/2} \times \mathcal{W}^{1/2}_\varepsilon, & \text{if } \varepsilon > 0, \\ H^{1/2}, & \text{if } \varepsilon = 0, \end{cases}$$

where

$$\mathcal{W}^{1/2}_\varepsilon = \{\eta \in \mathcal{M}^{1/2}_\varepsilon : \eta_s \in \mathcal{M}^{-1}_\varepsilon, \sup_{x\geq 1} x\mathbb{T}^\varepsilon_\eta(x) < \infty\}$$

is a Banach space endowed with the norm

$$\|\eta\|^2_{\mathcal{W}^{1/2}_\varepsilon} = \|\eta\|^2_{\mathcal{M}^{1/2}_\varepsilon} + \varepsilon\|\eta_s\|^2_{\mathcal{M}^{-1}_\varepsilon} + \sup_{x\geq 1} x\mathbb{T}^\varepsilon_\eta(x).$$

By [18, Lemma 5.5], $\mathcal{H}'_\varepsilon \Subset \mathcal{H}^0_\varepsilon$.

Lemma 8.2. *There exist* $\Lambda_j \geq 0$, $\lambda \in [0, 1/2)$ *and* $t^\star \geq t_1$ (*all independent of* $\varepsilon \in \mathcal{I}_\omega$) *such that the following conditions hold*.

(H1) *The map* $S_\varepsilon = S_\varepsilon(t^\star) : \mathcal{B}^1_\varepsilon \to \mathcal{B}^1_\varepsilon$ *satisfies* $S_\varepsilon z = L_\varepsilon z + K_\varepsilon z$, *where*

$$\|L_\varepsilon z_1 - L_\varepsilon z_2\|_{\mathcal{H}^0_\varepsilon} \leq \lambda\|z_1 - z_2\|_{\mathcal{H}^0_\varepsilon},$$
$$\|K_\varepsilon z_1 - K_\varepsilon z_2\|_{\mathcal{H}'_\varepsilon} \leq \Lambda_1\|z_1 - z_2\|_{\mathcal{H}^0_\varepsilon},$$

for every $z_1, z_2 \in \mathcal{B}^1_\varepsilon$.

(H2) *There holds*

$$\|S_\varepsilon^n z - \mathbb{L}_\varepsilon S_0^n \mathbb{P} z\|_{\mathcal{H}_\varepsilon^0} \le \Lambda_2^n \sqrt[8]{\varepsilon}, \quad \forall z \in \mathcal{B}_\varepsilon^1, \forall n \in \mathbb{N}.$$

(H3) *There holds*

$$\|S_\varepsilon(t)z - \mathbb{L}_\varepsilon S_0(t)\mathbb{P} z\|_{\mathcal{H}_\varepsilon^0} \le \Lambda_3 \sqrt[8]{\varepsilon}, \quad \forall z \in \mathcal{B}_\varepsilon^1, \forall t \in [t^\star, 2t^\star].$$

(H4) *The map*

$$z \mapsto S_\varepsilon(t)z : \mathcal{B}_\varepsilon^1 \to \mathcal{B}_\varepsilon^1$$

is Lipschitz continuous, with a Lipschitz constant independent of $t \in [t^\star, 2t^\star]$ *and of* ε. *Here,* $\mathcal{B}_\varepsilon^1$ *is endowed with the metric topology of* $\mathcal{H}_\varepsilon^0$.

(H5) *The map*

$$(t, z) \mapsto S_\varepsilon(t)z : [t^\star, 2t^\star] \times \mathcal{B}_\varepsilon^1 \to \mathcal{B}_\varepsilon^1$$

is 1/2*-Hölder continuous* (*with a constant that may depend on* ε). *Again,* $\mathcal{B}_\varepsilon^1$ *is endowed with the metric topology of* $\mathcal{H}_\varepsilon^0$.

Lemma 8.2, whose proof will be given in the next section, allows to apply the abstract result [4, Theorem A.2] which yields the thesis of Theorem 8.1, with (ii) replaced by the weaker statement

$$\operatorname{dist}_{\mathcal{H}_\varepsilon^0}(S_\varepsilon(t)\mathcal{B}_\varepsilon^1, \mathcal{E}_\varepsilon) \le M_0\, \mathrm{e}^{-\kappa t}, \quad \forall t \ge 0,$$

for some $M_0 > 0$. Finally, by means of (4.1), (6.1) and the above inequality, exploiting the transitivity of the exponential attraction property [7, Theorem 5.1] we recover (ii). The proof of Theorem 8.1 is then complete.

Remark 8.3. A closer look to the proof of (H2)–(H3) (see the next section) provides in fact a result that has an independent interest. Namely, for every $T > 0$ and every $R \ge 0$ there exist $C_{T,R} > 0$ and $C_R > 0$ such that, for every $z = (u_0, \eta_0) \in B_{\mathcal{H}_\varepsilon^1}(R)$, there hold

$$\sup_{t\in[0,T]} \|\mathbb{P}S_\varepsilon(t)z - S_0(t)\mathbb{P}z\|_{H^0} + \|\mathbb{P}S_\varepsilon(t)z - S_0(t)\mathbb{P}z\|_{L^2(0,T;H^1)} \le C_{T,R}\sqrt[8]{\varepsilon},$$

and

$$\|\mathbb{Q}_\varepsilon S_\varepsilon(t)z\|_{\mathcal{M}_\varepsilon^0} \le \|\eta_0\|_{\mathcal{M}_\varepsilon^0}\mathrm{e}^{-\delta t/4\varepsilon} + C_R\sqrt{\varepsilon}, \quad \forall t \ge 0.$$

These inequalities give a measure of the closeness (in terms of ε) of $S_\varepsilon(t)$ and $S_0(t)$ on finite-time intervals.

9. Proof of Lemma 8.2

For the sake of simplicity, we set $\omega = 1/2$.

9.1. Proof of (H1)

For any $z \in \mathcal{B}^1_\varepsilon$, we decompose the solution map as

$$(u(t), \eta^t) = S_\varepsilon(t)z = L_\varepsilon(t)z + K_\varepsilon(t)z,$$

where $L_\varepsilon(t)$ and $K_\varepsilon(t)$ are defined as in Section 5. It is apparent that, upon choosing $t^\star$ large enough, $L_\varepsilon = L_\varepsilon(t^\star)$ is a contraction, with a constant independent of ε. Concerning the other map, for $z_1, z_2 \in \mathcal{B}^1_\varepsilon$ we set

$$(w(t), \zeta^t) = K_\varepsilon(t)z_1 - K_\varepsilon(t)z_2.$$

Then (w, ζ) solves

$$\begin{cases} \partial_t w + \dfrac{1}{2}Aw + \displaystyle\int_0^\infty \mu_\varepsilon(s)A\zeta(s)\,\mathrm{d}s + B(u, u^1) + B(u^2, u) = 0, \\ \partial_t \zeta = T_\varepsilon \zeta + w, \end{cases}$$

supplemented with null initial data. Here, $u = u^1 - u^2$ and $u^i(t) = \mathbb{P}S_\varepsilon(t)z_i$. Multiplying the first equation by $A^{1/2}w$ in H^0 and the second by ζ in $\mathcal{M}^{1/2}_\varepsilon$, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}(\|A^{1/4}w\|^2 + \|\zeta\|^2_{\mathcal{M}^{1/2}_\varepsilon}) + \|A^{3/4}w\|^2 + \frac{\delta}{\varepsilon}\|\zeta\|^2_{\mathcal{M}^{1/2}_\varepsilon} \leq -2b(u, u^1, A^{1/2}w) - 2b(u^2, u, A^{1/2}w).$$

Since $H^{1/2} \hookrightarrow L^3$, and due to (7.8), there holds

$$\begin{aligned} -2b(u, u^1, A^{1/2}w) - 2b(u^2, u, A^{1/2}w) &\leq c\|u\|\|Au^1\|\|A^{3/4}w\| + c\|A^{1/2}u^2\|\|A^{1/2}u\|\|A^{3/4}w\| \\ &\leq \tfrac{1}{2}\|A^{3/4}w\|^2 + c\|Au^1\|^2\|u\|^2 + c\|A^{1/2}u\|^2. \end{aligned}$$

Therefore,

$$\frac{\mathrm{d}}{\mathrm{d}t}(\|A^{1/4}w\|^2 + \|\zeta\|^2_{\mathcal{M}^{1/2}_\varepsilon}) + \frac{1}{2}\|A^{3/4}w\|^2 \leq c\|Au^1\|^2\|u\|^2 + c\|A^{1/2}u\|^2.$$

Integrating on $(0, t^\star)$, in view of (4.1) and (7.8), it follows that

$$\|K_\varepsilon z_1 - K_\varepsilon z_2\|_{\mathcal{H}^{1/2}_\varepsilon} \leq c\|z_1 - z_2\|_{\mathcal{H}^0_\varepsilon},$$

having set $K_\varepsilon = K_\varepsilon(t^\star)$. Arguing exactly as in [4], we get the same estimate with $\mathcal{H}'_\varepsilon$ in place of $\mathcal{H}^{1/2}_\varepsilon$. To this aim, notice that the above integration also bears an integral control of $\|A^{3/4}w\|^2$.

9.2. Proof of (H2)–(H3)

These are straightforward consequence of the estimate

$$\sup_{z \in \mathcal{B}^1_\varepsilon} \|S_\varepsilon(t)z - \mathbb{L}_\varepsilon S_0(t)\mathbb{P}z\|_{\mathcal{H}^0_\varepsilon} \leq c\mathrm{e}^{-\delta t/4\varepsilon} + c\mathrm{e}^{ct}\sqrt[8]{\varepsilon}, \quad \forall t \geq 0. \tag{9.1}$$

To prove (9.1), we introduce the pair (u, η), where $u(t) = S_0(t)\mathbb{P}z$, and η is the solution to the problem

$$\begin{cases} \partial_t \eta = T_\varepsilon \eta + u, \\ \eta^0 = \eta_0. \end{cases}$$

Next, setting $(\hat{u}(t), \hat{\eta}^t) = S_\varepsilon(t)z$, the pair $(\overline{u}, \overline{\eta}) = (\hat{u} - u, \hat{\eta} - \eta)$ solves

$$\begin{cases} \partial_t \overline{u} + \frac{1}{2} A\overline{u} + \int_0^\infty \mu_\varepsilon(s) A\hat{\eta}(s)\,\mathrm{d}s = \frac{1}{2} Au - B(\hat{u}, \hat{u}) + B(u, u), \\ \partial_t \overline{\eta} = T_\varepsilon \overline{\eta} + \overline{u}, \end{cases} \tag{9.2}$$

with null initial data. Reasoning as in [4, Lemma 5.4], we see that

$$\max\{\|\hat{\eta}^t\|^2_{\mathcal{M}^0_\varepsilon}, \|\eta^t\|^2_{\mathcal{M}^0_\varepsilon}\} \le \|\eta_0\|^2_{\mathcal{M}^0_\varepsilon} \mathrm{e}^{-\delta t/2\varepsilon} + c\varepsilon, \tag{9.3}$$

for any $z \in \mathcal{B}^1_\varepsilon$. Moreover, multiplying the first equation in (9.2) by $\overline{u}$ in H^0 and the second by $\overline{\eta}$ in $\mathcal{M}^0_\varepsilon$, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}(\|\overline{u}\|^2 + \|\overline{\eta}\|^2_{\mathcal{M}^0_\varepsilon}) + \|A^{1/2}\overline{u}\|^2 \le -2b(\overline{u}, \hat{u}, \overline{u}) - 2\int_0^\infty \mu_\varepsilon(s)\langle A^{1/2}\eta(s), A^{1/2}\overline{u}\rangle\,\mathrm{d}s + \langle A^{1/2}u, A^{1/2}\overline{u}\rangle.$$

The last two terms on the right-hand side can be controlled by a positive function ψ satisfying (cf. [4, Lemma 5.5])

$$\int_0^t \psi(y)\,\mathrm{d}y \le c\mathrm{e}^{ct}\sqrt[4]{\varepsilon}.$$

Thus, we focus our attention on the trilinear form. From the Ladyzhenskaya inequality, there holds

$$-2b(\overline{u}, \hat{u}, \overline{u}) \le c\|\overline{u}\|^2_{L^4}\|A^{1/2}\hat{u}\| \le \tfrac{1}{2}\|A^{1/2}\overline{u}\|^2 + c\|A^{1/2}\hat{u}\|^2\|\overline{u}\|^2.$$

We conclude that

$$\frac{\mathrm{d}}{\mathrm{d}t}(\|\overline{u}\|^2 + \|\overline{\eta}\|^2_{\mathcal{M}^0_\varepsilon}) + \frac{1}{2}\|A^{1/2}\overline{u}\|^2 \le c\|A^{1/2}\hat{u}\|^2\|\overline{u}\|^2 + \psi.$$

In view of Corollary 5.3 and the Gronwall lemma, and thanks to (9.3), we obtain (9.1).

9.3. *Proof of (H4)–(H5)*

Notice that (H4) follows immediately from (4.1). Hence, calling $(u(t), \eta^t) = S_\varepsilon(t)z$, we are left to show that

$$\sup_{z\in\mathcal{B}^1_\varepsilon} \int_{t^\star}^{2t^\star} (\|\partial_t u(y)\|^2 + \|\partial_t \eta^y\|^2_{\mathcal{M}^0_\varepsilon})\,\mathrm{d}y \le Q_\varepsilon,$$

for some $Q_\varepsilon > 0$ (notice that Q_ε may depend on ε). The control of the first term in this inequality is a direct consequence of the first equation of Problem P_ε and (7.8). Concerning the second term, notice that, from the second equation of Problem P_ε, (7.8) and the representation formula for η, we obtain

$$\|\partial_t \eta\|^2_{\mathcal{M}^0_\varepsilon} \le 2\|T_\varepsilon \eta\|^2_{\mathcal{M}^0_\varepsilon} + 2\|u\|^2_{\mathcal{M}^0_\varepsilon} \le \frac{2}{\varepsilon}\|u\|^2_{\mathcal{L}^1_\varepsilon} + \frac{2\alpha}{\varepsilon}\|A^{1/2}u\|^2 \le \frac{c}{\varepsilon},$$

and the result follows integrating on $(t^\star, 2t^\star)$.

10. Conclusions

We are now in a position to interpret from a physical viewpoint the results obtained in the previous sections. We provide a comparison between the asymptotic behavior of a polymeric solution, modelled by a Jeffreys type fluid, and a Newtonian fluid, assuming a fixed value of the Reynolds number of the flow (for simplicity, we put $Re = 1$). When the polymeric solution is sufficiently dilute (say, $\omega \geq \omega_0$), then the contribution of the solvent prevails, and the longterm dynamics is close to a Newtonian flow, provided that the Weissenberg number is not too large ($\varepsilon \leq 1$). Otherwise, for higher values of the polymeric concentration ($\omega < \omega_0$), the non-Newtonian solution asymptotically behaves as a Newtonian fluid only if its Weissenberg number is small enough. In other words, the natural time Λ of the polymeric components must be smaller than the kinematic time of the flow and, accordingly, their elastic response must be less important than the viscous one. In particular, for any fixed $\omega < \omega_0$, there exists a limit value ε_0 (decreasing as ω^8 when the solvent concentration ω vanishes) such that, in the long time, the Newtonian dynamics is attained when the Weissenberg number of the flow is smaller than ε_0.

This result can be viewed as a contribution to give a rigorous statement of the following sentence [2, p. 226]: "*... one may also think (as we are inclined to do) that any real material behaving as a Newtonian fluid is simply a material with an extremely short natural time*". In this connection, Renardy first proved in [19] that some general theorems on the longtime behavior of a Newtonian fluid still hold for a viscoelastic fluid of Jeffreys type at low Weissenberg numbers. He analyzed the asymptotic stability of steady state solutions of the fully nonlinear model (2.7) with an exponential memory kernel proving, in particular, that linear stability implies nonlinear stability, provided that the Weissenberg number is sufficiently small, with a bound depending on the norm of the basic steady flow. It is worth noting that the results of [19], as well as the ones of the present paper, are valid for *any* Reynolds number, but they cannot be extended to the limit as ω tends to zero.

Finally, we stress that Remark 5.5 refers to a situation where the Reynolds number differs from 1. Precisely, for any fixed ω and ε, the Jeffreys type fluid asymptotically behaves as a Newtonian one provided that Re is sufficiently small or, equivalently, if the total viscosity ν is sufficiently large compared to $\rho U D$.

Acknowledgements

Research partially supported by the Italian MIUR PRIN Research Project *Modellizzazione Matematica ed Analisi dei Problemi a Frontiera Libera* and by the Italian MIUR FIRB Research Project *Analisi di Equazioni a Derivate Parziali, Lineari e Non Lineari: Aspetti Metodologici, Modellistica, Applicazioni.*

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