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Buckling and longterm dynamics of a nonlinear model for the extensible beam

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ABSTRACT

This work is focused on the longtime behavior of a nonlinear evolution problem describing the vibrations of an extensible elastic homogeneous beam resting on a viscoelastic foundation with stiffness $k > 0$ and positive damping constant. Buckling of solutions occurs as the axial load exceeds the first critical value, β_c, which turns out to increase piecewise-linearly with k. Under hinged boundary conditions and for a general axial load P, the existence of a global attractor, along with its characterization, is proved by exploiting a previous result on the extensible viscoelastic beam. As $P \leq \beta_c$, the stability of the straight position is shown for all values of k. But, unlike the case with null stiffness, the exponential decay of the related energy is proved if $P < \bar{\beta}(k)$, where $\bar{\beta}(k) \leq \beta_c(k)$ and the equality holds only for small values of k.

1. Introduction

1.1. Model equation

In this article, we investigate the longtime behavior of the following evolution problem

$$\begin{cases} \partial_{tt} u + \partial_{xxxx} u - \left(\beta + \int_0^1 |\partial_\xi u(\xi, \cdot)|^2 \mathrm{d}\xi\right) \partial_{xx} u = -ku - \delta \partial_t u + f, \\ u(0,t) = u(1,t) = \partial_{xx} u(0,t) = \partial_{xx} u(1,t) = 0, \\ u(x,0) = u_0(x), \qquad \partial_t u(x,0) = u_1(x), \end{cases} \tag{1.1}$$

in the unknown variable $u = u(x,t) : [0,1] \times \mathbb{R}^+ \to \mathbb{R}, \mathbb{R}^+ = [0,\infty)$, which represents the vertical deflection of the beam. For every $x \in [0,1]$, u_0, u_1 are assigned data. The real function $f = f(x)$ is the (given) lateral static load distribution and $-ku - \delta \partial_t u$ represents the (uniform) lateral action effected by the medium surrounding the beam. Finally, the parameter $\beta \in \mathbb{R}$ accounts for the axial force acting in the reference configuration: $\beta > 0$ when the beam is stretched, $\beta < 0$ when the beam is compressed. Usually, the axial load is referred as $P = -\beta$.

The solutions to problem (1.1) describe the transversal vibrations (in dimensionless variables) of an extensible elastic beam, which is assumed to have hinged ends and to rest on a viscoelastic foundation with stiffness $k > 0$ and damping constant $\delta > 0$. The geometric nonlinearity which is involved accounts for midplane stretching due to the elongation of the bar. A simplified version of this beam model has been adopted to study the vibration of railway track structures resting on a viscoelastic soil (see [1]). There, the elastic and damping properties of the rail bed are accounted for by continuously distributed or closely spaced spring–damper units.

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In recent years, an increasing attention was payed on the analysis of vibrations and post-buckling dynamics of nonlinear beam models, especially in connection with industrial applications. For a detailed overview, we refer the reader to [2] and references therein. Nowadays, the study of this subject has become of particular relevance in the analysis of micromachined beams [3,4] and microbridges [5].

It is worth noting that the static counterpart of problem (1.1) reduces to

$$\begin{cases} u'''' - \left(\beta + \int_0^1 |\partial_\xi u(\xi,\cdot)|^2 \, d\xi\right) u'' + ku = f, \\ u(0) = u(1) = u''(0) = u''(1) = 0. \end{cases} \tag{1.2}$$

Obviously, these steady-state equations does not change in connection with dynamical models accounting for any kind of additional damping, due to structural and/or external mechanical dissipation. When $k \equiv 0$ the investigation of the solutions to (1.2) and their stability, in dependence on β, represents a classical *nonlinear buckling problem* in the structural mechanics literature (see, for instance, [6–8]) which traces back to the pioneer article by Woinowsky-Krieger [9]. Numerical solutions for this problem are available in the literature (see, for instance, [10]). Recently, a careful analysis of the corresponding buckled stationary states and their stability properties was performed in [11] for all values of β. In [12], this analysis was improved to include a more general nonlinear term and a source f with a general shape. As far as we know, no similar analysis of (1.2) for the case $k > 0$ is present in the literature.

Neglecting the stiffness of the surrounding medium, $k \equiv 0$, the global dynamics of solutions to problem (1.1) has been first tackled by Hale [13], who proved the existence of a global attractor for a general β, relying on the existence of a suitable Lyapunov functional. The corresponding problem for an extensible *viscoelastic* beam has been addressed in [14], when $\delta > 0$, and in [15], when $\delta \equiv 0$. In spite of the difficulty which is represented by the geometric nonlinearity, in all these articles the existence of the global attractor, along with its optimal regularity, is obtained for a general value of β by means of an abstract operator setting. The analysis of the bending motion for an extensible *thermoelastic beam* is even more tangled since the dissipation is entirely contributed by the heat equation, where the Fourier heat conduction law is assumed. Neverthelesss, the existence of a regular global attractor can be shown even including into (1.1) the rotatory inertia term [16].

A common feature of previously quoted results is the following: if $f = 0$ the exponential decay of the energy is provided when $\beta > -\beta_c$, so that the unique null solution is exponentially stable if $P < \beta_c$. On the contrary, as the axial load P exceeds β_c the straight position loses stability and the beam buckles. So, when $\beta < -\beta_c$ a finite number of buckled solutions occurs and the global (exponential) attractor coincides with the unstable trajectories connecting them. For a general time-independent source term f, the number of buckled solutions may be infinite and the attractor coincides with the unstable set of the stationary points. The positive critical value β_c is named *Euler buckling load* and, in the purely mechanical case, it is equal to the square root of the first eingenvalue of the ∂_{xxxx} operator (which is referred as λ_1 in the sequel). In the thermoelastic case, because of the thermal expansion, the mean axial temperature of the beam also affects the value of β_c (see [16]).

1.2. Outline of the article

At a first sight, problem (1.1) with $k > 0$ looks like a slight modification of previously scrutinized models, where k vanishes. This is partially true. In particular, we remark that the restoring elastic force, acting on each point of the beam, opposes the buckling phenomenon. So, the Euler buckling limit β_c is no longer equal to $\sqrt{\lambda_1}$, but now turns into an increasing piecewise-linear function of k. When the lateral load f vanishes, the null solution is unique provided that $\beta > -\beta_c(k)$, and buckles when β exceeds this critical value. In general, as well as in the case $k = 0$, the set of buckled solutions is finite, but for some special positive values of k, called *resonant values*, infinitely many solutions may occur (see Theorem 1).

By paralleling the results for $k = 0$, the null solution is expected to be exponentially stable, when it is unique. Quite surprisingly, it is not so. For large values of k, the energy decays with a sub-exponential rate when $-\bar{\beta} > \beta > -\beta_c$ (see Theorem 6). In particular, for any fixed $k > \lambda_1$, the positive limiting value $\bar{\beta}(k)$ is smaller than the critical value $\beta_c(k)$, and the former overlaps the latter only if $0 \le k \le \lambda_1$. A picture of these functions, as k runs the positive axis, is given in Fig. 2.

The plan of the article is as follows. In Section 2, we discuss the general functional framework of (1.1) and exact solutions for the stationary post-buckling problem are presented for all $k > 0$, when $f = 0$. After formulating an abstract version of the dynamical problem, the existence of an absorbing set is addressed in Section 3. Some preliminary estimates and the exponential stability result are established in Section 4. The main result concerning the existence of a regular global attractor is stated in Section 5, where the asymptotic smoothing property of the semigroup generated by the abstract problem is proved by a suitable decomposition first proposed in [15].

Although we assume here that both ends of the beam are hinged, different boundary conditions for u are also physically significant, such as when both ends are clamped, or one end hinged and the other one clamped. On the contrary, the so-called cantilever boundary condition (one end clamped and the other one free) is not covered because it is pointedly inconsistent with the extensibility assumption of the model. Nevertheless, the hinged case we consider here is very special. Indeed, other boundary conditions lead to a completely different analysis that must take into consideration very special estimates for the complementary traces on the boundary, and only weaker forms of the regularity properties of solutions remain valid (see, for instance, [17]).

It is worth noting that several articles (see, for instance, [10,18,19]) are devoted to approximations as well as numerical simulations in the modelling of the deformations of extensible beams on elastic supports. In this connection, our article which exhibits exact solutions is of interest in order to fit computer applications. In particular, our model can be useful for engineering applications involving simply supported bridges subjected to moving vertical loads. For instance, it may be adapted to the study of lively footbridges [20,21].

Finally, we remark that our analysis is carried over an abstract version of the original problem which is independent of the space dimension, so that it could be extended to scrutinize shear deformations in plate models. The techniques of this work apply to plate models as well, without substantial changes.

In addition, our strategy can be generalized to the investigation of nonlinear dissipative models which describe the vibrations of extensible viscoelastic beams where the dissipative term derives from the internal viscoelastic dissipation (memory). Moreover, we are going to scrutinize the longterm damped dynamics of extensible elastic bridges suspended by flexible and elastic cables. In this model, the term $-ku$ is replaced by $-ku^+$ and it represents a restoring force due to the cables, which is different from zero only when they are being stretched.

2. Stationary solutions

Our aim is to analyze the multiplicity of solutions to the boundary value problem (1.2). Letting $L^2(0,1)$, the Hilbert space of square summable functions on $(0,1)$, the domain of the differential operator ∂_{xxxx} appearing in (1.2) is

$$\mathcal{D}(\partial_{xxxx}) = \{w \in H^4(0,1) : w(0) = w(1) = w''(0) = w''(1) = 0\}.$$

This operator is strictly positive selfadjoint with compact inverse, and its discrete spectrum is given by $\lambda_n = n^4\pi^4$, $n \in \mathbb{N}$. Thus, $\lambda_1 = \pi^4$ is the smallest eigenvalue. Besides, the following relation holds true

$$(\partial_{xxxx})^{1/2} = -\partial_{xx}, \qquad \mathcal{D}(-\partial_{xx}) = H^2(0,1) \cap H^1_0(0,1). \tag{2.1}$$

For every $k > 0$, let

$$\mu_n(k) = \frac{k}{n^2\pi^2} + n^2\pi^2, \qquad \beta_c(k) = \min_{n\in\mathbb{N}} \mu_n(k).$$

Assuming that $n_k \in \mathbb{N}$ be such that $\mu_{n_k} = \min_{n\in\mathbb{N}} \mu_n(k)$, then it satisfies

$$(n_k - 1)^2 n_k^2 \leq \frac{k}{\pi^4} < n_k^2(n_k+1)^2.$$

As a consequence, $\beta_c(k)$ is a piecewise-linear function of k (see Fig. 2). The set

$$\mathcal{R} = \{i^2 j^2 \pi^4 : i, j \in \mathbb{N}, i < j\}$$

is referred as the *resonant set*: when $k \in \mathcal{R}$ there exists at least a value $\mu_j(k)$, which is not simple. Indeed, $\mu_i = \mu_j$, $i \neq j$, provided that $k = i^2 j^2 \pi^4$ (resonant values). In the sequel, let $\mu_m(k)$ be the smallest value of $\{\mu_n\}_{n\in\mathbb{N}}$, which is not simple. Of course, the $\mu_n(k)$ are all simple and increasingly ordered with n whenever $k < 4\pi^4$. Given $k > 0$, for later convenience let $n_\star$ be the integer-valued function given by

$$n_\star(\beta) = |\mathcal{N}_\beta|, \qquad \mathcal{N}_\beta = \{n \in \mathbb{N} : \beta + \mu_n(k) < 0\},$$

where $|\mathcal{N}|$ stands for the cardinality of the set $\mathcal{N}$.

In the homogeneous case, we are able to establish the exact number of stationary solutions of (1.2) and their explicit form. In particular, we will show that there is always at least one solution, and at most a finite number of solutions, whenever the values of $\mu_n(k)$ not exceeding $-\beta$ are simple.

Theorem 1. *If $\beta \geq -\beta_c(k)$, then for every $k > 0$ system (1.2) with $f = 0$ has the null solution, corresponding to the straight equilibrium position. Otherwise:*

- *if $k \in \mathcal{R}$ and $\beta < -\mu_m(k)$, the smallest non simple eigenvalue, there are infinitely many solutions;*
- *whether $k \in \mathcal{R}$ and $-\mu_m(k) \leq \beta < -\beta_c(k)$, or $k \notin \mathcal{R}$ and $\beta < -\beta_c(k)$, then besides the null solution there are also $2n_\star(\beta)$ buckled solutions, namely*

$$u_n^\pm(x) = A_n^\pm \sin(n\pi x), \quad n = 1, 2, \ldots, n_\star \tag{2.2}$$

with

$$A_n^\pm = \pm\frac{1}{n\pi}\sqrt{-2\left[\beta + \mu_n(k)\right]}. \tag{2.3}$$

Proof. Clearly, $u = 0$ is a solution to (1.2) in the homogeneous case for all k and β. To find a nontrivial solution u, we put $h = \beta + \int_0^1 |u'(\xi)|^2 \, d\xi$, so that u solves the differential equation

$$u'''' - hu'' + ku = 0, \quad h \in \mathbb{R}, \ k > 0.$$

Letting $\lambda^2 = \chi$, the characteristic equation

$$\lambda^4 - h\lambda^2 + k = 0$$

admits solutions in the form

$$\chi_{1,2} = \frac{h \pm \sqrt{h^2 - 4k}}{2}. \tag{2.4}$$

As a consequence, taking into account the hinged boundary conditions, we obtain

- if $h \geq 2\sqrt{k}$, then $\chi_{1,2} \in \mathbb{R}^+$ and all corresponding values of λ are real, hence $u \equiv 0$;
- if $h \leq -2\sqrt{k}$, then $\chi_{1,2} \in \mathbb{R}^-$ and $\lambda_{1,2} = \pm\sqrt{|\chi_1|}\mathrm{i}$, $\lambda_{3,4} = \pm\sqrt{|\chi_2|}\mathrm{i}$; hence

$$u = 2\mathrm{i}a \sin \omega_1 x + 2\mathrm{i}b \sin \omega_2 x,$$

where a and b are suitable constants, while

$$\omega_1 = \sqrt{|\chi_1|} = n\pi, \qquad \omega_2 = \sqrt{|\chi_2|} = \ell\pi, \quad n, \ell \in \mathbb{N}. \tag{2.5}$$

Now, letting $2\,a = -\mathrm{i}\tilde{A}$ and $2\,b = -\mathrm{i}\tilde{B}$, we can write the solution into the form

$$u = \tilde{A} \sin n\pi x + \tilde{B} \sin \ell\pi x. \tag{2.6}$$

Moreover, from (2.4) and (2.5) we obtain

$$\frac{1}{2}\left(-h - \sqrt{h^2 - 4k}\right) = n^2\pi^2 \Rightarrow h = -\mu_n(k) = \beta + \int_0^1 \left|u'(\xi)\right|^2 \mathrm{d}\xi \tag{2.7}$$

$$\frac{1}{2}\left(-h + \sqrt{h^2 - 4k}\right) = \ell^2\pi^2 \Rightarrow h = -\mu_\ell(k) = \beta + \int_0^1 \left|u'(\xi)\right|^2 \mathrm{d}\xi \tag{2.8}$$

from which it follows $\mu_\ell(k) = \mu_n(k)$. In order to represent $\int_0^1 \left|u'(\xi)\right|^2 \mathrm{d}\xi$ in explicit form, we are lead to consider two occurrences.

- Let $k \notin \mathcal{R}$. In this case, from (2.7) and (2.8) it follows $\ell = n$ and, by (2.6), the following equality holds

$$\int_0^1 \left|u'(\xi)\right|^2 \mathrm{d}\xi = \frac{1}{2}\left(\tilde{A} + \tilde{B}\right)^2 n^2\pi^2 = -\beta - \mu_n(k),$$

so that letting $A = \tilde{A} + \tilde{B}$, (2.3) can be easily obtained. Of course, such nontrivial solutions exist if and only if $-\beta > \beta_c(k) = \min_{n\in\mathbb{N}} \mu_n(k)$.

- Let $k \in \mathcal{R}$. Then, $\mu_\ell(k) = \mu_n(k)$ for some $\ell < n$. In this case, from (2.6) we obtain the equality

$$\int_0^1 \left|u'(\xi)\right|^2 \mathrm{d}\xi = \frac{1}{2}\tilde{A}^2 n^2\pi^2 + \frac{1}{2}\tilde{B}^2\ell^2\pi^2, \quad \ell \neq n,$$

and, (2.7) and (2.8) cannot uniquely determine the values of $\tilde{A}$ and $\tilde{B}$. Accordingly, (2.6) represents infinitely many solutions provided that $-\beta > \mu_m(k)$. □

Remark 1. Assuming $k = 0$, we recover the results of [11] and [12].

When $k \notin \mathcal{R}$, the set of all stationary states is finite and will be denoted by $\mathcal{S}_0$. Depending on the values of k and β, the pairs of solutions branch from the unbuckled state $A_n^\pm = 0$ at the critical value $\beta = -\beta_c(k)$, i.e., the beam can buckle in either positive or negative directions of the transverse displacement. These branches exist for all $\beta < -\beta_c(k)$ and $A_n^\pm$ are monotone increasing functions of $|\beta|$. For each n, (2.3) admits real (buckled) solutions $A_n^\pm$ if and only if $\beta < -\mu_n$. When $k < 4\pi^4$, for any β in the interval

$$-\frac{k}{(n+1)^2\pi^2} - (n+1)^2\pi^2 < \beta < -\frac{k}{n^2\pi^2} - n^2\pi^2,$$

the set $\mathcal{S}_0$ contains exactly $2n_\star + 1$ stationary points: the null solution and the pairs of solutions represented by (2.2). These properties are sketched in Fig. 1 (see also [11]).

3. Absorbing set

We will consider an abstract version of problem (1.1). To this aim, let H_0 be a real Hilbert space, whose inner product and norm are denoted by $\langle\cdot,\cdot\rangle$ and $\|\cdot\|$, respectively. Let $A : \mathcal{D}(A) \Subset H_0 \to H_0$ be a strictly positive selfadjoint operator. We denote by $\lambda_1 > 0$, the first eigenvalue of A. For $\ell \in \mathbb{R}$, we introduce the scale of Hilbert spaces

$$H_\ell = \mathcal{D}(A^{\ell/4}), \qquad \langle u, v\rangle_\ell = \langle A^{\ell/4}u, A^{\ell/4}v\rangle, \qquad \|u\|_\ell = \|A^{\ell/4}u\|.$$

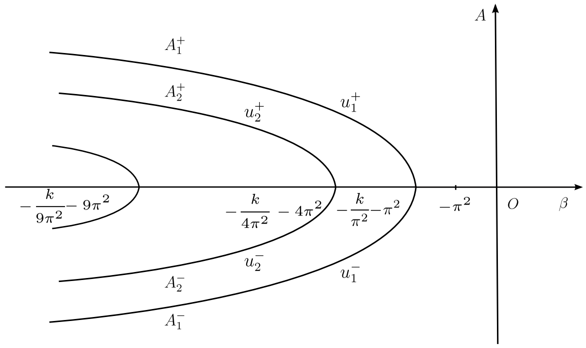

Fig. 1. A sketch of the nonlinear static response of the beam when $k < 4\pi^4$.

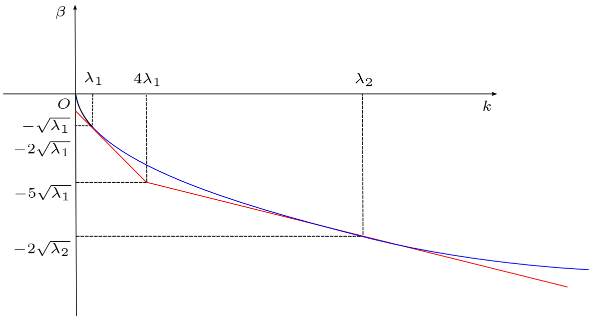

Fig. 2. A picture of the functions $-\bar{\beta}(k)$ and $-\beta_c(k)$. (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

In particular, $H_{\ell+1} \Subset H_\ell$ and the following scale of Poincaré inequalities holds

$$\sqrt{\lambda_1}\|u\|_\ell^2 \le \|u\|_{\ell+1}^2. \tag{3.1}$$

Finally, we define the product Hilbert spaces

$$\mathcal{H}_\ell = H_{\ell+2} \times H_\ell.$$

3.1. Abstract problem

For $\beta \in \mathbb{R}$ and $f \in H_0$, we investigate the following evolution equation on $\mathcal{H}_0$

$$\partial_{tt}u + Au + \left(\beta + \|u\|_1^2\right)A^{1/2}u + \delta\partial_t u + ku = f, \tag{3.2}$$

in the unknowns $u(t) : [0, \infty) \to H_2$ and $\partial_t u(t) : [0, \infty) \to H_0$, with initial conditions

$$(u(0), \partial_t u(0)) = (u_0, u_1) = z_0 \in \mathcal{H}_0.$$

Problem (1.1) is just a particular case of the abstract system (3.2), obtained by setting $H_0 = L^2(0, 1)$ and A the realization of ∂_{xxxx} in H_0.

Eq. (3.2) generates a strongly continuous semigroup (or dynamical system) $S(t)$ on $\mathcal{H}_0$: for any initial data $z_0 \in \mathcal{H}_0$, $S(t)z_0$ is the unique weak solution to (3.2), with related (twice the) energy given by

$$\mathcal{E}(t) = \|S(t)z_0\|_{\mathcal{H}_0}^2 = \|u(t)\|_2^2 + \|\partial_t u(t)\|^2.$$

Besides, $S(t)$ continuously depends on the initial data. We omit the proof of these facts, which can be demonstrated by means of a Galerkin procedure (e.g., following the lines of [14]). The crucial point in applying this technique is to have uniform energy estimates on any finite time-interval. As it will be apparent, these estimates are easily implied by the uniform inequalities on $(0, +\infty)$ established in the subsequent sessions.

Now, we prove the existence of the so-called absorbing set for the flow generated by problem (3.2), that is, a bounded set into which every orbit eventually enters. Such a set is defined as follows:

Definition 1. Let $B(0, R)$ be the open ball with center 0 and radius $R > 0$ in $\mathcal{H}_0$. A bounded set $\mathcal{B}_{\mathcal{H}_0} \subset \mathcal{H}_0$ is called an *absorbing set* for the semigroup $S(t)$ if, for any $R > 0$ and any initial value $z_0 \in B(0, R)$, there exists $t_0(R) > 0$ such that

$$z(t) \in \mathcal{B}_{\mathcal{H}_0} \quad \forall\, t \ge t_0,$$

where $z(t) = S(t)z_0$ is the solution starting from z_0.

The main result of this section follows from two lemmas, involving the functionals $\mathcal{L}(t)$ and $\Phi(t)$ defined as

$$\mathcal{L}(t) = \mathcal{E}(t) + \frac{1}{2}\left(\beta + \|u(t)\|_1^2\right)^2 + k\,\|u(t)\|^2 \geq \mathcal{E}(t) \geq 0, \tag{3.3}$$

$$\Phi(t) = \mathcal{L}(t) + \varepsilon\,\langle \partial_t u, u\rangle\,. \tag{3.4}$$

Lemma 2. *For all $t > 0$ and $z_0 \in \mathcal{H}_0$ with $\|z_0\|_{\mathcal{H}_0} \leq R$, there exists a positive constant C (depending on $\|f\|$ and R) such that*

$$\mathcal{E}(t) \leq C. \tag{3.5}$$

Proof. If we consider the functional

$$\mathcal{F}(t) = \mathcal{L}(t) - 2\,\langle f, u(t)\rangle\,,$$

from the energy identity

$$\frac{\mathrm{d}\mathcal{E}}{\mathrm{d}t} = -2\left\langle\left(\beta + \|u\|_1^2\right)A^{\frac{1}{2}}u + ku + \delta\,\partial_t u - f, \partial_t u\right\rangle$$

it easily follows the decreasing monotonicity of $\mathcal{F}$

$$\begin{aligned}\frac{\mathrm{d}\mathcal{F}}{\mathrm{d}t} &= \frac{\mathrm{d}\mathcal{E}}{\mathrm{d}t} + 2\left(\beta + \|u\|_1^2\right)\left\langle A^{\frac{1}{4}}u, A^{\frac{1}{4}}\partial_t u\right\rangle + 2k\,\langle u, \partial_t u\rangle - 2\,\langle f, \partial_t u\rangle \\ &= -2\delta\,\|\partial_t u\|^2 \leq 0\end{aligned}$$

and then

$$\mathcal{F}(t) \leq \mathcal{F}(0) \leq C_0(R, \|f\|).$$

Taking into account that

$$\|u(t)\|^2 \leq \frac{1}{\lambda_1}\,\|u(t)\|_2^2 \leq \frac{1}{\lambda_1}\mathcal{E}(t) = C_1\mathcal{E}(t),$$

we obtain the estimate

$$\mathcal{F}(t) \geq \mathcal{E}(t) - 2\,\langle f, u(t)\rangle \geq \mathcal{E}(t) - \frac{1}{\varepsilon}\,\|f\|^2 - \varepsilon\,\|u(t)\|^2 \geq (1 - \varepsilon C_1)\mathcal{E}(t) - \frac{1}{\varepsilon}\,\|f\|^2\,.$$

Finally, fixing $\varepsilon < \frac{1}{C_1}$, we have

$$\mathcal{E}(t) \leq \frac{1}{1 - \varepsilon C_1}\left(\mathcal{F}(0) + \frac{1}{\varepsilon}\,\|f\|^2\right) \leq \frac{1}{1 - \varepsilon C_1}\left(C_0(R, \|f\|) + \frac{1}{\varepsilon}\,\|f\|^2\right) = C. \quad \square$$

Lemma 3. *For any given $z \in \mathcal{H}_0$ and for any $t > 0$ and $\beta \in \mathbb{R}$, when ε is small enough there exist three positive constants, m_0, m_1 and m_2, independent of t such that*

$$m_0\mathcal{E}(t) \leq \Phi(t) \leq m_1\mathcal{E}(t) + m_2. \tag{3.6}$$

Proof. In order to prove the lower inequality we must observe that, by Young's inequality

$$|\langle \partial_t u, u\rangle| \geq -\frac{1}{2}\,\|\partial_t u\|^2 - \frac{1}{2}\,\|u\|^2\,;$$

hence, we obtain

$$\Phi(t) \geq \|u(t)\|_2^2 + \left(1 - \frac{\varepsilon}{2}\right)\|\partial_t u(t)\|^2 + \frac{1}{2}\left(\beta + \|u(t)\|_1^2\right)^2 + \left(k - \frac{\varepsilon}{2}\right)\,\|u(t)\|^2\,.$$

If we choose ε small enough to satisfy $\varepsilon < 2$ and $\varepsilon < 2k$, then we have

$$\Phi(t) \geq m_0\mathcal{L}(t) \geq m_0\mathcal{E}(t), \tag{3.7}$$

where $m_0 = \min\{1 - \frac{\varepsilon}{2}, 1 - \frac{\varepsilon}{2k}\}$.

The upper inequality can be obtained using the definition of Φ and applying the estimate

$$|\langle \partial_t u, u\rangle| \leq \frac{1}{2}\,\|\partial_t u\|^2 + \frac{1}{2\lambda_1}\,\|u\|_2^2\,. \tag{3.8}$$

First, we can write

$$\Phi(t) \leq \left[1 + \frac{1}{\lambda_1}\left(k + \frac{\varepsilon}{2}\right)\right]\|u(t)\|_2^2 + \left(\frac{\varepsilon}{2} + 1\right)\|\partial_t u(t)\|^2 + \frac{1}{2}\left(\beta + \|u(t)\|_1^2\right)^2\,.$$

Then, by (3.1) and Lemma 2 we infer

$$\left(\beta + \|u\|_1^2\right) \le |\beta| + \frac{1}{\sqrt{\lambda_1}} C = \bar{C}, \tag{3.9}$$

so that we finally obtain

$$\Phi(t) \le \left[2 + \tfrac{1}{\lambda_1}\left(k + \tfrac{\varepsilon}{2}\right) + \tfrac{\varepsilon}{2}\right] \mathcal{E}(t) + \tfrac{1}{2}\bar{C}^2 = m_1\, \mathcal{E}(t) + m_2. \quad \square$$

Theorem 4. *For any $\beta \in \mathbb{R}$, there exists an absorbing set in $\mathcal{H}_0$ for the dynamical system $(S(t), \mathcal{H}_0)$.*

Proof. By virtue of (3.3), we have

$$\begin{aligned}\frac{\mathrm{d}\mathcal{L}}{\mathrm{d}t} &= \frac{\mathrm{d}\mathcal{E}}{\mathrm{d}t} + 2\left(\beta + \|u\|_1^2\right)\left\langle A^{\frac{1}{4}} u, A^{\frac{1}{4}} \partial_t u\right\rangle + 2k\,\langle u, \partial_t u\rangle \\ &= -2\delta\, \|\partial_t u\|^2 + 2\,\langle f, \partial_t u\rangle\,.\end{aligned}$$

Moreover, by (3.4)

$$\begin{aligned}\frac{\mathrm{d}\Phi}{\mathrm{d}t} &= \frac{\mathrm{d}\mathcal{L}}{\mathrm{d}t} + \varepsilon\,\langle u, \partial_{tt} u\rangle + \varepsilon\, \|\partial_t u\|^2 = -\left(2\delta - \varepsilon\right)\|\partial_t u\|^2 + 2\,\langle f, \partial_t u\rangle \\ &\quad + \varepsilon\left[-\|u\|_2^2 - \delta\,\langle \partial_t u, u\rangle - k\,\|u\|^2 - \beta\,\|u\|_1^2 - \|u\|_1^4 + \langle f, u\rangle\right].\end{aligned}$$

A straightforward computation leads to the identity

$$\frac{\mathrm{d}\Phi}{\mathrm{d}t} + \varepsilon\Phi + 2\left(\delta - \varepsilon\right)\|\partial_t u\|^2 + \frac{\varepsilon}{2}\,\|u\|_1^4 = 2\,\langle f, \partial_t u\rangle + \varepsilon\left[\langle f, u\rangle - (\delta - \varepsilon)\,\langle \partial_t u, u\rangle\right] + \frac{\varepsilon}{2}\beta^2. \tag{3.10}$$

In the following, we estimate the terms in the right hand side (rhs) of the previous equality. By means of Hölder and Young's inequalities and (3.8), we have

$$\frac{\mathrm{d}\Phi}{\mathrm{d}t} + \varepsilon\Phi + \frac{1}{2}\left(3\delta - 5\varepsilon\right)\|\partial_t u\|^2 \le \frac{1}{2}\varepsilon\beta^2 + \left(\frac{1}{\varepsilon} + \frac{1}{2}\right)\|f\|^2 + \frac{\varepsilon^2(\delta - \varepsilon + 1)}{2\lambda_1}\,\|u\|_2^2\,.$$

Now, choosing $\varepsilon < \frac{3}{5}\,\delta$ and $\varepsilon < 1 + \delta$ we find

$$\frac{\mathrm{d}\Phi}{\mathrm{d}t} + \varepsilon\Phi \le \frac{1}{2}\varepsilon\beta^2 + \left(\frac{1}{\varepsilon} + \frac{1}{2}\right)\|f\|^2 + m\varepsilon^2\mathcal{E},$$

where

$$m = \frac{1}{2\lambda_1}\left(1 + \delta - \varepsilon\right) > 0.$$

Finally, from Lemma 3 we obtain

$$\frac{\mathrm{d}\Phi}{\mathrm{d}t} + \varpi\Phi \le \frac{1}{2}\varepsilon\beta^2 + \left(\frac{1}{\varepsilon} + \frac{1}{2}\right)\|f\|^2\,,$$

where $\varpi = \varepsilon\left(1 - \frac{m}{m_0}\varepsilon\right)$ is positive provided that $\varepsilon < \frac{m_0}{m}$. Since Φ is positive also when $\beta < 0$ provided that ε is chosen small enough, using Gronwall lemma it follows that

$$\Phi\,(t) \le \Phi\,(0)\,\mathrm{e}^{-\varpi\, t} + \frac{1}{2}\varepsilon\beta^2 + \left(\frac{1}{\varepsilon} + \frac{1}{2}\right)\|f\|^2\,, \tag{3.11}$$

and accordingly

$$\mathcal{E}(t) \le \frac{1}{m_0}\Phi(t) \le \Gamma_0(R)\mathrm{e}^{-\varpi t} + \Gamma_1(\beta, \|f\|), \tag{3.12}$$

where

$$\Gamma_0(R) = \frac{\Phi(0)}{m_0} \quad \text{and} \quad \Gamma_1(\beta, \|f\|) = \frac{1}{m_0}\left[\frac{1}{2}\varepsilon\beta^2 + \left(\frac{1}{\varepsilon} + \frac{1}{2}\right)\|f\|^2\right].$$

As a consequence, every ball $B\left(0, \bar{R}\right)$ in $\mathcal{H}_0$ with radius $\bar{R} > 1 + \Gamma_1(\beta, \|f\|)$ can be chosen as an absorbing set in that it verifies the following statement: *for all $z_0 = (u_0, u_1) \in B(0, R)$, there exists $t_0(R) = \frac{1}{\varpi}\log \Gamma_0(R)$ such that for any $t > t_0$, $z(t) \in B(0, \bar{R})$.* $\square$

4. Exponential stability

A direct proof of the exponential decay of the energy seems out of reach, so we exploit the equivalence between the energy $\mathcal{E}$ and the functional

$$\bar{\Phi} = \Phi - \frac{1}{2}\beta^2$$

which can be proved to be exponentially stable. The positivity of such a functional will be obtained as a direct corollary of the next Lemma 5.

Recalling Theorem 1, the set $\mathcal{S}_0$ of stationary solutions reduces to a singleton when

$$\beta \geq -\beta_c(k) = -\min_{n\in\mathbb{N}} \mu_n(k), \qquad \mu_n(k) = \sqrt{\lambda_n}\left[1 + \frac{k}{\lambda_n}\right], \quad \lambda_n = n^4\pi^4. \tag{4.1}$$

It is worth noting that $\beta_c(k)$ is a piecewise-linear function of k, in that $\beta_c(k) = \mu_1(k)$ when $0 < k < \sqrt{\lambda_1}\sqrt{\lambda_2} = 4\pi^4$, and in general

$$\beta_c(k) = \mu_n(k) \quad \text{when } \sqrt{\lambda_{n-1}}\sqrt{\lambda_n} < k < \sqrt{\lambda_n}\sqrt{\lambda_{n+1}}.$$

Unlike the case $k = 0$, the energy $\mathcal{E}(t)$ does not decay exponentially in the whole domain of the (β, k) plain where (4.1) is satisfied, but in a region which is strictly included in it.

For futher purposes, let

$$\bar{\beta}(k) = \begin{cases} \beta_c(k) & 0 < k \leq \lambda_1, \\ 2\sqrt{k} & k > \lambda_1. \end{cases}$$

A picture of this function is given in Fig. 2.

Lemma 5. *Let $\beta \in \mathbb{R}$, $k > 0$ and*

$$Lu = Au + \beta A^{\frac{1}{2}}u + ku.$$

There exists a real function $\nu = \nu(\beta, k)$ such that

$$\langle Lu, u\rangle \geq \nu \|u\|_2^2,$$

where $\nu(\beta, k) > 0$ if and only if $\beta > -\bar{\beta}(k)$.

Proof. Taking into account the inner product

$$\langle Lu, u\rangle = \|u\|_2^2 + \beta \|u\|_1^2 + k\|u\|^2,$$

we put

$$X = \|u\|_2; \qquad Y = \|u\|_1; \qquad Z = \|u\|$$

so that

$$\langle Lu, u\rangle = I(X, Y, Z) = X^2 + \beta Y^2 + kZ^2.$$

If $\beta \geq 0$, the desired inequality trivially holds true by choosing $\nu = 1$.

Letting $\beta < 0$, by means of the interpolation inequality $\|u\|_1^2 \leq \|u\|_2\|u\|$, we obtain

$$I(X, Y, Z) \geq X^2 + \beta XZ + kZ^2 = J(X, Z).$$

Hence, the thesis can be rewritten as follows: find $\nu > 0$ such that

$$J(X, Z)\,|_{D_0} \geq \nu X^2,$$

where $D_0 = \left\{(X, Z) : X \geq 0, Z \geq 0,\ 0 \leq Z \leq X/\sqrt{\lambda_1}\right\}$.

In order to prove this statement, we introduce the set

$$M = \left\{m \in \mathbb{R} : 0 \leq m \leq \frac{1}{\sqrt{\lambda_1}}\right\},$$

so that $D_0 = \{(X, Z) : Z = mX, X \geq 0 \text{ and } m \in M\}$ and the original problem reads: find $\nu > 0$ such that

$$J(X, Z)\,|_{Z=mX} \geq \nu_m X^2, \quad \forall m \in M$$

and

$$\nu = \inf_{m\in M} \nu_m > 0.$$

We first observe that

$$J(X,Z)\,|_{Z=mX} = \left(1+\beta m + km^2\right)X^2 = \eta(m)X^2$$

where $\eta(m)$ is a concave parabola. Hence, we have to find the region in the (β,k)-plane where η admits a strictly positive minimum on M, that is to say $\eta_{\min} = \nu > 0$. We shall prove that $\nu > 0$ if and only if $\beta > -\bar{\beta}(k)$. To this end, we split the proof into four steps.

- Step 1. We consider the region $R_1 = \left\{(\beta,k) : k > 0, -2\sqrt{k} < \beta < 0\right\}$. Since the discriminant of the parabola is negative, $\Delta_\eta = \beta^2 - 4k < 0$, the value $\eta(m)$ is strictly positive for all m in the closed interval M, and then $\nu > 0$.
- Step 2. Let $R_2 = \left\{(\beta,k) : 0 < k \le \lambda_1, -\sqrt{\lambda_1} - k/\sqrt{\lambda_1} < \beta \le -2\sqrt{k}\right\}$. Observing that $\lambda_1 < \sqrt{\lambda_1}\sqrt{\lambda_2}$, we infer that $\beta_c(k) = \mu_1 = \sqrt{\lambda_1} + k/\sqrt{\lambda_1}$. Now, since $\Delta_\eta = \beta^2 - 4k \ge 0$, there exist two solutions, $m_1, m_2 \in \mathbb{R}$, of $\eta(m) = 0$ so that η changes sign on $\mathbb{R}$. In particular, it must be negative inside the open interval (m_1, m_2) because of $k > 0$. On the other hand, η is positive at the ends of M. Indeed, $\eta(0) = 1$. In order to evaluate the sign of $\eta(1/\sqrt{\lambda_1})$ we let $k = \rho\,\lambda_1$, with $0 < \rho \le 1$, and we obtain

  $$\eta\left(\frac{1}{\sqrt{\lambda_1}}\right) = (1+\rho) + \frac{\beta}{\sqrt{\lambda_1}}.$$

  According to the definition of R_2

  $$-\sqrt{\lambda_1} - \rho\sqrt{\lambda_1} < \beta < -2\sqrt{\rho\lambda_1} \tag{4.2}$$

  therefore,

  $$\frac{\beta}{\sqrt{\lambda_1}} + (1+\rho) > 0,$$

  which implies $\eta\left(1/\sqrt{\lambda_1}\right) > 0$. Thus, we infer that either $(m_1, m_2) \subset M$, or (m_1, m_2) is external to M. In the sequel, we prove the latter occurrence by showing that the vertex of the parabola lies outside of M. Letting m_* the abscissa of the vertex, it satisfies $\eta'\,|_{m=m_*} = \beta + 2km_* = 0$ so that

  $$m_* = -\frac{\beta}{2k} = -\frac{\beta}{2\rho\lambda_1}.$$

  So, in order to have $\eta(m)\,|_M > 0$, it is enough to prove that

  $$-\frac{\beta}{2\rho\lambda_1} = m_* \ge \frac{1}{\sqrt{\lambda_1}},$$

  which is trivially true because from (4.2) we have

  $$\beta \le -2\sqrt{\rho}\sqrt{\lambda_1} \le -2\rho\sqrt{\lambda_1}.$$
- Step 3. Let $R_3 = \left\{(\beta,k) : k > \lambda_1, -\sqrt{\lambda_1} - k/\sqrt{\lambda_1} < \beta \le -2\sqrt{k}\right\}$. When $\beta < -2\sqrt{k}$, as before we have $\Delta_\eta = \beta^2 - 4k > 0$, so that η is negative valued in the open interval (m_1, m_2) delimited by solutions m_1, m_2 to the equation $\eta(m) = 0$. Nevertheless, in this case the vertex of the parabola lies inside M. Indeed, if we take $k = \sigma\lambda_1$, with $\sigma > 1$, the abscissa of the vertex satisfies

  $$0 < m_* = -\frac{\beta}{2\sigma\lambda_1} < \frac{1}{\sqrt{\lambda_1}},$$

  which holds true by virtue of the definition of R_3, in that

  $$-\beta < (1+\sigma)\sqrt{\lambda_1} < 2\sigma\sqrt{\lambda_1}.$$

  As a consequence, the minimum of η on M is $\eta(m_*) < 0$. When $\beta = -2\sqrt{k}$, we have $m_1 = m_2 = m_*$ and $\eta(m_*) = 0$, so that the minimum of η on M vanishes. In both cases, $\eta_{\min} = \nu$ is not positive.
- Step 4. We consider the set $R_4 = \left\{(\beta,k) : k > 0, \beta < -\sqrt{\lambda_1} - k/\sqrt{\lambda_1}\right\}$. In this case $\eta(0) = 1$, whilst

  $$\eta\left(\frac{1}{\sqrt{\lambda_1}}\right) = \frac{1}{\sqrt{\lambda_1}}\left(\sqrt{\lambda_1} + \beta + \frac{k}{\sqrt{\lambda_1}}\right) < 0$$

  and the minimum of η on M cannot be positive. □

We are now in a position to prove the following.

Theorem 6. *When $f = 0$, the solutions to (1.1) decay exponentially, i.e.*

$$\mathcal{E}(t) \le c_0\,\mathcal{E}(0)\mathrm{e}^{-ct}$$

with c_0 and c suitable positive constants, provided that $\beta > -\bar{\beta}(k)$.

Proof. Let $\bar{\Phi}$ be the functional obtained from Φ by letting $f = 0$ and neglecting the term $\frac{1}{2}\beta^2$, i.e.

$$\bar{\Phi}(t) = \|u(t)\|_2^2 + \|\partial_t u(t)\|^2 + \beta \|u(t)\|_1^2 + \frac{1}{2} \|u(t)\|_1^4 + k \|u(t)\|^2 + \varepsilon \langle \partial_t u(t), u(t) \rangle .$$

In view of applying Lemma 5, we remark that

$$\bar{\Phi} = \langle Lu, u \rangle + \|\partial_t u\|^2 + \frac{1}{2} \|u\|_1^4 + \varepsilon \langle u, \partial_t u \rangle .$$

The first step is to prove the equivalence between $\mathcal{E}$ and $\bar{\Phi}$, that is

$$c_1 \mathcal{E} \leq \bar{\Phi} \leq c_2 \mathcal{E}.$$

We split the proof into two parts.

- *First step*: $c_1 \mathcal{E} \leq \bar{\Phi}$.

  By virtue of Lemma 5 and (3.1), the following chain of inequalities holds provided that $\beta > -\bar{\beta}(k)$, which ensures the positivity of ν:

$$\bar{\Phi} \geq \left(\nu - \frac{\varepsilon}{2\lambda_1}\right) \|u\|_2^2 + \frac{1}{2} \|u\|_1^4 + \left(1 - \frac{\varepsilon}{2}\right) \|\partial_t u\|^2$$
$$\geq \left(\nu - \frac{\varepsilon}{2\lambda_1}\right) \|u\|_2^2 + \left(1 - \frac{\varepsilon}{2}\right) \|\partial_t u\|^2 \geq \min\left\{\nu - \frac{\varepsilon}{2\lambda_1}, 1 - \frac{\varepsilon}{2}\right\} \mathcal{E}.$$

If we choose $\varepsilon < \min\{2\nu\lambda_1, 2\}$ and we put $c_1 = \min\left\{\nu - \frac{\varepsilon}{2\lambda_1}, 1 - \frac{\varepsilon}{2}\right\}$, it follows

$$c_1 \mathcal{E} \leq \bar{\Phi}.$$

- *Second step*: $\bar{\Phi} \leq c_2 \mathcal{E}$.

  Using the expression of $\bar{\Phi}$, Young's inequality and (3.1), we can write the following chain of inequalities

$$\bar{\Phi} \leq \left(1 + \frac{k}{\lambda_1} + \frac{1}{2\lambda_1}\right) \|u\|_2^2 + \left(1 + \frac{\varepsilon^2}{2}\right) \|\partial_t u\|^2 + \beta \|u\|_1^2 + \frac{1}{2} \|u\|_1^4$$
$$\leq \left(2 + \frac{k}{\lambda_1} + \frac{1}{2\lambda_1} + \frac{\varepsilon^2}{2}\right) \mathcal{E} + \beta \|u\|_1^2 + \frac{1}{2} \|u\|_1^4 .$$

In particular, from (3.9) we obtain

$$\bar{\Phi} \leq \left(2 + \frac{k}{\lambda_1} + \frac{1}{2\lambda_1} + \frac{\varepsilon^2}{2}\right) \mathcal{E} + \bar{C} \|u\|_1^2$$
$$\leq \left(2 + \frac{k}{\lambda_1} + \frac{1}{2\lambda_1} + \frac{\varepsilon^2}{2} + \frac{\bar{C}}{\sqrt{\lambda_1}}\right) \mathcal{E} = c_2 \mathcal{E}.$$

The last step is to prove the exponential decay of $\bar{\Phi}$. To this aim, remembering the hypothesis $f = 0$, we can write the identity (3.10) in the following way:

$$\frac{d\bar{\Phi}}{dt} + \varepsilon\bar{\Phi} + 2(\delta - \varepsilon) \|\partial_t u\|^2 + \frac{\varepsilon}{2} \|u\|_1^4 + \varepsilon(\delta - \varepsilon) \langle \partial_t u, u \rangle = 0.$$

Exploiting Young's inequality, we obtain

$$\frac{d\bar{\Phi}}{dt} + \varepsilon\bar{\Phi} + \frac{3}{2}(\delta - \varepsilon) \|\partial_t u\|^2 \leq \frac{\varepsilon^2(\delta - \varepsilon)}{2\lambda_1} \|u\|_2^2$$

and choosing $\varepsilon < \delta$, it follows

$$\frac{d\bar{\Phi}}{dt} + \varepsilon\bar{\Phi} \leq \frac{\varepsilon^2(\delta - \varepsilon)}{2\lambda_1} \mathcal{E} \leq \frac{\varepsilon^2(\delta - \varepsilon)}{2c_1\lambda_1} \bar{\Phi}$$

and finally, if ε is small enough, we have

$$\frac{d\bar{\Phi}}{dt} + c\bar{\Phi} \leq 0, \tag{4.3}$$

where $c = \varepsilon\left[1 - \frac{\varepsilon(\delta-\varepsilon)}{2c_1\lambda_1}\right] > 0$. Eq. (4.3) implies

$$c_1\mathcal{E}(t) \le \bar{\Phi}(t) \le \bar{\Phi}(0)\,e^{-ct} \le c_2\,\mathcal{E}(0)\,e^{-ct}.$$

The thesis follows by letting $c_0 = \frac{c_2}{c_1}$. □

Remark 2. We stress that Theorem 6 holds even if $k = 0$. In this case, however, we have $\bar{\beta}(0) = -\beta_c(0) = -\sqrt{\lambda_1}$. Then the null solution is exponentially stable, if unique.

In Fig. 2, the piecewise-straight line is the bifurcation line $\beta = -\beta_c(k)$: below it, the system has multiple stationary solutions, above it, there exists only the null solution. The curve $\beta = -\bar{\beta}(k)$ is composed of the straight segment $\beta = \mu_1(k)$, when $0 < k \le \lambda_1$, and the parabola $\beta = -2\sqrt{k}$, if $k > \lambda_1$. It is worth noting that each segment composing the graphic of $-\beta_c$ is tangent to the parabola at $k = \lambda_n = n^4\pi^4$, $n \in \mathbb{N}$. In the region of the plain (β, k), which is bounded from below by $\beta > -\bar{\beta}(k)$ the exponential stability holds true. Whilst, in the area between the red and the blue line, when $k > \lambda_1$, we have asymptotic, but not exponential stability.

5. Global attractor

We now state the existence of a global attractor for $S(t)$, for any $\beta \in \mathbb{R}$ and $k \ge 0$. Recall that the global attractor $\mathcal{A}$ is the unique compact subset of $\mathcal{H}_0$, which is at the same time

(i) attracting:

$$\lim_{t\to\infty} \boldsymbol{\delta}(S(t)\mathcal{B}, \mathcal{A}) \to 0,$$

for every bounded set $\mathcal{B} \subset \mathcal{H}_0$;

(ii) fully invariant:

$$S(t)\mathcal{A} = \mathcal{A},$$

for every $t \ge 0$.

Here, $\boldsymbol{\delta}$ stands for the Hausdorff semidistance in $\mathcal{H}_0$, defined as (for $\mathcal{B}_1, \mathcal{B}_2 \subset \mathcal{H}_0$)

$$\boldsymbol{\delta}(\mathcal{B}_1, \mathcal{B}_2) = \sup_{z_1\in\mathcal{B}_1} \inf_{z_2\in\mathcal{B}_2} \|z_1 - z_2\|_{\mathcal{H}_0}.$$

We address the reader to the books [13,22] for a detailed presentation of the theory of attractors. We shall prove the following.

Theorem 7. *The semigroup $S(t)$ on $\mathcal{H}_0$ possesses a connected global attractor $\mathcal{A}$ bounded in $\mathcal{H}_2 = D(A) \times D(A^{\frac{1}{2}})$. Moreover, $\mathcal{A}$ coincides with the unstable manifold of the set $\mathcal{S}$ of the stationary states of $S(t)$, namely,*

$$\mathcal{A} = \left\{ \widetilde{z}(0) : \begin{array}{c} \widetilde{z} \text{ is a complete (bounded) trajectory of } S(t) : \\ \lim_{t\to\infty} \|\widetilde{z}(-t) - S\|_{\mathcal{H}_0} = 0 \end{array} \right\}.$$

The set $\mathcal{S}$ consists of all the vectors of the form $(\bar{u}, 0)$, where $\bar{u}$ is a (weak) solution to

$$A\bar{u} + \left(\beta + \|\bar{u}\|_1^2\right)A^{1/2}\bar{u} + k\bar{u} = f.$$

It is then apparent that $\mathcal{S}$ is bounded in $\mathcal{H}_0$. If $\mathcal{S}$ is finite, then

$$\mathcal{A} = \left\{ \widetilde{z}(0) : \begin{array}{c} \widetilde{z} \text{ is a complete (bounded) trajectory of } S(t) \\ \text{such that } \exists\, z_1, z_2 \in S : \\ \lim_{t\to\infty} \|\widetilde{z}(-t) - z_1\|_{\mathcal{H}_0} = \lim_{t\to\infty} \|\widetilde{z}(t) - z_2\|_{\mathcal{H}_0} = 0 \end{array} \right\}. \tag{5.1}$$

Remark 3. When $f = 0$, $k \ge 0$ and $\beta \ge -\beta_c(k)$, then $\mathcal{A} = \mathcal{S} = \{(0, 0)\}$. If $\beta < -\beta_c(k)$, then $\mathcal{S} = \mathcal{S}_0$ may be finite or infinite, according to Theorem 1. In the former case, (5.1) applies.

The existence of a Lyapunov functional, along with the fact that $\mathcal{S}$ is a bounded set, allows us to prove Theorem 7 by paralleling some arguments devised in [15].

By the interpolation inequality $\|u\|_1^2 \le \|u\|\|u\|_2$ and (3.1), it is clear that

$$\frac{1}{2}\|u\|_2^2 \le \|u\|_2^2 + \beta\|u\|_1^2 + \gamma\|u\|^2 \le m\|u\|_2^2, \tag{5.2}$$

for some $m = m(\beta, \gamma) \ge 1$, provided that $\gamma > 0$ is large enough. Now, choosing $\gamma = \mu + k$, where $k > 0$ is a fixed value, we assume μ large enough so that (5.2) holds true. Then, we decompose the solution $S(t)z$ into the sum (see [15])

$$S(t)z = L(t)z + K(t)z,$$

where

$$L(t)z = (v(t), \partial_t v(t)) \quad \text{and} \quad K(t)z = (w(t), \partial_t w(t))$$

solve the systems

$$\begin{cases} \partial_{tt} v + Av + (\beta + \|u\|_1^2)A^{1/2} v + \mu v + \delta \partial_t v + kv = 0, \\ (v(0), \partial_t v(0)) = z, \end{cases} \tag{5.3}$$

and

$$\begin{cases} \partial_{tt} w + Aw + (\beta + \|u\|_1^2)A^{1/2} w - \mu v + \delta \partial_t w + kw = f, \\ (w(0), \partial_t w(0)) = 0. \end{cases} \tag{5.4}$$

Having fixed the boundary set of initial data $B(0, R)$, Lemma 2 entails

$$\sup_{t\geq 0} \left\{ \|u(t)\|_2^2 + \|\partial_t u(t)\|^2 \right\} \leq C = C(R), \quad \forall z_0 \in B(0, R),$$

where $(u(t), \partial_t u(t)) = S(t)z_0$; this formula will be used many times in the subsequent proofs.

In order to parallel the procedure given in [15], we shall prove Theorem 7 as a consequence of the following five lemmas. We start showing the exponential decay of $L(t)z$ (see Lemmas 8 and 9) by means of a dissipation integral (see Lemma 10). Then, we prove the asymptotic smoothing property of $K(t)$, for initial data bounded by R (see Lemma 11). Finally, by collecting all these results, we provide the desired consequence (see Lemma 12).

Henceforth, let $R > 0$ be fixed and $\|z\|_{\mathcal{H}_0} \leq R$. In addition, C will denote a *generic* positive constant which depends (increasingly) only on R, unless otherwise specified, besides on the structural quantities of the system.

Lemma 8. *There is* $\omega = \omega(R) > 0$, *such that*

$$\|L(t)z\|_{\mathcal{H}_0} \leq C e^{-\omega t}.$$

Proof. Denoting

$$\mathcal{E}_0(t) = \|L(t)z\|_{\mathcal{H}_0}^2 = \|v(t)\|_2^2 + \|\partial_t v(t)\|^2,$$

for $\varepsilon > 0$ to be determined, we set

$$\Phi_0(t) = \mathcal{E}_0(t) + \beta\|v(t)\|_1^2 + (\mu + k)\|v(t)\|^2 + \|u(t)\|_1^2\|v(t)\|_1^2 + \varepsilon\langle \partial_t v(t), v(t)\rangle.$$

In light of Lemma 2 and inequalities (5.2), assuming that ε is small enough, we have the bounds

$$\frac{1}{4}\mathcal{E}_0 \leq \Phi_0 \leq C\mathcal{E}_0. \tag{5.5}$$

Now, we compute the time-derivative of Φ_0 along the solutions to system (5.3) and we obtain

$$\frac{d\Phi_0}{dt} + \varepsilon\Phi_0 + 2(\delta - \varepsilon)\|\partial_t v\|^2 = 2\langle \partial_t u, A^{1/2} u\rangle \|v\|_1^2 - \varepsilon(\delta - \varepsilon)\langle \partial_t v, v\rangle.$$

Using (3.5) and assuming ε small enough (in particular, $\varepsilon < \delta$), we control the rhs by

$$C\|\partial_t u\|\mathcal{E}_0 + \frac{\varepsilon}{8}\mathcal{E}_0 + (\delta - \varepsilon)\|\partial_t v(t)\|^2.$$

So, from (5.5), we obtain

$$\frac{d\Phi_0}{dt} + \frac{\varepsilon}{2}\Phi_0 + (\delta - \varepsilon)\|\partial_t v\|^2 \leq C\|\partial_t u\|\Phi_0.$$

Finally, the functional Φ_0 fulfills the differential inequality

$$\frac{d\Phi_0}{dt} + \frac{\varepsilon}{2}\Phi_0 \leq C\ \|\partial_t u\|\ \Phi_0. \tag{5.6}$$

The desired conclusion is entailed by applying Lemmas 9 and 10. □

Lemma 9 (*see Lemma 6.2, [15]*). *Let* $\varphi : [0, \infty) \to [0, \infty)$ *satisfy*

$$\varphi' + 2\varepsilon\varphi \leq g\varphi,$$

for some $\varepsilon > 0$ *and some positive function* g *such that*

$$\int_\tau^t g(y)dy \leq c_0 + \varepsilon(t - \tau), \quad \forall \tau \in [0, t],$$

with $c_0 \geq 0$. *Then, there exists* $c_1 \geq 0$ *such that*

$$\varphi(t) \leq c_1\varphi(0)e^{-\varepsilon t}.$$

Lemma 10. *For any* $\sigma > 0$ *small*

$$\int_\tau^t \|\partial_t u(y)\| \, \mathrm{d}y \le \sigma(t-\tau) + \frac{C_2}{\sigma},$$

for every $t \ge \tau \ge 0$.

Proof. For $\varepsilon \in (0, 1]$, we set

$$\Psi = \Phi - 2\langle f, u\rangle - \frac{1}{2}\beta^2,$$

where Φ is defined by (3.4). Taking the time- derivative of Ψ and using (3.10), we find

$$\frac{\mathrm{d}\Psi}{\mathrm{d}t} + \varepsilon\Psi + \frac{\varepsilon}{2}\|u\|_1^4 + 2(\delta-\varepsilon)\|\partial_t u\|^2 = -\varepsilon\left[\langle f, u\rangle + (\delta-\varepsilon)\langle\partial_t u, u\rangle\right]. \tag{5.7}$$

By virtue of Lemma 2, $\mathcal{E}$ is bounded and hence

$$\frac{\mathrm{d}\Psi}{\mathrm{d}t} + \varepsilon\Psi + (\delta-\varepsilon)\|\partial_t u\|^2 \le \varepsilon C.$$

Since Ψ is uniformly bounded by Lemma 3, we end up with

$$\frac{\mathrm{d}\Psi}{\mathrm{d}t} + (\delta-\varepsilon)\|\partial_t u\|^2 \le \varepsilon[C - \Psi] \le \varepsilon C.$$

Integrating this inequality on $[\tau, t]$, we get

$$(\delta-\varepsilon)\int_\tau^t \|\partial_t u(y)\|^2 \, \mathrm{d}y \le \varepsilon C(t-\tau) + \Psi(\tau) - \Psi(t).$$

Assuming $\varepsilon < \frac{\delta}{2}$, a further application of Lemma 3 entails

$$\int_\tau^t \|\partial_t u(y)\|^2 \, \mathrm{d}y \le \varepsilon C_1(t-\tau) + C_2.$$

Finally, by Hölder and Young's inequalities

$$\begin{aligned}\int_\tau^t \|\partial_t u(y)\| \, \mathrm{d}y &\le \sqrt{t-\tau}\Big(\varepsilon C_1(t-\tau) + C_2\Big)^{\frac{1}{2}} \\ &\le \sqrt{t-\tau}\left(\sqrt{\varepsilon C_1}\sqrt{t-\tau} + \sqrt{C_2}\right) \le \sqrt{\varepsilon C_1}(t-\tau) + \sqrt{C_2}\sqrt{t-\tau} \\ &\le 2\sqrt{\varepsilon C_1}(t-\tau) + \frac{C_2}{2\sqrt{\varepsilon C_1}} = \sigma(t-\tau) + \frac{C_2}{\sigma}\end{aligned}$$

where $\sigma = 2\sqrt{\varepsilon C_1}$. □

The next result provides the boundedness of $K(t)z$ in a more regular space.

Lemma 11. *The estimate*

$$\|K(t)z\|_{\mathcal{H}_2} \le C$$

holds for every $t \ge 0$.

Proof. We denote

$$\mathcal{E}_1(t) = \|K(t)z\|_{\mathcal{H}_2}^2 = \|w(t)\|_4^2 + \|\partial_t w(t)\|_2^2.$$

For $\varepsilon > 0$ small to be fixed later, we set

$$\Phi_1 = \mathcal{E}_1 + (\beta + \|u\|_1^2)\|w\|_3^2 + \varepsilon\langle\partial_t w, Aw\rangle - 2\langle f, Aw\rangle + k\|w\|_2^2.$$

The interpolation inequality

$$\|w\|_3^2 \le \|w\|_2\|w\|_4$$

and the fact that $\|w\|_2 \le C$ (which follows by comparison from (3.5) and Lemma 8) entail

$$\beta\|w\|_3^2 \ge -\frac{1}{2}\mathcal{E}_1 - C.$$

Therefore, provided that ε is small enough, we have the bounds

$$\frac{1}{2}\mathcal{E}_1 - C \leq \Phi_1 \leq C\mathcal{E}_1 + C.$$

Taking the time-derivative of Φ_1, we find

$$\frac{\mathrm{d}\Phi_1}{\mathrm{d}t} + \varepsilon\Phi_1 + 2(\delta-\varepsilon)\|\partial_t w\|_2^2 = 2\langle\partial_t u, A^{1/2}u\rangle\|w\|_3^2 + 2\mu\langle A^{1/2}v, A^{1/2}\partial_t w\rangle + \varepsilon\Big[\mu\langle A^{1/2}v, A^{1/2}w\rangle - (\delta-\varepsilon)\langle\partial_t w, Aw\rangle - \langle f, Aw\rangle\Big].$$

Using (3.5) and the above interpolation inequality, if ε is small enough, we control the rhs by

$$\frac{\varepsilon}{8}\mathcal{E}_1 + C\sqrt{\mathcal{E}_1} + C \leq \frac{\varepsilon}{4}\mathcal{E}_1 + \frac{C}{\varepsilon} \leq \frac{\varepsilon}{2}\Phi_1 + \frac{C}{\varepsilon}.$$

Hence, if ε is fixed small enough (in particular $\varepsilon < \delta$), we obtain

$$\frac{\mathrm{d}\Phi_1}{\mathrm{d}t} + \frac{\varepsilon}{2}\Phi_1 \leq C.$$

Since $\Phi_1(0) = 0$, from the Gronwall lemma and the controls satisfied by Φ_1, we obtain the desired estimate for $\mathcal{E}_1$. □

By collecting previous results, the following lemma can be applied to obtain the existence of the attractor $\mathcal{A}$.

Lemma 12 (*see [15], Lemma 4.3*). *Assume that, for every $R > 0$, there exists a positive function ψ_R vanishing at infinity and a compact set $\mathcal{K}_R \subset \mathcal{H}_0$ such that the semigroup $S(t)$ can be split into the sum $L(t) + K(t)$, where the one-parameter operators $L(t)$ and $K(t)$ fulfill*

$$\|L(t)z\|_{\mathcal{H}_0} \leq \psi_R(t) \quad \text{and} \quad K(t)z \in \mathcal{K}_R,$$

whenever $\|z\|_{\mathcal{H}_0} \leq R$ and $t \geq 0$. Then, $S(t)$ possesses a connected global attractor $\mathcal{A}$, which consists of the unstable manifold of the set $\mathcal{S}$.

References

[1] F.T.K. Au, Y.S. Cheng, Y.K. Cheung, Vibration analysis of bridges under moving vehicles and trains: An overview, Prog. Struct. Engng. Mater. 3 (2001) 299–304.
[2] A.H. Nayfeh, P.F. Pai, Linear and Nonlinear Structural Mechanics, Wiley-Interscience, New York, 2004.
[3] W. Fang, J.A. Wickert, Postbuckling of micromachined beams, J. Micromach. Microeng. 4 (1994) 116–122.
[4] M.I. Younis, A.H. Nayfeh, A study of the nonlinear response of a resonant microbeam to an electric actuation, Nonlinear Dynam. 31 (2003) 91–117.
[5] N. Lobontiu, E. Garcia, Mechanics of Microelectromechanical Systems, Kluwer, New York, 2005.
[6] J.M. Ball, Initial-boundary value problems for an extensible beam, J. Math. Anal. Appl. 42 (1973) 61–90.
[7] R.W. Dickey, Free vibrations and dynamic buckling of the extensible beam, J. Math. Anal. Appl. 29 (1970) 443–454.
[8] E.L. Reiss, B.J. Matkowsky, Nonlinear dynamic buckling of a compressed elastic column, Quart. Appl. Math. 29 (1971) 245–260.
[9] S. Woinowsky-Krieger, The effect of an axial force on the vibration of hinged bars, J. Appl. Mech. 17 (1950) 35–36.
[10] S.M. Choo, S.K. Chung, Finite difference approximate solutions for the strongly damped extensible beam equations, Appl. Math. Comput. 112 (2000) 11–32.
[11] A.H. Nayfeh, S.A. Emam, Exact solution and stability of postbuckling configurations of beams, Nonlinear Dynam. 54 (2008) 395–408.
[12] M. Coti Zelati, C. Giorgi, V. Pata, Steady states of the hinged extensible beam with external load, Math. Models Methods Appl. Sci., in press (doi:10.1142/S0218202510004143).
[13] J.K. Hale, Asymptotic Behavior of Dissipative Systems, Amer. Math. Soc., Providence, 1988.
[14] I. Bochicchio, Longtime behavior for nonlinear models of a viscoelastic beam, Ph.D. Thesis, Salerno, 2008.
[15] C. Giorgi, V. Pata, E. Vuk, On the extensible viscoelastic beam, Nonlinearity 21 (2008) 713–733.
[16] C. Giorgi, M.G. Naso, V. Pata, M. Potomkin, Global attractors for the extensible thermoelastic beam system, J. Differential Equations 246 (2009) 3496–3517.
[17] I. Lasiecka, R. Triggiani, Control Theory for Partial Differential Equations: Continuous and Approximation Theories. I, Cambridge University Press, Cambridge, 2000.
[18] S.M. Choo, S.K. Chung, R. Kannan, Finite element Galerkin solutions for the strongly damped extensible beam equations, J. Appl. Math. Comput. 9 (2002) 27–43.
[19] T.F. Ma, Existence results and numerical solutions for a beam equation with nonlinear boundary conditions, Appl. Numer. Math. 47 (2003) 189–196.
[20] F. Venuti, L. Bruno, N. Bellomo, Crowd dynamics on a moving platform: Mathematical modelling and application to lively footbridges, Math. Comput. Modelling 45 (2007) 252–269.
[21] F. Venuti, L. Bruno, Crowd-structure interaction in lively footbridges under synchronous lateral excitation: A literature review, Phys. Life Reviews 6 (2009) 176–206.
[22] R. Temam, Infinite-Dimensional Dynamical Systems in Mechanics and Physics, Springer, New York, 1988.