STABILITY OF ABSTRACT LINEAR THERMOELASTIC SYSTEMS WITH MEMORY

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An abstract linear thermoelastic system with memory is here considered. Existence, uniqueness, and continuous dependence results are given. In presence of regular and convex memory kernels, the system is shown to be exponentially stable. An application to the Kirchhoff plate equation is given.

1. Introduction

The aim of this paper is to study the asymptotic behavior in time of an abstract problem related to linear thermoelastic systems with memory. As in classical thermoelasticity, these systems consist of an elastic equation and a heat equation, which are coupled in such a way that the transfer between the mechanical energy and the heat energy is taken into account. Since the elastic behavior has a conservative character, dissipation is allowed by heat energy losses, only. Thermoelastic systems with memory can be viewed as particular models within the theory of linear thermoviscoelasticity, where both thermal and mechanical memory effects occur.[14,18,22]

Accounting for thermal memory effects only, the internal energy of the body depends on both the actual deformation and the past history of the temperature, whereas the heat flux law involves only the past history of the temperature gradient. On the other hand, invariance under time reversal and thermodynamical

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arguments[14] allow us to assume the material to have instantaneous response as far as the stress–strain relation is concerned.

In this framework, the thermal dissipation of the body is weaker than in the classical thermoelastic approach, based on the Fourier heat flux law. Moreover, the approach followed here involves perturbation of both the initial data and the past history. As a consequence, the energy decay is much more difficult to detect.

In particular, we are interested in well-posedness and exponential stability of an evolution problem arising in the two-dimensional theory of linear hereditary thermoelasticity. If we consider only small variations of the temperature and its gradient, we may suppose that the following linear system models temperature and vertical displacement evolution of a homogeneous, (thermally and elastically) isotropic Kirchhoff thin plate subject to thermal deformations and hereditary heat conduction law in a two-dimensional domain Ω:

$$\begin{aligned} \partial_{tt}u(t) + \Delta[\Delta u(t) + \vartheta(t)] = 0 \quad \text{in } \Omega \\ \partial_t \vartheta(t) + \vartheta(t) - \Delta u_t(t) + \int_0^\infty [\beta(\sigma)\vartheta(t-\sigma) - \kappa(\sigma)\Delta\vartheta(t-\sigma)]d\sigma = 0 \quad \text{in } \Omega \end{aligned} \tag{1.1}$$

for $t \in \mathbb{R}^+ = (0, +\infty)$, with initial data

$$\begin{aligned} u(0) &= u_0\,, \\ \partial_t u(0) &= v_0\,, \\ \vartheta(0) &= \vartheta_0\,, \\ \vartheta(-s) &= \vartheta_0(s)\,, \quad s \in \mathbb{R}^+ \end{aligned} \tag{1.2}$$

subject to boundary conditions

$$\begin{aligned} u(t) = \Delta u(t) + (1-\lambda)B_1 u(t) + \vartheta(t) = 0 \quad \text{on } \partial\Omega\,, \\ \int_0^s \vartheta(t-y)\,dy = 0 \quad \text{on } \partial\Omega\,, \quad s \in \text{ Supp}(\kappa) \end{aligned} \tag{1.3}$$

for every $t \in \mathbb{R}^+$, with

$$B_1 u = 2n_1 n_2 \partial_{x_1x_2} u - n_1^2 \partial_{x_1x_1} u - n_2^2 \partial_{x_2x_2} u\,, \tag{1.4}$$

where $x = (x_1, x_2)$, $n = (n_1, n_2)$ is the unit outward normal to $\partial\Omega$, and $0 < \lambda < 1/2$ is the Poisson ratio. The reader is referred to Sec. 5 for the derivation of the model.

Here, for simplicity, we put all the physical constants, except λ, equal to one. In addition, the kernels β and κ are assumed to be bounded convex functions vanishing at $+\infty$.

To formulate system (1.1) in a history space setting, we follow Giorgi, Marzocchi and Pata,[5] and we introduce a new variable, namely, the *summed past history* of ϑ up to time t which is defined by

$$\eta^t(x,s) = \int_0^s \vartheta(x, t-y)\,dy = \int_{t-s}^t \vartheta(x,y)\,dy \quad x \in \Omega,\ s \in \mathbb{R}^+\,.$$

One can easily check that η satisfies the first-order linear evolution equation

$$\partial_t \eta^t(s) + \partial_s \eta^t(s) = \vartheta(t) \quad \text{in } \Omega \times \mathbb{R}^+$$

for $t \in \mathbb{R}^+$, along with the boundary condition

$$\eta^t(0) = 0 \qquad \text{on } \Omega,\ \forall\, t \in \mathbb{R}^+\,,$$

and the initial condition

$$\eta^0 = \eta_0 \quad \text{in } \Omega \times \mathbb{R}^+\,,$$

where

$$\eta_0(s) = \int_0^s \vartheta_0(y)\, dy \quad \text{in } \Omega,\ s \geq 0$$

is the initial summed past history of ϑ.

For any $s \in \mathbb{R}^+$, we set for simplicity

$$\mu(s) = -\kappa'(s) \quad \text{and} \quad \nu(s) = -\beta'(s)$$

and we assume the following set of hypotheses:

(h1) $$\nu,\, \mu \in C^1(\mathbb{R}^+) \cap L^1(\mathbb{R}^+)\,,$$

(h2) $$\nu(s),\ \mu(s)\ \geq 0 \quad \forall\ s \in \mathbb{R}^+\,,$$

(h3) $$\nu'(s),\ \mu'(s) \leq 0 \quad \forall\ s \in \mathbb{R}^+\,.$$

A formal integration by parts yields[5]

$$\int_0^\infty \kappa(\sigma)\Delta\vartheta(t-\sigma)\, d\sigma = \int_0^\infty \mu(\sigma)\Delta\eta^t(\sigma)\, d\sigma$$

and

$$\int_0^\infty \beta(\sigma)\vartheta(t-\sigma)\, d\sigma = \int_0^\infty \nu(\sigma)\eta^t(\sigma)\, d\sigma\,.$$

The second relation of (1.3) transforms into (notice that $\mathrm{Supp}(\kappa) = \mathrm{Supp}(\mu)$)

$$\eta^t(s) = 0 \quad \text{on } \partial\Omega \times \mathrm{Supp}(\mu)$$

for every $t \in \mathbb{R}^+$. In view of this choice of variables, we can translate (1.1)–(1.3) into the following initial and boundary value problem.

Problem P. *Find the solution* (u, v, ϑ, η) *to the system*

$$\partial_t u(t) = v(t) \quad \textit{in } \Omega\,,$$

$$\partial_t v(t) + \Delta[\Delta u(t) + \vartheta(t)] = 0 \quad \textit{in } \Omega\,,$$

$$\partial_t \vartheta(t) + \vartheta(t) - \Delta v(t) + \int_0^\infty \nu(\sigma)\eta^t(\sigma)\, d\sigma - \int_0^\infty \mu(\sigma)\Delta\eta^t(\sigma)\, d\sigma = 0 \quad \textit{in } \Omega\,,$$

$$\partial_t \eta^t(s) + \partial_s \eta^t(s) = \vartheta(t) \quad \textit{in } \Omega \times \mathbb{R}^+\,,$$

for $t \in \mathbb{R}^+$, *which satisfies the initial and boundary conditions*

$$\begin{aligned} u(t) = \Delta u(t) + (1-\lambda)B_1 u(t) + \vartheta(t) = 0 &\quad on\ \partial\Omega \times \mathbb{R}^+ \,, \\ \eta^t(0) = 0 &\quad on\ \Omega \times \mathbb{R}^+ \,, \\ \eta^t(s) = 0 &\quad on\ \partial\Omega \times \mathrm{Supp}(\mu) \times \mathbb{R}^+ \,, \\ u(0) = u_0 &\quad in\ \Omega \,, \\ v(0) = v_0 &\quad in\ \Omega \,, \\ \vartheta(0) = \vartheta_0 &\quad in\ \Omega \,, \\ \eta^0 = \eta_0 &\quad in\ \Omega \times \mathbb{R}^+ \,. \end{aligned}$$

Recently, the asymptotic behavior of systems in linear viscoelasticity and thermoviscoelasticity was investigated by many authors. In his pioneering work,[3] Dafermos considered the thermoelastic problem involving no memory effects, proving the decay of the associated energy for every initial data, for u, ϑ satisfying Dirichlet–Dirichlet boundary conditions. Since then, much progress has been made to obtain the exponential decay rate of the energy for such equations. In this direction, we quote some works dealing with thermoelasic rods and plates.[1,10,11,15–17,19–21,23,25]

The plan of the paper is as follows. In Sec. 2 we consider a problem in an abstract setting, and we state well-posedness and exponential stability results. Section 3 is devoted to the proof of the theorems. It is worth mentioning that no assumptions on the exponential decay of the memory kernels are needed. In Sec. 4 we apply the abstract results to the plate equation. Finally, in Sec. 5, we present a derivation of the model.

2. Abstract Setting

Let A be a strictly positive self-adjoint operator on a Hilbert space $(V_0, \langle\cdot,\cdot\rangle, \|\cdot\|)$, of domain $\mathcal{D}(A) = V_2 \subset V_0$, such that $A \in \mathcal{L}(V_2, V_0)$ and $A^{-1} \in \mathcal{L}(V_0, V_2)$. It is well known that it is possible to define the powers A^s of A for $s \in \mathbb{R}$, and the space $V_s = \mathcal{D}(A^{s/2})$ turns out to be a Hilbert space with the inner product

$$\langle\cdot,\cdot\rangle_s = \langle A^{s/2}\cdot, A^{s/2}\cdot\rangle \,.$$

We denote by $\|\cdot\|_s$ the norm on V_s induced by the above inner product (in particular, $\|\cdot\|_0 = \|\cdot\|$). The injection $V_{s_1} \hookrightarrow V_{s_2}$ is dense and continuous whenever $s_1 \geq s_2$.

Let $\langle\cdot,\cdot\rangle_*$ be another inner product defined on V_2, which induces a norm $\|\cdot\|_*$ which is equivalent to $\|\cdot\|_2$. Actually, from the inverse mapping theorem, it would be enough to require that $\|\cdot\|_*$ be stronger (or weaker) than $\|\cdot\|_2$. This yields the existence of a strictly positive self-adjoint operator B on V_0 of domain $\mathcal{D}(B) = V_2$, $\mathcal{D}(B^{s/2}) = V_s$ for $s \in \mathbb{R}$, with $B \in \mathcal{L}(V_2, V_0)$ and $B^{-1} \in \mathcal{L}(V_0, V_2)$, such that

$$\langle\cdot,\cdot\rangle_* = \langle B\cdot, B\cdot\rangle \,.$$

Furthermore, $AB^{-1} \in \mathcal{L}(V_0)$ and $(AB^{-1})^{-1} = BA^{-1} \in \mathcal{L}(V_0)$. We agree to denote by V_* the space V_2 endowed with the inner product $\langle \cdot, \cdot \rangle_*$.

In view of (h1)–(h2), let

$$W = L^2_\nu(\mathbb{R}^+, V_0) \cap L^2_\mu(\mathbb{R}^+, V_1)$$

namely, the Hilbert space of V_0-valued functions on $\mathrm{Supp}(\nu) \cup \mathrm{Supp}(\mu) \subset \mathbb{R}^+$, which are V_1-valued on $\mathrm{Supp}(\mu)$, endowed with the inner product

$$\langle \varphi, \psi \rangle_W = \int_0^\infty \nu(\sigma) \langle \varphi(\sigma), \psi(\sigma) \rangle \, d\sigma + \int_0^\infty \mu(\sigma) \langle \varphi(\sigma), \psi(\sigma) \rangle_1 \, ds$$

and norm

$$\|\varphi\|^2_W = \langle \varphi, \varphi \rangle_W \, .$$

Finally, we introduce the Hilbert space

$$\mathcal{H} = V_* \times V_0 \times V_0 \times W$$

endowed with the usual inner product.

Setting $z(t) = (u(t), v(t), \vartheta(t), \eta^t)$ and $z_0 = (u_0, v_0, \vartheta_0, \eta_0) \in \mathcal{H}$, we consider linear evolution equation in $\mathcal{H}$:

$$\begin{cases} \dfrac{d}{dt} z(t) = L z(t) \, , \\ z(0) = z_0 \, . \end{cases} \tag{2.1}$$

The operator L is defined as

$$L \begin{pmatrix} u \\ v \\ \vartheta \\ \eta \end{pmatrix} = \begin{pmatrix} v \\ -B(Bu - (AB^{-1})^* \vartheta) \\ -\vartheta - Av - \displaystyle\int_0^\infty \nu(\sigma) \, \eta(\sigma) \, d\sigma - \int_0^\infty \mu(\sigma) A \eta(\sigma) \, d\sigma \\ \vartheta - \partial_s \eta \end{pmatrix},$$

where $(AB^{-1})^*$ is the adjoint of AB^{-1}, with domain

$$\mathcal{D}(L) = \left\{ z \in \mathcal{H} \;\middle|\; \begin{array}{l} Bu - (AB^{-1})^* \vartheta \in V_* \\ v \in V_* \\ \displaystyle\int_0^\infty \nu(\sigma) \, \eta(\sigma) \, d\sigma + \int_0^\infty \mu(\sigma) A \eta(\sigma) \, d\sigma \in V_0 \\ \vartheta \in W \\ \partial_s \eta \in W \\ \eta(0) = 0 \end{array} \right\}.$$

It is worthwhile to notice that, if $\mu \not\equiv 0$ (which means that $L^2_\mu(\mathbb{R}^+, V_1)$ does not reduce to zero), then $\vartheta \in W$ implies $\vartheta \in V_1$, and $Bu - (AB^{-1})^* \vartheta \in V_*$ implies in turn $u \in V_3$.

Existence, uniqueness and continuous dependence from initial data of the solutions to problem (2.1) is obtained showing that L is the infinitesimal generator of a C_0-semigroup of bounded linear operators (cf. Pazy[24] for the definitions).

Theorem 2.1. *Assume that the memory kernels ν and μ satisfy conditions* (h1)–(h3). *Then L is the infinitesimal generator of a C_0-semigroup $S(t) = e^{tL}$ of contractions on $\mathcal{H}$, i.e.*

$$\|S(t)\|_{\mathcal{B}(\mathcal{H})} \leq 1 \quad \forall\, t \geq 0\,,$$

where $\mathcal{B}(\mathcal{H})$ is the space of bounded linear operators on $\mathcal{H}$.

We recall that a C_0-semigroup is said to be *exponentially stable* if there exist two constants $M \geq 1$ and $\varepsilon > 0$ such that

$$\|S(t)\|_{\mathcal{B}(\mathcal{H})} \leq Me^{-\varepsilon t}\,.$$

Theorem 2.2. *Assume that the memory kernels ν and μ satisfy conditions* (h1)–(h3). *Then the semigroup $S(t)$ associated to problem* (2.1) *is exponentially stable.*

3. Proof of the Theorems

Proof of Theorem 2.1. We first show that L is dissipative. Indeed, for every $z \in \mathcal{D}(L)$ we have

$$\begin{aligned}\langle Lz, z\rangle_{\mathcal{H}} &= -\|\vartheta\|^2 - \langle \partial_s\eta, \eta\rangle_W \\ &= -\|\vartheta\|^2 - \frac{1}{2}\int_0^\infty \nu(\sigma)\frac{d}{d\sigma}\|\eta(\sigma)\|^2\,d\sigma - \frac{1}{2}\int_0^\infty \mu(\sigma)\frac{d}{d\sigma}\|\eta(\sigma)\|_1^2\,d\sigma \\ &= -\|\vartheta\|^2 + \frac{1}{2}\int_0^\infty \nu'(\sigma)\|\eta(\sigma)\|^2\,d\sigma + \frac{1}{2}\int_0^\infty \mu'(\sigma)\|\eta(\sigma)\|_1^2\,d\sigma \leq 0\,, \end{aligned} \tag{3.1}$$

thanks to (h3). The boundary term of the above integration by parts equals zero, since $\eta(0) = 0$ (see Giorgi, Marzocchi and Pata[5] for justification).

Next, we show that the operator $I - L$ is onto, where I is the identity operator on $\mathcal{H}$. Let $z^* = (u^*, v^*, \vartheta^*, \eta^*) \in \mathcal{H}$, and consider the equation

$$(I - L)z = z^*$$

which, written in components, reads

$$u - v = u^*\,, \tag{3.2}$$

$$v + B(Bu - (AB^{-1})^*\vartheta) = v^*\,, \tag{3.3}$$

$$2\vartheta + Av + \int_0^\infty \nu(\sigma)\eta(\sigma)\,d\sigma + \int_0^\infty \mu(\sigma)A\eta(\sigma)\,d\sigma = \vartheta^*\,, \tag{3.4}$$

$$\eta - \vartheta + \partial_s\eta = \eta^*\,. \tag{3.5}$$

Integration of (3.5) with respect to s bears

$$\eta(s) = \eta(s;\vartheta) = (1 - e^{-s})\vartheta + \int_0^s e^{y-s}\eta^*(y)\,dy\,. \tag{3.6}$$

Substituting (3.2) into (3.3), and (3.2) and (3.6) into (3.4), we obtain the following system:

$$\begin{cases} u + B(Bu - (AB^{-1})^*\vartheta) = u^* + v^* \\ c_\nu \vartheta + c_\mu A\vartheta + Au = \vartheta^* + Au^* - \displaystyle\int_0^\infty \nu(\sigma) \int_0^\sigma e^{y-\sigma} \eta^*(y)\, dy\, d\sigma \\ \qquad\qquad - \displaystyle\int_0^\infty \mu(\sigma) \int_0^\sigma e^{y-\sigma} A\eta^*(y)\, dy\, d\sigma\,, \end{cases} \tag{3.7}$$

where

$$c_\nu = 2 + \int_0^\infty \nu(\sigma)(1 - e^{-\sigma})\, d\sigma \quad \text{and} \quad c_\mu = \int_0^\infty \mu(\sigma)(1 - e^{-\sigma})\, d\sigma$$

are non-negative constants in force of (h1)–(h2).

Assume now $c_\mu > 0$ (corresponding to $\mu \not\equiv 0$). It is not hard to check that the right-hand sides of (3.7) belong to V_0 and V_{-1}, respectively. Indeed (see Giorgi, Naso and Pata[7] for a similar calculation),

$$\left\| \int_0^\infty \mu(\sigma) \int_0^\sigma e^{y-\sigma} A\eta^*(y)\, dy\, d\sigma \right\|_{-1} \leq \int_0^\infty \mu(y) \|\eta^*(y)\|_1\, dy < \infty\,.$$

We associate to (3.7) the following bilinear form on $V_* \times V_1$:

$$b((u,\vartheta),(\tilde{u},\tilde{\vartheta})) = \langle u, \tilde{u}\rangle + \langle u, \tilde{u}\rangle_* - \langle \vartheta, \tilde{u}\rangle_1 + c_\nu \langle \vartheta, \tilde{\vartheta}\rangle + c_\mu \langle \vartheta, \tilde{\vartheta}\rangle_1 + \langle u, \tilde{\vartheta}\rangle_1\,.$$

Hence, by means of Lax–Milgram theorem (see, e.g., Evans[4]), the elliptic problem (3.7) admits a unique (weak) solution $(\hat{u}, \hat{\vartheta}) \in V_* \times V_1$. Thus the vector $\hat{z} = (\hat{u}, \hat{v}, \hat{\vartheta}, \hat{\eta})$, with $\hat{v} = \hat{u} - u^*$ and $\hat{\eta}(s) = \eta(s; \hat{\vartheta})$, solves Eqs. (3.2)–(3.5). Since $\hat{\vartheta} \in V_1$, from (3.5) we have that $\hat{\eta} \in W$. From (3.2) we also get that $\hat{v} \in V_*$. Hence, comparing (3.4), $\int_0^\infty \nu(\sigma)\hat{\eta}(\sigma)\, d\sigma + \int_0^\infty \mu(\sigma) A\hat{\eta}(\sigma)\, d\sigma \in V_0$, whereas (3.3) entails $A\hat{u} - \hat{\vartheta} \in V_*$. Finally, $\partial_s \eta \in W$ by comparison in (3.5), and the equality $\hat{\eta}(0) = 0$ follows from (3.6). Thus we conclude that $\hat{z} \in \mathcal{D}(L)$.

Concerning the case $c_\mu = 0$, it is possible to solve the second equation of (3.7) with respect to ϑ, and substitute the result in the first equation, so obtaining

$$u + B(Bu + d_\nu (AB^{-1})^* Au) = u^* + v^* + Bf \tag{3.8}$$

having set $d_\nu = 1/c_\nu$ and

$$f = d_\nu\, (AB^{-1})^* \left(\vartheta^* + Au^* - \int_0^\infty \nu(\sigma) \int_0^\sigma e^{y-\sigma} \eta^*(y)\, dy\, d\sigma \right) \in V_0\,.$$

Notice that $u^* + v^* + Bf \in V_{-2}$. The elliptic equation

$$u + B(Bu + d_\nu (AB^{-1})^* Au) = g$$

admits a unique (weak) solution in V_2 whenever $g \in V_{-2}$. Indeed, the associated bilinear form on V_* is given by

$$b(u, \tilde{u}) = \langle u, \tilde{u}\rangle + \langle u, \tilde{u}\rangle_* + d_\nu \langle u, \tilde{u}\rangle_2\,.$$

We conclude that (3.8) has a solution $\hat{u} \in V_*$. Again, the vector $\hat{z} = (\hat{u}, \hat{v}, \hat{\vartheta}, \hat{\eta})$, with $\hat{\vartheta} = -d_\nu A\hat{u} + (B^{-1}A)^* f$, $\hat{v} = \hat{u} - u^*$ and $\hat{\eta}(s) = \eta(s; \hat{\vartheta})$, solves Eqs. (3.2)–(3.5). Moreover, comparing (3.5), we get at once that $\hat{\eta} \in W$ (since in this case $W = L^2_\nu(\mathbb{R}^+, V_0)$). Proceeding as in the previous case we complete the other checks.

Since L is dissipative and $\mathrm{Range}(\omega I - L) = \mathcal{H}$ for some $\omega > 0$, from the classical Lumer–Phillips theorem,[24] L is the infinitesimal generator of a C_0-semigroup of contractions $S(t)$. □

Proof of Theorem 2.2. In the sequel, we will consider the *complexifications* of the Hilbert spaces used so far, and the *complexification* of the operator L (see Giorgi, Naso and Pata[7] for more details). To avoid a cumbersome notation, we will keep the same symbol for both L and the Hilbert spaces.

We show that the operator $i\beta - L$ is uniformly bounded below as $\beta \in \mathbb{R} \setminus [-\sigma, \sigma]$ for some $\sigma > 0$. This property, in force of a slight generalization of Lemma 2.6 in Giorgi, Naso and Pata[7] (see also Curtain and Zwart,[2] Theorem 5.1.5), assures the exponential decay of the semigroup $S(t)$.

We proceed by contradiction, and assume that the assertion is false. Then there exist sequences $\beta_n \in \mathbb{R}$ and $z_n = (u_n, v_n, \vartheta_n, \eta_n) \in \mathcal{D}(L)$ with

$$\|z_n\|^2_{\mathcal{H}} = \|u_n\|^2_* + \|v_n\|^2 + \|\vartheta_n\|^2 + \|\eta_n\|^2_W = 1 \quad \forall\, n \in \mathbb{N} \tag{3.9}$$

and

$$|\beta_n| \geq \sigma \quad \forall\, n \in \mathbb{N} \quad \text{for some } \sigma > 0 \tag{3.10}$$

such that

$$\lim_{n\to\infty} \|(i\beta_n - L)z_n\|_{\mathcal{H}} = 0$$

which is equivalent to

$$i\beta_n u_n - v_n \underset{n\to\infty}{\longrightarrow} 0 \quad \text{in } V_*\,, \tag{3.11}$$

$$i\beta_n v_n + B(Bu_n - (AB^{-1})^*\vartheta_n) \underset{n\to\infty}{\longrightarrow} 0 \quad \text{in } V_0\,, \tag{3.12}$$

$$\begin{aligned} i\beta_n\vartheta_n + \vartheta_n + Av_n &+ \int_0^\infty \nu(\sigma)\,\eta_n(\sigma)\,d\sigma \\ &+ \int_0^\infty \mu(\sigma)\,A\eta_n(\sigma)\,d\sigma \underset{n\to\infty}{\longrightarrow} 0 \quad \text{in } V_0\,, \end{aligned} \tag{3.13}$$

$$i\beta_n\eta_n - \vartheta_n + \partial_s\eta_n \underset{n\to\infty}{\longrightarrow} 0 \quad \text{in } W\,. \tag{3.14}$$

Repeating the calculations leading to (3.1),

$$\begin{aligned} &\|\vartheta_n\|^2 + \langle \partial_s\eta_n, \eta_n\rangle_W \\ &\quad = \|\vartheta_n\|^2 - \frac{1}{2}\int_0^\infty \nu'(\sigma)\|\eta_n(\sigma)\|^2\,d\sigma - \frac{1}{2}\int_0^\infty \mu'(\sigma)\|\eta_n(\sigma)\|^2_1\,d\sigma \end{aligned}$$

$$= -\Re\langle Lz_n, z_n\rangle_{\mathcal{H}}$$

$$= \Re\langle (i\beta_n - L)z_n, z_n\rangle_{\mathcal{H}} \underset{n\to\infty}{\longrightarrow} 0\,.$$

Therefore

$$\|\vartheta_n\|^2 \underset{n\to\infty}{\longrightarrow} 0 \tag{3.15}$$

and

$$\langle \partial_s \eta_n, \eta_n\rangle_W \underset{n\to\infty}{\longrightarrow} 0\,. \tag{3.16}$$

Recall that, from the equivalence of the norms, there exists $C \geq 1$ such that

$$\frac{1}{C}\,\|w\|_* \leq \|w\|_2 \leq C\,\|w\|_*\,, \quad \forall\, w \in V_*\,.$$

Thus from (3.10) and (3.11) we get

$$\frac{1}{\beta_n}\|v_n\|_2 \leq \frac{C}{\beta_n}\|v_n\|_* \leq K$$

for some $K > 0$ independent of $n \in \mathbb{N}$. The inner product in V_0 of (3.13) and ϑ_n/β_n entails

$$i\|\vartheta_n\|^2 + \frac{1}{\beta_n}\|\vartheta_n\|^2 + \frac{1}{\beta_n}\langle Av_n, \vartheta_n\rangle + \frac{1}{\beta_n}\langle \eta_n, \vartheta_n\rangle_W \underset{n\to\infty}{\longrightarrow} 0\,. \tag{3.17}$$

Since

$$\frac{1}{\beta_n}\langle Av_n, \vartheta_n\rangle \leq \frac{1}{\beta_n}\|v_n\|_2\|\vartheta_n\| \leq K\|\vartheta_n\| \underset{n\to\infty}{\longrightarrow} 0\,,$$

using (3.15) we conclude from (3.17) that

$$\frac{1}{\beta_n}\langle \eta_n, \vartheta_n\rangle_W \underset{n\to\infty}{\longrightarrow} 0\,. \tag{3.18}$$

Thus, taking the inner product in W of (3.14) and η_n/β_n we are led to

$$i\|\eta_n\|_W^2 - \frac{1}{\beta_n}\langle \vartheta_n, \eta_n\rangle_W + \frac{1}{\beta_n}\langle \partial_s\eta_n, \eta_n\rangle_W \underset{n\to\infty}{\longrightarrow} 0$$

and from (3.10), (3.16) and (3.18) we find the relation

$$\|\eta_n\|_W^2 \underset{n\to\infty}{\longrightarrow} 0\,. \tag{3.19}$$

The inner products in V_0 of (3.11) and v_n, and of (3.12) and u_n, appealing to (3.15), give the two convergences

$$i\beta_n\langle u_n, v_n\rangle - \|v_n\|^2 \underset{n\to\infty}{\longrightarrow} 0$$

and

$$i\beta_n\langle v_n, u_n\rangle + \|u_n\|_*^2 \underset{n\to\infty}{\longrightarrow} 0\,.$$

Adding the first relation to the complex conjugate of the second one, we get

$$\|u_n\|_*^2 - \|v_n\|^2 \underset{n\to\infty}{\longrightarrow} 0\,.$$

On the other hand, from (3.9), (3.15) and (3.19),

$$\|u_n\|_*^2 + \|v_n\|^2 \xrightarrow[n\to\infty]{} 1$$

and we end up with

$$\|u_n\|_*^2 \xrightarrow[n\to\infty]{} \frac{1}{2} \quad \text{and} \quad \|v_n\|^2 \xrightarrow[n\to\infty]{} \frac{1}{2}\,. \tag{3.20}$$

We complete the proof finding a contradiction with (3.20). The inner product in V_0 of (3.12) and $A^{-1}\vartheta_n$, yields

$$i\beta_n\langle v_n, A^{-1}\vartheta_n\rangle + \langle Bu_n, BA^{-1}\vartheta_n\rangle - \|\vartheta_n\|^2 \xrightarrow[n\to\infty]{} 0\,.$$

But

$$|\langle Bu_n, BA^{-1}\vartheta_n\rangle| \le \|u_n\|_*\|A^{-1}\vartheta_n\|_* \le C\,\|u_n\|_*\|\vartheta_n\|\,.$$

Thus the last two terms of the above relation go to zero, due to (3.15), and we get that

$$i\beta_n\langle v_n, A^{-1}\vartheta_n\rangle \xrightarrow[n\to\infty]{} 0\,.$$

Finally, taking the inner product in V_0 of (3.13) and $A^{-1}v_n$, we obtain

$$i\beta_n\langle\vartheta_n, A^{-1}v_n\rangle + \langle\vartheta_n, A^{-1}v_n\rangle + \|v_n\|^2 + \langle\eta_n, A^{-1}v_n\rangle_W \xrightarrow[n\to\infty]{} 0$$

and in force of (3.15) and (3.16), we get

$$\|v_n\|^2 \xrightarrow[n\to\infty]{} 0$$

which is a contradiction. □

4. Application to the Plate Equation

Set $V_0 = L^2(\Omega)$, $V_1 = H_0^1(\Omega)$ and $V_2 = H^2(\Omega)\cap H_0^1(\Omega)$, and let $A = -\Delta$, the Laplace operator with Dirichlet boundary condition. On V_2 introduce the inner product

$$\langle u, w\rangle_* = \int_\Omega [\partial_{x_1x_1}u\partial_{x_1x_1}w + \partial_{x_2x_2}u\partial_{x_2x_2}w + \lambda(\partial_{x_1x_1}u\partial_{x_2x_2}w + \partial_{x_2x_2}u\partial_{x_1x_1}w)$$
$$+ 2(1-\lambda)\partial_{x_1x_2}u\partial_{x_1x_2}w]\,dx_1\,dx_2$$

which defines the strictly positive self-adjoint operator B. We recall that for every u, ϑ, w smooth enough, such that $w = 0$ on $\partial\Omega$, the following Green's formula holds (see Lagnese[12]):

$$\int_\Omega (\Delta^2 u)w\,d\Omega = \langle u, w\rangle_* - \int_{\partial\Omega}[\Delta u + (1-\lambda)B_1u]\partial_n w\,d(\partial\Omega) \tag{4.1}$$

with B_1 as in (1.4), and

$$\int_\Omega (\Delta\vartheta)w\,d\Omega = \int_\Omega (\Delta w)\vartheta\,d\Omega - \int_{\partial\Omega}\vartheta\,\partial_n w\,d(\partial\Omega)\,. \tag{4.2}$$

Hence, exploiting the boundary conditions (1.3), we can give the following:

Definition 4.1. A quadruplet $z = (u, v, \vartheta, \eta) \in C([0, +\infty), \mathcal{H})$ is said to be a variational solution to problem **P** if

$$\langle \partial_t u, \tilde{u} \rangle - \langle v, \tilde{u} \rangle = 0\,,$$

$$\langle \partial_t v, \tilde{v} \rangle + \langle u, \tilde{v} \rangle_* - \langle \vartheta, A\tilde{v} \rangle = 0\,,$$

$$\langle \partial_t \vartheta, \tilde{\vartheta} \rangle + \langle \vartheta, \tilde{\vartheta} \rangle + \langle \vartheta, A\tilde{\vartheta} \rangle + \langle \eta, \tilde{\vartheta} \rangle_W = 0\,,$$

$$\langle \partial_t \eta + \partial_s \eta, \tilde{\eta} \rangle_W - \langle \vartheta, \tilde{\eta} \rangle_W = 0\,,$$

for every $\tilde{u} \in V_0$, $\tilde{v} \in V_2$, $\tilde{\vartheta} \in V_2$, $\tilde{\eta} \in W \cap L^2_\mu(\mathbb{R}^+, V_2)$, and almost every $t \in \mathbb{R}^+$ (with abuse of notation the brackets are also used to denote duality products); and

$$\begin{aligned} u(0) &= u_0 \in V_2 && \text{a.e. in } \Omega\,, \\ v(0) &= v_0 \in V_0 && \text{a.e. in } \Omega\,, \\ \vartheta(0) &= \vartheta_0 \in V_0 && \text{a.e. in } \Omega\,, \\ \eta^0 &= \eta_0 \in W && \text{a.e. in } \Omega \times \mathbb{R}^+\,. \end{aligned}$$

A quick look at (4.1) and (4.2) shows that in fact

$$\langle Au - \vartheta, Aw \rangle = \langle Bu - (AB^{-1})^* \vartheta, Bw \rangle\,, \quad \forall\, w \in V_2\,. \tag{4.3}$$

Thus, in force of (4.3), the variational solutions to problem **P** are given by the semigroup solutions of (2.1), to which, in the hypotheses (h1)–(h3), Theorems 2.1 and 2.2 apply.

5. The Model Equation

We consider a thin plate of uniform thickness $d \ll 1$. When the plate is in equilibrium, we assume it occupies a fixed bounded domain $\mathcal{D} \subset \mathbb{R}^3$ placed in a reference frame $x = (x_1, x_2, x_3)$, which has a middle surface midway between its faces in a region $\Omega \subset \mathbb{R}^2$ of the plane $x_3 = 0$.

In order to describe the thermomechanical behavior of the plate, we first formulate proper constitutive equations in the framework of the Gurtin–Pipkin's linear theory on heat conduction with memory.[9] Then, by means of the Kirchhoff strain-displacement relations, we deduce model equations for the plate, as performed in the works of Lagnese.[12,13] Both approaches rely on the basic assumption that variations of the absolute temperature Θ are small compared to some uniform reference temperature Θ_0, namely

$$\left| \frac{\Theta - \Theta_0}{\Theta_0} \right| \ll 1\,.$$

The resulting system models temperature and vertical displacement evolution of a homogeneous, (thermally and elastically) isotropic Kirchhoff plate subject to thermal deformations and hereditary heat conduction law.

According to Lagnese,[13] we assume that the plate is composed by an *isotropic* (mechanically and thermally) *linear thermoelastic* material. As a consequence, the *stress–strain* law is given by

$$\mathbf{S}(x,t) = \mathbb{L}[\mathbf{E}(x,t) - \mathbf{E}^{\theta}(x,t)]\,, \tag{5.1}$$

where the *elastic strain* $\mathbf{E}$ and the *thermal strain* $\mathbf{E}^{\theta}$ are second-order tensors, as well as the *stress* $\mathbf{S}$. The isotropic fourth order tensor

$$\mathbb{L} = l\,\mathbf{I} \otimes \mathbf{I} + 2m\,\mathbb{I} \tag{5.2}$$

involves two constants l and m given by

$$l = \frac{E\,\lambda}{(1-2\lambda)(1+\lambda)}\,, \quad m = \frac{E}{2(1+\lambda)}\,,$$

where $E > 0$ is the Young elastic modulus, and $0 < \lambda < 1/2$ is the Poisson ratio. In small displacement theory, $\mathbf{E}$ is given by

$$\mathbf{E}(x,t) = \frac{1}{2}(\nabla\mathbf{U}(x,t) + (\nabla\mathbf{U})^{\mathrm{T}}(x,t)) \tag{5.3}$$

where $\mathbf{U}$ is the *displacement vector*, and

$$\mathbf{E}^{\theta}(x,t) = \gamma\,\theta(x,t)\,\mathbf{I}\,,$$

where $\theta = \Theta - \Theta_0$ denotes the *temperature variation*, and $\gamma > 0$ is the *coefficient of thermal expansion*. Substitution of the above relation in (5.1) entails

$$\mathbf{S}(x,t) = \mathbb{L}\,\mathbf{E}(x,t) - \frac{\gamma E}{1-2\lambda}\,\theta(x,t)\,\mathbf{I}\,.$$

As we shall see later, this choice is perfectly compatible with the presence of thermal hereditary terms into the heat flux and internal energy.

On the other hand, following Gurtin and Pipkin,[9] the linearized constitutive equation of the *heat flux vector* $\mathbf{q}$ for a thermally isotropic conductor with memory is given by

$$\mathbf{q}(x,t) = -\int_0^{\infty} k(\sigma)\nabla\theta(x,t-\sigma)\,d\sigma\,, \tag{5.4}$$

where $k : \mathbb{R}^+ \to \mathbb{R}$ is the *heat flux memory kernel*. Unfortunately, here we are not allowed to use the corresponding linearized expression of the internal energy and its balance equation. Indeed, the theory of Gurtin and Pipkin only applies to rigid heat conductor, so that we must resort to some generalization if small deformations are taken into account. Therefore we take advantage of the thermodynamically consistent theory of linear thermoviscoelasticity proposed by Lazzari and Vuk.[14] There, the usual energy balance equation is replaced by

$$\rho_0 h(x,t) = -\nabla \cdot \mathbf{q}(x,t) + \rho_0 r(x,t)\,, \tag{5.5}$$

where h is the *thermal power*, which denotes the rate of heat absorption per unit of volume, $\rho_0 > 0$ is the density of the medium, and r is the external heat supply per unit of mass.

As pointed out by Gurtin,[8] in order to comply the property of *invariance under time-reversal* the thermal power-strain kernel, accounting for hereditary contributions to mechanical dissipation, must be equal to the stress-temperature kernel, which is involved in a memory-based stress response. Here, because of the instantaneous constitutive relation (5.1), the time-reversal property is automatically fulfilled assuming for h the following linearized constitutive equation:

$$h(x,t) = \frac{\Theta_0}{\rho_0}\mathbf{B}\colon \partial_t \mathbf{E}(x,t) + c\partial_t\theta(x,t) + \int_0^\infty a(\sigma)\partial_t\theta(x,t-\sigma)\,d\sigma\,, \tag{5.6}$$

where $\mathbf{B}$ is a symmetric second-order tensor, $c > 0$ is the *specific heat* of the body, and $a : \mathbb{R}^+ \to \mathbb{R}$ is the *thermal memory kernel.* Regarding the kernels k and a, we assume that they are smooth enough and summable on $\mathbb{R}^+$. Also, we require that k', a'' are both non-positive, $a(0) > 0$, and k, a' vanish at infinity.

The particular form of $\mathbf{S}$ and h leads to the lack of memory in the coupling terms of the thermoelastic system (see Giorgi and Naso[6] for a more detailed justification of this choice). According to Lazzari and Vuk,[14] thermodynamic compatibility between (5.1) and (5.6) is fulfilled provided that

$$\mathbf{B} = \frac{\gamma E}{1-2\lambda}\mathbf{I}\,, \tag{5.7}$$

where $\gamma \neq 0$ is a coupling constant between thermal and mechanical evolution. Without loss of generality γ can be taken strictly positive. It is worth observing that if $\gamma = 0$, the thermal power h reduces to the time derivative of the Gurtin–Pipkin's internal energy for rigid conductors, and the heat equation (5.5) uncouples from the motion equation.

Henceforth, we denote by $u_i(x_1, x_2)$, $i = 1, 2, 3$, the components of the displacement vector of the points of the middle surface Ω of the plate which have coordinates $(x_1, x_2, 0)$ at equilibrium. If d is the uniform thickness of the plate, let $\Omega^- = \Omega \times \{-d/2\}$ and $\Omega^+ = \Omega \times \{d/2\}$ denote its faces, and $\Gamma = \partial\Omega \times (-d/2, d/2)$ its edge, where $\partial\Omega$ is the boundary of Ω.

As customary in thin plate theory, we assume that the transverse normal stress is negligible compared to other stresses, namely $S_{33} = 0$. This allows E_{33} to be expressed as a function of E_{11} and E_{22}. In addition, integration with respect to x_3 is carried out, the stretching components u_1, u_2 uncouple from the bending component $u = u_3$ of the plate displacement, so that both the *strain energy* and the *kinetic energy* split into two parts, accordingly.

Finally, neglecting transverse shear effects, we suppose that $\mathbf{u} = (u_1, u_2, u)$ is related to $\mathbf{U}$ by the approximate relations

$$U_1 = u_1 - x_3\partial_{x_1}u\,, \quad U_2 = u_2 - x_3\partial_{x_2}u\,, \quad U_3 = u\,.$$

By virtue of (5.3), this assumption leads to the strain-displacement relations of the

Kirchhoff model, namely

$$\begin{cases} E_{11} = \partial_{x_1} u_1 - x_3 \partial_{x_1 x_1} u \,, \\ E_{22} = \partial_{x_2} u_2 - x_3 \partial_{x_2 x_2} u \,, \\ E_{12} = \dfrac{1}{2}[\partial_{x_2} u_1 + \partial_{x_1} u_2 - 2x_3 \partial_{x_1 x_2} u] \,, \\ E_{13} = E_{23} = E_{33} = 0 \,. \end{cases} \tag{5.8}$$

In addition, consistency with the absence of transverse shear requires that the plate be subject to an external distribution of loads per unit of mass $\mathbf{b} = (b_1, b_2, b_3)$ with b_1 and b_2 independent of x_3.

Introducing the *modulus of flexural rigidity*

$$D = \frac{Ed^3}{12\rho_0(1-\lambda^2)}$$

and the *thermal resultant distribution*

$$\vartheta(x_1, x_2, t) = \frac{12}{d^3} \int_{-d/2}^{d/2} x_3 \theta(x_1, x_2, x_3, t)\, dx_3 \tag{5.9}$$

the motion equation for the bending component of the plate can be obtained exactly as in Lagnese[13] (see Chap. I, Sec. 6), and reads

$$d\partial_{tt} u - \frac{d^3}{12} \Delta \partial_{tt} u + D\Delta \left[\Delta u + \gamma \frac{1+\lambda}{2} \vartheta \right] = f \,, \tag{5.10}$$

where the dependence on $(x_1, x_2, t) \in \Omega \times \mathbb{R}^+$ is understood and not written, Δ denotes the two-dimensional Laplacian, and

$$f(x_1, x_2, t) = \int_{-d/2}^{d/2} b_3(x_1, x_2, x_3, t)\, dx_3 \,.$$

Equation (5.10) involves the unknown field ϑ whose dynamics will be introduced on the basis of the energy balance equation. To this end, we first substitute (5.4), (5.6) and (5.7) into (5.5), to get

$$\begin{aligned} \partial_t \theta(x,t) + \beta_0 \mathbf{I} \colon \partial_t \mathbf{E}(x,t) + \int_0^\infty \alpha(s) \partial_t \theta(x, t-\sigma)\, d\sigma \\ - \int_0^\infty \kappa(\sigma) \Delta_3 \theta(x, t-\sigma)\, d\sigma = \frac{1}{c} r(x,t) \,, \end{aligned} \tag{5.11}$$

where Δ_3 denotes the three-dimensional Laplacian, and

$$\beta_0 = \frac{\Theta_0 \gamma E}{\rho_0 c(1-2\lambda)} \,, \quad \kappa(s) = \frac{k(s)}{\rho_0 c} \,, \quad \alpha(s) = \frac{a(s)}{c} \,.$$

Then we insert relations (5.8) of the Kirchhoff model into (5.11); we multiply the result by $12x_3/d^3$ and integrate in x_3 from $-d/2$ to $d/2$. Recalling (5.9), and setting

$$g(x_1, x_2, t) = \frac{12}{d^3 c} \int_{-d/2}^{d/2} x_3 r(x_1, x_2, x_3, t)\, dx_3$$

and

$$p(x_1, x_2, t) = \frac{12}{d^3} \int_{-d/2}^{d/2} x_3 \left[\int_0^\infty \kappa(\sigma) \partial_{x_3 x_3} \theta(x_1, x_2, x_3, t - \sigma) \, d\sigma \right] dx_3$$

we end up with

$$\partial_t \vartheta(t) - \beta_0 \Delta \partial_t u(t) + \int_0^\infty \alpha(\sigma) \partial_t \vartheta(t - \sigma) \, d\sigma - \int_0^\infty \kappa(\sigma) \Delta \vartheta(t - \sigma) \, d\sigma = g(t) + p(t) \,, \tag{5.12}$$

where the dependence on (x_1, x_2) is understood. Now, integration by parts with respect to x_3 yields

$$\begin{aligned} p(t) &= \frac{12}{d^3} \int_0^\infty \kappa(\sigma)(x_3 \partial_{x_3} \theta(t - \sigma) - \theta(t - \sigma)) \Big|_{x_3 = -d/2}^{x_3 = d/2} d\sigma \\ &= \frac{6}{d^2} (\mathbf{q} \cdot \mathbf{n}|_{\Omega^+} - \mathbf{q} \cdot \mathbf{n}|_{\Omega^-}) - \frac{12}{d^3} \int_0^\infty \kappa(\sigma) \theta(t - \sigma) \Big|_{x_3 = -d/2}^{x_3 = d/2} d\sigma \,, \end{aligned}$$

where $\mathbf{n}|_{\Omega^-}$ and $\mathbf{n}|_{\Omega^+}$ denote the unit outward normals to the lower and upper faces of the plate, respectively. Concerning the first term of the right-hand side of the above relation, a quite natural boundary condition for the plate is that the heat flux vanishes across its faces, namely

$$\mathbf{q} \cdot \mathbf{n}|_{\Omega^+} = \mathbf{q} \cdot \mathbf{n}|_{\Omega^-} = 0 \,. \tag{5.13}$$

In order to evaluate the second term, for every $x_3 \in (-d/2, d/2)$ and $(x_1, x_2) \in \Omega$ we make the following assumption (which is justified because of the thinness of the plate):

$$\theta(x_1, x_2, x_3, t) = \vartheta_0(x_1, x_2, t) + \vartheta_1(x_1, x_2, t) \, x_3 + \vartheta_2(x_1, x_2, t) x_3^2 + o(d^3) \,.$$

In force of (5.9) it follows that

$$\vartheta_1 = \vartheta + o(d) \quad \text{and} \quad \theta|_{x_3 = -d/2}^{x_3 = -d/2} = d\vartheta + o(d^3) \,. \tag{5.14}$$

Hence, (5.13) and (5.14) lead to the approximate equality

$$p(t) = -\frac{12}{d^2} \int_0^\infty \kappa(\sigma) \vartheta(t - \sigma) \, d\sigma \,.$$

A similar result can be obtained assuming a uniform environmental temperature and applying Newton's law of cooling instead of adiabatic boundary condition on the lower and upper faces (see, for instance, Lagnese,[13] p. 30). As a consequence, (5.12) becomes

$$\partial_t \vartheta(t) - \beta_0 \Delta \partial_t u(t) + \int_0^\infty \alpha(\sigma) \partial_t \vartheta(t - \sigma) \, d\sigma - \int_0^\infty \kappa(\sigma) \Delta \vartheta(t - \sigma) \, d\sigma + \frac{12}{d^2} \int_0^\infty \kappa(\sigma) \vartheta(t - \sigma) \, d\sigma = g(t) \,. \tag{5.15}$$

According to the assumed regularity of kernel α, we observe that

$$\int_0^\infty \alpha(\sigma)\partial_t\vartheta(t-\sigma)\,d\sigma = \alpha(0)\vartheta(t) + \int_0^\infty \alpha'(\sigma)\vartheta(t-\sigma)\,d\sigma$$

and substitution into (5.15) bears

$$\begin{aligned}&\partial_t\vartheta(t) + \alpha_0\vartheta(t) - \beta_0\Delta\partial_t u(t)\\ &\qquad + \int_0^\infty [\beta(\sigma)\vartheta(t-\sigma) - \kappa(\sigma)\Delta\vartheta(t-\sigma)]\,d\sigma = g(t)\,,\end{aligned} \tag{5.16}$$

where

$$\beta(s) = \alpha'(s) + \frac{12}{d^2}\kappa(s) \quad \text{and} \quad \alpha_0 = \alpha(0) > 0\,.$$

Finally, we turn our attention to the boundary conditions. If the plate is simply supported along its edge Γ, then the corresponding boundary conditions on u and ϑ along $\partial\Omega$ are as follows (see Lagnese,[13] Eq. (2.20))

$$u(t) = \Delta u(t) + (1-\lambda)B_1 u(t) + \gamma\frac{1+\lambda}{2}\vartheta(t) = 0 \tag{5.17}$$

with B_1 as in (1.4).

In order to derive a boundary condition relative to the thermal environment, we assume the plate to reside in a medium of uniform temperature (measured from the reference temperature). Accounting for hereditary conductors, and assuming that the environmental temperature equals the reference one, Newton's law of cooling can be generalized as follows:

$$k(y)\partial_n\theta(t-y) = -\lambda_0\theta(t-y) \quad \text{on } \Gamma$$

for almost every $y \in \text{Supp}(k)$, where $\lambda_0 \geq 0$ is the conductivity constant of the edge of the plate, and ∂_n denotes the derivative along the outward unit normal n introduced in (1.4). Clearly, $\text{Supp}(k) = \text{Supp}(\kappa)$. Multiplying this relation times $12x_3/(\rho_0 d^3 c)$, and integrating with respect to x_3 over $[-d/2, d/2]$, we get, in view of (5.9),

$$\kappa(y)\partial_n\vartheta(t-y) = -\frac{\lambda_0}{\rho_0 c}\vartheta(t-y) \quad \text{on } \partial\Omega\,, \quad \text{a.e. } y \in \text{Supp}(\kappa)\,.$$

Observing that if $s \in \text{Supp}(\kappa)$ then $[0,s] \subset \text{Supp}(\kappa)$, we can integrate over $[0,s]$ to obtain

$$-\frac{\rho_0 c}{\lambda_0}\int_0^s \kappa(y)\partial_n\vartheta(t-y)\,dy = \int_0^s \vartheta(t-y)\,dy \quad \text{on } \partial\Omega\,, \quad s \in \text{Supp}(\kappa)\,.$$

Finally, letting $\lambda_0 \to \infty$, i.e. assuming that the edge has infinite thermal conductivity, we conclude that

$$\int_0^s \vartheta(t-y)\,dy = 0 \quad \text{on } \partial\Omega\,, \quad s \in \text{Supp}(\kappa)\,. \tag{5.18}$$

Collecting Eqs. (5.10) (neglecting rotatory inertia, namely, taking the term $d^3\Delta\partial_{tt}u/12$ equal to zero) and (5.16)–(5.18), we are led to the following boundary value problem, which resumes the thermomechanical evolution of the plate:

$$d\partial_{tt}u(t) + D\Delta\left[\Delta u(t) + \gamma\frac{1+\lambda}{2}\vartheta(t)\right] = f(t) \quad \text{in } \Omega\,,$$

$$\partial_t\vartheta(t) + \alpha_0\vartheta(t) - \beta_0\Delta\partial_t u(t) + \int_0^\infty [\beta(\sigma)\vartheta(t-\sigma) - \kappa(\sigma)\Delta\vartheta(t-\sigma)]\,d\sigma = g(t) \quad \text{in } \Omega\,,$$

$$u(t) = \Delta u(t) + (1-\lambda)B_1 u(t) + \gamma\frac{1+\lambda}{2}\vartheta(t) = 0 \quad \text{on } \partial\Omega\,,$$

$$\int_0^s \vartheta(t-y)\,dy = 0 \quad \text{on } \partial\Omega\,, \quad s \in \mathrm{Supp}(\kappa)\,.$$

Clearly, if no external forces and heat supplies are applied to the plate, the functions f and g vanish.

Remark 5.1. When smooth kernels μ and ν are involved, the resulting system describes heat propagation with a thermal damping weaker than in the classical approach based on the Fourier law. However, some additional thermal dissipation is supported by the hereditary term into the constitutive Eq. (5.5) for the thermal power h. This term, and especially $\alpha(0)$, plays a crucial role in our main stability result. From a physical point of view, it compensates the absence of heat outflow due to the adiabatic boundary condition on the lower and upper faces. This is why we conjecture that exponential stability of the plate fails to hold when the memory term is neglected in the expression of h.

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