International Journal of Pure and Applied Mathematics
Volume 58 No. 1 2010, 61-76

LONGTIME BEHAVIOR FOR OSCILLATIONS OF AN EXTENSIBLE VISCOELASTIC BEAM WITH ELASTIC EXTERNAL SUPPLY

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Abstract: This work is focused on a nonlinear equation describing the oscillations of an extensible viscoelastic beam with fixed ends, subject to distributed elastic external force. For a general axial load β, the existence of a finite/infinite set of stationary solutions and buckling occurrence are scrutinized. The exponential stability of the straight position is discussed. Finally, the related dynamical system in the history space framework is shown to possess a regular global attractor.

AMS Subject Classification: 35B40, 35B41, 37B25, 74G60, 74H40, 74K10
Key Words: extensible viscoelastic beam, Lyapunov functional, exponential stability, global attractor

1. Introduction

In this paper we analyze the asymptotic behavior of the following nonlinear dissipative evolution problem

Received: December 15, 2009

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$$\partial_{tt}u + \partial_{xxxx}u + \int_0^\infty \mu(s)\partial_{xxxx}[u(t) - u(t-s)]ds - \big(\beta + \|\partial_x u\|^2_{L^2(0,1)}\big)\partial_{xx}u$$
$$= -ku + f, \quad (1)$$

in the unknown variable $u = u(x,t) : [0,1] \times \mathbb{R} \to \mathbb{R}$, which describes the vibrations of an extensible viscoelastic beam of unitary natural length. The real function $f = f(x)$ is the lateral (static) load distribution, the term $-ku$ represents the lateral action effected by the elastic foundation and the real parameter β denotes the axial force acting in the reference configuration (positive in stretching, negative in compression). The *memory kernel* μ is a nonnegative absolutely continuous function on $\mathbb{R}^+ = (0,\infty)$ (hence, differentiable almost everywhere) such that

$$\mu'(s) + \delta\mu(s) \leq 0, \qquad \int_0^\infty \mu(s)ds = \kappa, \tag{2}$$

for some $\delta > 0$ and $\kappa > 0$. In particular, μ has an exponential decay of rate δ at infinity.

Precisely, the model is obtained by combining the pioneering ideas of Woinowsky-Krieger [23] with the theory of Drozdov and Kolmanovskii [14], i.e. taking into account the geometric nonlinearity due to the deflection, which produces an elongation of the bar, and the energy loss due to the internal dissipation of the material, which translates into a linear viscoelastic response in bending.

In our analysis, we assume that both ends of the beam are hinged; namely, for every $t \in \mathbb{R}$

$$u(0,t) = u(1,t) = \partial_{xx}u(0,t) = \partial_{xx}u(1,t) = 0. \tag{3}$$

Moreover, because of integro-differential nature of (1), the *past history* of u (which need not fulfill the equation for negative times) is assumed to be known. Hence, the initial condition reads

$$u(x,t) = u_\star(x,t), \qquad (x,t) \in [0,1] \times (-\infty, 0], \tag{4}$$

where $u_\star : [0,1] \times (-\infty,0] \to \mathbb{R}$ is a given function.

In order to apply the theory of strongly continuous semigroups, we recast the original problem as a differential system in the *history space framework*. To this end, following Dafermos [11], we introduce the *relative displacement history*

$$\eta^t(x,s) = u(x,t) - u(x,t-s),$$

so that equation (1) turns into

$$\begin{cases} \partial_{tt}u + \partial_{xxxx}u + \displaystyle\int_0^\infty \mu(s)\partial_{xxxx}\eta(s)ds - \big(\beta + \|\partial_x u\|^2_{L^2(0,1)}\big)\partial_{xx}u = -ku + f, \\ \partial_t\eta = -\partial_s\eta + \partial_t u. \end{cases} \tag{5}$$

Accordingly, the initial condition (4) becomes

$$\begin{cases} u(x,0) = u_0(x), & x \in [0,1], \\ \partial_t u(x,0) = u_1(x), & x \in [0,1], \\ \eta^0(x,s) = \eta_0(x,s), & (x,s) \in [0,1] \times \mathbb{R}^+, \end{cases} \tag{6}$$

where we set

$$u_0(x) = u_\star(x,0), \qquad u_1(x) = \partial_t u_\star(x,0), \qquad \eta_0(x,s) = u_\star(x,0) - u_\star(x,-s).$$

As far as the boundary conditions are concerned, (3), for every $t \geq 0$, translates into

$$\begin{cases} u(0,t) = u(1,t) = \partial_{xx}u(0,t) = \partial_{xx}u(1,t) = 0, \\ \eta^t(0,s) = \eta^t(1,s) = \partial_{xx}\eta^t(0,s) = \partial_{xx}\eta^t(1,s) = 0, \\ \eta^t(x,0) = \lim_{s\to 0} \eta^t(x,s) = 0. \end{cases} \tag{7}$$

It is worth noting that the static counterpart of problem (1) reduces to

$$\begin{cases} u'''' - \Big(\beta + \|\partial_x u\|^2_{L^2(0,1)}\Big)\, u'' + ku = f, \\ u(0) = u(1) = u''(0) = u''(1) = 0. \end{cases} \tag{8}$$

When $k \equiv 0$ the investigation of the solutions to (8) and their stability, in dependence on β, represents a classical *nonlinear buckling problem* (see, for instance, [2, 13, 20]). Its numerical solutions are available in [7], while their connection with industrial applications is discussed in [15]. Recently, a careful analysis of the corresponding buckled stationary states was performed in [19] for all values of β. In [9] this analysis was improved to include a source f with a general shape.

For every $k > 0$ and vanishing sources, exact solutions to (8) can be found in [6], and at a first sight, this case looks like a slight modification of previously scrutinized models where k vanishes. This is partially true. Indeed, the restoring elastic force, acting on each point of the beam, opposes the buckling phenomenon and increases the critical Euler buckling value β_c, which is no longer equal to $\sqrt{\lambda_1}$, the root square of the first eingenvalue of the ∂_{xxxx} operator, but turns out to be a *piecewise* linear function of k. When the lateral load f vanishes, the null solution is unique provided that $\beta \geq -\beta_c(k)$ (see Theorem 2), and buckles when β exceeds this critical value, as well as in the case $k = 0$.

On the contrary, for some special positive values of k, called *resonant values*, infinitely many solutions may occur.

Moreover, in the case $k \equiv 0$, if f vanishes, the exponential decay of the energy is provided when $\beta > -\lambda_1$, so that the unique null solution is exponentially stable. On the contrary, as the axial load $\beta \leq -\lambda_1$ the straight position loses stability and the beam buckles. So, a finite number of buckled solutions occurs and the global attractor coincides with the unstable trajectories connecting them.

By paralleling the results for $k = 0$, the null solution is expected to be exponentially stable, when it is unique. Quite surprisingly, it is not so. For large values of k, the energy decays with a sub-exponential rate when $-\bar{\beta} > \beta > -\beta_c$ (see Theorem 8). In particular, for any fixed $k > \lambda_1$, the positive limiting value $\bar{\beta}(k)$ is smaller than the critical value $\beta_c(k)$, and the first overlaps the latter only if $0 \leq k \leq \lambda_1$.

The motion equation (1) with $\mu = k = f = 0$ turns out to be conservative and has been considered for hinged ends in [2, 12], with particular reference to well-posedness results. Adding an external viscous damping term $\delta\partial_t u$ ($\delta > 0$) to this conservative model, stability properties of the unbuckled (trivial) states have been established in [3, 13] and, more formally, in [20]. In this case, the global dynamics of solutions for a general β has been tackled first in [18], where some regularity of the attractor is obtained provided that δ is large enough. When the stiffness of the surrounding medium is neglected ($k = 0$), the existence of a regular attractor was proved for extensible Kirchhoff beam [10], extensible viscoelastic [17] and thermo-elastic [16] beams. A similar result for an extensible elastic beam resting on a viscoelastic foundation was obtained in [6]. This strategy can be generalized to the investigation of nonlinear dissipative models which describe the vibrations of extensible thermoelastic beams [4], and to the analysis of the long term damped dynamics of extensible elastic bridges suspended by flexible and elastic cables [5]. In the last case the term $-ku$ is replaced by $-ku^+$ and it represents a restoring force due to the cables. Moreover, our approach may be adapted to the study of simply supported bridges subjected to moving vertical load [22].

The final result of this work concerns the existence of a regular global attractor for all values of the real parameter β. The main difficulty comes from the very weak dissipation exhibited by the model, entirely contributed by the memory term. So, the existence of the global attractor is stated through the existence of a Lyapunov functional and the asymptotic smoothing property of the semigroup generated by the abstract problem via a suitable decomposition first proposed in [17].

2. The Abstract Setting

In this section we will consider an abstract version of problem (5)-(7). To this aim, let $(H_0, \langle\cdot,\cdot\rangle, \|\cdot\|)$ be a real Hilbert space, and let $A : \mathcal{D}(A) \subset H_0 \to H_0$ be a strictly positive selfadjoint operator. For $\ell \in \mathbb{R}$, we introduce the scale of Hilbert spaces

$$H_\ell = \mathcal{D}(A^{\ell/4}), \qquad \langle u, v\rangle_\ell = \langle A^{\ell/4}u, A^{\ell/4}v\rangle, \qquad \|u\|_\ell = \|A^{\ell/4}u\|.$$

In particular, $H_{\ell+1} \subset H_\ell$ and the generalized Poincaré inequalities hold

$$\sqrt{\lambda_1}\,\|u\|_\ell^2 \le \|u\|_{\ell+1}^2, \quad \forall u \in H_{\ell+1}, \tag{9}$$

where $\lambda_1 > 0$ is the first eigenvalue of A.

Given μ satisfying (2), we consider the L^2-weighted spaces

$$\mathcal{M}_\ell = L^2_\mu(\mathbb{R}^+, H_{\ell+2}), \;\; \langle \eta, \xi\rangle_{\ell,\mu} = \int_0^\infty \mu(s)\langle \eta(s), \xi(s)\rangle_{\ell+2} ds, \;\; \|\eta\|^2_{\ell,\mu} = \langle \eta, \eta\rangle_{\ell,\mu}$$

along with the infinitesimal generator of the right-translation semigroup on $\mathcal{M}_0$, that is, the linear operator

$$T\eta = -D\eta, \qquad \mathcal{D}(T) = \{\eta \in \mathcal{M}_0 : D\eta \in \mathcal{M}_0, \; \eta(0) = 0\},$$

where D stands for the distributional derivative, and $\eta(0) = \lim_{s\to 0}\eta(s)$ in H_2.

Besides, we denote by $\mathcal{M}^1_\ell$ the weighted Sobolev spaces

$$\mathcal{M}^1_\ell = H^1_\mu(\mathbb{R}^+, H_{\ell+2}) = \{\eta \in \mathcal{M}_\ell : D\eta \in \mathcal{M}_\ell\}, \qquad \|\eta\|^2_{\mathcal{M}^1_\ell} = \|\eta\|^2_{\ell,\mu} + \|D\eta\|^2_{\ell,\mu}.$$

Moreover, the functional

$$\mathcal{J}(\eta) = -\int_0^\infty \mu'(s)\|\eta(s)\|_2^2 ds$$

is finite provided that $\eta \in \mathcal{D}(T)$. From the assumption (2) on μ,

$$\|\eta\|^2_{0,\mu} \le \frac{1}{\delta}\mathcal{J}(\eta). \tag{10}$$

Finally, we define the product Hilbert spaces

$$\mathcal{H}_\ell = H_{\ell+2} \times H_\ell \times \mathcal{M}_\ell.$$

For $\beta \in \mathbb{R}$ and $f \in H_0$, we investigate the evolution system on $\mathcal{H}_0$ in the unknowns $u(t) : [0,\infty) \to H_2$, $\partial_t u(t) : [0,\infty) \to H_0$ and $\eta^t : [0,\infty) \to \mathcal{M}_0$

$$\begin{cases} \partial_{tt}u + Au + \displaystyle\int_0^\infty \mu(s)A\eta(s)ds + \big(\beta + \|u\|_1^2\big)A^{1/2}u = -ku + f, \\ \partial_t\eta = T\eta + \partial_t u, \end{cases} \tag{11}$$

with initial conditions

$$(u(0), u_t(0), \eta^0) = (u_0, u_1, \eta_0) = z \in \mathcal{H}_0.$$

Remark 1. Problem (5)-(7) is just a particular case of the abstract system (11), obtained by setting $H_0 = L^2(0,1)$ and

$$A = \partial_{xxxx}, \qquad \mathcal{D}(\partial_{xxxx}) = \{w \in H^4(0,1) : w(0) = w(1) = w''(0) = w''(1) = 0\}.$$

This operator is strictly positive selfadjoint with compact inverse, and its discrete spectrum is given by $\lambda_n = n^4\pi^4$, $n \in \mathbb{N}$. Thus, $\lambda_1 = \pi^4$ is the smallest eigenvalue. Besides, the peculiar relation $(\partial_{xxxx})^{1/2} = -\partial_{xx}$ holds true, with $\mathcal{D}(-\partial_{xx}) = H^2(0,1) \cap H_0^1(0,1)$.

Besides, system (11) generates a strongly continuous semigroup (or dynamical system) $S(t)$ on $\mathcal{H}_0$ which continuously depends on the initial data: for any initial data $z \in \mathcal{H}_0$, $S(t)z$ is the unique weak solution to (11), with related (twice the) energy given by

$$\mathcal{E}(t) = \|S(t)z\|^2_{\mathcal{H}_0} = \|u(t)\|_2^2 + \|\partial_t u(t)\|^2 + \|\eta^t\|^2_{0,\mu}.$$

We omit the proof of these facts, which can be demonstrated either by means of a Galerkin procedure or with a standard fixed point method. In both cases, it is crucial to have uniform energy estimates on any finite time-interval.

3. Steady States

In the concrete problem (5)-(7), taking for simplicity $f = 0$, the stationary solutions $(u, 0, 0)$ solve the boundary value problem

$$\begin{cases} u'''' - \big(\beta + \|u'\|^2_{L^2(0,1)}\big)u'' + ku = 0, \\ u(0) = u(1) = u''(0) = u''(1) = 0. \end{cases} \tag{12}$$

Our aim is to analyze the multiplicity of such solutions.

For every $k > 0$, let

$$\mu_n(k) = \frac{k}{n^2\pi^2} + n^2\,\pi^2\,, \qquad \beta_c(k) = \min_{n\in\mathbb{N}} \mu_n(k).$$

Assuming that $n_k \in \mathbb{N}$ be such that $\mu_{n_k} = \min\limits_{n\in\mathbb{N}} \mu_n(k)$, then it satisfies

$$(n_k - 1)^2 n_k^2 \le \frac{k}{\pi^4} < n_k^2(n_k+1)^2\,.$$

As a consequence, $\beta_c(k)$ is a piecewise-linear function of k.

We consider the *resonant set*

$$\mathcal{R} = \{i^2 j^2 \pi^4 : i, j \in \mathbb{N}, i < j\}.$$

When $k \in \mathcal{R}$ there exists at least a value $\mu_j(k)$ which is not simple and $\mu_i = \mu_j$, $i \neq j$, provided that $k = i^2 j^2 \pi^4$ (resonant values). In the sequel, let $\mu_m(k)$ be the smallest value of $\{\mu_n\}_{n \in \mathbb{N}}$ which is not simple. Of course, the $\mu_n(k)$ are all simple and increasingly ordered with n whenever $k < 4\pi^4$. Given $k > 0$, let $n_\star$ be the integer-valued function given by

$$n_\star(\beta) = |\mathcal{N}_\beta|\,, \qquad \mathcal{N}_\beta = \{n \in \mathbb{N} : \beta + \mu_n(k) < 0\},$$

where $|\mathcal{N}|$ stands for the cardinality of the set $\mathcal{N}$.

In the homogeneous case, we are able to establish the exact number of stationary solutions and their explicit form. In particular, we show that there is always at least one solution, and at most a finite number of solutions, whenever the values of $\mu_n(k)$ not exceeding $-\beta$ are simple.

Theorem 2. (see [6]) *If $\beta \geq -\beta_c(k)$, then for every $k > 0$ system* (12) *has only the null solution, depicting the straight equilibrium position. Otherwise:*

— if $k \in \mathcal{R}$ and $\beta < -\mu_m(k)$, the smallest non simple eigenvalue, there are infinitely many solutions;

— if either $k \in \mathcal{R}$ and $-\mu_m(k) \leq \beta < -\beta_c(k)$, or $k \notin \mathcal{R}$ and $\beta < -\beta_c(k)$, then besides the null solution there are also $2n_\star(\beta)$ buckled solutions, namely

$$u_n^{\pm}(x) \;=\; A_n^{\pm}\, \sin(n\pi x)\,, \quad n = 1, 2, \ldots, n_\star \tag{13}$$

with

$$A_n^{\pm} = \pm\, \frac{1}{n\,\pi} \sqrt{-\,2\;[\beta + \mu_n(k)]}. \tag{14}$$

Remark 3. Assuming $k = 0$ we recover the results of [9].

When $k \notin \mathcal{R}$ the set of all stationary states is finite. Depending on the value of k and β, the solutions branch in the pairs from the unbuckled state $A_n^{\pm} = 0$ at the critical value $\beta = -\beta_c(k)$, i.e., the beam can buckle in either the positive or negative directions of the transverse displacement. These branches exist for all $\beta < -\beta_c(k)$ and $A_n^{\pm}$ are monotone increasing functions of $|\beta|$. For each n, (14) admits real (buckled) solutions $A_n^{\pm}$ if and only if $\beta < -\mu_n$. When $k < 4\pi^4$, for any β in the interval

$$-\frac{k}{(n+1)^2\,\pi^2} - (n+1)^2\,\pi^2 < \beta < -\frac{k}{n^2\,\pi^2} - n^2\,\pi^2$$

the set $\mathcal{S}_0$ of the stationary solutions contains exactly $2n_\star + 1$ stationary points: the null solution and the solutions represented by (13).

When $k \in \mathcal{R}$ the set $\mathcal{S}_0$ contains an infinite numbers of solutions and all

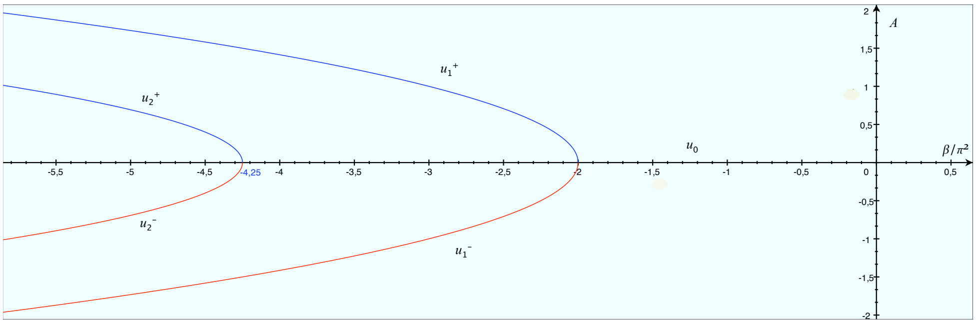

Figure 1: The bifurcation picture when $k = \pi^4$

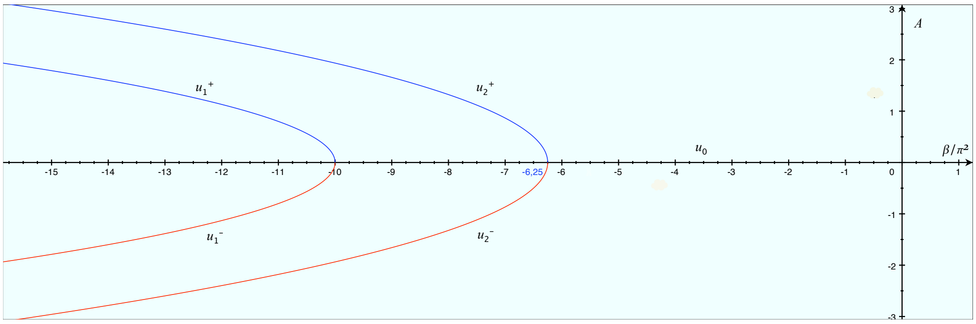

Figure 2: The bifurcation picture when $k = 9\pi^4$

the resonant values are obtained by solving

$$n^2\pi^2 + \frac{k}{n^2\pi^2} = m^2\pi^2 + \frac{k}{m^2\pi^2}\,, \qquad m, n \in \mathbb{N}\,,\ n > m\,.$$

The smallest value k is then equal to $4\pi^4$ and occurs when $n = 2$, $m = 1$. In the sequel we present a sketch of different bifurcation pictures occurring when $k = \pi^4$ (see Figure 1), and $k = 9\pi^4$ (see Figure 2).

4. The Lyapunov Functional

It is well known that the absorbing set gives a first rough estimate of the dissipativity of the system. In addition, it is the preliminary step to scrutinize its asymptotic dynamics and hence to prove the existence of a global attractor. Unfortunately, when the dissipation is very weak, a direct proof via explicit energy estimates might be very hard to find. For a quite general class of the so-called *gradient systems* it is possible to use an alternative approach appealing to the existence of a Lyapunov functional. This technique has been adopted in [17].

Definition 4. The Lyapunov functional is a function $\mathcal{L} \in C(\mathcal{H}_0, \mathbb{R})$ satisfying the following conditions:

(i) $\mathcal{L}(z) \to +\infty$ if and only if $\|z\|_{\mathcal{H}_0} \to +\infty$;

(ii) $\mathcal{L}(S(t)z)$ is nonincreasing for any $z \in \mathcal{H}_0$;

(iii) $\mathcal{L}(S(t)z) = \mathcal{L}(z)$ for all $t > 0$ implies that $z \in \mathcal{S}$.

Proposition 5. *The function*

$$\mathcal{L}(t) \;=\; \mathcal{E}(t) + \frac{1}{2}\left(\beta + \|u(t)\|_1^2\right)^2 + k\,\|u(t)\|^2 - 2\,\langle f,\, u(t)\rangle$$

is a Lyapunov functional for $S(t)$.

Proof. The continuity of $\mathcal{L}$ and assertion (i) above are clear. Using (11), we obtain quite directly the inequality

$$\frac{d}{dt}\mathcal{L}(S(t)z) \leq -\delta\|\eta^t\|_{0,\mu}^2, \tag{15}$$

which proves the monotonicity of $\mathcal{L}$ along the trajectories departing from z. Finally, if $\mathcal{L}(S(t)z)$ is constant in time, we have that $\eta^t = 0$ for all t, which implies that $u(t)$ is constant. Hence, $z = S(t)z = (u_0, 0, 0)$ for all t, that is, $z \in \mathcal{S}$. □

The existence of a Lyapunov functional ensures that

Lemma 6. *For all $t > 0$ and initial data $z \in \mathcal{H}_0$, with $\|z\|_{\mathcal{H}_0} \leq R$, there exists a positive constant C (depending on $\|f\|$ and R) such that*

$$\mathcal{E}(t) \leq C.$$

Proof. Inequality (15) ensures that

$$\mathcal{L}(t) \leq \mathcal{L}(0) \leq C_0(R, \|f\|)\ .$$

Moreover, taking into account that

$$\|u(t)\|^2 \leq \frac{1}{\lambda_1}\,\|u(t)\|_2^2 \leq \frac{1}{\lambda_1}\mathcal{E}(t) = C_1\mathcal{E}(t)$$

we obtain the estimate

$$\mathcal{L}(t) \geq \mathcal{E}(t) - 2\,\langle f\,,\, u(t)\rangle \geq \mathcal{E}(t) - \frac{1}{\varepsilon}\,\|f\|^2 - \varepsilon\,\|u(t)\|^2 \geq (1-\varepsilon C_1)\mathcal{E}(t) - \frac{1}{\varepsilon}\,\|f\|^2\,.$$

Finally, fixing $\varepsilon < \frac{1}{C_1}$, we have

$$\mathcal{E}(t) \leq \frac{1}{1-\varepsilon C_1}(\mathcal{L}(0) + \frac{1}{\varepsilon}\,\|f\|^2) \leq \frac{1}{1-\varepsilon C_1}\left(C_0(R, \|f\|) + \frac{1}{\varepsilon}\,\|f\|^2\right) = C. \quad \square$$

Moreover, defining the following functional

$$\Phi(t) \;=\; E(t) + \varepsilon \,\langle \partial_t u \,,\, u \rangle \,,$$

where $E(t) \;=\; \mathcal{E}(t) + \frac{1}{2}\left(\beta + \|u(t)\|_1^2\right)^2 + k\,\|u(t)\|^2$, we can prove

Lemma 7. *For any given $z \in \mathcal{H}_0$ and for any $t > 0$ and $\beta \in \mathbb{R}$, when ε is small enough there exist three positive constants, m_0, m_1 and m_2, independent of t such that*

$$m_0\, \mathcal{E}(t) \le \Phi(t) \le m_1\, \mathcal{E}(t) \,+\, m_2.$$

Proof. In order to prove the lower inequality we must observe that, by Young inequality

$$|\langle \partial_t u \,,\, u \rangle| \ge -\frac{1}{2}\,\|\partial_t u\|^2 - \frac{1}{2}\,\|u\|^2 \;\;;$$

hence, we obtain

$$\Phi(t) \;\ge\; \|u(t)\|_2^2 + \left(1 - \frac{\varepsilon}{2}\right)\|\partial_t u(t)\|^2 + \frac{1}{2}\left(\beta + \|u(t)\|_1^2\right)^2 \\ + \left(k \,-\, \frac{\varepsilon}{2}\right)\|u(t)\|^2 + \|\eta^t\|_{0,\mu}^2 \;.$$

If we choose ε small enough to satisfy $\varepsilon < 2$ and $\varepsilon < 2k$, then we have

$$\Phi(t) \;\ge m_0\, E(t) \ge m_0\, \mathcal{E}(t), \tag{16}$$

where $m_0 \;=\; \min\{1 - \frac{\varepsilon}{2}, 1 - \frac{\varepsilon}{2k}\}$.

The upper inequality can be obtained using the definition of Φ and applying the estimate

$$|\langle \partial_t u \,,\, u \rangle| \le \frac{1}{2}\,\|\partial_t u\|^2 + \frac{1}{2\lambda_1}\,\|u\|_2^2 \;. \tag{17}$$

First, we can write

$$\Phi(t) \le \left[1 + \frac{1}{\lambda_1}\left(k + \frac{\varepsilon}{2}\right)\right]\|u(t)\|_2^2 + \left(\frac{\varepsilon}{2} + 1\right)\|\partial_t u(t)\|^2 \\ + \frac{1}{2}\left(\beta + \|u(t)\|_1^2\right)^2 + \|\eta^t\|_{0,\mu}^2.$$

Then, by (9) and Lemma 6 we infer

$$\left(\beta + \|u\|_1^2\right) \le |\beta| + \frac{1}{\sqrt{\lambda_1}}\,C = \bar{C}, \tag{18}$$

so that we finally obtain

$$\Phi(t) \le \left[2 + \frac{1}{\lambda_1}\left(k + \frac{\varepsilon}{2}\right) + \frac{\varepsilon}{2}\right]\mathcal{E}(t) + \frac{1}{2}\bar{C}^2 \;=\; m_1\, \mathcal{E}(t) + m_2. \qquad \square$$

As a byproduct, we deduce the existence of a bounded absorbing set $\mathcal{B}_0$, chosen to be the ball of $\mathcal{H}_0$ centered at zero of radius $R_0 = 1 + \sup\{\|z\|_{\mathcal{H}_0} : \mathcal{L}(z) \le K\}$, where $K = 1 + \sup_{z_0 \in \mathcal{S}} \mathcal{L}(z_0)$. Note that R_0 can be explicitly calculated in terms of the structural quantities of our system.

5. Exponential Stability

Recalling Theorem 2, the set $\mathcal{S}_0$ of stationary solutions reduces to a singleton when

$$\beta \ge -\beta_c(k) = -\min_{n\in\mathbb{N}} \mu_n(k), \quad \mu_n(k) = \sqrt{\lambda_n}\left[1 + \frac{k}{\lambda_n}\right], \quad \lambda_n = n^4\pi^4. \tag{19}$$

It is worth noting that $\beta_c(k)$ is a piecewise-linear function of k, in that $\beta_c(k) = \mu_1(k)$ when $0 < k < \sqrt{\lambda_1}\sqrt{\lambda_2} = 4\pi^4$, and in general

$$\beta_c(k) = \mu_n(k) \quad \text{when} \quad \sqrt{\lambda_{n-1}}\sqrt{\lambda_n} < k < \sqrt{\lambda_n}\sqrt{\lambda_{n+1}}.$$

Unlike the case $k = 0$, the energy $\mathcal{E}(t)$ does not decay exponentially in the whole domain of the (β, k) plain where (19) is satisfied, but in a region which is strictly included in it.

More precisely, let

$$\bar{\beta}(k) = \begin{cases} \beta_c(k), & 0 < k \le \lambda_1, \\ 2\sqrt{k}, & k > \lambda_1, \end{cases}$$

the following result holds

Theorem 8. *When $f = 0$, the solutions to (1) decay exponentially, i.e.*

$$\mathcal{E}(t) \le c_0\,\mathcal{E}(0)\,e^{-ct}$$

with c_0 and c suitable positive constants, if and only if $\beta > -\bar{\beta}(k)$.

Using the same strategy bolstered in [5, 6], the proof of this theorem is a direct consequence of the following lemma.

Lemma 9. (see [6]) *Let $\beta \in \mathbb{R}$, $k > 0$ and*

$$Lu = Au + \beta A^{\frac{1}{2}}u + k\,u\ .$$

There exists a real function $\nu = \nu(\beta, k)$ such that

$$\langle Lu\,,\, u\rangle \ge \nu\,\|u\|_2^2\,,$$

where $\nu(\beta, k) > 0$ if and only if $\beta > -\bar{\beta}(k)$.

Remark 10. We stress that Theorem 8 holds even if $k = 0$. In this case however we have $\bar{\beta}(0) = -\beta_c(0) = -\sqrt{\lambda_1}$. Then the null solution is exponentially stable, if unique.

6. The Global Attractor

Now we state the existence of a global attractor for $S(t)$, for any $\beta \in \mathbb{R}$ and $k \geq 0$. We recall that the global attractor $\mathcal{A}$ is the unique compact subset of $\mathcal{H}_0$ which is at the same time, *fully invariant*, i.e. $S(t)\mathcal{A} = \mathcal{A}$, for every $t \geq 0$ and *attracting*, i.e.

$$\lim_{t\to\infty} \boldsymbol{\delta}(S(t)\mathcal{B}, \mathcal{A}) \to 0,$$

for every bounded set $\mathcal{B} \subset \mathcal{H}_0$, where $\boldsymbol{\delta}$ stands for the Hausdorff semidistance in $\mathcal{H}_0$ (see [1], [18], [21]).

We shall prove the following

Theorem 11. *The semigroup $S(t)$ on $\mathcal{H}_0$ possesses a connected global attractor $\mathcal{A}$ bounded in $\mathcal{H}_2$, whose third component is included in $\mathcal{D}(T)$, bounded in $\mathcal{M}_0^1$ and pointwise bounded in H_4. Moreover, $\mathcal{A}$ coincides with the unstable manifold of the set $\mathcal{S}$ of the stationary points of $S(t)$, namely,*

$$\mathcal{A} = \left\{ \widetilde{z}\,(0) : \begin{array}{c} \widetilde{z} \text{ is a complete (bounded) trajectory of } S(t) : \\ \lim_{t\to\infty} \|\widetilde{z}(-t) - \mathcal{S}\|_{\mathcal{H}_0} = 0 \end{array} \right\}.$$

The set $\mathcal{S}$ of all stationary solutions consists of the vectors of the form $(u, 0, 0)$, where u is a (weak) solution to the equation

$$Au + \left(\beta + \|u\|_1^2\right) A^{1/2} u + ku = f.$$

It is then apparent that $\mathcal{S}$ is bounded in $\mathcal{H}_0$. If $\mathcal{S}$ is finite, then

$$\mathcal{A} = \left\{ \tilde{z}(0) : \lim_{t\to\infty} \|\tilde{z}(-t) - z_1\|_{\mathcal{H}_0} = \lim_{t\to\infty} \|\tilde{z}(t) - z_2\|_{\mathcal{H}_0} = 0 \right\}, \tag{20}$$

for some $z_1, z_2 \in \mathcal{S}$.

Remark 12. When $f = 0$, $k \geq 0$ and $\beta \geq -\beta_c(k)$, then $\mathcal{A} = \mathcal{S} = \{(0,0,0)\}$. If $\beta < -\beta_c(k)$, then $\mathcal{S} = \mathcal{S}_0$ may be finite or infinite, according to Theorem 2. In the former case, (20) applies.

The existence of a Lyapunov functional, along with the fact that $\mathcal{S}$ is a bounded set, allow us prove the existence of the attractor exploiting a general result from [8], tailored for our particular case.

Lemma 13. (see [8]) *Assume that, for every $R > 0$, there exist a positive function ψ_R vanishing at infinity and a compact set $\mathcal{K}_R \subset \mathcal{H}_0$ such that the semigroup $S(t)$ can be split into the sum $L(t) + K(t)$, where the one-parameter operators $L(t)$ and $K(t)$ fulfill*

$$\|L(t)z\|_{\mathcal{H}_0} \leq \psi_R(t) \qquad \text{and} \qquad K(t)z \in \mathcal{K}_R,$$

whenever $\|z\|_{\mathcal{H}_0} \le R$ *and* $t \ge 0$. *Then,* $S(t)$ *possesses a connected global attractor* $\mathcal{A}$*, which consists of the unstable manifold of the set* $\mathcal{S}$.

The proof of Theorem 11 will be carried out be showing a suitable asymptotic compactness property of the semigroup, obtained exploiting a particular decomposition of $S(t)$ devised in [17].

By the interpolation inequality

$$\|u\|_1^2 \le \|u\|\|u\|_2,$$

it is clear that, provided that $\gamma > 0$ is large enough,

$$\frac{1}{2}\|u\|_2^2 \le \|u\|_2^2 + \beta\|u\|_1^2 + \gamma\|u\|^2 \le m\|u\|_2^2, \tag{21}$$

for some $m = m(\beta,\gamma) \ge 1$. Now, choosing $\gamma = \alpha + k$, where $k > 0$ is a fixed value, we assume α large enough so that (21) holds true. Let $R > 0$ be fixed and $\|z\|_{\mathcal{H}_0} \le R$. Paralleling the procedure given in [17], we decompose the solution $S(t)z$ into the sum

$$S(t)z = L(t)z + K(t)z,$$

where

$$L(t)z = (v(t), \partial_t v(t), \xi^t) \qquad \text{and} \qquad K(t)z = (w(t), \partial_t w(t), \zeta^t)$$

solve the systems

$$\begin{cases} \partial_{tt} v + Av + \int_0^\infty \mu(s)A\xi(s)ds + (\beta + \|u\|_1^2)A^{1/2}v + \alpha v + kv = 0, \\ \partial_t \xi = T\xi + \partial_t v, \\ (v(0), \partial_t v(0), \xi^0) = z, \end{cases} \tag{22}$$

and

$$\begin{cases} \partial_{tt} w + Aw + \int_0^\infty \mu(s)A\zeta(s)ds + (\beta + \|u\|_1^2)A^{1/2}w - \alpha v + kw = f, \\ \partial_t \zeta = T\zeta + \partial_t w, \\ (w(0), \partial_t w(0), \zeta^0) = 0. \end{cases} \tag{23}$$

Then, Theorem 11 is proved as a consequence of the following lemmas.

Lemma 14. *There is* $\omega = \omega(R) > 0$ *such that*

$$\|L(t)z\|_{\mathcal{H}_0} \le Ce^{-\omega t}.$$

It shows the exponential decay of $L(t)z$ by means of a dissipation integral (see Lemma 5.2 of [17]).

Lemma 15. *The estimate*

$$\|K(t)z\|_{\mathcal{H}_2} \le C$$

holds for every $t \ge 0$.

It shows the asymptotic smoothing property of $K(t)$ in a more regular space, for initial data bounded by R (see Lemma 6.3 of [17]).

The proof of these two lemmas can be done following the guidelines of [17], remembering that in definition of the coefficient α in the functional Φ_0 (see [17] p. 726) also the elastic coefficient k must be taken into account.

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