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ON A DOUBLY NONLINEAR PHASE-FIELD MODEL FOR FIRST-ORDER TRANSITIONS WITH MEMORY

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Abstract. Solid-liquid transitions in thermal insulators and weakly conducting media are modeled through a phase-field system with memory. The evolution of the phase variable φ is ruled by a balance law which takes the form of a Ginzburg-Landau equation. A thermodynamic approach is developed starting from a special form of the internal energy and a nonlinear hereditary heat conduction flow of Coleman-Gurtin type. After some approximation of the energy balance, the absolute temperature θ obeys a doubly nonlinear "heat equation" where a third-order nonlinearity in φ appears in place of the (customarily constant) latent-heat. The related initial and boundary value problem is then formulated in a suitable setting and its well–posedness and stability is proved.

1. Introduction

In this paper we investigate the well-posedness of the system

$$\varphi_t - \kappa\,\Delta\varphi + F'(\varphi) + \frac{\theta}{\theta_c}\,G'(\varphi) = 0 \tag{1.1}$$

$$\partial_t\Big(\alpha(\theta) - \frac{1}{\theta_c}\,G(\varphi)\Big) - k_0\Delta\theta - \int_0^\infty k(s)\Delta\theta(t-s)\,ds = R(\theta), \tag{1.2}$$

where ∂_t or the subscript t denote partial differentiation with respect to time. Here, $\varphi \in [0,1]$ is the phase-field variable and θ the absolute temperature. We propose this system as a phase-field model for first-order solid-liquid transitions. It generalizes [4] and accounts for thermal memory effects in

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weakly heat conducting media as well as insulators where thermal conductivity increases linearly with temperature (see, for instance, [14]). In the sequel, the full model is developed by regarding separately the two equations of the system.

First, we introduce the order parameter of the transition as a phase-field $\varphi(x,t)$ which changes smoothly in space. In addition, its evolution equation is regarded as a balance law of the structure order (see [11]). To fix ideas we let φ increase with the structure order so that $\varphi = 0$ in the less–ordered phase (liquid). As customary in temperature-induced transitions, this balance law takes the form of the Ginzburg-Landau equation (1.1), where θ_c is the temperature transition value and $W = F + G$ provides the so-called "double–well" potential. Usually, it is a fourth–order polynomial in φ which has two wells of equal depth located at $\varphi = 0$ and $\varphi = 1$. On the contrary, the choice of $G(\varphi)$ is quite arbitrary and related to the latent heat L. In particular, G is required to satisfy

$$G(1) - G(0) = L\,,$$

where $L > 0$ gives the heat absorbed during the transition from solid to liquid phase (melting latent heat). In the Caginalp model [9], for instance, G is linear in φ. Penrose and Fife in [20] proposed a different class of models where G has quadratic growth. Some solidification models (see, for instance, [21]) assume that G has degree three or five, but there is no reason to rule out G to be even. The choice of G with an odd degree is motivated by making more efficient numerical simulations in the small undercooling regime (see [13]). On the contrary, an even function G induces some boundedness property on the phase variable (see Proposition 2.1). Here, according to [11], we assume both F and G to be fourth-order polynomials

$$F(\varphi) = L(3\,\varphi^4 - 4\,\varphi^3) \quad \text{and} \quad G(\varphi) = L(3\varphi^4 - 8\varphi^3 + 6\,\varphi^2)\,, \tag{1.3}$$

so that $W(\varphi) = 6L\,\varphi^2(1-\varphi)^2$ and $F(0) = G(0) = 0\,,\ F(1) = -G(1) = -L\,,$ $F'(0) = F'(1) = G'(0) = G'(1) = 0\,.$ Since $L > 0$, this choice accounts for first–order phase transition phenomena and solutions to (1.1) fulfill the following properties:

- $\varphi(x,t) = 0$ and $\varphi(x,t) = 1$ are equilibrium solutions (*i.e.,* local minimizers of the free energy) whatever may be the value of θ,
- at $\theta = \theta_c$ both solutions are stable (*i.e.,* absolute minima); a change of stability occurs when $\theta < \theta_c$ or $\theta > \theta_c$.

It is worth noting that these statements cannot be satisfied when the degree of F' and G' is less than two. In particular, neither Caginalp nor Penrose–Fife models agree with the first property. What is more, the choice (1.3) enables us to prove the bound $0 \le \varphi(x,t) \le 1$ provided that $0 \le \varphi(x,0) \le 1$ and $\theta(x,t) > 0$ for all $t > 0$ (see [11]).

The evolution equation for the temperature is obtained from the energy balance law, as usual. After assuming a special expression for the internal power of the structure order, the heat equation takes the form

$$\partial_t \left(e - F(\varphi) - \frac{\kappa}{2}|\nabla\varphi|^2 \right) - \frac{\theta}{\theta_c}\partial_t G(\varphi) - (\partial_t\varphi)^2 = -\nabla\cdot\mathbf{q} + r. \tag{1.4}$$

In order to derive (1.2) we assume here a heat conduction law of the Coleman-Gurtin type (see [10]) so that the heat flux vector $\mathbf{q}$ depends both on the present value of the temperature gradient $\nabla\theta(t)$ and its past history up to $t > 0$, namely,

$$\mathbf{q}(t) = -k_0\theta(t)\nabla\theta(t) - \theta(t)\int_0^\infty k(s)\nabla\theta^t(s)\,ds\,, \qquad k_0 > 0\,, \tag{1.5}$$

where $\theta^t(s) = \theta(t-s)$. When the memory term is neglected ($k \equiv 0$) the nonlinear Fourier heat conduction law considered in [4] is recovered,

$$\mathbf{q}(t) = -k_0\theta(t)\nabla\theta(t)\,.$$

On the other hand, when

$$k(s) = \epsilon\,(k_0 - k_1)\exp(-\epsilon s)\,, \qquad k_0 > k_1 > 0$$

(1.5) is equivalent to the rate–type constitutive equation

$$\partial_t\mathbf{p} + \epsilon\,\mathbf{p} = -k_0\nabla\partial_t\theta - \epsilon\,k_1\nabla\theta\,, \qquad \mathbf{q} = \theta\,\mathbf{p}$$

which models the heat flux in a (nonlinear) Jeffreys-type rigid conductor at rest.

According to (1.5), the free energy density ψ is assumed to split into two parts: the first, ψ_1, independent of $\nabla\theta^t$, the last, ψ_2, which only depends on $\nabla\theta^t$. In the special case $\alpha(\theta) = c\ln\theta$ equation (1.2) is obtained from the energy balance by means of a suitable choice of ψ_1 and ψ_2 in agreement with thermodynamics. In particular, after replacing (1.5) into (1.4) and multiplying the result by $1/\theta$, some approximations are needed to obtain (1.2).

A lot of papers deal with the heat equation with memory, namely (1.2) where $\alpha(\theta) = c\theta$ and $G = 0$. The model was proposed in [10] and lately studied by many authors (see e.g. [15] and references therein), including its asymptotic behavior in the history space setting [16, 17].

Well-posedness results for the full system (1.1)–(1.2) have been scrutinized recently in the literature assuming $\alpha(\theta) = c\,\theta$ and letting G to have at most quadratic growth. In addition, F is assumed to be – or include – the indicator function $I_{[0,1]}$ of the closed interval $[0,1]$, in order to force the phase variable to sit between 0 and 1. In particular, when a quadratic nonlinearity for G is involved and $\alpha(\theta) = c\,\theta$, the problem has been studied first in [8], where existence and uniqueness results for weak and smooth solutions have been proved via energy methods. More recently, a system like (1.1)–(1.2) was proposed in [5] as a particular case of a more general phase-field model with thermal memory where the usual energy balance is replaced by an entropy balance law. Using a different thermodynamic approach, a thorough investigation of (1.1)–(1.2) has been carried out in [6, 7] assuming $\alpha(\theta) = \ln\theta$, $G(\varphi) = L\,\varphi$ and $F = I_{[0,1]}$. The main results proved therein, that is existence, uniqueness and asymptotic behavior of solutions, jointly with positivity of the temperature field, strongly depend on the linear form of G.

The aim of this paper is to prove well–posedness and stability of the Cauchy-Neumann-Dirichlet initial-boundary value problem generated by (1.1)-(1.2) with $\alpha(\theta) = \ln\theta$ and a fourth-order nonlinearity for G. This is achieved by means of a procedure introduced in [4], where we prove well-posedness of a similar system without memory ($k \equiv 0$). The main difficulty here is due to the different state identification at time $t > 0$, namely $(\varphi(t), \theta(t), \nabla\theta^t)$. Indeed, the presence of a convolution term would prevent us from applying any maximum principle to prove the positivity of the temperature component of the solution. This positivity result can be achieved here only by virtue of the choice $\alpha(\theta) = \ln\theta$ in (1.2). Even if we take advantage of some results proven in [6], the novelty of our paper consists in obtaining the boundedness of the phase-field solution directly in the customary setting of a double well potential, with no recurse to the interval indicator function and its subdifferential, which are rather typical of the Stefan problem with a sharp interface. What is more, this procedure enables us to handle nonlinearity in G of degree higher than two.

The plan is as follows. In Section 2, we introduce the model and we formulate the problem with proper initial and boundary conditions. In Section 3 we prove the existence result. First we construct a sequence G_ε of functions with lipschitz derivative, approaching G and behaving as well as G in $(0,1)$. When G_ε is considered in place of G, exploiting a recent result by Colli *et* *al.* [6], we obtain existence, uniqueness and positivity of the temperature for all solutions $(\varphi_\varepsilon(t), \theta_\varepsilon(t), \nabla\theta_\varepsilon^t)$. This allows us to apply Proposition 2.1 and ensures that $0 \leq \varphi_\varepsilon \leq 1$ holds for all $\varepsilon > 0$. Finally, we pass to the limit as $\varepsilon \to 0$ and prove both uniform convergence and uniqueness of solutions via energy methods. Section 4 is devoted to show uniqueness of the solution. Finally, in Section 5 we prove the stability of solutions by means of energy estimates and Gronwall lemma.

2. Phase transition model

Following the point of view of [11] and [12], every phase transition can be interpreted as a change in the order structure inside the material. The variable which measures the internal order structure is a scalar function φ which takes its values between 0 and 1, such that $\varphi = 0$ corresponds to the less ordered phase and $\varphi = 1$ describes the more ordered state. We denote respectively by K, $\mathbf{p}$ and σ the density, the flux and the supply of the internal order structure.

The balance of the structure order considered in [11] and [12] leads to the following local equation

$$\rho K = \nabla \cdot \mathbf{p} + \rho\sigma, \tag{2.1}$$

where ρ is the mass density. We assume in particular the constitutive equations

$$K = \varphi_t + f(\varphi) + \frac{\theta}{\theta_c} g(\varphi) \tag{2.2}$$

$$\mathbf{p} = \kappa \nabla\varphi, \tag{2.3}$$

where θ is the absolute temperature, θ_c is the critical temperature of the phase transition, κ is a positive constant, f, g are two functions characterizing the first-order phase transition.

Therefore, equation (2.1) assumes the form

$$\rho\Big[\varphi_t + f(\varphi) + \frac{\theta}{\theta_c} g(\varphi)\Big] = \kappa \triangle\varphi + \rho\sigma. \tag{2.4}$$

Henceforth, we suppose that the density ρ is constant and for sake of simplicity we let $\rho = 1$. Moreover, we assume $\sigma = 0$. Accordingly, (2.4) reduces to

$$\varphi_t + f(\varphi) + \frac{\theta}{\theta_c} g(\varphi) = \kappa \triangle\varphi. \tag{2.5}$$

In order to deduce the equation for the temperature, we consider the energy balance law

$$e_t = \mathcal{P}^i - \nabla\cdot\mathbf{q} + r\,, \tag{2.6}$$

where e is the internal energy, $\mathbf{q}$ is the heat flux, r is the heat supply and $\mathcal{P}^i$ is the internal structure order power, defined as

$$\mathcal{P}^i = K\varphi_t + \mathbf{p}\cdot\nabla\varphi_t. \tag{2.7}$$

By defining F, G such that $F'(\varphi) = f(\varphi)$, $G'(\varphi) = g(\varphi)$, $F(0) = G(0) = 0$, the internal power is given by

$$\mathcal{P}^i = \varphi_t^2 + \partial_t F(\varphi) + \frac{\theta}{\theta_c}\partial_t G(\varphi) + \frac{\kappa}{2}\partial_t(|\nabla\varphi|^2).$$

Thus, from (2.6) we obtain

$$\partial_t\Big[e - F(\varphi) - \frac{\kappa}{2}|\nabla\varphi|^2\Big] - \frac{\theta}{\theta_c}\partial_t G(\varphi) - \varphi_t^2 = -\nabla\cdot\mathbf{q} + r. \tag{2.8}$$

In this paper we deal with weakly conducting materials with fading thermal memory, so that the heat flux is assumed to obey the constitutive equation (see [12])

$$\begin{aligned}\mathbf{q}(t) &= -k_0\theta(t)\nabla\theta(t) - \theta(t)\int_0^\infty k(s)\nabla\theta^t(s)ds \\ &= -k_0\theta(t)\nabla\theta(t) + \theta(t)\int_0^\infty k'(s)\nabla\tilde{\theta}^t(s)ds,\end{aligned} \tag{2.9}$$

where $k_0 > 0$, $k \in W^{2,1}(0,\infty)\cap H^1(0,\infty)$ and

$$\begin{aligned}\nabla\theta^t(s) &= \nabla\theta(t-s), \\ \nabla\tilde{\theta}^t(s) &= \int_0^s \nabla\theta(t-\tau)d\tau = \int_{t-s}^t \nabla\theta(\tau)d\tau.\end{aligned}$$

Substitution into (2.8) yields

$$\begin{aligned}&\partial_t\left[e - F(\varphi) - \frac{\kappa}{2}|\nabla\varphi|^2\right] - \frac{\theta}{\theta_c}\partial_t G(\varphi) - \varphi_t^2 \\ &= k_0\theta\triangle\theta + k_0|\nabla\theta|^2 - \theta\int_0^\infty k'(s)\triangle\tilde{\theta}^t(s)ds - \nabla\theta\cdot\int_0^\infty k'(s)\nabla\tilde{\theta}^t(s)ds + r.\end{aligned} \tag{2.10}$$

We examine the restrictions imposed to the constitutive equations by the principles of thermodynamics. To this aim we identify the state of the system at time $t > 0$ with the quadruplet $(\theta(t), \varphi(t), \nabla\varphi(t), \nabla\tilde{\theta}^t)$ and suppose that the thermodynamical potentials depend on the state. Denoting by η the entropy function, the second law of thermodynamics, expressed in terms of the Clausius-Duhem inequality, gives

$$\eta_t \geq -\nabla \cdot \left(\frac{\mathbf{q}}{\theta}\right) + \frac{r}{\theta}.$$

Hence, relations (2.6), (2.7) imply

$$\theta\eta_t \geq e_t - K\varphi_t - \mathbf{p} \cdot \nabla\varphi_t + \frac{1}{\theta}\mathbf{q} \cdot \nabla\theta.$$

By introducing the free energy $\psi = e - \theta\eta$, in view of constitutive equations (2.2) and (2.3), the previous inequality leads to

$$\psi_t + \eta\theta_t - \varphi_t^2 - \Big[f(\varphi) + \frac{\theta}{\theta_c}g(\varphi)\Big]\varphi_t - \kappa\nabla\varphi \cdot \nabla\varphi_t + \frac{1}{\theta}\mathbf{q} \cdot \nabla\theta \leq 0. \tag{2.11}$$

Hereafter, we suppose the free energy depending on the state variables as

$$\psi(\theta, \varphi, \nabla\varphi, \nabla\tilde{\theta}^t) = \psi_1(\theta, \varphi, \nabla\varphi) + \psi_2(\nabla\tilde{\theta}^t),$$

where the functionals ψ_1, ψ_2 satisfy

$$\psi_1(0, 0, \mathbf{0}) = 0, \qquad \psi_2(\mathbf{0}) = 0.$$

Then from (2.9) and (2.11), we obtain the inequality

$$\begin{aligned} \partial_t\psi_1 + \partial_t\psi_2 \;\; &\leq \;\; -\eta\theta_t + \varphi_t^2 + \Big[f(\varphi) + \frac{\theta}{\theta_c}g(\varphi)\Big]\varphi_t + \kappa\nabla\varphi \cdot \nabla\varphi_t \\ &+ \;\; k_0|\nabla\theta|^2 - \nabla\theta \cdot \int_0^\infty k'(s)\nabla\tilde{\theta}^t(s)ds. \end{aligned} \tag{2.12}$$

The arbitrariness of $\theta_t, \varphi_t, \nabla\varphi_t$ implies the relations

$$\eta = -\frac{\partial\psi_1}{\partial\theta}, \qquad f(\varphi) + \frac{\theta}{\theta_c}g(\varphi) = \frac{\partial\psi_1}{\partial\varphi}, \qquad \kappa\nabla\varphi = \frac{\partial\psi_1}{\partial\nabla\varphi}.$$

Accordingly we deduce the representations

$$\psi_1(\theta, \varphi, \nabla\varphi) = -C(\theta) + F(\varphi) + \frac{\theta}{\theta_c}G(\varphi) + \frac{\kappa}{2}|\nabla\varphi|^2 \tag{2.13}$$

$$\eta(\theta, \varphi) = c(\theta) - \frac{1}{\theta_c}G(\varphi), \tag{2.14}$$

where $c(\theta)$ is any function of the temperature and $C'(\theta) = c(\theta)$.

Therefore, inequality (2.12) reduces to

$$\partial_t\psi_2 \leq \varphi_t^2 + k_0|\nabla\theta|^2 - \nabla\theta \cdot \int_0^\infty k'(s)\nabla\tilde{\theta}^t(s)ds. \tag{2.15}$$

There are many functionals ψ_2 satisfying (2.15) (see [2]). Here we consider the Graffi free energy

$$\psi_2(t) = \psi_G(t) = -\frac{1}{2}\int_0^\infty k'(s)|\nabla\tilde{\theta}^t(s)|^2 ds, \tag{2.16}$$

where the kernel k is such that

$$k'(s) \leq 0, \qquad k''(s) \geq 0. \tag{2.17}$$

By differentiating (2.16) with respect to time, we obtain

$$\partial_t\psi_G(t) = -\int_0^\infty k'(s)\nabla\tilde{\theta}^t(s)\cdot\left[\nabla\theta(t) - \nabla\theta^t(s)\right] ds$$

and an integration by parts yields

$$\partial_t\psi_G(t) = -\nabla\theta(t)\cdot\int_0^\infty k'(s)\nabla\tilde{\theta}^t(s)ds - \frac{1}{2}\int_0^\infty k''(s)|\nabla\tilde{\theta}^t(s)|^2 ds. \tag{2.18}$$

Substitution into (2.15) leads to the inequality

$$0 \leq \varphi_t^2(t) + k_0|\nabla\theta(t)|^2 + \frac{1}{2}\int_0^\infty k''(s)|\nabla\tilde{\theta}^t(s)|^2 ds, \tag{2.19}$$

which is satisfied in view of (2.17).

We conclude this section with the introduction of the system of differential equations which describe the evolution of this first order phase model. In this framework, we choose

$$c(\theta) = \alpha(1 + \ln\theta)\,,$$

with $\alpha > 0$ and

$$f(\varphi) = -12L\varphi^2(1-\varphi), \qquad g(\varphi) = 12L\varphi(1-\varphi)^2.$$

Therefore equation (2.5) reads

$$\varphi_t = \kappa\triangle\varphi + 12L\varphi^2(1-\varphi) - \frac{12L}{\theta_c}\theta\varphi(1-\varphi)^2. \tag{2.20}$$

Moreover, owing to (2.13) and (2.14), the internal energy is written as

$$e = \psi + \theta\eta = \alpha\theta + F(\varphi) + \frac{\kappa}{2}|\nabla\varphi|^2 - \frac{1}{2}\int_0^\infty k'(s)|\nabla\tilde{\theta}^t(s)|^2 ds.$$

Substitution into (2.10) and use of (2.18) provide

$$\begin{aligned}\alpha\theta_t - 12L\frac{\theta}{\theta_c}\varphi(1-\varphi)^2\varphi_t &= \varphi_t^2 + k_0\theta\triangle\theta + k_0|\nabla\theta|^2 \\ &- \theta\int_0^\infty k'(s)\triangle\tilde{\theta}^t(s)ds + \frac{1}{2}\int_0^\infty k''(s)|\nabla\tilde{\theta}^t(s)|^2 ds + r.\end{aligned}$$

In the sequel we consider an approximation of this equation, namely

$$\alpha\theta_t - 12L\frac{\theta}{\theta_c}\varphi(1-\varphi)^2\varphi_t = k_0\theta\triangle\theta - \theta\int_0^\infty k'(s)\triangle\tilde{\theta}^t(s)ds\,, \tag{2.21}$$

which is obtained by ignoring the terms

$$\varphi_t^2 + k_0|\nabla\theta|^2 + \frac{1}{2}\int_0^\infty k''(s)|\nabla\tilde{\theta}^t(s)|^2 ds.$$

This could be justified by requiring that φ_t^2 and $|\nabla\theta^t(s)|^2$, $s \geq 0$, are negligible and assuming the condition

$$-\int_0^\infty s^2 k''(s)ds = M < \infty.$$

Indeed, a direct computation proves the inequality

$$\begin{aligned} k_0|\nabla\theta(t)|^2 - \int_0^\infty k''(s)|\nabla\tilde{\theta}^t(s)|^2 ds &\leq k_0|\nabla\theta(t)|^2 + M\sup_{s>0}|\nabla\theta^t(s)|^2 \\ &\leq (k_0+M)\sup_{s\geq 0}|\nabla\theta^t(s)|^2. \end{aligned}$$

It is worth noting that under such an approximation the thermodynamic consistence of the model is guaranteed anyway, since relation (2.19) still holds as an equality.

Finally, by multiplying (2.21) by $1/\theta$, we obtain the "heat equation"

$$\alpha(\ln\theta)_t - \frac{12L}{\theta_c}\varphi(1-\varphi)^2\varphi_t = k_0\triangle\theta - \int_0^\infty k'(s)\triangle\tilde{\theta}^t(s)ds + \frac{r}{\theta}. \tag{2.22}$$

Equations (2.20) and (2.22) are completed with initial and boundary conditions. Due to the presence of a memory term in (2.22), we prescribe in Ω, the domain occupied by the material,

$$\varphi(x,0) = \varphi_0\,, \qquad \theta(x,s) = \theta^0(x,s)\,,\ s \leq 0\,, \tag{2.23}$$

jointly with a Neumann boundary condition for the phase field and a nonhomogeneous Dirichlet condition for the temperature

$$\nabla\varphi\cdot\mathbf{n}|_{\partial\Omega} = 0\,, \qquad \theta|_{\partial\Omega} = \theta_\Gamma, \tag{2.24}$$

where $\mathbf{n}$ denotes the unit outward normal to $\partial\Omega$, the boundary of Ω.

As already proved in [11], the IBV problem (2.20)-$(2.23)_1$-$(2.24)_1$ ensures the boundedness of the phase field between 0 and 1 as a consequence of the positivity of the temperature θ.

Proposition 2.1. (see [11], Theorem 2) *Given* $\theta \geq 0$ *in* $\Omega \times (0,T)$*, if* φ *is a solution of* (2.20), $(2.23)_1$ *and* $(2.24)_1$ *with initial datum* φ_0 *satisfying* $0 \leq \varphi_0 \leq 1$ *a.e. in* Ω*, then* $0 \leq \varphi \leq 1$*, a.e. in* $\Omega \times (0,T)$.

It is worth noting that the same result holds when the function $g(\varphi) = 12L\varphi(1-\varphi)^2$ is replaced by $\mu(\varphi)$, where μ is any continuous function such that $r\mu(r) \geq 0$, for all $r \in \mathbb{R}$.

3. Existence of the solutions

For convenience, we introduce here some notation and recall some standard inequalities to be used in the sequel. As usual, we denote by $L^p(\Omega)$ and $H^k(\Omega)$, respectively, the Lebesgue and Sobolev spaces endowed with the standard norms $\|\cdot\|_{L^p(\Omega)}$ and $\|\cdot\|_{H^k(\Omega)}$. In particular $\|\cdot\|$ stands for the $L^2(\Omega)-$norm. Let $H^1_0(\Omega)$ denote the closure in $H^1(\Omega)$ of the space $C^\infty_0(\Omega)$ of smooth functions with compact support. For each $v \in H^1_0(\Omega)$ the Poincaré inequality

$$\lambda_0(\Omega)\|v\|^2 \leq \|\nabla v\|^2 \tag{3.1}$$

holds. Finally, let us denote $Q_t = \Omega \times (0,t)$, for any $t \in [0,T]$.

If a, b are two functions of t, we denote by $*$ the convolution product, namely,

$$a * b(t) = \int_0^t a(s)b(t-s)ds.$$

By differentiating with respect to time, we obtain the identity

$$(a * b)_t = a(0)b + a_t * b. \tag{3.2}$$

If $a \in L^1(0,t)$, $b \in L^2(Q_t)$, the Young theorem provides

$$\|a * b\|_{L^2(Q_t)} \leq \|a\|_{L^1(0,t)}\|b\|_{L^2(Q_t)}. \tag{3.3}$$

Now let us introduce some assumptions on the data. For simplicity the boundary datum θ_Γ is supposed to be time independent. Moreover we require

- $(H1)$ $k, k' \in L^1(0,+\infty)$;
- $(H2)$ $r \in L^\infty(Q_T)$;
- $(H3)$ $\varphi_0 \in H^1(\Omega)$, $\quad 0 \leq \varphi_0 \leq 1$ a.e. in Ω, $\quad \theta_0 \in L^\infty(\Omega)$;
- $(H4)$ $\theta_\Gamma \in H^{1/2}(\partial\Omega) \cap L^\infty(\partial\Omega)$;
- $(H5)$ there exist two positive constants θ_* and θ^* such that $\theta_* \leq \theta_\Gamma \leq \theta^*, \theta_* \leq \theta_0 \leq \theta^*$.

In order to deal with homogeneous boundary conditions, let us consider the function $\theta_{\mathcal{H}} \in H^1(\Omega)$ solution to problem

$$\begin{cases} \triangle\theta_{\mathcal{H}} = 0, & \text{in } \Omega \\ \theta_{\mathcal{H}}|_{\partial\Omega} = \theta_\Gamma \,. \end{cases}$$

In view of $(H4), (H5)$, $\theta_{\mathcal{H}}$ satisfies the following inequalities

$$\theta_* \leq \theta_{\mathcal{H}} \leq \theta^*, \qquad \text{a.e. in } \Omega \tag{3.4}$$

$$\|\theta_{\mathcal{H}}\|_{H^1(\Omega)} \leq c\|\theta_\Gamma\|_{H^{1/2}(\partial\Omega)} \,. \tag{3.5}$$

Let us denote by u the difference

$$u = \theta - \theta_{\mathcal{H}}. \tag{3.6}$$

Accordingly u satisfies a homogeneous Dirichlet boundary condition and owing to (3.1) the inequality

$$\|u\| \leq \|u\|_{H^1(\Omega)} \leq \lambda_1(\Omega)\|\nabla u\| \tag{3.7}$$

holds.

Now we perform a further approximation of equation (2.22), by assuming that the temperature θ is close to $\theta_{\mathcal{H}}$, for every $t > 0$. Accordingly, by means of the approximation

$$\frac{\theta_{\mathcal{H}}}{\theta} \approx 1 - \frac{1}{\theta_{\mathcal{H}}}(\theta - \theta_{\mathcal{H}}) = 2 - \frac{\theta}{\theta_{\mathcal{H}}}\,,$$

equation (2.22) is written as

$$\alpha(\ln\theta)_t - \frac{12L}{\theta_c}\varphi(1-\varphi)^2\varphi_t = k_0\triangle\theta - \int_0^\infty k'(s)\triangle\tilde{\theta}^t(s)ds + \frac{r}{\theta_{\mathcal{H}}}\Big(2 - \frac{\theta}{\theta_{\mathcal{H}}}\Big). \tag{3.8}$$

Moreover, in order to distinguish the dependence on the past history θ^0 in the previous equation, we split the heat flux as

$$\mathbf{q}(t) = -k_0\theta(t)\nabla\theta(t) - \theta(t)(k * \nabla\theta(t)) + \mathbf{q}_H(t),$$

where

$$\mathbf{q}_H(t) = -\theta(t)\int_0^\infty k(t+s)\nabla\theta^0(s)ds.$$

Thus, we prove existence of solutions to problem

$$(P)\qquad \begin{cases} \varphi_t = \kappa\triangle\varphi + 12L\varphi^2(1-\varphi) - \dfrac{12L}{\theta_c}\theta\varphi(1-\varphi)^2 \\ \alpha(\ln\theta)_t - \dfrac{12L}{\theta_c}\varphi(1-\varphi)^2\varphi_t = k_0\triangle\theta + k * \triangle\theta + r_H - \dfrac{r}{\theta_{\mathcal{H}}^2}\theta \\ \nabla\varphi\cdot\mathbf{n}|_{\partial\Omega} = 0, \qquad \theta|_{\partial\Omega} = \theta_\Gamma \\ \varphi(x,0) = \varphi_0(x), \qquad \theta(x,0) = \theta_0, \end{cases}$$

where

$$r_H = -\nabla\cdot\left(\frac{\mathbf{q}_H}{\theta}\right) + \frac{2r}{\theta_{\mathcal{H}}} = \int_0^\infty k(t+s)\triangle\theta^0(s)ds + \frac{2r}{\theta_{\mathcal{H}}}$$

is a known function of x and t.

Let us require the following additional assumptions:

$(H6)$ $r_H \in L^2(Q_T)$;

$(H7)$ there exists $\delta > 1$ such that $b = \inf\{r(x,t) : (x,t) \in Q_T\} \geq -\frac{k_0\theta_*^2}{\delta\lambda_1(\Omega)}$.

We introduce here a family of approximating problems (P_ε), $\varepsilon \in (0,1]$. More precisely, for each $\varepsilon \in (0,1]$, we denote by g_ε the Yosida regularization of the function $g(\varphi) = 12L\varphi(\varphi-1)^2$ and consider the problem

$$(P_\varepsilon)\qquad \begin{cases} \varphi_t = \kappa\triangle\varphi + 12L\varphi^2(1-\varphi) - g_\varepsilon(\varphi)\dfrac{\theta}{\theta_c} \\ \alpha(\ln\theta)_t - \dfrac{1}{\theta_c}g_\varepsilon(\varphi)\varphi_t = k_0\triangle\theta + k * \triangle\theta + r_H - \dfrac{r}{\theta_{\mathcal{H}}^2}\theta \\ \nabla\varphi\cdot\mathbf{n}|_{\partial\Omega} = 0, \qquad \theta|_{\partial\Omega} = \theta_\Gamma \\ \varphi(x,0) = \varphi_0(x), \qquad \theta(x,0) = \theta_{0\varepsilon}, \end{cases}$$

where the initial data $\theta_{0\varepsilon}$ are chosen with the following properties

$$\theta_{0\varepsilon} \in H^1(\Omega), \qquad \theta_* \leq \theta_{0\varepsilon} \leq \theta^*, \text{ for any } \varepsilon \in (0,1]$$

and

$$\theta_{0\varepsilon} \to \theta_0 \text{ in } L^2(\Omega) \text{ and a.e. in } \Omega \text{ as } \varepsilon \to 0.$$

The existence of a solution $(\varphi_\varepsilon, \theta_\varepsilon)$ to (P_ε) is obtained with the same technique used in [6], where the authors prove the existence of a solution (φ, θ)

to problem

$$(P') \quad \begin{cases} \varphi_t = \kappa \triangle \varphi - \xi - \sigma'(\varphi) - \lambda'(\varphi)\dfrac{\theta}{\theta_c} \\ \alpha(\ln\theta)_t - \dfrac{1}{\theta_c}\lambda'(\varphi)\varphi_t = k_0 \triangle \theta + k * \triangle\theta + R \\ \xi \in \beta(\varphi) \\ \nabla\varphi \cdot \mathbf{n}|_{\partial\Omega} = 0, \qquad \theta|_{\partial\Omega} = \theta_\Gamma \\ \varphi(x,0) = \varphi_0(x), \qquad \theta(x,0) = \theta_0 \end{cases}$$

such that

$$\begin{aligned} \varphi &\in L^2(0,T,H^2(\Omega)) \cap H^1(0,T,L^2(\Omega)) \\ \theta &\in L^2(0,T,H^1(\Omega)), \quad \theta > 0 \text{ a.e. in } Q_T \\ \ln\theta &\in L^\infty(0,T,L^2(\Omega)) \cap H^1(0,T,H^{-1}(\Omega)). \end{aligned}$$

Here λ' and σ' are Lipschitz continuous functions and β is a maximal monotone graph. With the identifications

$$\lambda' = g_\varepsilon, \qquad \beta(\varphi) = 12L(\varphi^3 - \varphi^2 + \varphi), \qquad \sigma'(\varphi) = -12L\varphi\,,$$

λ', σ' are Lipschitz continuous functions and β is monotone. Since the additional term $\frac{r}{\theta_{\mathcal{H}}^2}\theta$ is linear in θ, existence of solutions to (P_ε) can be proved as in [6] with minor modifications. Accordingly, the following theorem holds.

Theorem 3.1. *For any $\varepsilon \in (0,1]$, problem (P_ε) admits at least a solution $(\varphi_\varepsilon, \theta_\varepsilon)$ such that*

$$\begin{aligned} \varphi_\varepsilon &\in L^2(0,T,H^2(\Omega)) \cap H^1(0,T,L^2(\Omega)) \\ \theta_\varepsilon &\in L^2(0,T,H^1(\Omega)), \quad \theta_\varepsilon > 0 \text{ a.e. in } Q_T \\ \ln\theta_\varepsilon &\in L^\infty(0,T,L^2(\Omega)) \cap H^1(0,T,H^{-1}(\Omega)). \end{aligned}$$

Remark 3.1. Since $\theta_\varepsilon > 0$ and $\varphi_\varepsilon g_\varepsilon(\varphi_\varepsilon) \geq 0$, owing to Proposition 2.1, each solution of (P_ε) satisfies

$$0 \leq \varphi_\varepsilon \leq 1, \qquad \text{a.e. in } Q_T\,. \tag{3.9}$$

We prove now that a solution $(\varphi_\varepsilon, \theta_\varepsilon)$ of problem (P_ε) converges to a solution (φ, θ) of (P) as $\varepsilon \to 0$. To this purpose we deduce some a priori estimates which ensure the boundedness of $(\varphi_\varepsilon, \theta_\varepsilon)$ uniformly with respect to ε.

In the subsequent inequalities we denote by c any positive constant independent of ε. Moreover, we will repeatedly use the Hölder and Young inequalities with suitable choices of the constants.

Lemma 3.1. *If* $(\varphi_\varepsilon, \theta_\varepsilon)$ *is a solution to* (P_ε)*, there exists a constant* c *such that*

$$\|\varphi_\varepsilon\|_{L^\infty(0,T,H^1(\Omega))\cap H^1(0,T,L^2(\Omega))} + \|u_\varepsilon\|_{L^2(0,T,H^1_0(\Omega))} + \|\theta_\varepsilon\|_{L^\infty(0,T,L^1(\Omega))} \le c. \tag{3.10}$$

Proof. Let us multiply the first equation of (P_ε) by $\varphi_{\varepsilon t}$, the second by $u_\varepsilon = \theta_\varepsilon - \theta_{\mathcal{H}}$ and integrate over Q_t. We obtain

$$\int_{Q_t} \left(\varphi_{\varepsilon t}^2 + k_0|\nabla u_\varepsilon|^2\right) dxds \tag{3.11}$$
$$+ \int_\Omega \Big[\frac{\kappa}{2}|\nabla\varphi_\varepsilon(t)|^2 + 3L\varphi_\varepsilon^4(t) + G_\varepsilon(\varphi_\varepsilon(t))\frac{\theta_{\mathcal{H}}}{\theta_c} + \alpha\theta_\varepsilon(t)\Big]dx = I_1 + I_2 + I_3\,,$$

where

$$I_1 = \int_\Omega \left[4L\varphi_\varepsilon^3(t) + \alpha\theta_{\mathcal{H}}\ln\theta_\varepsilon(t)\right] dx$$
$$I_2 = \int_{Q_t} \Big[-(k * \nabla u_\varepsilon)\cdot\nabla u_\varepsilon + \Big(r_H - \frac{r}{\theta_{\mathcal{H}}}\Big)u_\varepsilon - \frac{r}{\theta_{\mathcal{H}}^2}u_\varepsilon^2\Big]dxds$$
$$I_3 = \int_\Omega \Big[\frac{\kappa}{2}|\nabla\varphi_0|^2 + 3L\varphi_0^4 - 4L\varphi_0^3 + G_\varepsilon(\varphi_0)\frac{\theta_{\mathcal{H}}}{\theta_c} + \alpha\theta_{0\varepsilon} - \alpha\theta_{\mathcal{H}}\ln\theta_{0\varepsilon}\Big]dx$$

and

$$G_\varepsilon(r) = \int_0^r g_\varepsilon(s)ds.$$

From inequalities (3.4) and (3.9) we deduce

$$I_1 \le c + c\int_\Omega \ln\theta_\varepsilon(t)dx \le c + \frac{\alpha}{2}\int_\Omega \theta_\varepsilon(t)dx.$$

In view of identity (3.2) we obtain

$$\begin{aligned} I_2 \quad \le \quad & \int_{Q_t} |[k(0)(1 * \nabla u_\varepsilon) + k' * 1 * \nabla u_\varepsilon]\cdot\nabla u_\varepsilon|dxds \\ & + \Big[\|r_H\|_{L^2(Q_t)} + \frac{1}{\theta_*}\|r\|_{L^2(Q_t)}\Big]\|u_\varepsilon\|_{L^2(Q_t)} - b\int_{Q_t}\frac{u_\varepsilon^2}{\theta_{\mathcal{H}}^2}dxds, \end{aligned}$$

so that by means of (3.3) and (3.7) we prove

$$I_2 \le (|k(0)| + \|k'\|_{L^1(0,T)})\|1 * \nabla u_\varepsilon\|_{L^2(Q_t)}\|\nabla u_\varepsilon\|_{L^2(Q_t)}$$
$$+ \lambda_1(\Omega)\Big[\|r_H\|_{L^2(Q_t)} + \frac{1}{\theta_*}\|r\|_{L^2(Q_t)}\Big]\|\nabla u_\varepsilon\|_{L^2(Q_t)} + \max\{0, -b\}\frac{1}{\theta_*^2}\|u_\varepsilon\|^2_{L^2(Q_t)}.$$

By virtue of $(H7)$, it follows

$$I_2 \le \frac{k_0}{\delta}\|\nabla u_\varepsilon\|^2_{L^2(Q_t)} + c\int_0^t \|\nabla u_\varepsilon\|^2_{L^2(Q_s)}ds + c.$$

Finally, assumptions $(H3)$ and (3.4) prove the boundedness of I_3.

Substitution into (3.11) yields

$$\|\varphi_{\varepsilon t}\|^2_{L^2(Q_t)} + k_0\frac{(\delta-1)}{\delta}\|\nabla u_\varepsilon\|^2_{L^2(Q_t)} + \frac{\kappa}{2}\|\nabla\varphi_\varepsilon(t)\|^2 + 3L\|\varphi_\varepsilon(t)\|^4_{L^4(\Omega)}$$
$$+\int_\Omega \Big[G_\varepsilon(\varphi_\varepsilon(t))\frac{\theta_{\mathcal{H}}}{\theta_c} + \frac{\alpha}{2}\theta_\varepsilon(t)\Big]dx \le c + c\int_0^t \|\nabla u_\varepsilon\|^2_{L^2(Q_s)}ds.$$

Therefore, Gronwall's inequality implies (3.10). □

Remark 3.2. By comparison with (P_ε), using (3.10), we prove the inequalities

$$\|\triangle\varphi_\varepsilon\|_{L^2(Q_T)} \le c, \tag{3.12}$$
$$\|(\ln\theta_\varepsilon)_t\|_{L^2(0,T,H^{-1}(\Omega))} \le c. \tag{3.13}$$

Lemma 3.2. *Let $(\varphi_\varepsilon, \theta_\varepsilon)$ be a solution to (P_ε), then*

$$\|\ln\theta_\varepsilon\|_{L^2(Q_T)} \le c. \tag{3.14}$$

Proof. Let us integrate the second equation of the problem (P_ε) in the time interval $(0,t)$, thus obtaining

$$\alpha(\ln\theta_\varepsilon - \ln\theta_{0\varepsilon}) - \frac{1}{\theta_c}[G_\varepsilon(\varphi_\varepsilon) - G_\varepsilon(\varphi_0)] \tag{3.15}$$
$$= k_0 * \triangle\theta_\varepsilon + 1 * k * \triangle\theta_\varepsilon + 1 * \Big(r_H - \frac{r}{\theta_{\mathcal{H}}} - \frac{r}{\theta^2_{\mathcal{H}}}u_\varepsilon\Big).$$

By subtracting to both sides $\alpha\ln\theta_{\mathcal{H}}$, multiplying by $-\triangle u_\varepsilon = -\triangle\theta_\varepsilon$ and integrating in Q_t we deduce

$$\int_{Q_t}\alpha\nabla(\ln\theta_\varepsilon - \ln\theta_{\mathcal{H}})\cdot\nabla\theta_\varepsilon dxds + \frac{k_0}{2}\|1 * \triangle u_\varepsilon(t)\|^2$$
$$= \int_{Q_t}\Big[\alpha(\ln\theta_{\mathcal{H}} - \ln\theta_{0\varepsilon})\triangle u_\varepsilon - \frac{1}{\theta_c}[G_\varepsilon(\varphi_\varepsilon) - G_\varepsilon(\varphi_0)]\triangle u_\varepsilon$$
$$-(1 * k * \triangle u_\varepsilon)\triangle u_\varepsilon - 1 * \Big(r_H - \frac{r}{\theta_{\mathcal{H}}} - \frac{r}{\theta^2_{\mathcal{H}}}u_\varepsilon\Big)\triangle u_\varepsilon\Big]dxds.$$

Hence,

$$\int_{Q_t}\frac{\alpha}{\theta_\varepsilon}|\nabla\theta_\varepsilon|^2dxds + \frac{k_0}{2}\|1 * \triangle u_\varepsilon(t)\|^2 = \int_\Omega \alpha(\ln\theta_{\mathcal{H}} - \ln\theta_{0\varepsilon})1 * \triangle u_\varepsilon(t)dx$$

$$+\int_{Q_t}\Big\{\frac{\alpha}{\theta_{\mathcal{H}}}\nabla\theta_{\mathcal{H}}\cdot\nabla\theta_\varepsilon-\frac{1}{\theta_c}[G_\varepsilon(\varphi_\varepsilon)-G_\varepsilon(\varphi_0)]\triangle u_\varepsilon$$
$$-(1*k*\triangle u_\varepsilon)\triangle u_\varepsilon-1*\Big(r_H-\frac{r}{\theta_{\mathcal{H}}}-\frac{r}{\theta_{\mathcal{H}}^2}u_\varepsilon\Big)\triangle u_\varepsilon\Big\}dxds. \tag{3.16}$$

In order to estimate the right hand side of the previous equation notice that relations (3.4), (3.5) and (3.10) imply

$$\alpha\int_\Omega(\ln\theta_{\mathcal{H}}-\ln\theta_{0\varepsilon})1*\triangle u_\varepsilon(t)dx\le\alpha\|\ln\theta_{\mathcal{H}}-\ln\theta_{0\varepsilon}\|\|1*\triangle u_\varepsilon(t)\|$$
$$\le c+\frac{k_0}{16}\|1*\triangle u_\varepsilon(t)\|^2,$$
$$\int_{Q_t}\frac{\alpha}{\theta_{\mathcal{H}}}\nabla\theta_{\mathcal{H}}\cdot\nabla\theta_\varepsilon dxds$$
$$\le\frac{\alpha}{\theta_*}\|\nabla\theta_{\mathcal{H}}\|_{L^2(Q_T)}(\|\nabla u_\varepsilon\|_{L^2(Q_T)}+\|\nabla\theta_{\mathcal{H}}\|_{L^2(Q_T)})\le c.$$

Moreover, an integration by parts yields

$$\frac{1}{\theta_c}\int_{Q_t}[G_\varepsilon(\varphi_\varepsilon)-G_\varepsilon(\varphi_0)]\triangle u_\varepsilon dxdt$$
$$=\frac{1}{\theta_c}\int_\Omega[G_\varepsilon(\varphi_\varepsilon(t))-G_\varepsilon(\varphi_0)](1*\triangle u_\varepsilon)(t)dx-\frac{1}{\theta_c}\int_{Q_t}g_\varepsilon(\varphi_\varepsilon)\varphi_{\varepsilon t}(1*\triangle u_\varepsilon)dxds,$$

so that (3.10) provides the estimate

$$\frac{1}{\theta_c}\int_{Q_t}[G_\varepsilon(\varphi_\varepsilon)-G_\varepsilon(\varphi_0)]\triangle u_\varepsilon dxds$$
$$\le\frac{1}{\theta_c}\|G_\varepsilon(\varphi_\varepsilon(t))-G_\varepsilon(\varphi_0)\|\|1*\triangle u_\varepsilon(t)\|+c\int_0^t\|\varphi_{\varepsilon t}(s)\|\|1*\triangle u_\varepsilon(s)\|ds$$
$$\le c+\frac{k_0}{16}\|1*\triangle u_\varepsilon(t)\|^2+c\int_0^t\|1*\triangle u_\varepsilon(s)\|^2ds.$$

In view of the identity (3.2) we deduce

$$\int_{Q_t}(k*1*\triangle u_\varepsilon)\triangle u_\varepsilon dxds$$
$$=\int_\Omega(k*1*\triangle u_\varepsilon)(t)(1*\triangle u_\varepsilon)(t)dx-\int_{Q_t}(k*1*\triangle u_\varepsilon)_t(1*\triangle u_\varepsilon)dxds$$
$$\le\frac{k_0}{16}\|(1*\triangle u_\varepsilon)(t)\|^2+c\|(k*1*\triangle u_\varepsilon)(t)\|^2$$

$$+\int_{Q_t}[|k(0)|(1*\triangle u_\varepsilon)^2+|(k'*1*\triangle u_\varepsilon)(1*\triangle u_\varepsilon)|]dxds.$$

Hence, owing to (3.3) we obtain

$$\int_{Q_t}(k*1*\triangle u_\varepsilon)\triangle u_\varepsilon dxds \le \frac{k_0}{16}\|(1*\triangle u_\varepsilon)(t)\|^2$$
$$+c(\|k\|^2_{L^2(0,T)}+|k(0)|+\|k'\|_{L^1(0,T)})\int_0^t\|1*\triangle u_\varepsilon(s)\|^2ds.$$

Finally, last integral in (3.16) can be estimated as

$$\int_{Q_t}1*\Big(r_H-\frac{r}{\theta_{\mathcal{H}}}-\frac{r}{\theta^2_{\mathcal{H}}}u_\varepsilon\Big)\triangle u_\varepsilon dxds$$
$$=\int_\Omega 1*\Big(r_H-\frac{r}{\theta_{\mathcal{H}}}-\frac{r}{\theta^2_{\mathcal{H}}}u_\varepsilon\Big)(t)(1*\triangle u_\varepsilon)(t)dx$$
$$-\int_{Q_t}\Big(r_H-\frac{r}{\theta_{\mathcal{H}}}-\frac{r}{\theta^2_{\mathcal{H}}}u_\varepsilon\Big)(1*\triangle u_\varepsilon)dxds$$
$$\le c\Big[\|r_H\|^2_{L^2(Q_T)}+\|r\|^2_{L^2(Q_T)}+\|r\|^2_{L^\infty(Q_T)}\|u_\varepsilon\|^2_{L^2(Q_T)}\Big]$$
$$+\frac{k_0}{16}\|(1*\triangle u_\varepsilon)(t)\|^2+\int_0^t\|1*\triangle u_\varepsilon(s)\|^2ds.$$

Substitution of the previous inequalities into (3.16) leads to

$$\int_{Q_t}\frac{\alpha}{\theta_\varepsilon}|\nabla\theta_\varepsilon|^2dxds+\frac{k_0}{4}\|(1*\triangle u_\varepsilon)(t)\|^2\le c\int_0^t\|1*\triangle u_\varepsilon(t)\|^2dt+c.$$

Therefore, Gronwall's inequality yields

$$\|1*\triangle u_\varepsilon\|^2_{L^\infty(0,T,L^2(\Omega))}\le c$$

and substituting into (3.15) we obtain

$$\|\ln\theta_\varepsilon\|^2_{L^\infty(0,T,L^2(\Omega))}\le c.$$

Hence (3.14) holds. □

The a priori estimates proved in the previous lemmas allow to pass to the limit as $\varepsilon\to 0$ and to obtain a solution (φ,θ) of problem (P).

Theorem 3.2. *If assumptions* $(H1)-(H7)$ *are satisfied, problem* (P) *admits at least a solution* (φ,θ) *such that*

$$\varphi \in L^2(0,T,H^2(\Omega))\cap H^1(0,T,L^2(\Omega))$$
$$\theta \in L^2(0,T,H^1(\Omega)),\quad \theta>0 \text{ a.e. in } Q_T$$

$$\ln\theta \quad \in \quad L^\infty(0,T,L^2(\Omega))\cap H^1(0,T,H^{-1}(\Omega)).$$

Moreover, $0\le\varphi\le 1$ *a.e. in* Q_T*.*

Proof. Inequalities (3.10), (3.12), (3.13) and (3.14), prove, up to subsequences, the following convergences

$$\varphi_\varepsilon\to\varphi \quad \text{weakly in } L^2(0,T,H^2(\Omega)) \text{ and in } H^1(0,T,L^2(\Omega)) \tag{3.17}$$

$$u_\varepsilon\to u \quad \text{weakly in } L^2(0,T,H_0^1(\Omega)) \tag{3.18}$$

$$\ln\theta_\varepsilon\to\ell \quad \text{weakly in } L^2(Q_T) \text{ and in } H^1(0,T,H^{-1}(\Omega)) \tag{3.19}$$

and in view of (3.6)

$$\theta_\varepsilon\to\theta \quad \text{weakly in } L^2(0,T,H^1(\Omega))\,. \tag{3.20}$$

Therefore, the compact embeddings $H^2(\Omega)\hookrightarrow H^1(\Omega)$, $L^2(\Omega)\hookrightarrow H^{-1}(\Omega)$ yield

$$\begin{array}{l}\varphi_\varepsilon\to\varphi \quad \text{strongly in } L^2(0,T,H^1(\Omega)),\\ \ln\theta_\varepsilon\to\ell \quad \text{strongly in } L^2(0,T,H^{-1}(\Omega)).\end{array}$$

Accordingly, ([19, p.12])

$$12L\varphi_\varepsilon^2(1-\varphi_\varepsilon)\to 12L\varphi^2(1-\varphi)=g(\varphi) \quad \text{weakly in } L^2(Q_T).$$

From the inequality

$$\begin{aligned}|g_\varepsilon(\varphi_\varepsilon)-g(\varphi)| &\le |g_\varepsilon(\varphi_\varepsilon)-g_\varepsilon(\varphi)|+|g_\varepsilon(\varphi)-g(\varphi)|\\ &\le c|\varphi_\varepsilon-\varphi|+|g_\varepsilon(\varphi)-g(\varphi)|,\end{aligned}$$

we deduce $g_\varepsilon(\varphi_\varepsilon)\to g(\varphi)$ strongly in $L^2(Q_T)$ and thanks to (3.17), (3.20) we obtain

$$\begin{array}{l}g_\varepsilon(\varphi_\varepsilon)\theta_\varepsilon\to 12L\varphi^2(1-\varphi)\theta \quad \text{weakly in } L^2(Q_T)\\ g_\varepsilon(\varphi_\varepsilon)\varphi_{\varepsilon t}\to 12L\varphi^2(1-\varphi)\varphi_t \quad \text{weakly in } L^2(Q_T).\end{array}$$

In order to prove the convergence of the logarithmic term, we observe that

$$\int_{Q_T}\theta_\varepsilon\ln\theta_\varepsilon dxdt=\int_{Q_T}(u_\varepsilon+\theta_{\mathcal H})\ln\theta_\varepsilon dxdt\to\langle\ell,u\rangle+\int_{Q_T}\theta_{\mathcal H}\ell dxdt,$$

where $\langle\cdot,\cdot\rangle$ denotes the duality pairing between $L^2(0,T,H_0^1(\Omega))$ and $L^2(0,T,H^{-1}(\Omega))$. Hence, $\ell=\ln\theta$ a.e. in Q_T (see [3, p.42]).

Finally, Proposition 2.1 proves the boundedness of φ. □

4. Uniqueness of the solution

This section is devoted to prove uniqueness of the solution to problem (P). In particular we state a theorem which shows the continuous dependence on the data.

Theorem 4.1. *Let (φ_1, θ_1), (φ_2, θ_2) be two solutions of the problem (P) with data φ_{01}, θ_{01}, $\theta_{\Gamma 1}$, r_{H1}, r_1 and $\varphi_{02}, \theta_{02}, \theta_{\Gamma 2}, r_{H2}, r_2$ satisfying $(H2)-(H7)$, then the following inequality holds*

$$\begin{aligned}
&\|\varphi_1(t)-\varphi_2(t)\|^2 + \|\nabla\varphi_1 - \nabla\varphi_2\|^2_{L^2(Q_t)} \\
&+ \int_{Q_t} (\ln\theta_1 - \ln\theta_2)(\theta_1-\theta_2)dxds + \|1 * \nabla(u_1-u_2)(t)\|^2 \\
&\le M[\|r_1-r_2\|_{L^\infty(Q_t)} + \|r_{H1}-r_{H2}\|^2_{L^2(Q_t)} + \|d_{01}-d_{02}\|^2 \\
&\quad + \|\theta_{\Gamma 1}-\theta_{\Gamma 2}\|^2_{H^{1/2}(\partial\Omega)} + \|\theta_{\Gamma 1}-\theta_{\Gamma 2}\|_{H^{1/2}(\partial\Omega)}]\,, \qquad (4.1)
\end{aligned}$$

where

$$d_{0i} = \alpha\ln\theta_{0i} - \frac{1}{\theta_c}\Big(\frac{1}{4}\varphi_{0i}^4 - \frac{2}{3}\varphi_{0i}^3 + \frac{1}{2}\varphi_{0i}^2\Big), \qquad i=1,2$$

and $M>0$ depends on $\varphi_{0i}, \theta_{0i}, \theta_{\Gamma i}, r_{Hi}, r_i$, $i=1,2$.

Proof. Let us integrate the second equation of (P) over the time interval $(0,t)$

$$\begin{aligned}
&\alpha\ln\theta - k_0 * \triangle\theta - 1 * k * \triangle\theta \qquad (4.2)\\
&= \frac{L}{\theta_c}\Big(3\varphi^4 - 8\varphi^3 + 6\varphi^2\Big) + 1 * \Big(r_H - \frac{r}{\theta^2_{\mathcal{H}}}\theta\Big) - \frac{L}{\theta_c}\Big(3\varphi_0^4 - 8\varphi_0^3 + 6\varphi_0^2\Big).
\end{aligned}$$

By letting

$\varphi = \varphi_1 - \varphi_2$, $\theta = \theta_1 - \theta_2$, $\theta_{\mathcal{H}} = \theta_{\mathcal{H}1} - \theta_{\mathcal{H}2}$, $r_H = r_{H1} - r_{H2}$, $d_0 = d_{01} - d_{02}$,

from the first equation of (P) and (4.2), we obtain

$$\begin{aligned}
\varphi_t - \kappa\triangle\varphi &= 12L[\varphi_1^2(1-\varphi_1) - \varphi_2^2(1-\varphi_2)] \\
&\quad - \frac{12L}{\theta_c}[\theta_1\varphi_1(1-\varphi_1)^2 + \theta_2\varphi_2(1-\varphi_2)^2], \qquad (4.3)
\end{aligned}$$

$$\begin{aligned}
&\alpha(\ln\theta_1 - \ln\theta_2) - k_0 * \triangle\theta - 1 * k * \triangle\theta \\
&= \frac{L}{\theta_c}\Big[3(\varphi_1^4 - \varphi_2^4) - 8(\varphi_1^3 - \varphi_2^3) + 6(\varphi_1^2 - \varphi_2^2)\Big] \\
&+ 1 * r_H - 1 * \Big(\frac{r_1}{\theta^2_{\mathcal{H}1}}\theta_1 - \frac{r_2}{\theta^2_{\mathcal{H}2}}\theta_2\Big) + d_0\,. \qquad (4.4)
\end{aligned}$$

Multiplication of (4.3) by φ and (4.4) by $u = u_1 - u_2 = \theta - \theta_{\mathcal{H}}$ provides

$$\frac{1}{2}\|\varphi(t)\|^2 + \kappa\|\nabla\varphi\|^2_{L^2(Q_t)} + \alpha\int_{Q_t}(\ln\theta_1 - \ln\theta_2)\theta dxds$$
$$+\frac{k_0}{2}\|1 * \nabla u(t)\|^2 = J_1 + J_2 + J_3 + J_4, \tag{4.5}$$

where

$$J_1 = 12L\int_{Q_t}\Big[\varphi_1^2(1-\varphi_1) - \varphi_2^2(1-\varphi_2) - \frac{\theta_{\mathcal{H}1}}{\theta_c}\varphi_1(1-\varphi_1)^2 + \frac{\theta_{\mathcal{H}2}}{\theta_c}\varphi_2(1-\varphi_2)^2\Big]\varphi dxds$$

$$J_2 = \frac{L}{\theta_c}\int_{Q_t}\Big[3(\varphi_1^4 - \varphi_2^4) - 8(\varphi_1^3 - \varphi_2^3) + 6(\varphi_1^2 - \varphi_2^2)\Big]udxds - \frac{12L}{\theta_c}\int_{Q_t}[u_1\varphi_1(1-\varphi_1)^2 - u_2\varphi_2(1-\varphi_2)^2]\varphi dxds$$

$$J_3 = \int_{Q_t}\Big[1 * k * \nabla u\cdot\nabla u - 1 * \Big(\frac{r_1}{\theta^2_{\mathcal{H}1}}u_1 - \frac{r_2}{\theta^2_{\mathcal{H}2}}u_2 + \frac{r_1}{\theta_{\mathcal{H}1}} - \frac{r_2}{\theta_{\mathcal{H}2}}\Big)u\Big]dxds$$

$$J_4 = \int_{Q_t}[(1 * r_H + d_0)u - \alpha(\ln\theta_1 - \ln\theta_2)\theta_{\mathcal{H}}]\,dxds.$$

Let us estimate each integral J_i, $i = 1, ..., 4$. Concerning J_1, we have

$$J_1 = 12L\int_{Q_t}\Big[(\varphi_1+\varphi_2)\varphi^2 - (\varphi_1^2 + \varphi_1\varphi_2 + \varphi_2^2)\varphi^2 - \frac{1}{\theta_c}\varphi_1(1-\varphi_1)^2\varphi\theta_{\mathcal{H}} - \frac{\theta_{\mathcal{H}2}}{\theta_c}(1+\varphi_1^2+\varphi_1\varphi_2+\varphi_2^2)\varphi^2 + \frac{2\theta_{\mathcal{H}2}}{\theta_c}(\varphi_1+\varphi_2)\varphi^2\Big]dxds$$
$$\leq 12L\int_{Q_t}\Big[(\varphi_1+\varphi_2)\varphi^2 - \frac{1}{\theta_c}\varphi_1(1-\varphi_1)^2\theta_{\mathcal{H}}\varphi + \frac{2\theta_{\mathcal{H}2}}{\theta_c}(\varphi_1+\varphi_2)\varphi^2\Big]dxds.$$

The boundedness of φ_1, φ_2 and inequalities (3.4), (3.5) imply

$$\begin{aligned} J_1 &\leq c\|\varphi\|^2_{L^2(Q_t)} + c\int_0^t\Big[\|\theta_{\mathcal{H}}\| + \|\varphi(s)\|\Big]\|\varphi(s)\|ds \\ &\leq c\|\varphi\|^2_{L^2(Q_t)} + c\|\theta_\Gamma\|^2_{H^{1/2}(\partial\Omega)}. \end{aligned} \tag{4.6}$$

The second integral J_2 can be written as

$$\begin{aligned} J_2 &= \frac{1}{\theta_c}\int_{Q_t}\big[-u_1\left(9\varphi_1^2 + 6\varphi_1\varphi_2 + 3\varphi_2^2 - 16\varphi_1 - 8\varphi_2 + 6\right) \\ &\quad + u_2\left(9\varphi_2^2 + 6\varphi_1\varphi_2 + 3\varphi_1^2 - 16\varphi_2 - 8\varphi_1 + 6\right)\big]\,\varphi^2 dxds. \end{aligned}$$

Hence, Proposition 2.1 yields

$$
\begin{aligned}
J_2 &\le c\int_{Q_t}(|u_1|+|u_2|)\varphi^2 dxds \qquad (4.7)\\
&\le \nu\int_0^t \|\nabla\varphi(s)\|^2 ds + c\int_0^t [\|u_1(s)\|^2_{H^1(\Omega)}+\|u_2(s)\|^2_{H^1(\Omega)}]\|\varphi(s)\|^2 ds
\end{aligned}
$$

for each $\nu > 0$.

Now we consider the third integral J_3. We obtain

$$
\begin{aligned}
J_3 &= \int_{Q_t} 1*k*\nabla u\cdot\nabla u dxds\\
&-\int_{Q_t} 1*\Big[\frac{r_1}{\theta^2_{\mathcal{H}1}}u+\frac{u_2}{\theta^2_{\mathcal{H}2}}r-\frac{(\theta_{\mathcal{H}1}+\theta_{\mathcal{H}2})r_2u_2}{\theta^2_{\mathcal{H}1}\theta^2_{\mathcal{H}2}}\theta_{\mathcal{H}}+\frac{r}{\theta_{\mathcal{H}1}}-\frac{r_2}{\theta_{\mathcal{H}1}\theta_{\mathcal{H}2}}\theta_{\mathcal{H}}\Big]udxds.
\end{aligned}
$$

An integration by parts yields

$$
\begin{aligned}
J_3 &= \int_\Omega (1*k*\nabla u)(t)\cdot(1*\nabla u)(t)dx-\int_{Q_t}\partial_t(k*1*\nabla u)(1*\nabla u)dxds\\
&-\int_\Omega 1*\left[\frac{r_1}{\theta^2_{\mathcal{H}1}}u+\frac{u_2}{\theta^2_{\mathcal{H}2}}r-\frac{(\theta_{\mathcal{H}1}+\theta_{\mathcal{H}2})r_2u_2}{\theta^2_{\mathcal{H}1}\theta^2_{\mathcal{H}2}}\theta_{\mathcal{H}}+\frac{r}{\theta_{\mathcal{H}1}}-\frac{r_2}{\theta_{\mathcal{H}1}\theta_{\mathcal{H}2}}\right](t)\\
&\qquad\qquad\times(1*u)(t)dx\\
&+\int_{Q_t} 1*\left[\frac{r_1}{\theta^2_{\mathcal{H}1}}u+\frac{u_2}{\theta^2_{\mathcal{H}2}}r-\frac{(\theta_{\mathcal{H}1}+\theta_{\mathcal{H}2})r_2u_2}{\theta^2_{\mathcal{H}1}\theta^2_{\mathcal{H}2}}\theta_{\mathcal{H}}+\frac{r}{\theta_{\mathcal{H}1}}-\frac{r_2}{\theta_{\mathcal{H}1}\theta_{\mathcal{H}2}}\theta_{\mathcal{H}}\right]\\
&\qquad\qquad\times(1*u)dxds\,.
\end{aligned}
$$

Hence, by using inequalities (3.2) and (3.4) we have

$$
\begin{aligned}
J_3 \quad \le \quad & \|1*k*\nabla u(t)\|\|1*\nabla u(t)\|+|k(0)|\|1*\nabla u\|^2_{L^2(Q_t)}\\
&+\|k'*1*\nabla u\|_{L^2(Q_t)}\|1*\nabla u\|_{L^2(Q_t)}\\
&+\frac{1}{\theta_*}\left[\|1*u(t)\|+\|1*u\|_{L^2(Q_t)}\right]\Big[\frac{1}{\theta_*}\|u_2\|_{L^2(Q_t)}\|r\|_{L^\infty(Q_t)}\\
&+\frac{2\theta^*}{\theta_*^3}\|r_2\|_{L^\infty(Q_t)}\|u_2\|_{L^2(0,t,L^4(\Omega))}\|\theta_{\mathcal{H}}\|_{L^4(\Omega)}\\
&+\|r\|_{L^2(Q_t)}+\frac{1}{\theta_*}\|r_2\|_{L^\infty(Q_t)}\|\theta_{\mathcal{H}}\|_{L^2(Q_t)}\Big]\\
&-\int_\Omega\int_0^t\frac{r_1}{\theta^2_{\mathcal{H}1}}uds(1*u)(t)dx+\int_\Omega\int_0^t\frac{r_1}{\theta^2_{\mathcal{H}1}}u(1*u)dsdx. \qquad (4.8)
\end{aligned}
$$

A further integration by parts in the last two terms and $(H7)$ provide

$$
\begin{aligned}
&-\int_\Omega\int_0^t \frac{r_1}{\theta^2_{\mathcal{H}1}} u ds (1*u)(t) dx + \int_\Omega\int_0^t \frac{r_1}{\theta^2_{\mathcal{H}1}} u(1*u) dx ds \\
&= \int_\Omega \Big\{ -\frac{r_1}{2\theta^2_{\mathcal{H}1}}[(1*u)(t)]^2 + \int_0^t \frac{r_{1t}}{\theta_{\mathcal{H}1}}(1*u) ds (1*u)(t) \\
&- \int_0^t \frac{r_{1t}}{2\theta^2_{\mathcal{H}1}}[(1*u)(s)]^2 ds \Big\} dx \\
&\leq \max\{0, -\frac{b}{2\theta_*^2}\}\|(1*u)(t)\|^2 \\
&+c\|r_{1t}\|_{L^\infty(Q_t)}\left[\|(1*u)(t)\| + \|1*u\|_{L^2(Q_t)}\right]\|1*u\|_{L^2(Q_t)}\,.
\end{aligned}
$$

Substitution into (4.8) and use of (3.5), (3.7), (3.10) yield

$$
\begin{aligned}
J_3 \leq\ & c\left[\|1*\nabla u\|^2_{L^2(Q_t)} + \|r\|^2_{L^\infty(Q_t)} + \|\theta_\Gamma\|^2_{H^{1/2}(\partial\Omega)}\right] \\
&+ \left(\nu + \frac{k_0}{2\delta}\right)\|1*\nabla u(t)\|^2
\end{aligned} \tag{4.9}
$$

for each $\nu > 0$.

Finally, last integral J_4 can be controlled by using (3.5), (3.14), as

$$
\begin{aligned}
J_4 \quad \leq \quad & \nu\|1*\nabla u(t)\|^2 + c(\|r_H\|^2_{L^2(Q_t)} + \|1*\nabla u(t)\|^2_{L^2(Q_t)} + \|d_0\|^2) \\
& +\alpha(\|\ln\theta_1\|_{L^1(0,T,L^2(\Omega))} + \|\ln\theta_2\|_{L^1(0,T,L^2(\Omega))})\|\theta_{\mathcal{H}}\| \\
\leq \quad & \nu\|1*\nabla u(t)\|^2 + c(\|r_H\|^2_{L^2(Q_t)} + \|1*\nabla u(t)\|^2_{L^2(Q_t)} \\
& +\|d_0\|^2 + \|\theta_\Gamma\|_{H^{1/2}(\partial\Omega)}).
\end{aligned} \tag{4.10}
$$

Collecting (4.6), (4.7), (4.9), (4.10) and choosing the constant ν, from (4.5) we deduce

$$
\begin{aligned}
&\frac{1}{2}\|\varphi(t)\|^2 + \frac{\kappa}{2}\|\nabla\varphi\|^2_{L^2(Q_t)} + \alpha\int_{Q_t}(\ln\theta_1 - \ln\theta_2)\theta dx ds \\
&\quad + \frac{k_0}{4}\left(1 - \frac{1}{\delta}\right)\|1*\nabla u(t)\|^2 \\
&\leq \int_0^t \left[\phi_1(s)\|\varphi(s)\|^2 + c\|1*\nabla u(s)\|^2\right] dx ds + \phi_2(t)\,,
\end{aligned}
$$

where

$$
\begin{aligned}
\phi_1(t) &= c[1 + \|u_1(t)\|^2_{H^1(\Omega)} + \|u_2(t)\|^2_{H^1(\Omega)}] \\
\phi_2(t) &= c[\|\theta_\Gamma\|^2_{H^{1/2}(\partial\Omega)} + \|r\|^2_{L^\infty(Q_t)} + \|r_H\|^2_{L^2(Q_t)} + \|d_0\|^2 + \|\theta_\Gamma\|_{H^{1/2}(\partial\Omega)}]\,.
\end{aligned}
$$

Therefore, Gronwall's inequality provides

$$\|\varphi(t)\|^2 + \|\nabla\varphi\|^2_{L^2(Q_t)} + \int_{Q_t}(\ln\theta_1 - \ln\theta_2)\theta dxds + \|1 * \nabla u(t)\|^2$$
$$\leq \int_0^t \phi_1(s)\phi_2(s)e^{\int_s^t \phi_1(\tau)d\tau}ds + \phi_2(t),$$

so that (4.1) holds. □

5. Stability

In this section we deduce some energy estimates that prove the stability of the solution to problem (P). To this aim we assume

$$\mathcal{E}(\varphi, \nabla\varphi, \theta, \nabla\tilde{u}) = \int_\Omega \Big[\frac{\kappa}{2}|\nabla\varphi|^2 + 3L\frac{\theta_{\mathcal{H}}}{\theta_c}\varphi^2\left(\varphi^2+2\right) + \alpha\theta\Big]dx$$
$$- \int_\Omega\int_0^\infty \frac{k'(s)}{2}|\nabla\tilde{u}(s)|^2 dsdx$$

which is positive definite because of the inequalities

$$\kappa, \alpha, L, \theta_c > 0 \qquad \text{and} \qquad \theta_{\mathcal{H}} \geq \theta_* > 0 \ \text{ a.e. in } \Omega\,,\ k' \leq 0 \ \text{ in } \mathbb{R}^+\,.$$

Theorem 5.1. *Let* $(H2)$ *and* $(H7)$ *be respectively replaced by the stronger conditions*

$(H2')$ $\quad r \in L^1 \cap L^\infty(\mathbb{R}^+, L^\infty(\Omega))\,,$

$(H7')$ $\quad b = \inf\{r(x,t) : (x,t) \in \Omega\times\mathbb{R}^+\} \geq -\dfrac{k_0\,\theta_*^2}{\delta\lambda_1(\Omega)}$ *for some* $\delta > 1$.

Then for all $t > 0$ *there exist* m_0, $m_1 > 0$ *such that* $\mathcal{E}(t) \leq m_0\,\mathcal{E}(0) + m_1$.

Proof. Let us multiply (2.20) by φ_t and (3.8) by u. By integrating over Ω we obtain

$$\|\varphi_t\|^2 + k_0\|\nabla u\|^2 + \int_\Omega\int_0^\infty k(s)\nabla u^t(s)ds\cdot\nabla u dx$$
$$+ \int_\Omega \Big[\kappa\nabla\varphi\cdot\nabla\varphi_t - 12L\varphi^2(1-\varphi)\varphi_t \tag{5.1}$$
$$+12L\frac{\theta_{\mathcal{H}}}{\theta_c}\varphi(1-\varphi)^2\varphi_t + \alpha\theta_t + \alpha(\ln\theta)_t\theta_{\mathcal{H}}\Big]dx = \int_\Omega \frac{r}{\theta_{\mathcal{H}}}\Big(2 - \frac{\theta}{\theta_{\mathcal{H}}}\Big)udx.$$

After an integration by parts, the term involving the kernel memory can be written as

$$\int_0^\infty k(s)\nabla u^t(s)ds\cdot\nabla u = -\int_0^\infty k'(s)\nabla\tilde{u}^t(s)ds\cdot\nabla u$$

$$= -\partial_t \int_0^\infty \frac{k'(s)}{2} |\nabla \tilde{u}^t(s)|^2 ds + \int_0^\infty \frac{k''(s)}{2} |\nabla \tilde{u}^t(s)|^2 ds.$$

Substitution into (5.1) yields

$$\|\varphi_t\|^2 + k_0 \|\nabla u\|^2 + \int_0^\infty \frac{k''(s)}{2} \|\nabla \tilde{u}^t(s)\|^2 ds + \frac{d\Phi}{dt} = \int_\Omega \frac{r}{\theta_{\mathcal{H}}} \Big(2 - \frac{\theta}{\theta_{\mathcal{H}}}\Big) u \, dx, \quad (5.2)$$

where

$$\Phi = \mathcal{E} + \int_\Omega \Big[3L\varphi^4 - 4L\Big(1 + \frac{2\theta_{\mathcal{H}}}{\theta_c}\Big)\varphi^3 + c_1\Big] dx + \int_\Omega (c_1 - \alpha\theta_{\mathcal{H}} \ln \theta) \, dx$$

and c_1 is a suitable positive constant. Now, we choose c_1 (large) and c_0 (small) such that

$$\int_\Omega \Big[3L\varphi^4 - 4L\Big(1 + \frac{2\theta_{\mathcal{H}}}{\theta_c}\Big)\varphi^3 + c_1\Big] dx \geq c_0 \int_\Omega 3L \frac{\theta_{\mathcal{H}}}{\theta_c} \varphi^2 \left(\varphi^2 + 2\right) dx \geq 0.$$

Moreover, the inequality $\ln \theta \leq \sqrt{\theta}$ and the bounds (3.4) imply

$$\int_\Omega [\alpha\theta - \alpha\theta_{\mathcal{H}} \ln \theta + c_1] dx \geq \int_\Omega [\alpha\theta - \alpha\theta^* \sqrt{\theta} + c_1] dx \geq c_0 \int_\Omega \alpha\theta \, dx > 0$$

provided that c_1 is sufficiently large and c_0 is sufficiently small. Accordingly, the functional Φ turns out to be positive and such that

$$c_0 \mathcal{E} \leq \Phi \leq c_2 \mathcal{E} + c_3 \, .$$

In the case $r \equiv 0$, from (5.2) we have $d\Phi/dt \leq 0$ and then

$$\mathcal{E}(t) \leq \frac{1}{c_0} \Phi(t) \leq \frac{1}{c_0} \Phi(0) \leq \frac{c_2}{c_0} \mathcal{E}(0) + \frac{c_3}{c_0} \, . \quad (5.3)$$

When $r \neq 0$, from $(H7')$ the right-hand side of (5.2) can be estimated by

$$\begin{aligned} \int_\Omega \frac{r(t)}{\theta_{\mathcal{H}}} \Big(2 - \frac{\theta(t)}{\theta_{\mathcal{H}}}\Big) u(t) \, dx &= \int_\Omega r(t) \Big(\frac{\theta(t)}{\theta_{\mathcal{H}}} - 1 - \frac{u^2(t)}{\theta_{\mathcal{H}}^2}\Big) dx \\ &\leq \|r(t)\|_{L^\infty(\Omega)} \Big[\int_\Omega \frac{\theta(t)}{\theta_{\mathcal{H}}} dx + |\Omega|\Big] + \max\{0, -\frac{b}{\theta_*^2}\} \|u(t)\|^2 \\ &\leq \frac{1}{\theta_*} \|r(t)\|_{L^\infty(\Omega)} \Big[\int_\Omega \theta(t) dx + \theta_* |\Omega|\Big] + \frac{k_0}{\delta} \|\nabla u(t)\|^2 \, . \end{aligned}$$

By substituting into (5.2) and using (5.3), we obtain

$$\frac{d\Phi(t)}{dt} \leq \frac{1}{\alpha\theta_*} \|r(t)\|_{L^\infty(\Omega)} \left[\mathcal{E}(t) + \alpha\theta_* |\Omega|\right] \leq \rho(t) [\Phi(t) + c] \, ,$$

where $\rho(t) = \|r(t)\|_{L^\infty(\Omega)}/c_0\alpha\theta_*$ and $c = c_0\alpha\theta_*|\Omega|$. In view of $(H2')$, $\rho \in L^1(\mathbb{R}^+)$ and by the Gronwall lemma we deduce

$$\Phi(t) \le \exp\Big(\int_\tau^t \rho(\xi)\,d\xi\Big)\Phi(\tau) + M_1\,.$$

and then

$$\Phi(t) \le M_0\Phi(0) + M_1\,,$$

where $M_0 = \int_0^\infty \rho(\xi)\,d\xi$. Hence

$$\mathcal{E}(t) \le \frac{1}{c_0}\Phi(t) \le \frac{M_0}{c_0}\Phi(0) + \frac{M_1}{c_0} \le \frac{M_0c_2}{c_0}\mathcal{E}(0) + \frac{M_0c_3}{c_0} + \frac{M_1}{c_0}\,.$$

and the thesis follows by letting

$$m_0 = \frac{M_0c_2}{c_0} \qquad \text{and} \qquad m_1 = \frac{M_0c_3}{c_0} + \frac{M_1}{c_0}\,.$$

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References

[1] S. Aizicovici and V. Barbu, *Existence and asymptotic results for a system of integro-partial differential equations*, NoDEA Nonlinear Differential Equations Appl., 3 (1996), 1–18.

[2] G. Amendola and S. Carillo, *Thermal work and minimum free energy in a heat conductor with memory,* Quart. J. Mech. Appl. Math., 57 (2004), 429–446.

[3] V. Barbu, "Nonlinear Semigroups and Differential Equations in Banach Spaces," Noordhoff International Publishing, Leiden, 1976.

[4] V. Berti, M. Fabrizio, and C. Giorgi, *Well-posedness for solid-liquid phase transitions with a fourth-order nonlinearity,* Physica D, 236 (2007), 13–21.

[5] E. Bonetti, P. Colli, and M. Fremond, *A phase field model with thermal memory governed by the entropy balance,* Math. Models Methods Appl. Sci., 13 (2003), 1565–1588.

[6] E. Bonetti, P. Colli, M. Fabrizio, and G. Gilardi, *Global solution to a singular integrodifferential system related to the entropy balance,* Nonlinear Anal. TMA, 66 (2007), 1949–1979.

[7] E. Bonetti, P. Colli, M. Fabrizio, and G. Gilardi, *Modelling and long-time behaviour for phase transitions with entropy balance and thermal memory conductivity,* Discrete Contin. Dyn. Syst. Ser. B, 6 (2006), 1001–1026.

[8] G. Bonfanti and F. Luterotti, *Global solution to a phase-field model with memory and quadratic nonlinearity*, Adv. Math. Sci. Appl., 9 (1999), 523–538.

[9] G. Caginalp, *An analysis of a phase field model of a free boundary,* Arch. Rational Mech. Anal., 92 (1986), 205–245.

[10] B.D. Coleman and M.E. Gurtin, *Equipresence and constitutive equations for rigid heat conductors,* Z. Angew. Math. Phys., 18 (1967), 199–208.

[11] M. Fabrizio, *Ginzburg-Landau equations and first and second order phase transitions,* Internat. J. Engrg. Sci., 44 (2006), 529-539.

[12] M. Fabrizio, C. Giorgi, and A. Morro, *A continuum theory for first-order phase transitions based on the balance of structure order,* Math. Methods Appl. Sci., DOI: 10.1002/mma.930

[13] A. Karma and W.J. Rappel, *Quantitative phase-field modeling of dendritic growth in two and three dimensions,* Phys. Rev. E., 57 (1998), 4323–4349.

[14] C. Kittel, "Introduction to Solid State Physics," J.Wiley & Sons, New York, 1968.

[15] C. Giorgi and G. Gentili, *Thermodynamic properties and stability for the heat flux equation with linear memory,* Quart. Appl. Math., 51 (1993), 343–362.

[16] C. Giorgi, A. Marzocchi, and V. Pata, *Uniform attractors for a non-autonomous semilinear heat equation with memory,* Quart. Appl. Math., 58 (2000), 661–683.

[17] C. Giorgi, A. Marzocchi, and V. Pata, Asymptotic behavior of a semilinear problem in heat conduction with memory, NoDEA Nonlinear Differential Equations Appl. Vol. 5 (1998), 333–354.

[18] C. Giorgi and V. Pata, *Asymptotic behavior of a nonlinear hyperbolic heat equation with memory,* NoDEA Nonlinear Differential Equations Appl., 8 (2001), 157–171.

[19] J.L. Lions, "Quelques methodes de resolution des problemes aux limites non lineaires", Dunod-Gautheir Villars, Paris, 1969.

[20] O. Penrose and P. C. Fife, *On the relation between the standard phase-field model and a "thermodynamically consistent" phase-field model,* Physica D, 69 (1993), 107–113.

[21] S.L. Wang, R. F. Sekerka, A. A. Wheeler, B. T. Murray, S. R. Coriell, R. J. Braun, and G. B. McFadden, *Thermodynamically consistent phase-field models for solidification,* Physica D, 69 (1993), 189–200.