# UNIFORM ATTRACTORS FOR A NON-AUTONOMOUS SEMILINEAR HEAT EQUATION WITH MEMORY

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**Abstract.** In this paper we investigate the asymptotic behavior, as time tends to infinity, of the solutions of an non-autonomous integro-partial differential equation describing the heat flow in a rigid heat conductor with memory. Existence and uniqueness of solutions is provided. Moreover, under proper assumptions on the heat flux memory kernel and on the magnitude of nonlinearity, the existence of uniform absorbing sets and of a global uniform attractor is achieved. In case of quasiperiodic dependence of time of the external heat supply, the above attractor is shown to have finite Hausdorff dimension.

## 0. Introduction

Let $\Omega \subset \mathbb{R}^3$ be a fixed bounded domain occupied by a rigid, isotropic, homogeneous heat conductor with linear memory. We consider the following integro-partial differential equation, which is derived in the framework of the well-established theory of heat flow with memory due to Coleman & Gurtin [8]:

$$\begin{aligned}
&c_0 \frac{\partial}{\partial t}\theta - k_0 \Delta\theta - \int_{-\infty}^{t} k(t-s)\Delta\theta(s)\, ds + g(\theta) = h \qquad \text{on } \Omega \times (\tau, +\infty) \\
&\theta(x,t) = 0 \qquad\qquad x \in \partial\Omega \quad t > \tau \\
&\theta(x,\tau) = \theta_0(x) \qquad x \in \Omega
\end{aligned} \tag{0.1}$$

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where $\theta : \Omega \times \mathbb{R} \to \mathbb{R}$ is the *temperature variation field* relative to the equilibrium reference value, $k : \mathbb{R}^+ \to \mathbb{R}$ is the *heat flux memory kernel*, and the constants $c_0$ and $k_0$ denote the *specific heat* and the *instantaneous conductivity*, respectively. The function $h : \Omega \times [\tau, +\infty) \to \mathbb{R}$ is a time-dependent external heat source, whereas $g : \mathbb{R} \to \mathbb{R}$ is a nonlinear heat supply.

System (0.1) was studied in [14], assuming a time-independent external heat source $h$ and a polynomial nonlinearity $g$. Along the line of the procedure suggested by Dafermos in his pioneer work [11], we introduce the new variables

$$\theta^t(x,s) = \theta(x,t-s) \qquad s \geq 0$$

and

$$\eta^t(x,s) = \int_0^s \theta^t(x,y)\,dy = \int_{t-s}^t \theta(x,y)\,dy \qquad s \geq 0.$$

Assuming $k(\infty) = 0$, performing a change of variable, and setting

$$\mu(s) = -k'(s)$$

formal integration by parts transforms the above system (0.1) into

$$\begin{aligned}
&c_0 \frac{\partial}{\partial t}\theta - k_0 \Delta\theta - \int_0^\infty \mu(s)\Delta\eta^t(s)\,ds + g(\theta) = h \qquad \text{on } \Omega \times (\tau, +\infty) \\
&\frac{\partial}{\partial t}\eta^t(x,s) = \theta(x,t) - \frac{\partial}{\partial s}\eta^t(x,s) \qquad x \in \Omega \quad t > \tau \quad s > 0 \\
&\theta(x,t) = 0 \qquad x \in \partial\Omega \quad t > \tau \\
&\theta(x,\tau) = \theta_0(x) \qquad x \in \Omega \\
&\eta^\tau(x,s) = \eta_0(x,s) \qquad x \in \Omega \quad s > 0
\end{aligned} \tag{0.2}$$

where the term

$$\eta^\tau(x,s) = \int_0^s \theta^\tau(x,y)\,dy = \int_{\tau-s}^\tau \theta(x,y)\,dy \qquad s \geq 0$$

is the prescribed *initial integrated past history* of $\theta(x,t)$, which does not depend on $\theta_0(x)$, and is assumed to vanish on $\partial\Omega$, as well as $\theta(x,t)$. As a consequence it follows that

$$\eta^t(x,s) = 0 \qquad x \in \partial\Omega \quad t > \tau \quad s > 0.$$

Indeed, the above assertion is obvious if $\tau \leq t - s$, and if $\tau > t - s$ we can write

$$\eta^t(x,s) = \eta_0(x, \tau + s - t) + \int_\tau^t \theta(x,y)\,dy.$$

In the sequel we agree to denote by $\partial_t$ or more simply by $_t$ derivation with respect to $t$, and by the *prime* derivation with respect to $s$.

The constitutive quantities $c_0$, $k_0$ and $\mu$ are required to verify the following set of hypotheses:

(h1) $c_0 > 0 \qquad k_0 > 0$

(h2) $\mu \in C^1(\mathbb{R}^+) \cap L^1(\mathbb{R}^+) \qquad \mu(s) \geq 0 \qquad \mu'(s) \leq 0 \quad \forall\, s \in \mathbb{R}^+$

(h3) $\mu'(s) + \delta\mu(s) \leq 0 \qquad \forall\, s \in \mathbb{R}^+$ and some $\delta > 0$.

Notice that (h3) implies the exponential decay of $\mu(s)$. Nevertheless, it allows $\mu(s)$ to have a singularity at $s = 0$, whose order is less than 1, since $\mu(s)$ is a non-negative $L^1$-function.

Now, taking for simplicity $c_0 = 1$, and denoting $z(t) = (\theta(t), \eta^t)$, $z_0 = (\theta_0, \eta_0)$, and setting

$$Lz = (k_0\Delta\theta + \int_0^\infty \mu(s)\Delta\eta(s)\, ds, \theta - \eta')$$

and

$$G(z) = (h - g(\theta), 0)$$

problem (0.2) assumes the compact form

$$\begin{aligned} &z_t = Lz + G(z) \\ &z(x,t) = 0 \qquad x \in \partial\Omega \quad t > \tau \\ &z(x,\tau) = z_0. \end{aligned} \tag{0.3}$$

We recall that existence, uniqueness and stability of the linear problem corresponding to (0.1) (i.e., with $g \equiv 0$) have been investigated by several authors (see, e.g., [13,15,18,19]). Related results which include phase transition phenomena are in [3,9]. On the other hand, when no memory effect occur, long-time behavior of semilinear parabolic problems like (0.1) with $k \equiv 0$ have been widely studied, both for autonomous and non-autonomous equations (see, e.g., [5,6,7,22]). However, the main aim of this paper is to study the role played by the memory term as time tends to infinity. Results concerning asymptotic behavior of solutions for semilinear problems in presence of nontrivial terms of convolution type, involving the principal part of the differential operator, can be found in [1,12,14,16].

In this paper, due to the time-dependence of $h$, the evolutive system (0.3) of differential equations is non-autonomous. Therefore, in order to study its asymptotic behavior, we have to introduce the notion of *process*, which is a generalization of the semigroup of operators on a Hilbert space.

**Definition 0.1.** A two-parameter family $\{U(t,\tau)\}_{t\geq\tau,\,\tau\in\mathbb{R}}$ of operators on a Hilbert space $\mathcal{H}$ is said to be a *process* if the following hold:

(i) $U(\tau,\tau)$ is the identity map on $\mathcal{H}$ for any $\tau \in \mathbb{R}$;

(ii) $U(t,s)U(s,\tau) = U(t,\tau)$ for any $t \geq s \geq \tau$;

(iii) $U(t,\tau)x \to x$ as $t \downarrow \tau$ for any $x \in \mathcal{H}$ and any $\tau \in \mathbb{R}$.

In Theorem 2.1 and Theorem 2.2 below we show that, given $z_0$ in a suitable Hilbert space $\mathcal{H}$, under proper conditions on $g$ and $h$, there exists a unique solution $z(t)$ in $\mathcal{H}$

to problem (0.3), which continuously depends on $z_0$. Thus we can define the family of processes $U_h(t,\tau)$, depending on the functional parameter $h$, and acting on $\mathcal{H}$, as

$$U_h(t,\tau)z_0 = z(t) \tag{0.4}$$

where $z(t)$ is the solution at time $t$ of (0.3) with initial data $z_0$ given at time $\tau$. The parameter $h$ is usually called the *symbol* of the process. Notice that if $h$ is time-independent, the process $U_h(t,\tau)$ reduces to a semigroup by setting

$$S_h(t) = U_h(t,0).$$

The study of the long-time behavior of the family of processes $U_h(t,\tau)$ will be carried out in Section 3 and Section 4, where, using the techniques of [5,6], we prove the existence of absorbing sets in $\mathcal{H}$ and in a smaller space $\mathcal{V}$, and of a global attractor. In all cases the objects are uniform as $h$ is allowed to move in a suitable functional space. Finally, in Section 5 we show that for a particular choice of the symbol space (namely for an external heat supply with a quasiperiodic dependence on time) the Hausdorff dimension of the uniform attractor is finite.

## 1. Functional Setting and Notation

Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with Lipschitz boundary. With usual notation, we introduce the spaces $L^p$, $H^k$ and $H_0^k$ acting on $\Omega$. Let $\langle\cdot,\cdot\rangle$ and $\|\cdot\|$ denote the $L^2$-inner product and $L^2$-norm, respectively, and let $\|\cdot\|_p$ denote the $L^p$-norm. With abuse of notation, we use $\langle\cdot,\cdot\rangle$ to denote also the duality between $L^p$ and its dual space $L^q$. We recall Poincaré inequality

$$\lambda_0\|v\|^2 \le \|\nabla v\|^2 \qquad \forall\, v \in H_0^1 \tag{1.1}$$

and

$$\gamma_0\|\nabla v\|^2 \le \|\Delta v\|^2 \qquad \forall\, v \in H^2 \cap H_0^1 \tag{1.2}$$

for some $\lambda_0, \gamma_0 > 0$, where (1.2) is obtained from (1.1) and Young inequality (see Lemma 1.2 below). In force of (1.1)-(1.2), the inner products on $H_0^1$ and $H^2 \cap H_0^1$ can be defined in the following manner:

$$\langle u,v\rangle_{H_0^1} = \langle\nabla u,\nabla v\rangle$$

and

$$\langle u,v\rangle_{H^2\cap H_0^1} = \langle\Delta u,\Delta v\rangle$$

In view of (h2), let $L^2_\mu(\mathbb{R}^+,L^2)$ be the Hilbert space of $L^2$-valued functions on $\mathbb{R}^+$, endowed with the inner product

$$\langle\varphi,\psi\rangle_\mu = \int_0^\infty \mu(s)\langle\varphi(s),\psi(s)\rangle\,ds$$

Similarly on $L^2_\mu(\mathbb{R}^+, H^1_0)$ and $L^2_\mu(\mathbb{R}^+, H^2 \cap H^1_0)$, respectively, we have the inner products

$$\langle \varphi, \psi \rangle_{L^2_\mu(\mathbb{R}^+, H^1_0)} = \langle \nabla\varphi, \nabla\psi \rangle_\mu$$

and

$$\langle \varphi, \psi \rangle_{L^2_\mu(\mathbb{R}^+, H^2 \cap H^1_0)} = \langle \Delta\varphi, \Delta\psi \rangle_\mu.$$

Finally we introduce the Hilbert spaces

$$\mathcal{H} = L^2 \times L^2_\mu(\mathbb{R}^+, H^1_0)$$

and

$$\mathcal{V} = H^1_0 \times L^2_\mu(\mathbb{R}^+, H^2 \cap H^1_0)$$

which are respectively endowed with the inner products

$$\langle w_1, w_2 \rangle_{\mathcal{H}} = \langle \psi_1, \psi_2 \rangle + \langle \nabla\varphi_1, \nabla\varphi_2 \rangle_\mu$$

and

$$\langle w_1, w_2 \rangle_{\mathcal{V}} = \langle \nabla\psi_1, \nabla\psi_2 \rangle + \langle \Delta\varphi_1, \Delta\varphi_2 \rangle_\mu$$

where $w_i = (\psi_i, \varphi_i) \in \mathcal{H}$ or $\mathcal{V}$ for $i = 1, 2$.

We will also consider spaces of functions defined on an (possibly infinite) interval $I$ with values in a Banach space $X$ such as $C(I, X)$, $L^p(I, X)$ and $H^k(I, X)$, with the usual norms.

To describe the asymptotic behavior of the solutions of our system we need to introduce the space $\mathcal{T}^p_b(\mathbb{R}, X)$ of $L^p_{\text{loc}}$-*translation bounded* functions with values in a Banach space $X$, namely

$$\mathcal{T}^p_b(\mathbb{R}, X) = \left\{ f \in L^p_{\text{loc}}(\mathbb{R}, X) : \|f\|_{\mathcal{T}^p_b(\mathbb{R}, X)} = \sup_{\xi \in \mathbb{R}} \left( \int_\xi^{\xi+1} \|f(y)\|^p_X \, dy \right)^{\frac{1}{p}} < \infty \right\}.$$

In an analogous manner, given $\tau \in \mathbb{R}$, we define the space $\mathcal{T}^p_b([\tau, +\infty), X)$.

**Definition 1.1.** A function $f \in L^p_{\text{loc}}(\mathbb{R}, X)$ is said to be *translation compact* in $L^p_{\text{loc}}(\mathbb{R}, X)$, and we write $f \in \mathcal{T}^p_c(\mathbb{R}, X)$, if the *hull* of $f$, defined as

$$H(f) = \overline{\{f(\cdot + r)\}_{r \in \mathbb{R}}}^{L^p_{\text{loc}}(\mathbb{R}, X)}$$

is compact in $L^p_{\text{loc}}(\mathbb{R}, X)$.

The reader is referred to [7] for a more detailed presentation of the subject. Here we just highlight that $\mathcal{T}^p_c(\mathbb{R}, X) \subset \mathcal{T}^p_b(\mathbb{R}, X)$. Moreover

$$\|\varphi\|_{\mathcal{T}^p_b(\mathbb{R}, X)} \leq \|f\|_{\mathcal{T}^p_b(\mathbb{R}, X)} \quad \forall\, \varphi \in H(f).$$

We also remark that the class $\mathcal{T}^p_c(\mathbb{R}, X)$ is quite general. For example, it contains $L^q(\mathbb{R}, X)$ for all $q \geq p$, the constant $X$-valued functions, and the class of *almost periodic* functions (see [2]).

We now recall some technical results which will be needed in the course of the investigation.

**Lemma 1.2.** (Young inequality). *Let $a$, $b \geq 0$ be given. Then for every $\epsilon > 0$, and for every $1 < p, q < \infty$ such that $\frac{1}{p} + \frac{1}{q} = 1$, the inequality*

$$ab \leq \epsilon a^p + K(\epsilon, p, q) b^q$$

*holds with*

$$K(\epsilon, p, q) = \frac{1}{q}(\epsilon p)^{-\frac{q}{p}}. \tag{1.3}$$

**Lemma 1.3.** (Gagliardo-Nirenberg). *Let $2 < p \leq 6$. Then there exists $c > 0$ such that the inequality*

$$\|u\|_p \leq c\|\nabla u\|^\nu \|u\|^{1-\nu}$$

*holds for all $u \in H_0^1$, with*

$$\nu = \frac{3}{2}\left[\frac{p-2}{p}\right].$$

Here and in the sequel, for $\tau \in \mathbb{R}$, we denote $\mathbb{R}_\tau = [\tau, +\infty)$. The following Gronwall-type lemma is a modification of Lemme A.5 in [4] (see [20] for a detailed proof).

**Lemma 1.4.** *Let $\phi$ be a non-negative, absolutely continuous function on $\mathbb{R}_\tau$, $\tau \in \mathbb{R}$, which satisfies for some $\epsilon > 0$ and $0 \leq \sigma < 1$ the differential inequality*

$$\frac{d}{dt}\phi(t) + \epsilon\phi(t) \leq \Lambda + m_1(t)\, \phi(t)^\sigma + m_2(t) \qquad a.e.\ t \in \mathbb{R}_\tau$$

*where $\Lambda \geq 0$, and $m_1$ and $m_2$ are non-negative locally summable functions on $\mathbb{R}_\tau$. Then*

$$\begin{aligned}\phi(t) \leq & \frac{1}{1-\sigma}\left[\phi(\tau)\, e^{-\epsilon(t-\tau)} + \frac{\Lambda}{\epsilon}\right] + \left[\int_\tau^t m_1(y)\, e^{-\epsilon(1-\sigma)(t-y)} dy\right]^{\frac{1}{1-\sigma}} \\ & + \frac{1}{1-\sigma}\int_\tau^t m_2(y)\, e^{-\epsilon(t-y)} dy\end{aligned}$$

*for any $t \in \mathbb{R}_\tau$.*

The easy proof of the next result is left to the reader.

**Lemma 1.5.** *Let $m \in \mathcal{T}_b^1(\mathbb{R}_\tau, \mathbb{R}^+)$ for some $\tau \in \mathbb{R}$. Then, for every $\epsilon > 0$,*

$$\int_\tau^t m(y)\, e^{-\epsilon(t-y)} dy \leq C(\epsilon)\|m\|_{\mathcal{T}_b^1(\mathbb{R}_\tau, \mathbb{R}^+)}$$

*where*

$$C(\epsilon) = \frac{e^\epsilon}{1-e^{-\epsilon}}. \tag{1.4}$$

We conclude the section with a lemma which will be needed in the last part of the paper (for the proof see [22], p. 300–303).

**Lemma 1.6.** *There exists a positive constant* $\kappa$ *such that, for any given* $m$ *vectors* $\{\varphi_1, \ldots, \varphi_m\}$ *in* $H_0^1$ *which are orthonormal in* $L^2$*, it follows that*

$$\sum_{j=1}^{m} \|\nabla \varphi_j\|^2 \geq \kappa m^{\frac{5}{3}}.$$

## 2. Existence and Uniqueness

In this section we establish existence and uniqueness results for problem (0.3). Unlike in [14], where the nonlinearity is assumed to be an odd degree polynomial on any order, here we are interested in considering more general nonlinear terms. Of course we have to pay the price of requiring additional properties. Here below is a list of conditions on $g$ which will be used throughout the paper.

**Conditions on the nonlinear term** $g$**.** Let $g \in C(\mathbb{R})$, and assume that there exist non-negative constants $c_j$, $j = 1, 2, 3, 4, 5$, and $\beta > 0$ such that

(g1) $|g(u)| \leq c_1(1 + |u|^\beta)$

(g2) $u \cdot g(u) \geq -c_2 + c_3|u|^{\beta+1}$

(g3) $|g(u) - g(v)| \leq c_4|u - v|(1 + |u|^\gamma + |v|^\gamma)$ with $\gamma = \max\{\beta - 1, 0\}$

(g4) $g \in C^1(\mathbb{R})$ and $g'(u) \geq -c_5$.

Clearly, if the nonlinearity is weak, the above conditions can be relaxed. For instance, if $\beta \leq 1$ then $c_3 = 0$, if $\beta < 1$ then (g2) is not needed at all.

**Theorem 2.1.** *Assume* (h1)-(h2)*, and let* (g1)-(g2) *hold for some* $\beta > 0$*. For any initial time* $\tau \in \mathbb{R}$*, given*

$$h \in L^1_{\text{loc}}(\mathbb{R}_\tau, L^2) + L^2_{\text{loc}}(\mathbb{R}_\tau, H^{-1}) \qquad \textit{and} \qquad z_0 = (\theta_0, \eta_0) \in \mathcal{H}$$

*there exists a function* $z = (\theta, \eta)$*, with*

$$\theta \in L^\infty([\tau, T], L^2) \cap L^2([\tau, T], H_0^1) \cap L^{\beta+1}([\tau, T], L^{\beta+1}) \qquad \forall\, T > \tau$$
$$\eta \in L^\infty([\tau, T], L^2_\mu(\mathbb{R}^+, H_0^1)) \qquad \forall\, T > \tau$$

*such that*

$$z_t = Lz + G(z)$$

*in the weak sense, and*

$$z|_{t=\tau} = z_0.$$

*Furthermore*

$$z \in C([0,T],\mathcal{H}) \qquad \forall\, T > \tau.$$

Proof of the above result re-casts exactly the Faedo-Galerkin scheme used in [14]. The only difference here depends on the presence of a time dependent heat supply $h \in L^1_{\rm loc}(\mathbb{R}_\tau, L^2) + L^2_{\rm loc}(\mathbb{R}_\tau, H^{-1})$, which can be easily handled making use of the Gronwall lemma.

We just recall two relations from [14] which will be needed later. The first is obtained taking the inner product in $\mathcal{H}$ of $(0.3)_1$ and $z = (\theta,\eta)$, applying the divergence theorem, performing an integration by parts, and using (h2), whereas the second one is obtained integrating the first one from $\tau$ to $T$, along with proper estimates.

$$\frac{d}{dt}\|z\|^2_{\mathcal{H}} + 2k_0\|\nabla\theta\|^2 - \int_0^\infty \mu'(s)\|\nabla\eta(s)\|^2\,ds = -2\langle g(\theta),\theta\rangle + 2\langle h,\theta\rangle \tag{2.1}$$

$$\int_\tau^T \|\theta(y)\|^{\beta+1}_{\beta+1}\,dy \le C\left(\|\theta_0\|^2 + T\right) \tag{2.2}$$

for some $C > 0$ independent of $T$.

In order to obtain uniqueness results for (0.3), further restrictions on $g$ are needed. We shall provide two uniqueness results under different hypotheses.

**Theorem 2.2.** *In the hypotheses of Theorem 2.1, assume that either* (g3) *with* $\beta \le \frac{7}{3}$ *or* (g4) *hold. Then the solution* $z(t)$ *to* (0.3) *is unique, and the mapping*

$$z_0 \mapsto z(t) \in C(\mathcal{H},\mathcal{H}) \qquad \forall\, t \in [\tau,T].$$

*Proof.* Suppose that $z_1 = (\theta_1,\eta_1)$ and $z_2 = (\theta_2,\eta_2)$ are two solutions of (0.3) with initial data $z_{10}$ and $z_{20}$, respectively, and set $\tilde z = (\tilde\theta,\tilde\eta) = z_1 - z_2$ and $\tilde z_0 = z_{10} - z_{20}$. Taking the difference of $(0.3)_1$ with $z_1$ and $z_2$ in place of $z$, and taking the product by $\tilde z$ in $\mathcal{H}$, we get (repeating the argument leading to (2.1))

$$\frac{d}{dt}\|\tilde z\|^2_{\mathcal{H}} + 2k_0\|\nabla\tilde\theta\|^2 \le -2\langle g(\theta_1) - g(\theta_2),\tilde\theta\rangle \tag{2.3}$$

where we used (h2) to delete the integral term.
Suppose first that (g3) holds with $\beta \le \frac{7}{3}$. Since

$$2|\langle g(\theta_1) - g(\theta_2),\tilde\theta\rangle| \le 2c_4\int_\Omega |\tilde\theta|^2\,[1 + |\theta_1|^\gamma + |\theta_2|^\gamma]\,dx$$

applying the generalized Hölder inequality with $p$, $q \ge 1$ such that

$$\frac{1}{6} + \frac{1}{p} + \frac{1}{q} = 1$$

and in virtue of the continuity of the embedding $H_0^1 \hookrightarrow L^6$,

$$\begin{aligned} 2|\langle g(\theta_1)-g(\theta_2),\tilde\theta\rangle| &\le 2c_4\,\|\tilde\theta\|_6\,\|\tilde\theta\|_p\,\|1+|\theta_1|^\gamma+|\theta_2|^\gamma\|_q \\ &\le c_6\,\|\nabla\tilde\theta\|\,\|\tilde\theta\|_p\,\|1+|\theta_1|^\gamma+|\theta_2|^\gamma\|_q \end{aligned} \tag{2.4}$$

for some $c_6>0$. We now consider two cases. If $\gamma\le 1$, then choose $p=2$ and $q=3$ in (2.4), and define

$$m_\gamma(t)=\frac{c_6^2}{8k_0}\|1+|\theta_1(t)|^\gamma+|\theta_2(t)|^\gamma\|_3^2. \tag{2.5}$$

Since $3\gamma\le\gamma+2$, and $\theta_1,\theta_2\in L^{\beta+1}([\tau,T],L^{\beta+1})$, it is clear that $m_\gamma\in L^1([\tau,T])$, and in force of (2.2), $\|m_\gamma\|_{L^1([\tau,T])}$ remains bounded as $z_{10}$ and $z_{20}$ run in a bounded set. By Young inequality,

$$2|\langle g(\theta_1)-g(\theta_2),\tilde\theta\rangle|\le 2k_0\|\nabla\tilde\theta\|^2+m_\gamma\|\tilde\theta\|^2. \tag{2.6}$$

If $1<\gamma\le\frac{4}{3}$, let

$$p=\frac{6\gamma+12}{10-\gamma}\qquad\text{and}\qquad q=\frac{\gamma+2}{\gamma}$$

in (2.4). From Lemma 1.3,

$$\|\tilde\theta\|_p\le c\|\nabla\tilde\theta\|^\nu\|\tilde\theta\|^{1-\nu}$$

with

$$\nu=2\left[\frac{\gamma-1}{\gamma+2}\right].$$

Therefore (2.4) enhances to

$$2|\langle g(\theta_1)-g(\theta_2),\tilde\theta\rangle|\le cc_6\,\|\nabla\tilde\theta\|^{1+\nu}\,\|\tilde\theta\|^{1-\nu}\,\|1+|\theta_1|^\gamma+|\theta_2|^\gamma\|_q \tag{2.7}$$

Define

$$m_\gamma(t)=(cc_6)^{\frac{2}{1-\nu}}K\left(2k_0,\tfrac{2}{1+\nu},\tfrac{2}{1-\nu}\right)\,\|1+|\theta_1|^\gamma+|\theta_2|^\gamma\|_q^{\frac{2}{1-\nu}} \tag{2.8}$$

with $K$ as in (1.3). Notice that $\gamma q=\gamma+2$, and $\frac{2\gamma}{1-\nu}\le\gamma+2$ for $\gamma\le\frac{4}{3}$. Being $\theta_1,\theta_2\in L^{\gamma+2}([\tau,T],L^{\gamma+2})$, we conclude that $m_\gamma\in L^1([\tau,T])$, and again from (2.2), $\|m_\gamma\|_{L^1([\tau,T])}$ remains bounded as $z_{10}$ and $z_{20}$ run in a bounded set. Applying Lemma 1.2 to (2.7) we get that inequality (2.6) holds also in this case. Thus (2.3) turns into

$$\frac{d}{dt}\|\tilde z\|_{\mathcal H}^2\le m_\gamma\|\tilde\theta\|^2\le m_\gamma\|\tilde z\|_{\mathcal H}^2$$

and Gronwall lemma then yields

$$\|\tilde z(t)\|_{\mathcal H}^2\le\|\tilde z_0\|_{\mathcal H}^2\,\exp\left[\int_\tau^t m_\gamma(y)\,dy\right]$$

which implies the result.

Assume then (g4), and notice that

$$g(\theta_1) - g(\theta_2) = \tilde{\theta} \int_0^1 g'(\lambda\theta_1 + (1-\lambda)\theta_2)\, d\lambda.$$

Therefore

$$-2\langle g(\theta_1) - g(\theta_2), \tilde{\theta}\rangle = -2 \int_\Omega |\tilde{\theta}|^2 \left[\int_0^1 g'(\lambda\theta_1 + (1-\lambda)\theta_2)\, d\lambda\right] dx \leq 2c_5 \|\tilde{\theta}\|^2$$

and (2.3) becomes

$$\frac{d}{dt}\|\tilde{z}\|^2_{\mathcal{H}} \leq 2c_5\|\tilde{\theta}\|^2 \leq 2c_5\|\tilde{z}\|^2_{\mathcal{H}}.$$

Using again Gronwall lemma we obtain

$$\|\tilde{z}(t)\|^2_{\mathcal{H}} \leq \|\tilde{z}_0\|^2_{\mathcal{H}}\, e^{2c_5(t-\tau)}$$

which concludes the proof. □

## 3. Existence of Uniform Absorbing Sets in $\mathcal{H}$ and in $\mathcal{V}$

Let $\mathcal{E}$ be the Hilbert space into which move all orbits of problem (0.3), namely

$$U_h(t,\tau) : \mathcal{E} \to \mathcal{E} \qquad \mathcal{E} = \mathcal{H} \text{ or } \mathcal{E} = \mathcal{V}$$

with $U_h(t,\tau)$ given by (0.4). The aim of this section is to prove in either case the existence of a bounded absorbing set, which is uniform as $h$ runs into a given functional set, typically a complete metric space. Such a set is sometimes called the *symbol space.*

In the sequel $F$ will always denote a symbol space. We also agree to call $B_{\mathcal{E}}(0,R)$ the open ball in $\mathcal{E}$ with center 0 and radius $R > 0$.

**Definition 3.1.** A set $\mathcal{B}_0 \subset \mathcal{E}$ is said to be *uniformly absorbing* (with respect to $h \in F$) for the family $\{U_h(t,\tau),\, h \in F\}$ if for any bounded set $\mathcal{B} \subset \mathcal{E}$ there exists $t^* = t^*(\mathcal{B})$ such that

$$\bigcup_{h\in F} U_h(t,\tau)\mathcal{B} \subset \mathcal{B}_0 \qquad \forall\, t \geq \tau + t^* \quad \forall\, \tau \in \mathbb{R}.$$

**Theorem 3.2.** *Assume* (h1)-(h3), *and let* (g1)-(g2), *and either* (g3) *with* $\beta \leq \frac{7}{3}$, *or* (g4) *hold. Let*

$$F \subset \mathcal{T}_b^1(\mathbb{R}, L^2) + \mathcal{T}_b^2(\mathbb{R}, H^{-1})$$

*be a bounded set. Then there exists a bounded, uniformly absorbing set in $\mathcal{H}$ for the family $\{U_h(\tau,t),\, h \in F\}$.*

*Proof.* Let

$$\Phi = \sup_{h\in F} \|h\|_{\mathcal{T}_b^1(\mathbb{R},L^2)+\mathcal{T}_b^2(\mathbb{R},H^{-1})} = \sup_{h\in F}\left[\inf_{h=h_1+h_2}\left\{\|h_1\|_{\mathcal{T}_b^1(\mathbb{R},L^2)} + \|h_2\|_{\mathcal{T}_b^2(\mathbb{R},H^{-1})}\right\}\right]$$

and let $h = h_1 + h_2$ be a decomposition of $h$. In force of (g2) and Young inequality we get

$$-2\langle g(\theta), \theta\rangle + 2\langle h, \theta\rangle \leq 2c_2|\Omega| - 2c_3\|\theta\|_{\beta+1}^{\beta+1} + 2\|h_1\|\|\theta\| + \frac{k_0}{2}\|\nabla\theta\|^2 + \frac{2}{k_0}\|h_2\|_{H^{-1}}^2.$$

Thus, using (h3) and (1.1), and denoting

$$\epsilon_0 = \min\{\lambda_0 k_0, \delta\} \qquad \text{and} \qquad \Lambda = 2\max\left\{c_2|\Omega|, 1, \frac{1}{k_0}\right\}$$

equation (2.1) turns into

$$\frac{d}{dt}\|z\|_{\mathcal{H}}^2 + \epsilon_0\|z\|_{\mathcal{H}}^2 + k_0\|\nabla\theta\|^2 + 2c_3\|\theta\|_{\beta+1}^{\beta+1} \leq \Lambda\left[1 + \|h_1\|\|z\|_{\mathcal{H}} + \|h_2\|_{H^{-1}}^2\right]. \tag{3.1}$$

Therefore, setting

$$\Psi = \Psi(\epsilon_0, \Lambda, \Phi) = \frac{2\Lambda}{\epsilon_0} + \Lambda^2\Phi^2 C^2\left(\tfrac{\epsilon_0}{2}\right) + 2\Lambda\Phi^2 C(\epsilon_0) \tag{3.2}$$

with $C$ as in (1.4), Lemma 1.4 with $\sigma = \frac{1}{2}$ and Lemma 1.5 entail

$$\|z(t)\|_{\mathcal{H}}^2 \leq 2\|z(t_0)\|_{\mathcal{H}}^2 e^{-\epsilon_0(t-t_0)} + \Psi \qquad \forall\, t \geq t_0 \tag{3.3}$$

for any given $t_0 \geq \tau$. In particular,

$$\|z(t)\|_{\mathcal{H}}^2 \leq 2\|z_0\|_{\mathcal{H}}^2 e^{-\epsilon_0(t-\tau)} + \Psi \qquad \forall\, t \geq \tau \tag{3.4}$$

from which it follows at once that every ball $B_{\mathcal{H}}(0,\rho)$, with radius $\rho > \sqrt{\Psi}$, is a uniformly absorbing set in $\mathcal{H}$ as $h \in F$. □

Observe that, in the above proof, condition (g3) with $\beta \leq \frac{7}{3}$ or (g4) are used only to formulate the result in term of process (and therefore uniqueness of solution is required). If we relax (g3) and (g4), the very same uniform bound holds for a (not necessarily unique) solution of (0.3).

In order to get an absorbing set in $\mathcal{V}$ we have to strengthen the hypothesis on the symbol space.

**Theorem 3.3.** *Assume* (h1)-(h3)*, and let* (g1)-(g2)*, and either* (g3) *with* $\beta < \frac{7}{3}$*, or* (g4) *hold. Let*

$$F \subset \mathcal{T}_b^2(\mathbb{R}, L^2)$$

*be a bounded set. Then there exists a bounded, uniformly absorbing set in* $\mathcal{V}$ *for the family* $\{U_h(\tau,t),\, h \in F\}$.

*Proof.* Let

$$\Phi = \sup_{h\in F}\|h\|_{\mathcal{T}_b^2(\mathbb{R},L^2)}.$$

Observe that (3.1)-(3.3) still hold (with $h_1 = h$ and $h_2 \equiv 0$). Hence integration of (3.1) on the interval $[t, t+1]$, with $t \geq \tau$, and Young inequality lead to the estimate

$$\begin{aligned}\int_t^{t+1} & \Big[k_0\|\nabla\theta(y)\|^2 + 2c_3\|\theta(y)\|_{\beta+1}^{\beta+1}\Big]\,dy \\ & \leq \|z(t)\|_{\mathcal{H}}^2 + \Lambda + \int_t^{t+1} \Lambda\,\|h(y)\|\,\|z(y)\|_{\mathcal{H}}\,dy \\ & \leq (2+\Lambda)\|z(t_0)\|_{\mathcal{H}}^2 + \Psi + \Lambda + \frac{1}{2}\Lambda\Phi^2 + \frac{1}{2}\Lambda\Psi \qquad \forall\, t \geq t_0\end{aligned} \tag{3.5}$$

for any given $t_0 \geq \tau$. To achieve uniform estimates involving the existence of a bounded uniformly absorbing set in $\mathcal{V}$ we multiply $(0.2)_1$ by $-\Delta\theta$ with respect to the inner product of $L^2$, and the laplacian of $(0.2)_2$ by $\Delta\eta$ with respect to the inner product of $L^2_\mu(\mathbb{R}^+, L^2)$. Adding the two terms, and performing an integration by parts, we obtain

$$\frac{d}{dt}\|z\|_{\mathcal{V}}^2 + 2k_0\|\Delta\theta\|^2 - \int_0^\infty \mu'(s)\|\Delta\eta(s)\|^2\,ds = 2\langle g(\theta) - h, \Delta\theta\rangle. \tag{3.6}$$

Young inequality entails

$$2|\langle h, \Delta\theta\rangle| \leq \frac{k_0}{2}\|\Delta\theta\|^2 + \frac{2}{k_0}\|h\|^2. \tag{3.7}$$

Concerning the term $2\langle g(\theta), \Delta\theta\rangle$, assume first that (g3) holds with $\beta < \frac{7}{3}$. Young inequality and (g1) then give

$$\begin{aligned}2\langle g(\theta), \Delta\theta\rangle & \leq \frac{k_0}{2}\|\Delta\theta\|^2 + \frac{2c_1^2}{k_0}\|1 + |\theta|^\beta\|^2 \\ & \leq \frac{k_0}{2}\|\Delta\theta\|^2 + c_7 + c_8\|\theta\|_{2\gamma+2}^{2\gamma+2}\end{aligned}$$

for some $c_7$, $c_8 > 0$. From Lemma 1.3 (recall that $\gamma = \max\{\beta - 1, 0\}$),

$$\|\theta\|_{2\gamma+2}^{2\gamma+2} \leq c\|\nabla\theta\|^{3\gamma}\,\|\theta\|^{2-\gamma}$$

hence, introducing

$$\sigma = \frac{1}{2}\max\{3\gamma - 2, 0\} < 1$$

we get

$$2\langle g(\theta), \Delta\theta\rangle \leq \frac{k_0}{2}\|\Delta\theta\|^2 + c_7 + cc_8(1 + \|\theta\|^2)(1 + \|\nabla\theta\|^2)\|\nabla\theta\|^{2\sigma}. \tag{3.8}$$

Consider next the case when (g4) holds. Since $\theta|_{\partial\Omega} = 0$, the continuity of $g$ and the Green formula yield

$$\int_{\partial\Omega} g(\theta)\nabla\theta\cdot\mathbf{n}\,d\sigma = \int_{\partial\Omega} g(0)\nabla\theta\cdot\mathbf{n}\,d\sigma = \int_\Omega g(0)\Delta\theta\,dx$$

where $\mathbf{n}$ is the outward pointing normal vector. Thus, using again the Green formula, in force of Young inequality and (g4),

$$
\begin{aligned}
2\langle g(\theta), \Delta\theta\rangle &= 2\int_\Omega g(0)\,\Delta\theta\,dx - 2\int_\Omega g'(\theta)\nabla\theta\cdot\nabla\theta\,dx \\
&\leq \frac{2}{k_0}g^2(0)|\Omega| + \frac{k_0}{2}\|\Delta\theta\|^2 + 2c_5\|\nabla\theta\|^2.
\end{aligned} \tag{3.9}
$$

Clearly the above computations are justified in a Faedo-Galerkin scheme. Therefore in either case, setting

$$\varphi_1 = cc_8(1+\|\theta\|^2)(1+\|\nabla\theta\|^2)$$

and

$$\varphi_2 = \frac{2}{k_0}\|h\|^2 + c_7 + \frac{2}{k_0}g^2(0)|\Omega| + 2c_5\|\nabla\theta\|^2$$

from (3.7)-(3.9) we conclude that

$$2\langle g(\theta) - h, \Delta\theta\rangle \leq \varphi_1\,\|\nabla\theta\|^{2\sigma} + \varphi_2 + k_0\|\Delta\theta\|^2.$$

Finally, in virtue of (h3) and (1.2), and denoting

$$\epsilon_1 = \min\{\gamma_0 k_0, \delta\}$$

we obtain from (3.6) the inequality

$$\frac{d}{dt}\|z\|^2_{\mathcal{V}} + \epsilon_1\|z\|^2_{\mathcal{V}} \leq \varphi_1\,\|z\|^{2\sigma}_{\mathcal{V}} + \varphi_2.$$

From (3.3) and (3.5), there exist two positive constants $K_1$ and $K_2$ (depending on $\Phi$) such that

$$\int_t^{t+1}[\varphi_1(y) + \varphi_2(y)]\,dy \leq K_1\,\|z(t_0)\|^4_{\mathcal{H}} + K_2 \qquad \forall\, t\geq t_0$$

for any given $t_0 \geq \tau$. Hence Lemma 1.4 and Lemma 1.5 lead to

$$
\begin{aligned}
\|z(t)\|^2_{\mathcal{V}} \leq& \frac{1}{1-\sigma}\,\|z(t_0)\|^2_{\mathcal{V}}e^{-\epsilon_1(t-\tau)} + \Big[C\left(\epsilon_1(1-\sigma)\right)\left(K_1\,\|z(t_0)\|^4_{\mathcal{H}} + K_2\right)\Big]^{\frac{1}{1-\sigma}} \\
&+ \frac{1}{1-\sigma}\,C(\epsilon_1)\left(K_1\,\|z(t_0)\|^4_{\mathcal{H}} + K_2\right) \qquad \forall\, t\geq t_0
\end{aligned} \tag{3.10}
$$

for any given $t_0 \geq \tau$. Let now $z_0 \in B(0,R)$ in $\mathcal{V}$. Recalling (1.1)-(1.2),

$$\|z_0\|_{\mathcal{H}} \leq R_1 = R\max\left\{\frac{1}{\lambda_0}, \frac{1}{\gamma_0}\right\}.$$

and (3.10) applied for $t_0 = \tau$ yields

$$\|z(t)\|^2_{\mathcal{V}} \leq \frac{1}{1-\sigma}\,R^2 + \xi(R_1) \qquad \forall\, t\geq\tau \tag{3.11}$$

having defined the function

$$\xi(r) = \left[C\,(\epsilon_1\,(1-\sigma))\,(K_1\,r^4 + K_2)\right]^{\frac{1}{1-\sigma}} + \frac{1}{1-\sigma}\,C(\epsilon_1)\,(K_1\,r^4 + K_2).$$

Notice that $\xi(r)$ is increasing in $r$. Set now, with reference to (3.2), $\rho > \sqrt{\Psi}$. According to Theorem 3.2 there exists $t_R \geq 0$ such that $\|z(t)\|_{\mathcal{H}} \leq \rho$ whenever $t \geq \tau + t_R$. Thus from (3.10)-(3.11) we get that

$$\|z(t)\|_{\mathcal{V}}^2 \leq \frac{R^2 + \xi(R_1)(1-\sigma)}{(1-\sigma)^2}\,e^{-\epsilon_1(t-\tau)} + \xi(\rho) \qquad \forall\, t \geq \tau + t_R.$$

Therefore we get that every ball $B_{\mathcal{V}}(0,\rho')$ , with $\rho' > \sqrt{\xi(\sqrt{\Psi})}$, is a uniformly absorbing set in $\mathcal{V}$ as $h \in F$. □

## 4. Existence of a Uniform Attractor

We begin recalling some definitions due to Haraux [17].

**Definition 4.1.** A set $\mathcal{K} \subset \mathcal{H}$ is said to be *uniformly attracting* for the family $\{U_h(t,\tau),\, h \in F\}$ if for any $\tau \in \mathbb{R}$ and any bounded set $\mathcal{B} \subset \mathcal{H}$

$$\lim_{t\to\infty}\left[\sup_{h\in F}\operatorname{dist}(U_h(t,\tau)\mathcal{B},\mathcal{K})\right] = 0 \tag{4.1}$$

where

$$\operatorname{dist}(\mathcal{B}_1,\mathcal{B}_2) = \sup_{z_1\in\mathcal{B}_1}\inf_{z_2\in\mathcal{B}_2}\|z_1 - z_2\|_{\mathcal{H}}$$

denotes the semidistance of two sets $\mathcal{B}_1$ and $\mathcal{B}_2$ in $\mathcal{H}$. A family of processes that possesses a uniformly attracting compact set is said to be *uniformly asymptotically compact.*

**Definition 4.2.** A closed set $\mathcal{A} \subset \mathcal{H}$ is said to be a *uniform attractor* for the family $\{U_h(t,\tau),\, h \in F\}$ if it is at the same time uniformly attracting and contained in every closed uniformly attracting set.

The above minimality property replaces the invariance property that characterizes the attractors of semigroups. It is also clear from the definition that the uniform attractor of a family of processes is unique.

The fundamental results of Chepyzhov and Vishik (see [5,6]) that we are going to exploit read as follows.

**Theorem 4.3.** *Assume that $F$ is a compact metric space and that there exists a continuous semigroup $T(t)$ acting on it, which satisfies the translation equality*

$$U_h(t+s,\tau+s) = U_{T(s)h}(t,\tau) \qquad \forall\, h \in F. \tag{4.2}$$

*Assume also that $U_h(t,\tau)$ is continuous as a map $\mathcal{H}\times F\to\mathcal{H}$, for every $\tau\in\mathbb{R}$ and $t\geq\tau$. Then if the family $\{U_h(t,\tau),\ h\in F\}$ is uniformly asymptotically compact it possesses a compact uniform attractor given by*

$$\mathcal{A}=\left\{\begin{array}{l} z(0) \text{ such that } z(t) \text{ is any bounded complete} \\ \text{trajectory of } U_h(t,\tau) \text{ for some } h\in\mathcal{A}(F)\end{array}\right\}$$

*where $\mathcal{A}(F)$ is the attractor of the semigroup $T(t)$ on $F$.*

The existence of $\mathcal{A}(F)$ is assured from the well-known theorems about the attractors of semigroups (see, e.g., [22]).

**Theorem 4.4.** *The set $\tilde{\mathcal{A}}=\mathcal{A}\times\mathcal{A}(F)\subset\mathcal{H}\times F$ is the attractor of the semigroup $S(t)$ acting on $\mathcal{H}\times F$ defined by*

$$S(t)(z_0,h)=(U_h(t,0)z_0,T(t)h)\,. \tag{4.3}$$

From the translation equality (4.2) it is immediate to verify that $S(t)$ is a semigroup. Using standard techniques, one could prove directly the existence of an attractor $\tilde{\mathcal{A}}$ for $S(t)$. The peculiarity of Theorem 4.3 and Theorem 4.4 stands in the characterization of $\tilde{\mathcal{A}}$ as $\mathcal{A}\times\mathcal{A}(F)$.

In view of the above results, we write the solution $z=(\theta,\eta)$ to (0.3) as $z=z_L+z_N$, with $z_L=(\theta_L,\eta_L)$ and $z_N=(\theta_N,\eta_N)$, where $z_L$ solves the linearized homogeneous system, and $z_N$ is the solution of the nonlinear system with null initial data, namely,

$$\begin{aligned} &\partial_t z_L = Lz_L \\ &z_L|_{\partial\Omega}=0 \\ &z_L(0)=z_0 \end{aligned} \tag{4.4}$$

and

$$\begin{aligned} &\partial_t z_N = Lz_N+G(z) \\ &z_N|_{\partial\Omega}=0 \\ &z_N(0)=0. \end{aligned} \tag{4.5}$$

It is apparent that the solution $z_L$ to (4.4) fulfills the uniform estimate (3.4) with $\Psi=0$, namely,

$$\|z_L(t)\|^2_{\mathcal{H}}\leq 2\|z_0\|^2_{\mathcal{H}}e^{-\epsilon_0(t-\tau)} \qquad \forall\, t\geq\tau. \tag{4.6}$$

Since

$$\|z_N(t)\|^2_{\mathcal{H}}\leq 2\|z(t)\|^2_{\mathcal{H}}+2\|z_L(t)\|^2_{\mathcal{H}}$$

we have also

$$\|z_N(t)\|^2_{\mathcal{H}}\leq 8\|z_0\|^2_{\mathcal{H}}e^{-\epsilon_0(t-\tau)}+2\Psi \qquad \forall\, t\geq\tau. \tag{4.7}$$

For further reference, we denote by $\eta_N^t(s;\tau,z_0,h)$ the second component of the solution $z_N$ to (4.5) at time $t$ with initial time $\tau$ with $z(\tau)=z_0$ and symbol $h$. Observe that $\eta_N$ can be computed explicitly from (4.5) as follows:

$$\eta_N^t(s)=\begin{cases}\displaystyle\int_0^s \theta_N(t-y)\,dy & 0\le s\le t-\tau\\ \displaystyle\int_0^{t-\tau} \theta_N(t-y)\,dy & s>t-\tau.\end{cases}\tag{4.8}$$

Our goal is to build a compact uniformly attracting set for the process. In the sequel, let $\rho>0$ be fixed such that $B_{\mathcal{H}}(0,\rho)$ is a uniformly absorbing set in $\mathcal{H}$ for $U_h(t,\tau)$, as $h\in F$ (whose existence is assured by Theorem 3.2). Moreover, with reference to Definition 3.1, let $t_\rho=t^*(B_{\mathcal{H}}(0,\rho))$.

**Lemma 4.5.** *Assume* (h1)-(h3), *and let* (g1)-(g3) *with* $\beta\le\frac{5}{3}$ *hold. Let* $F$ *and* $\Phi$ *as in Theorem 3.3. Then there exists a positive constant* $\Gamma$*, depending on* $\Phi$*, such that*

$$\|z_N(t)\|_{\mathcal{V}}^2\le\Gamma\,(1+\|z(t_0)\|_{\mathcal{H}})^4\qquad\forall\,t\ge t_0\tag{4.9}$$

*for any* $t_0\ge\tau$, $\tau\in\mathbb{R}$. *Moreover such a* $\Gamma$ *does not depend on the particular initial time* $\tau$ *chosen.*

*Proof.* We parallel the proof of Theorem 3.3 (with $z_N$ in place of $z$), the only difference being the evaluation of the term $2\langle g(\theta),\Delta\theta_N\rangle$. Indeed in this case, using Young inequality, (g1), and Lemma 1.3, we obtain

$$\begin{aligned}2\langle g(\theta),\Delta\theta_N\rangle&\le\frac{k_0}{2}\|\Delta\theta_N\|^2+\frac{2c_1^2}{k_0}\|1+|\theta|^\beta\|^2\\&\le\frac{k_0}{2}\|\Delta\theta_N\|^2+c_7+c_8\|\theta\|_{2\gamma+2}^{2\gamma+2}\\&\le\frac{k_0}{2}\|\Delta\theta_N\|^2+c_7+cc_8\|\nabla\theta\|^{3\gamma}\|\theta\|^{2-\gamma}\\&\le\frac{k_0}{2}\|\Delta\theta_N\|^2+c_7+cc_8(1+\|\nabla\theta\|^2)(1+\|\theta\|^2)\end{aligned}$$

since $\beta\le\frac{5}{3}$. Thus, denoting

$$\varphi=\frac{2}{k_0}\|h\|^2+c_7+cc_8(1+\|\nabla\theta\|^2)(1+\|\theta\|^2)$$

we conclude that

$$\frac{d}{dt}\|z_N\|_{\mathcal{V}}^2+\epsilon_1\|z_N\|_{\mathcal{V}}^2\le\varphi.$$

From (3.3) and (3.5) there exist two positive constants $K_3$ and $K_4$ (depending on $\Phi$) such that

$$\int_t^{t+1}\varphi(y)\,dy\le K_3\,\|z(t_0)\|_{\mathcal{H}}^4+K_4\qquad\forall\,t\ge t_0$$

for any given $t_0 \geq \tau$. Hence Lemma 1.4 and Lemma 1.5 lead to

$$\|z_N(t)\|_{\mathcal{V}}^2 \leq C(\epsilon_1)\,(K_3\,\|z(t_0)\|_{\mathcal{H}}^4 + K_4)$$

as claimed. □

**Lemma 4.6.** *Assume* (h1)-(h3), *and* (g1)-(g3) *with* $\beta \leq \frac{5}{3}$, *and let* $F$ *and* $\Phi$ *be as in Theorem 3.3. Denote*

$$\mathcal{M} = \bigcup_{h\in F} \bigcup_{z_0\in B_{\mathcal{H}}(0,\rho)} \bigcup_{\tau\in\mathbb{R}} \bigcup_{t\geq\tau+t_\rho} \eta_N^t(\cdot;\tau,z_0,h).$$

*Then* $\mathcal{M}$ *is relatively compact in* $L^2_\mu(\mathbb{R}^+, H^1_0)$.

*Proof.* It is clear from Lemma 4.5 that $\mathcal{M}$ is bounded in $L^2_\mu(\mathbb{R}^+, H^2\cap H^1_0)$. Let then $\eta_N^t \in \mathcal{M}$. The derivative of (4.8) yields

$$\frac{\partial}{\partial s}\,\eta_N^t(s) = \begin{cases} \theta_N(t-s) & 0\leq s\leq t-\tau \\ 0 & s > t-\tau. \end{cases} \tag{4.10}$$

Thus (4.7) and (4.10) entail

$$\int_0^\infty \mu(s)\left\|\frac{\partial}{\partial s}\,\eta_N^t(s)\right\|^2 ds = \int_0^{t-\tau} \mu(s)\|\theta_N(t-s)\|^2\,ds \leq \big(8\rho^2+2\Psi\big)\|\mu\|_{L^1(\mathbb{R}^+)}.$$

So we conclude that $\mathcal{M}$ is bounded in $L^2_\mu(\mathbb{R}^+, H^2\cap H^1_0)\cap H^1_\mu(\mathbb{R}^+, L^2)$. Moreover, from (4.8) and (4.9), with $t_0=\tau$, it is easy to check that for every $\eta\in\mathcal{M}$

$$\|\nabla\eta(s)\|^2 \leq s^2\Gamma(1+\rho)^4 \in L^1_\mu(\mathbb{R}^+)$$

in force of the exponential decay of $\mu$. The proof is completed applying the following result from [21]:

*Let* $\mu \in C(\mathbb{R}^+)\cap L^1(\mathbb{R}^+)$ *be a non-negative function, such that if* $\mu(s_0)=0$ *for some* $s_0\in\mathbb{R}^+$ *then* $\mu(s)=0$ *for every* $s>s_0$. *Let* $B_0$, $B$, $B_1$ *be three Banach spaces, with* $B_0$ *and* $B_1$ *reflexive, such that*

$$B_0 \hookrightarrow B \hookrightarrow B_1$$

*the first injection being compact. Let* $\mathcal{M}\subset L^2_\mu(\mathbb{R}^+, B)$ *satisfy the following hypotheses:*

(i) $\mathcal{M}$ *is bounded in* $L^2_\mu(\mathbb{R}^+, B_0)\cap H^1_\mu(\mathbb{R}^+, B_1)$

(ii) $\sup_{\eta\in\mathcal{M}} \|\eta(s)\|_B^2 \leq h(s) \quad \forall\, s\in\mathbb{R}^+$ *for some* $h\in L^1_\mu(\mathbb{R}^+)$.

*Then* $\mathcal{M}$ *is relatively compact in* $L^2_\mu(\mathbb{R}^+, B)$. □

**Proposition 4.7.** *Assume* (h1)-(h3), *and let* (g1)-(g3) *with* $\beta\leq\frac{5}{3}$ *hold. Let* $F$ *and* $\Phi$ *as in Theorem 3.3. Then there exists a compact, uniformly attracting set for the family* $\{U_h(t,\tau),\, h\in F\}$.

*Proof.* Denote by $\overline{\mathcal{M}}$ the closure of $\mathcal{M}$ in $L^2_\mu(\mathbb{R}^+, H^1_0)$, and introduce the set

$$\mathcal{K} = B_{H^2\cap H^1_0}(0, \Gamma(1+\rho)^4) \times \overline{\mathcal{M}} \subset \mathcal{H}.$$

From the compact embedding $H^2 \cap H^1_0 \hookrightarrow H^1_0$ and Lemma 4.6, $\mathcal{K}$ is compact in $\mathcal{H}$. We need to show the uniform attraction property. Let $\mathcal{B} \subset \mathcal{H}$ be a bounded set, with $R = \sup_{z\in\mathcal{B}} \|z\|_{\mathcal{H}}$, and let $t^* = t^*(\mathcal{B})$ such that, for every $h \in F$,

$$U_h(t,\tau)\mathcal{B} \subset B_{\mathcal{H}}(0,\rho) \qquad \forall\, t \geq \tau + t^*.$$

Let then $t > t_\rho + t^* + \tau$, and set $\hat{t} = t - t_\rho - t^* - \tau > 0$. Using the process properties we get that

$$\begin{aligned} U_h(\hat{t} + t_\rho + t^* + \tau, \tau)\mathcal{B} &= U_h(\hat{t} + t_\rho + t^* + \tau, t^* + \tau)U_h(t^* + \tau, \tau)\mathcal{B} \\ &\subset U_h(\hat{t} + t_\rho + t^* + \tau, t^* + \tau)B_{\mathcal{H}}(0,\rho). \end{aligned}$$

Pick any $z(t) \in U_h(t,\tau)\mathcal{B}$, for $t > t_\rho + t^* + \tau$. Applying (4.9) with $t_0 = \tau + t^*$ we get

$$\|\Delta\theta_N(t)\|^2 \leq \|z_N(t)\|^2_{\mathcal{V}} \leq \Gamma\left(1 + \|z(\tau + t^*)\|_{\mathcal{H}}\right)^4 \leq \Gamma\,(1+\rho)^4.$$

It is then apparent that $z_N(t) \in \mathcal{K}$. Therefore, from (4.6),

$$\inf_{m\in\mathcal{K}} \|z(t) - m\|_{\mathcal{H}} \leq \|z_L(t)\|_{\mathcal{H}} \leq \sqrt{2}Re^{-\frac{\epsilon_0}{2}(t-\tau)} \qquad \forall\, t > t_\rho + t^* + \tau.$$

Being the above inequality independent of $h \in F$, we conclude that

$$\sup_{h\in F} \operatorname{dist}(U_h(t,\tau)\mathcal{B}, \mathcal{K}) \leq \sqrt{2}Re^{-\frac{\epsilon_0}{2}(t-\tau)} \qquad \forall\, t > t_\rho + t^* + \tau$$

hence (4.1) holds, and the result is proved. □

We are now ready to state the main result of the section.

**Theorem 4.8.** *Assume* (h1)-(h3), *and let* (g1)-(g3) *with* $\beta \leq \frac{5}{3}$ *hold. Furthermore, let*

$$f \in \mathcal{T}^2_c(\mathbb{R}, L^2).$$

*Then there exists a compact uniform attractor* $\mathcal{A}$ *for the family* $\{U_h(t,\tau),\, h \in H(f)\}$ *given by*

$$\mathcal{A} = \left\{ \begin{array}{l} z(0) \text{ such that } z(t) \text{ is any bounded complete} \\ \text{trajectory of } U_h(t,\tau) \text{ for some } h \in H(f) \end{array} \right\}.$$

*Proof.* We apply Theorem 4.3 with $F = H(f)$ and

$$T(t)h(s) = h(s+t) \qquad h \in H(f)$$

(i.e., $T(t)$ is the *translation semigroup* acting on $H(f)$). It is then immediate to verify the translation equality (4.2). From Proposition 4.7, with $\Phi = \|f\|_{\mathcal{T}_b^2(\mathbb{R},L^2)}$, we get that the family $\{U_h(t,\tau),\ h \in H(f)\}$ is uniformly asymptotically compact. The proof of the $(\mathcal{H} \times H(f), \mathcal{H})$-continuity is practically identical to the proof of Theorem 2.2, and is therefore omitted. □

Notice that the attractor of the semigroup $T(t)$ on $H(f)$ coincides with the entire space $H(f)$.

In the course of the investigation we showed that $B_{\mathcal{H}}(0,\rho) \times H(f)$ is a bounded absorbing set for the semigroup $S(t)$ defined in (4.3). In particular, $B_{\mathcal{H}}(0,\rho) \times H(f)$ is connected in $\mathcal{H} \times H(f)$. Indeed, it is immediate to see that $\{f(\cdot + r)\}_{r\in\mathbb{R}}$ is path connected, and therefore its closure, i.e., $H(f)$ is connected. Then by [17], Proposition 5.2.7, the attractor $\tilde{\mathcal{A}}$ of $S(t)$ is connected, and so is its projection on $\mathcal{H}$. We summarize this discussion in the next corollary.

**Corollary 4.9.** *The uniform attractor $\mathcal{A}$ for the family $\{U_h(t,\tau),\ h \in H(f)\}$ given by Theorem 4.8 is connected.*

**Remark 4.10.** The restriction $\beta \leq \frac{5}{3}$ is due to the presence of the memory. In fact, in the particular case when the memory kernel vanishes, we reduce to the semilinear equation

$$\theta_t - k_0\Delta\theta + g(\theta) = h$$

and using the uniform Gronwall lemma (see, e.g., [22]), it is easy to show that the above results hold for $\beta < \frac{7}{3}$.

## 5. Hausdorff Dimension of the Uniform Attractor

In previous Theorem 4.3 and Theorem 4.4 it is shown that the uniform attractor $\mathcal{A}$ of the family $\{U_h(t,\tau),\ h \in H(f)\}$ is the projection on $\mathcal{H}$ of the attractor $\tilde{\mathcal{A}}$ of the corresponding semigroup $S(t)$, acting on $\mathcal{H}\times H(f)$. Therefore the Hausdorff dimensions of these sets satisfy the inequality $\dim_{\mathrm{H}} \mathcal{A} \leq \dim_{\mathrm{H}} \tilde{\mathcal{A}}$. In this section, along the line of [5], we show that $\dim_{\mathrm{H}} \tilde{\mathcal{A}} < \infty$ (and thus $\dim_{\mathrm{H}} \mathcal{A} < \infty$) when the external heat source has a quasiperiodic dependence on time.

We recall that the *Hausdorff dimension* of a subset $\mathcal{X}$ of a metric space $\mathcal{E}$ is defined by

$$\dim_{\mathrm{H}} \mathcal{X} = \sup\left\{ d > 0 : \sup_{\varepsilon>0} \inf_{C_\epsilon} \sum_{i\in I} r_i^d < +\infty \right\}$$

where $C_\epsilon = \{B_i(r_i)\}_{i\in I}$ is a covering of $\mathcal{X}$ of balls of radii $r_i \leq \epsilon$.

We now state the fundamental result from [10] concerning the Hausdorff dimensions of fully invariant sets. We need first two definitions.

**Definition 5.1.** Let $\mathcal{E}$ be a Hilbert space, $\mathcal{L}(\mathcal{E})$ the space of continuous linear operators from $\mathcal{E}$ to $\mathcal{E}$, $\mathcal{X} \subset \mathcal{E}$, and $S$ a (nonlinear) continuous map from $\mathcal{X}$ into $\mathcal{E}$. Then $S$ is said to be *uniformly quasidifferentiable* on $\mathcal{X}$ if for any $u \in \mathcal{X}$ there exists $S'(u) \in \mathcal{L}(\mathcal{E})$ (the quasidifferential of $S$ at $u$ with respect to $\mathcal{X}$) such that

$$\|Su - Sv - S'(u)(u-v)\|_{\mathcal{E}} \leq \sigma(\|u-v\|_{\mathcal{E}})\|u-v\|_{\mathcal{E}} \qquad \forall\, v \in \mathcal{X}$$

where $\sigma : \mathbb{R} \to \mathbb{R}^+$ is independent on $u$, and $\sigma(y) \to 0$ as $y \to 0^+$. The operator $S'(u)$ might not be unique.

**Definition 5.2.** Let $M$ be a linear operator on a Hilbert space $\mathcal{E}$. For any $m \in \mathbb{N}$ the *m-dimensional trace* of $M$ is defined as

$$\mathrm{Tr}_m M = \sup_Q \sum_{j=1}^{m} \langle M u_j, u_j \rangle_{\mathcal{E}}$$

where the supremum ranges over all possible orthogonal projections $Q$ in $\mathcal{E}$ on the $m$-dimensional space $Q\mathcal{E}$ belonging to the domain of $M$, and $\{u_1, \ldots, u_m\}$ is a orthonormal basis of $Q\mathcal{E}$.

The following result holds.

**Theorem 5.3.** *Let there be given a Hilbert space $\mathcal{E}$, and let $\mathcal{X} \subset \mathcal{E}$ be a compact fully invariant set for $S(t)$, i.e. $S(t)\mathcal{X} = \mathcal{X}$ for all $t \geq 0$. Assume also that $S(t)$ is uniformly quasidifferentiable on $\mathcal{X}$ for all $t \geq 0$, and*

$$\sup_{u_0 \in \mathcal{X}} \|S'(t, u_0)\|_{L(\mathcal{E})} \leq C(t) < \infty \qquad \forall\, t \geq 0$$

*where $S'(t, u_0)$ is the quasidifferential of $S(t)$ at $u_0$. It is also assumed that $S'(t, u_0)$ is generated by the equation in variation*

$$\begin{aligned} U_t &= M(u)U \\ U(0) &= U_0 \end{aligned}$$

*that is, $S'(t, u)U_0 = U(t)$ with $u(t) = S(t)u_0$. Introducing the number $q_m$ by the formula*

$$q_m = \liminf_{T \to \infty} \sup_{u \in \mathcal{X}} \left\{ \frac{1}{T} \int_0^T \mathrm{Tr}_m M(S(t)u)\, dt \right\}$$

*if there exists $m$ such that $q_m < 0$, then $\dim_{\mathrm{H}} \mathcal{X} \leq m$.*

As anticipated at the beginning of the section, we shall consider a *quasiperiodic* external heat supply, i.e., a function $f : \Omega \times \mathbb{R} \to \mathbb{R}$ of the form

$$f(x, t) = \Phi(x, \Lambda t) = \Phi(x, \lambda_1 t, \ldots, \lambda_K t)$$

where $\Phi(x, \omega) \in C^1(\mathbf{T}^K, L^2)$ is a $2\pi$-periodic function of $\omega$ on the $K$-dimensional torus $\mathbf{T}^K$, and $\Lambda = (\lambda_1, \ldots, \lambda_K)$ are rationally independent numbers. For further reference we agree to denote

$$\Phi'(x, \omega) = \left( \frac{\partial}{\partial \omega_1} \Phi(x, \omega), \ldots, \frac{\partial}{\partial \omega_K} \Phi(x, \omega) \right) \qquad \omega = (\omega_1, \ldots, \omega_K).$$

It is immediate to check that $f \in \mathcal{T}_c^2(\mathbb{R}, L^2)$, and $h \in H(f)$ if and only if

$$h(x, t) = \Phi(x, \Lambda t + \omega_0) \qquad \omega_0 \in \mathbf{T}^K.$$

Therefore $H(f)$ might be identified with $\mathbf{T}^K$, and the translation semigroup acting on $H(f)$ is equivalent to the translation semigroup $T(t)$ on $\mathbf{T}^K$, defined by

$$T(t)\omega_0 = [\Lambda t + \omega_0] = (\Lambda t + \omega_0)(\text{mod } 2\pi)^K.$$

In the remaining of the paper we shall denote $\mathcal{E} = \mathcal{H} \times \mathbf{T}^K$, and we consider the semigroup $S(t)$ acting on $\mathcal{E}$ given by (4.3), with $T(t)$ as above. Clearly, with reference to the previous section, the set $\tilde{\mathcal{A}} = \mathcal{A} \times \mathbf{T}^K$ (being $\mathcal{A}$ the uniform attractor of the family $\{U_\omega(t,\tau),\, \omega \in \mathbf{T}^K\}$) is the attractor of $S(t)$. For every $w_0 = (z_0, \omega_0) \in \mathcal{E}$, the vector $w(t) = S(t)w_0 = (z(t), \omega(t))$ is the solution of the differential equation

$$\begin{aligned} z_t &= Lz + G(z,\omega) \\ \omega_t &= \Lambda \\ z(0) &= z_0 \\ \omega(0) &= \omega_0. \end{aligned} \tag{5.1}$$

where

$$G(z,\omega) = (\Phi(\omega) - g(\theta), 0).$$

In the above formula and in the sequel, without further warning, we use the complete decomposition of $w$, i.e., $w = (\theta, \eta, \omega) = (\Pi_1 w, \Pi_2 w, \Pi_3 w)$, where $\Pi_j$, $j = 1,2,3$, denote the projections on $L^2$, $L^2_\mu(\mathbb{R}^+, H^1_0)$, and $\mathbf{T}^K$, respectively. We also define the operators $A$ (linear) and $F$ on $\mathcal{E}$ by $Aw = (Lz, 0)$ and $F(w) = (G(z,\omega), 0)$.

**Theorem 5.4.** *Assume* (h1)-(h2), *and let* (g1)-(g3) *with* $\beta \le \frac{7}{3}$ *and* $g \in C^1(\mathbb{R})$ *hold. Then the semigroup* $S(t)$ *acting on* $\mathcal{E}$ *is uniformly quasidifferentiable on any bounded set* $\mathcal{X}$ *which is invariant for* $S(t)$, *i.e.,* $S(t)\mathcal{X} \subset \mathcal{X}$ *for every* $t \ge 0$, *and the quasidifferential* $S'(t, w_0)$ *at the point* $w_0 = (z_0, \omega_0) = (\theta_0, \eta_0, \omega_0)$ *satisfies the variation equation*

$$\begin{aligned} W_t &= AW + F'(w)W \\ W(0) &= W_0 = (Z_0, \Sigma_0) = (\Theta_0, H_0, \Sigma_0) \end{aligned} \tag{5.2}$$

*where*

$$S'(t, w_0)W_0 = W(t) = (Z(t), \Sigma(t)) = (\Theta(t), H^t, \Sigma(t))$$

$$w(t) = (z(t), \omega(t)) = (\theta(t), \eta^t, \omega(t)) = S(t)w_0$$

*and*

$$F'(w)W = (-g'(\theta)\Theta + \Phi'(\omega)\Sigma, 0, 0)$$

*being* $F'$ *the Fréchet differential of* $F$. *Furthermore*

$$\sup_{w_0 \in \mathcal{X}} \|S'(t, w_0)\|_{L(\mathcal{E})} \le C(t) < \infty \qquad \forall\, t \ge 0$$

*Proof.* Let $w = (z, \omega)$ and $w^* = (z^*, \omega^*)$ be solutions to system (5.1) with initial data $w_0$ and $w_0^*$, respectively, with $w_0,\, w_0^* \in \mathcal{X}$. The difference $\tilde{w} = w^* - w$ satisfies the problem

$$\begin{aligned} &\tilde{w}_t = A\tilde{w} + F(w^*) - F(w) \\ &\tilde{w}(0) = \tilde{w}_0 = w_0^* - w_0. \end{aligned} \tag{5.3}$$

Arguing as in the proof of Theorem 2.2, taking the inner product in $\mathcal{E}$ of (5.3) and $\tilde{w}$, we get

$$\frac{d}{dt}\|\tilde{w}\|_{\mathcal{E}}^2 \le m_\gamma \|\tilde{w}\|_{\mathcal{E}}^2 + 2|\langle \Phi(\omega^*) - \Phi(\omega), \tilde{\theta}\rangle|$$

with $m_\gamma$ given by (2.5) if $\gamma \le 1$, and by (2.8) if $1 < \gamma \le \frac{4}{3}$. Due to the fact that $\mathcal{X}$ is invariant for $S(t)$ and is bounded in $\mathcal{E}$, from (2.2) it is clear that

$$\sup_{\theta^*, \theta \in \Pi_1 \mathcal{X}} m_\gamma \in L^1([0,T]) \qquad \forall\, T > 0.$$

Since

$$|\langle \Phi(\omega^*) - \Phi(\omega), \tilde{\theta}\rangle| \le \|\Phi(\omega^*) - \Phi(\omega)\| \|\tilde{\theta}\| \le \|\Phi'\|_{L^\infty(\mathbf{T}^K, L^2)} \|\tilde{\omega}\|_{\mathbf{T}^K} \|\tilde{\theta}\|$$

using Young inequality and defining $r_\gamma = m_\gamma + \|\Phi'\|_{L^\infty(\mathbf{T}^K, L^2)}$, in virtue of Gronwall lemma we conclude that

$$\|\tilde{w}(t)\|_{\mathcal{E}}^2 \le \|\tilde{w}_0\|_{\mathcal{E}}^2 \exp\left[\int_0^t r_\gamma(y)\, dy\right] \le \|\tilde{w}_0\|_{\mathcal{E}}^2 C(T) \qquad \forall\, t \le T \tag{5.4}$$

where $C(T) < \infty$ for all $T > 0$. Let now $W$ be the solution to (5.2) with initial data $W(0) = \tilde{w}_0$. Indeed, it is easy to see that the linear (non-autonomous) problem (5.2) with initial data $W(0) = \tilde{w}_0$ possesses a unique solution $W \in C([0,T], \mathcal{E})$ for all $T > 0$. Our goal is to show that $W(t) = S'(t, w_0)\tilde{w}_0$. Denote

$$\varphi = w^* - w - W = \tilde{w} - W.$$

Clearly $\varphi$ satisfies

$$\begin{aligned} &\varphi_t = A\varphi + F'(w)\varphi + h(w^*, w) \\ &\varphi(0) = 0 \end{aligned} \tag{5.5}$$

where

$$h(w^*, w) = F(w^*) - F(w) - F'(w)\tilde{w}.$$

Due to the differentiability assumptions on $g$ and $\Phi$, there exists $\sigma : \mathbb{R} \to \mathbb{R}^+$, $\sigma(y) \to 0$ as $y \to 0^+$, such that

$$\frac{\|h(w^*, w)\|_{\mathcal{E}}}{\|\tilde{w}\|_{\mathcal{E}}} = \sigma(\|\tilde{w}\|_{\mathcal{E}}).$$

We take the inner product in $\mathcal{E}$ of (5.5) and $\varphi$. It is easily seen that the estimate for the term $\langle F'(w)\varphi, \varphi\rangle_{\mathcal{E}}$ can be carried out exactly as above, whereas, by Young inequality,

$$2\langle h(w^*, w), \varphi\rangle_{\mathcal{E}} \le \|h(w^*, w)\|_{\mathcal{E}}^2 + \|\varphi\|_{\mathcal{E}}^2.$$

Thus we get

$$\frac{d}{dt}\|\varphi\|^2_{\mathcal{E}} \le (r_\gamma + 1)\|\varphi\|^2_{\mathcal{E}} + \|h(w^*, w)\|^2_{\mathcal{E}}.$$

Gronwall lemma and (5.4) entail

$$\begin{aligned} \|\varphi(t)\|^2_{\mathcal{E}} &\le \left[\int_0^t \|h(w^*(y), w(y))\|^2_{\mathcal{E}}\, dy\right] \exp\left[\int_0^t (r_\gamma(y) + 1)\, dy\right] \\ &\le \sup_{s\in[0,T]} \sigma^2(\|\tilde{w}(s)\|_{\mathcal{E}}) \left[\int_0^t \|\tilde{w}(y)\|^2_{\mathcal{E}}\, dy\right] e^T C(T) \\ &\le \sup_{s\in[0,T]} \sigma^2(\|\tilde{w}(s)\|_{\mathcal{E}})\|\tilde{w}_0\|^2_{\mathcal{E}}\tilde{C}(T) \qquad \forall\, t \le T \end{aligned} \tag{5.6}$$

with $\tilde{C}(T) = T\, e^T (C(T))^2$. From (5.4) we also get that

$$\lim_{\tilde{w}_0\to 0} \sup_{s\in[0,T]} \sigma^2(\|\tilde{w}(s)\|_{\mathcal{E}}) = 0.$$

Then we conclude that, for every $T > 0$

$$\lim_{\tilde{w}_0\to 0} \frac{\|\varphi(T)\|^2_{\mathcal{E}}}{\|\tilde{w}_0\|^2_{\mathcal{E}}} = 0$$

which gives the required uniform quasidifferentiability . Finally, taking the inner product of (5.2) and $W$, and performing calculations analogous to those leading to (5.6), we obtain the estimate

$$\|W(T)\|^2_{\mathcal{E}} \le \|W_0\|^2_{\mathcal{E}} C(T) \qquad \forall\, T \ge 0$$

which yields the last assertion of the theorem. □

It is immediate to verify that Theorem 5.4 still holds if we replace condition (g3) with $\beta \le \frac{7}{3}$ with condition (g4).

We now state our result about the dimension of the attractor.

**Theorem 5.5** *Assume* (h1)-(h2)*, and let* (g1)-(g3) *with* $\beta \le \frac{5}{3}$ *and* $g \in C^1(\mathrm{I\!R})$ *hold. Then the attractor* $\tilde{\mathcal{A}}$ *of the semigroup* $S(t)$ *acting on* $\mathcal{E}$ *has finite Hausdorff dimension.*

*Proof.* let $w_0 \in \tilde{\mathcal{A}}$ (so that $w \in \tilde{\mathcal{A}}$). Let $W = (\Theta, H, \Sigma)$ be a unitary vector belonging to the domain of $A + F'(w)$. Then

$$\langle \big(A + F'(w)\big)W, W\rangle_{\mathcal{E}} = \langle AW, W\rangle_{\mathcal{E}} - \langle g'(\theta)\Theta, \Theta\rangle + \langle \Phi'(\omega)\Sigma, \Theta\rangle.$$

From a direct calculation (see also [14]),

$$\langle AW, W\rangle_{\mathcal{E}} \le -k_0\|\nabla\Theta\|^2 - \frac{\delta}{2}\|\nabla H\|^2_\mu.$$

An application of the generalized Hölder inequality entails

$$-\langle g'(\theta)\Theta, \Theta\rangle \le \|\Theta\|_6\|\Theta\|\|g'(\theta)\|_3.$$

Since $\beta \leq 5/3$, the term $\|g'(\theta)\|_3^2$ is uniformly bounded as $w_0 \in \tilde{\mathcal{A}}$. Thus, exploiting the embedding $H_0^1 \hookrightarrow L^6$, and using Young inequality, we get

$$-\langle g'(\theta)\Theta, \Theta\rangle \leq \frac{k_0}{2}\|\nabla\Theta\|^2 + c_9\|\Theta\|^2.$$

for some $c_9 > 0$. Finally, being $\|\Phi'\|_{L^\infty(\mathbf{T}^K, L^2)} < \infty$,

$$\langle \Phi'(\omega)\Sigma, \Theta\rangle \leq c_{10}\|\Theta\|^2 + c_{10}\|\Sigma\|_{\mathbf{T}^K}^2$$

for some $c_{10} > 0$. Adding the pieces together, setting $c_{11} = c_9 + c_{10}$, we have

$$\langle \big(A + F'(w)\big)W, W\rangle_{\mathcal{E}} \leq -\frac{k_0}{2}\|\nabla\Theta\|^2 - \frac{\delta}{2}\|\nabla H\|_\mu^2 + c_{11}\|\Theta\|^2 + c_{10}\|\Sigma\|_{\mathbf{T}^K}^2.$$

Therefore we conclude that $A + F'(w) \leq M$, where $M$ is the diagonal operator acting on $L^2 \oplus L_\mu^2(\mathbb{R}^+, H_0^1) \oplus \mathbf{T}^K$ defined by

$$M = \begin{pmatrix} \frac{k_0}{2}\Delta + c_{11}I & 0 & 0 \\ 0 & -\frac{\delta}{2}I & 0 \\ 0 & 0 & c_{10}I \end{pmatrix}.$$

From the definition of $\mathrm{Tr}_m$, it is apparent that $\mathrm{Tr}_m\big(A + F'(w)\big) \leq \mathrm{Tr}_m(M)$. Since $M$ is diagonal, it is easy to see that

$$\mathrm{Tr}_m(M) = \sup_Q \sum_{j=1}^m \langle MW_j, W_j\rangle_{\mathcal{E}}$$

where the supremum is taken over the projections $Q$ of the form $Q_1 \oplus Q_2 \oplus Q_3$. This amounts to considering vectors $W_j$ where only one of the three components is non-zero (and in fact of norm one in its space). Choose then $m > k$, and let $n_1$, $n_2$, $n_3$ be the numbers of vectors $W_j$ of the form $(\Theta, 0, 0)$, $(0, H, 0)$, and $(0, 0, \Sigma)$, respectively. Notice that, since $\mathbf{T}^K$ is $k$-dimensional, $n_3 \leq k$. Thus, applying Lemma 1.6, we get

$$\mathrm{Tr}_m(M) \leq -\frac{k_0}{2}\kappa n_1^{\frac{5}{3}} + c_{11}n_1 - \frac{\delta}{2}n_2 + c_{10}n_3$$

which gives at once

$$q_m \leq -\frac{k_0}{2}\kappa n_1^{\frac{5}{3}} + c_{11}n_1 - \frac{\delta}{2}n_2 + c_{10}k.$$

Since as $m$ goes to infinity either $n_1$ or $n_2$ (or both) go to infinity, it is clear that there exists $m_0$ such that $q_{m_0} < 0$. Thus the desired conclusion follows from Theorem 5.3 and Theorem 5.4. □

**Corollary 5.6** *Assume* (h1)-(h2), *and let* (g1)-(g3) *with* $\beta \leq \frac{5}{3}$ *and* $g \in C^1(\mathbb{R})$ *hold. Let* $f \in C^1(\mathbb{R}, L^2)$ *be quasiperiodic in time. Then the uniform attractor* $\mathcal{A}$ *of the family* $\{U_h(t,\tau),\ h \in H(f)\}$ *given by Theorem 4.7 has finite Hausdorff dimension.*

## References


[1] S. Aizicovici & V. Barbu, *Existence and asymptotic results for a system of integro-partial differential equations*, NoDEA Nonlinear Differential Equations Appl. **3**, 1–18 (1996)

[2] L. Amerio & G. Prouse, *Abstract almost periodic functions and functional equations*, Van Nostrand, New York (1971)

[3] G. Bonfanti, P. Colli, M. Grasselli & F. Luterotti, *Nonsmooth kernels in a phase relaxation problem with memory*, Nonlinear Anal. **32**, 455–465 (1998)

[4] H. Brézis, *Operateurs maximaux monotones*, North-Holland, Amsterdam (1973)

[5] V.V. Chepyzhov & M.I. Vishik, *Nonautonomous evolution equations and their attractors*, Russian J. Math. Phys. **1**, 165–190 (1993)

[6] V.V. Chepyzhov & M.I. Vishik, *Attractors of non-autonomous dynamical systems and their dimension*, J. Math. Pures Appl. **73**, 279–333 (1994)

[7] V.V. Chepyzhov & M.I. Vishik, *Non-autonomous evolutionary equations with translation compact symbols and their attractor*, C.R. Acad. Sci. Paris Sér. I Math. **321**, 153–158 (1995)

[8] B.D. Coleman & M.E. Gurtin, *Equipresence and constitutive equation for rigid heat conductors*, Z. Angew. Math. Phys. **18**, 199–208 (1967)

[9] P. Colli & M. Grasselli, *Phase transition problems in materials with memory*, J. Integral Equations Appl. **5**, 1–22 (1993)

[10] P. Constantine, C. Foias & R. Temam, *Attractors representing turbulent flows*, Mem. Amer. Math. Soc. **53**, n.314 (1993)

[11] C.M. Dafermos, *Asymptotic stability in viscoelasticity*, Arch. Rational Mech. Anal. **37**, 297–308 (1970)

[12] W. Desch & R.K. Miller, *Exponential stabilization of Volterra integrodifferential equations in Hilbert spaces*, J. Differential Equations **70**, 366–389 (1987)

[13] G. Gentili & C. Giorgi, *Thermodynamic properties and stability for the heat flux equation with linear memory*, Quart. Appl. Math. **51**, 342–362 (1993)

[14] C. Giorgi, A. Marzocchi & V. Pata, *Asymptotic behavior of a semilinear problem in heat conduction with memory*, NoDEA Nonlinear Differential Equations Appl., to appear

[15] H. Grabmüller, *On linear theory of heat conduction in materials with memory*, Proc. Roy. Soc. Edinburgh **A-76**, 119–137 (1976-77)

[16] M. Grasselli & V. Pata, *Long time behavior of a homogenized model in viscoelastodynamics*, Discrete Contin. Dynam. Systems **4**, 339–358 (1998)

[17] A. Haraux, *Systèmes dynamiques dissipatifs et applications*, Masson, Paris (1990)

[18] R.K. Miller, *An integrodifferential equation for rigid heat conductors with memory*, J. Math. Anal. Appl. **66**, 331–332 (1978)

[19] J.W. Nunziato, *On heat conduction in materials with memory*, Quart. Appl. Math. **29**, 187–204 (1971)

[20] V. Pata, G. Prouse & M.I. Vishik, *Traveling waves of dissipative non-autonomous hyperbolic equations in a strip*, Adv. Differential Equations, to appear

[21] V. Pata, A. Zucchi, *Attractors for a damped hyperbolic equations with linear memory*, submitted

[22] R. Temam, *Infinite-dimensional dynamical systems in mechanics and physics*, Springer-Verlag, New York (1988)

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