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Asymptotic behavior of a semilinear problem in heat conduction with memory *

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Abstract

This paper is devoted to existence, uniqueness and asymptotic behavior, as time tends to infinity, of the solutions of an integro-partial differential equation arising from the theory of heat conduction with memory, in presence of a temperature-dependent heat supply. A linearized heat flux law involving positive instantaneous conductivity is matched with the energy balance, to generate an autonomous semilinear system subject to initial history and Dirichlet boundary conditions. Existence and uniqueness of solution is provided. Moreover, under proper assumptions on the heat flux memory kernel, the existence of absorbing sets in suitable function spaces is achieved.

Key words. Heat equation, integro-partial differential equations, existence and uniqueness, asymptotic behavior, absorbing set.

AMS subject classifications. 35B40, 35K05, 45K05.

1 Introduction and setting of the problem

In this paper we investigate the asymptotic behavior of the solutions of a semilinear problem describing the heat flow in a rigid, isotropic, homogeneous heat conductor with linear memory. The nonlinear source term has to comply some dissipativeness

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condition, even if it can exhibit antidissipative behaviors for low temperatures. Such a nonlinear heat supply might describe, for instance, temperature-dependent radiative phenomena (see, e.g., [18]). In addition, a non-Fourier constitutive law for the heat flux is considered here. The resulting linearized model is derived in the framework of the well-established theory of heat flow with memory due to Coleman & Gurtin [5].

Let $\Omega \subset \mathbb{R}^N$ be a fixed bounded domain with Lipschitz boundary occupied by a rigid heat conductor. If we consider only small variations of the absolute temperature and temperature gradient from equilibrium reference values, we may suppose that the *internal energy* $e : \Omega \times \mathbb{R} \to \mathbb{R}$ and the *heat flux vector* $\mathbf{q}(x,t) : \Omega \times \mathbb{R} \to \mathbb{R}^N$ are described by the following constitutive equations:

$$\begin{aligned} e(x,t) &= e_0 + c_0\theta(x,t) \\ \mathbf{q}(x,t) &= -k_0\nabla\theta(x,t) - \int_{-\infty}^{t} k(t-s)\nabla\theta(x,s)\,ds \end{aligned} \tag{1.1}$$

where $\theta : \Omega \times \mathbb{R} \to \mathbb{R}$ is the *temperature variation field* relative to the equilibrium reference value, $k : \mathbb{R}^+ \to \mathbb{R}$ is the *heat flux memory kernel*, whose properties will be specified later, and the constants e_0, c_0 and k_0 denote the internal energy at equilibrium, the *specific heat* and the *instantaneous conductivity*, respectively.

We consider the *energy balance* equation

$$e_t + \nabla \cdot \mathbf{q} = r \qquad \text{where} \qquad {}_t = \frac{\partial}{\partial t}$$

and we assume that a nonlinear temperature dependent heat source r is involved, namely,

$$r(x,t) = h(x,t) - g(\theta).$$

At first glance, one may be tempted to redefine the source term to include the contribution of temperature values taken in the past. In this framework, subject to Dirichlet boundary conditions, the problem reads as follows:

$$\begin{aligned} &c_0\theta_t - k_0\Delta\theta - \int_0^t k(t-s)\Delta\theta(s)\,ds + g(\theta) = f \qquad \text{on } \Omega \times \mathbb{R}^+ \\ &\theta(x,t) = 0 \qquad\qquad x \in \partial\Omega \quad t > 0 \\ &\theta(x,0) = \theta_0(x) \qquad x \in \Omega \end{aligned} \tag{1.2}$$

where the causal function f contains the temperature independent heat supply h and the term due to the *past history* of θ from $-\infty$ to 0^-, more precisely,

$$f(x,t) = h(x,t) + \int_{-\infty}^{0} k(t-s)\Delta\theta(x,s)\,ds \qquad x \in \Omega \quad t \geq 0.$$

The above is a non-autonomous term, unless the past history of the temperature vanishes and h is independent of time. Unfortunately, even in this particular case, the dynamical system (1.2) is non-autonomous. Indeed, the family of operators mapping the initial value θ_0 into the solution $\theta(t)$ of (1.2) does not match the usual semigroup properties. This is due to the presence of the convolution term, which generally renders the solution value at time t depending on the whole function up to t.

In order to overcome these difficulties, a different formulation can be attained introducing the new variables

$$\theta^t(x,s) = \theta(x,t-s) \qquad s \geq 0$$

and

$$\eta^t(x,s) = \int_0^s \theta^t(x,\tau)\,d\tau = \int_{t-s}^t \theta(x,\tau)\,d\tau \qquad s \geq 0.$$

Assuming $k(\infty) = 0$, a change of variable and a formal integration by parts yield

$$\int_{-\infty}^t k(t-s)\nabla\theta(s)\,ds = -\int_0^\infty k'(s)\nabla\eta^t(s)\,ds.$$

Hence $(1.1)_2$ reads

$$\mathbf{q}(x,t) = -k_0\nabla\theta(x,t) + \int_0^\infty k'(s)\nabla\eta^t(x,s)\,ds$$

where, here and in the sequel, the *prime* denotes derivation with respect to variable s. Setting

$$\mu(s) = -k'(s)$$

the above choice of variables leads to the following system:

$$\begin{aligned}
&c_0\theta_t(t) - k_0\Delta\theta(t) - \int_0^\infty \mu(s)\Delta\eta^t(s)\,ds + g(\theta(t)) = h(t) \qquad \text{on } \Omega\times\mathbb{R}^+\\
&\eta^t_t(s) = \theta(t) - \frac{\partial}{\partial s}\eta^t(s) \qquad \text{on } \Omega\times\mathbb{R}^+\times\mathbb{R}^+\\
&\theta(x,t) = \eta^t(x,s) = 0 \qquad x\in\partial\Omega \quad t,s>0\\
&\theta(x,0) = \theta_0(x) \qquad x\in\Omega\\
&\eta^0(x,s) = \eta_0(x,s) \qquad x\in\Omega \quad s>0
\end{aligned} \tag{1.3}$$

The term

$$\eta^0(x,s) = \int_{-s}^0 \theta(x,\tau)\,d\tau$$

is the *initial integrated past history* of θ, which is assumed to vanish on $\partial\Omega$, as well as θ. When h is independent of time, the initial-boundary value problem

(1.3) is really an autonomous dynamical system with respect to the unknown pair $(\theta(t),\eta^t)$. In particular, its asymptotic behavior can be treated by usual methods in the framework of the semigroup theory (see, e.g., [27]). Because of this feature, our attention will be restricted to (1.3). For sake of simplicity we further suppose the nonlinear part of the heat supply, $g:\mathbb{R}\to\mathbb{R}$, to be a polynomial of odd degree with positive leading coefficient:

$$g(\theta)=\sum_{k=1}^{2p} g_{2p-k}\theta^{k-1} \qquad g_0>0 \quad p\in\mathbb{N}. \tag{1.4}$$

In fact, no significant change in the proofs of the results presented here is required if we consider, more generally, a continuously differentiable function g on $\mathbb{R}$ satisfying

$$\begin{aligned} &\text{(i)} && |g(u)|\le k_1(1+|u|^\beta)\\ &\text{(ii)} && u\cdot g(u)\ge -k_2+k_3|u|^{\beta+1}\\ &\text{(iii)} && g'(u)\ge -k_4 \end{aligned}$$

for some $\beta>0$ and $k_j\ge 0$, $j=1,2,3,4$.

In view of the evolution problem (1.3), the constitutive quantities c_0, k_0 and μ are required to verify the following set of hypotheses.

(h1) $c_0>0 \qquad k_0>0$

(h2) $\mu\in C^1(\mathbb{R}^+)\cap L^1(\mathbb{R}^+) \qquad \mu(s)\ge 0 \qquad \mu'(s)\le 0 \quad \forall\, s\in\mathbb{R}^+$

(h3) $\mu'(s)+\delta\mu(s)\le 0 \quad \forall\, s\in\mathbb{R}^+$ and some $\delta>0$.

In the sequel we shall assume for simplicity $c_0=1$. Restriction (h2) can be equivalently expressed requiring that $k(s)$ is a bounded, positive, non-increasing, convex function of class C^2 vanishing at infinity. This is a sufficient assumption in order to make a linearly hereditary rigid conductor compatible with thermodynamics and fading memory (see [11]). Moreover, from (h2) it easily follows that $\mu(s)\ge 0$ for every $s\in\mathbb{R}^+$, and

$$k(0)=\int_0^\infty \mu(s)\,ds \qquad \text{is finite and nonnegative.} \tag{1.5}$$

By Gronwall inequality, (h3) implies that $\mu(s)$ decays exponentially for $s>\delta$. This condition ensures the exponential stability of the solutions of the linearized system and can hardly be weakened. Nevertheless, (h3) allows $\mu(s)$ to have a singularity at $s=0$, whose order is less than 1, since $\mu(s)$ is a non-negative L^1-function. For instance, a weakly singular kernel of the following type

$$\mu(s)=\frac{e^{-\delta s}}{s^\gamma} \qquad 0\le\gamma<1$$

is allowed. Such a kernel is associated with a fractional derivative model modified by an exponential decay factor.

Now, denoting

$$z(t) = (\theta(t), \eta^t)$$

$$z_0 = (\theta_0, \eta_0)$$

and setting

$$\mathcal{L}z = (k_0 \Delta\theta + \int_0^\infty \mu(s)\Delta\eta(s)\, ds, \theta - \eta')$$

and

$$\mathcal{G}(z) = (h - g(\theta), 0)$$

problem (1.3) assumes the compact form

$$\begin{aligned} & z_t = \mathcal{L}z + \mathcal{G}(z) \\ & z(x,t) = 0 \qquad x \in \partial\Omega \quad t > 0 \\ & z(x,0) = z_0 \end{aligned} \tag{1.6}$$

Existence, uniqueness and stability of the linear problem corresponding to (1.2) (i.e., with $g \equiv 0$) have been investigated by several authors (e.g., Grabmüller [14], Miller [19], Nunziato [20], Slemrod [23,24]). More recently, Gentili & Giorgi [12] revised the subject on the basis of thermodynamical arguments, and Colli, Grasselli and coworkers [4,6] extended such results to include phase transition phenomena. Existence of a strong solution of the nonlinear problem has been proved by Barbu & Malik [3], Crandall, Londen & Nohel [7], and Londen & Nohel [17], assuming nonlinear terms of the form of maximal monotone (possibly multivalued) operators. In particular, in [3] it is also proved the uniqueness of the solution.

On the other hand, the introduction of new variables leading to (1.3) parallels the procedure followed by Dafermos in his pioneer work [8] to achieve exponential stability in linear viscoelasticity. Along this line we also mention the works of Barbu [2] and Petzeltová [22]. In particular, in [22], using the standard tools of semigroup theory, it is provided existence, uniqueness and continuous dependence of the solution of the linear problem, whereas the analysis of the complete equation is restricted to the usual framework (i.e., the past history is not treated as a variable of the equation). More interesting is [2], where an equation with a maximal monotone nonlinearity is investigated. Here the idea is to re-cast the equation in the new framework, and prove that the new system still exhibits a maximal monotone nonlinear part. Henceforth the solution is provided by means of the theory of nonlinear equation of monotone type. Finally we quote the paper of Staffans [25], who introduced special semigroups of operator in connection with finite and infinite delay (see also Desch & Miller [10]).

In our work the nonlinear term is required to comply some dissipativeness conditions (i.e., monotonicity for large θ). Nonetheless, it may display some antidissipative behavior for low temperatures. For instance, a nonlinearity of the form $g(\theta) = \theta^3 - \theta$ is allowed. Lack of monotonicity of the nonlinear term does not allow us to exploit the machinery of maximal monotone operators, and therefore, in order to find the solution, a Faedo-Galerkin scheme is adopted.

Notice that the positiveness of the instantaneous conductivity k_0 assumed in (h1) leads to an equation which is parabolic if the convolution term is neglected. In this connection, longtime behavior for semilinear parabolic problems like (1.2), but lacking in the memory term ($k \equiv 0$), was studied by Temam in [27]. On the other hand, because of (1.5), it is worth stressing certain similarity between a particular case of (1.2) and some semilinear equations without memory. Indeed, with a particular choice of the kernel, problem (1.2) can be transformed as follows: *Let*

$$\Theta(t) = \int_0^t \theta(s)\, ds$$

denote the primitive of the unknown function θ*, and* k *be such that* $k(t) = k > 0$ *on* $[0, T]$ *but vanishes elsewhere. Then, for all* $t < T$*, problem* (1.2) *takes the form*

$$\begin{aligned}
&c_0 \Theta_{tt} - k_0 \Delta \Theta_t - k \Delta \Theta + g(\Theta_t) = h \\
&\Theta(x,t) = 0 \qquad x \in \partial\Omega \quad t > 0 \\
&\Theta(x,0) = 0 \qquad x \in \Omega \\
&\Theta_t(x,0) = \theta_0 \qquad x \in \Omega.
\end{aligned} \tag{1.7}$$

In this case, with suitable assumptions on the nonlinear function g, Ghidaglia & Marzocchi [13] proved the existence of a global attractor for the solutions of (1.7). From the above discussion it is expected for $(1.2)_1$ a behavior not far from that of a strongly damped wave equation. For general purpose, however, the terms due to convolution of the solution with non-constant memory kernel k cannot be ignored, and the above quoted results fail to be valid.

Asymptotic behavior of solutions for semilinear problems in presence of nontrivial terms of convolution type involving the principal part of the differential operator is treated, for instance, in [2,3,7,17,22]. Aizicovici & Barbu [1] considered the asymptotic properties of the solutions of a non-Fourier phase field model; and Grasselli & Pata [15] studied the asymptotic properties of the solutions of a family of differential equations with memory.

As far as we know, though, the analysis of the asymptotic behavior of the solution of the equation together with its past history has not been treated by many authors. Here we just quote [2,22] and the paper of Dafermos & Slemrod [9]. The problem of finding an absorbing set and a global attractor for the couple solution-past history of equations of this type, apparently, has not been treated at all by anyone.

Longtime behavior of problem (1.6) is treated here by means of techniques close to those employed in [27]. In present case, though, some difficulties arise since (1.6) is a system in which one of the equations is non-standard, and the usual procedures in the theory of semilinear parabolic equations do not apply.

In Section 2 we prove existence, uniqueness and continuity of solutions $z = z(t)$ with values in a suitable weighted Hilbert space $\mathcal{H}$. At this stage, assumptions (h1) and (h2) only are required. Further regularity of solutions yields existence, uniqueness and continuity in $\mathcal{V}$, a weighted Hilbert space which is (non-compactly) embedded into $\mathcal{H}$. In addition, the special structure of system (1.6) allows the introduction of a semigroup of continuous operators $S(t) : H \to H$, where $H = \mathcal{H}$ or $H = \mathcal{V}$, such that $z(t) = S(t)z_0$.

In Section 3 we show the existence of absorbing sets in $\mathcal{H}$ and $\mathcal{V}$, by virtue of some a priori estimates on the solution which heavily rely on (h3). This condition, which seems to be unavoidable, implies the exponential decay of the energy norm of the linearized solution. No bounds on the degree of the polynomial nonlinearity is required. Such an asymptotic behavior is a common feature in semilinear dissipative systems. Nevertheless, study of system (1.6) requires a non-trivial analysis due to the presence in the nonlinear source (1.4) of antidissipative terms of lower degree which may add instability.

Finally, we stress that our investigation of problem (1.6) is carried out on the basis of a general theory involving semigroups, which applies to nonlinear autonomous dynamical systems. This approach does not work in connection with a time-dependent source term $h(x, t)$, where one has to introduce the notion of *process*. Moreover, a further detailed study of the asymptotic behavior of solutions could reveal interesting properties. For instance, it is not known whether attractors of a semilinear integro-partial differential equation such as $(1.2)_1$ exist and have finite dimension. All these issues fall into the scope of a forthcoming paper.

2 Existence and uniqueness of solutions

First, we introduce some notation. Unless otherwise specified, it is understood that we consider spaces of functions acting on the domain Ω. Let $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$ denote the L^2-inner product and L^2-norm, respectively, and let $\| \cdot \|_p$ denote the L^p-norm. Accordingly, let $\langle \cdot, \cdot \rangle_{2,m}$ and $\| \cdot \|_{2,m}$, $m = 1, 2$, be the inner product and norm of H_0^1 and $H^2 \cap H_0^1$, respectively. We recall that for every $v \in H_0^1$ the Poincaré inequality

$$\lambda_0(\Omega)\|v\|^2 \leq \|\nabla v\|^2 \tag{2.1}$$

holds. If $v \in H^2 \cap H_0^1$, Poincaré and Young inequalities yield

$$\gamma_0(\Omega)\|\nabla v\|^2 \leq \|\Delta v\|^2. \tag{2.2}$$

Thus, in force of (2.1)–(2.2), we set

$$\langle\cdot,\cdot\rangle_{2,1} = \langle\nabla\cdot,\nabla\cdot\rangle \qquad \text{and} \qquad \langle\cdot,\cdot\rangle_{2,2} = \langle\Delta\cdot,\Delta\cdot\rangle$$

In view of (h2), let $L^2_\mu(\mathbb{R}^+, L^2)$ be the Hilbert space of functions $\varphi : \mathbb{R}^+ \to L^2$ endowed with the inner product

$$\langle\varphi_1,\varphi_2\rangle_\mu = \int_0^\infty \mu(s)\langle\varphi_1(s),\varphi_2(s)\rangle\, ds$$

and let $\|\varphi\|_\mu$ denote the corresponding norm. In a similar manner we introduce the inner products $\langle\cdot,\cdot\rangle_{m,\mu}$ and relative norms $\|\cdot\|_{m,\mu}$ $(m = 1, 2)$ on $L^2_\mu(\mathbb{R}^+, H^1_0)$ and $L^2_\mu(\mathbb{R}^+, H^2 \cap H^1_0)$ as

$$\langle\cdot,\cdot\rangle_{1,\mu} = \langle\nabla\cdot,\nabla\cdot\rangle_\mu \qquad \text{and} \qquad \langle\cdot,\cdot\rangle_{2,\mu} = \langle\Delta\cdot,\Delta\cdot\rangle_\mu.$$

We will also consider, with standard notation, spaces of functions defined on an interval I with values in a Banach space X such as $C(I, X)$, $L^p(I, X)$ and $H^{m,p}(I, X)$, with the usual norms. Finally we introduce the Hilbert spaces

$$\mathcal{H} = L^2 \times L^2_\mu(\mathbb{R}^+, H^1_0)$$

and

$$\mathcal{V} = H^1_0 \times L^2_\mu(\mathbb{R}^+, H^2 \cap H^1_0)$$

which are respectively endowed with the inner products

$$\langle w_1, w_2\rangle_{\mathcal{H}} = \langle\psi_1,\psi_2\rangle + \langle\varphi_1,\varphi_2\rangle_{1,\mu}$$

and

$$\langle w_1, w_2\rangle_{\mathcal{V}} = \langle\psi_1,\psi_2\rangle_{2,1} + \langle\varphi_1,\varphi_2\rangle_{2,\mu}$$

where $w_i = (\psi_i, \varphi_i) \in \mathcal{H}$ or $\mathcal{V}$ for $i = 1, 2$. The norm induced on $\mathcal{H}$ is the so-called *energy norm* and reads

$$\|(\psi,\varphi)\|^2_{\mathcal{H}} = \|\psi\|^2 + \int_0^\infty \mu(s)\|\nabla\varphi(s)\|^2\, ds.$$

We are now ready to state the existence and uniqueness result for problem (1.6).

Theorem *Assume* (1.4) *and* (h1)-(h2)*, and let*

$$h \in L^2 \qquad \textit{and} \qquad z_0 = (\theta_0, \eta_0) \in \mathcal{H}.$$

Then there exists a unique function $z = (\theta, \eta)$, *with*

$$\begin{aligned} &\theta \in L^\infty([0,T], L^2) \cap L^2([0,T], H_0^1) \cap L^{2p}([0,T], L^{2p}) \qquad \forall\, T > 0 \\ &\eta \in L^\infty([0,T], L^2_\mu(\mathbb{R}^+, H_0^1)) \qquad \forall\, T > 0 \end{aligned} \tag{2.3}$$

such that

$$z_t = \mathcal{L}z + \mathcal{G}(z) \tag{2.4}$$

in the weak sense, and

$$z|_{t=0} = z_0.$$

Furthermore

$$z \in C([0,T], \mathcal{H}) \qquad \forall\, T > 0$$

and the mapping

$$z_0 \mapsto z(t) \in C(\mathcal{H}, \mathcal{H}) \qquad \forall\, t \in [0,T].$$

If we also assume $z_0 \in \mathcal{V}$, *then*

$$\begin{aligned} &\theta \in L^\infty([0,T], H_0^1) \cap L^2([0,T], H^2 \cap H_0^1) \qquad \forall\, T > 0 \\ &\eta \in L^\infty([0,T], L^2_\mu(\mathbb{R}^+, H^2 \cap H_0^1)) \qquad \forall\, T > 0 \end{aligned}$$

and

$$z \in C([0,T], \mathcal{V}) \qquad \forall\, T > 0.$$

Proof

We follow a standard Faedo-Galerkin method. We recall that there exists a smooth orthonormal basis $\{\omega_j\}_{j=1}^\infty$ of L^2 which is also orthogonal in H_0^1. Typically one takes a complete set of normalized eigenfunctions for $-\Delta$ in H_0^1, such that $-\Delta\omega_j = \nu_j\omega_j$, being ν_j the eigenvalue corresponding to ω_j. Next we want to select a orthonormal basis $\{\zeta_j\}_{j=1}^\infty$ of $L^2_\mu(\mathbb{R}^+, H_0^1)$ which also belongs to $\mathcal{D}(\mathbb{R}^+, H_0^1)$. Here and in the sequel, $\mathcal{D}(I, X)$ is the space of infinitely differentiable X-valued function with compact support in $I \subset \mathbb{R}$, whose dual space is the distribution space on I with values in X^* (dual of X), denoted by $\mathcal{D}'(I, X^*)$. To this purpose we choose vectors of the form $l_k\omega_j$ $(k, j = 1, \dots \infty)$, where $\{l_j\}_{j=1}^\infty$ is a orthonormal basis $L^2_\mu(\mathbb{R}^+)$ which is also in $\mathcal{D}(\mathbb{R}^+)$.

We divide the proof in 6 steps.

Step 1 (Faedo-Galerkin scheme). Fix $T > 0$. Given an integer n, denote by P_n and Q_n the projections on the subspaces

$$\mathrm{Span}\{\omega_1, \dots, \omega_n\} \subset H_0^1 \qquad \text{and} \qquad \mathrm{Span}\{\zeta_1, \dots, \zeta_n\} \subset L^2_\mu(\mathbb{R}^+, H_0^1)$$

respectively. We look for a function $z_n = (\theta_n, \eta_n)$ of the form

$$\theta_n(t) = \sum_{j=1}^{n} a_j(t)\,\omega_j \qquad \text{and} \qquad \eta_n^t(s) = \sum_{j=1}^{n} b_j(t)\,\zeta_j(s)$$

satisfying

$$\begin{aligned} &\langle \partial_t z_n, (\omega_k, \zeta_j)\rangle_{\mathcal{H}} = \langle \mathcal{L} z_n, (\omega_k, \zeta_j)\rangle_{\mathcal{H}} + \langle \mathcal{G}(z_n), (\omega_k, \zeta_j)\rangle_{\mathcal{H}} \\ &z_n|_{t=0} = (P_n\theta_0, Q_n\eta_0). \end{aligned} \tag{2.5}$$

for a.e. $t \leq T$, for every $k, j = 0, \dots n$, where ω_0 and ζ_0 are the zero vectors in the respective spaces. Taking (ω_k, ζ_0) and (ω_0, ζ_k) in (2.5), and applying the divergence theorem to the term

$$\langle \int_0^\infty \Delta\eta_n(s)\,ds, \omega_k\rangle$$

we get a system of ODE in the variables $a_k(t)$ and $b_k(t)$ of the form

$$\begin{aligned} &\frac{d}{dt} a_k = -\nu_k a_k - \sum_{j=1}^{n} b_j \langle \zeta_j, \omega_k\rangle_{1,\mu} + \langle h, \omega_k\rangle - \langle g(\theta_n), \omega_k\rangle \\ &\frac{d}{dt} b_k = \sum_{j=1}^{n} a_j \langle \omega_j, \zeta_k\rangle_{1,\mu} - \sum_{j=1}^{n} b_j \langle \zeta_j', \zeta_k\rangle_{1,\mu} \end{aligned} \tag{2.6}$$

subject to the initial conditions

$$\begin{aligned} a_k(0) &= \langle \theta_0, \omega_k\rangle \\ b_k(0) &= \langle \eta_0, \zeta_k\rangle_{1,\mu}. \end{aligned} \tag{2.7}$$

According to standard existence theory for ODE there exists a continuous solution of (2.6)–(2.7) on some interval $(0, T_n)$. The a priori estimates that follow imply that in fact $T_n = +\infty$.

Step 2 (Energy estimates). Multiplying the first equation of (2.6) by a_k and the second by b_k, summing over k and adding the results, we get

$$\frac{1}{2}\frac{d}{dt}\|z_n\|_{\mathcal{H}}^2 = \langle \mathcal{L} z_n, z_n\rangle_{\mathcal{H}} + \langle \mathcal{G}(z_n), z_n\rangle_{\mathcal{H}}. \tag{2.8}$$

Since by the divergence theorem

$$\langle \int_0^\infty \mu(s)\Delta\eta_n(s)\,ds\,,\,\theta_n\rangle = -\int_0^\infty \mu(s)\int_\Omega \nabla\eta_n(s)\cdot\nabla\theta_n\,dx\,ds = -\langle \eta_n, \theta_n\rangle_{1,\mu}$$

we have that

$$\langle \mathcal{L} z_n, z_n\rangle_{\mathcal{H}} = -k_0\|\nabla\theta_n\|^2 - \langle \eta_n', \eta_n\rangle_{1,\mu}. \tag{2.9}$$

Using Young inequality, from (1.4) there exists a constant $b_0 > 0$ such that

$$g(\theta) \cdot \theta \geq \frac{1}{2} g_0 \theta^{2p} - b_0$$

so that it follows

$$\begin{aligned} \langle \mathcal{G}(z_n), z_n \rangle_{\mathcal{H}} &= -\langle g(\theta_n) - h, \theta_n \rangle \\ &\leq -\frac{1}{2} g_0 \|\theta_n\|_{2p}^{2p} + b_0 |\Omega| + \frac{\epsilon}{2} \|\theta_n\|^2 + \frac{1}{2\epsilon} \|h\|^2. \end{aligned} \tag{2.10}$$

Setting

$$\alpha_0 = \frac{1}{2} k_0 \lambda_0 \qquad \text{and} \qquad \Lambda = 2 b_0 |\Omega| + \frac{1}{\alpha_0} \|h\|^2$$

and choosing $\epsilon = \alpha_0$, (2.1), (2.6), (2.8), (2.9) and (2.10) entail

$$\frac{d}{dt} \|z_n\|_{\mathcal{H}}^2 + 2 \langle \eta_n', \eta_n \rangle_{1,\mu} + \alpha_0 \|\theta_n\|^2 + k_0 \|\nabla \theta_n\|^2 + g_0 \|\theta_n\|_{2p}^{2p} \leq \Lambda. \tag{2.11}$$

Integration by parts and (h2) bear

$$2 \langle \eta_n', \eta_n \rangle_{1,\mu} = - \int_0^\infty \mu'(s) \|\nabla \eta_n(s)\|^2 \, ds \geq 0. \tag{2.12}$$

Thus the term $2\langle \eta_n', \eta_n \rangle_{1,\mu}$ in (2.11), as well as $\alpha_0 \|\theta_n\|^2$, can be neglected, and we have

$$\frac{d}{dt} \|z_n\|_{\mathcal{H}}^2 + k_0 \|\nabla \theta_n\|^2 + g_0 \|\theta_n\|_{2p}^{2p} \leq \Lambda. \tag{2.13}$$

Integration on $(0, t)$, $t \in (0, T)$ leads to the following estimate:

$$\|z_n(t)\|_{\mathcal{H}}^2 + \int_0^t \left[k_0 \|\nabla \theta_n(\tau)\|^2 + g_0 \|\theta_n(\tau)\|_{2p}^{2p} \right] d\tau \leq \|z_0\|_{\mathcal{H}}^2 + \Lambda T.$$

In particular, we see that

$$\begin{aligned} &\theta_n \quad \text{is bounded in} \quad L^\infty([0,T], L^2) \cap L^2([0,T], H_0^1) \cap L^{2p}([0,T], L^{2p}) \\ &\eta_n \quad \text{is bounded in} \quad L^\infty([0,T], L^2_\mu(\mathbb{R}^+, H_0^1)). \end{aligned} \tag{2.14}$$

Up to passing to a subsequence, there exists a function $z = (\theta, \eta)$ such that

$$\begin{aligned} &\theta_n \rightharpoonup \theta \quad \text{weakly-star in} \quad L^\infty([0,T], L^2) \\ &\theta_n \rightharpoonup \theta \quad \text{weakly in} \quad L^2([0,T], H_0^1) \\ &\theta_n \rightharpoonup \theta \quad \text{weakly in} \quad L^{2p}([0,T], L^{2p}) \\ &\eta_n \rightharpoonup \eta \quad \text{weakly-star in} \quad L^\infty([0,T], L^2_\mu(\mathbb{R}^+, H_0^1)). \end{aligned} \tag{2.15}$$

Step 3 (Passage to limit). For a fixed integer m choose a function

$$u=(\sigma,\xi)\in\mathcal{D}((0,T),H_0^1\cap L^{2p})\times\mathcal{D}((0,T),\mathcal{D}(\mathbb{R}^+,H_0^1))$$

of the form

$$\sigma(t)=\sum_{j=1}^m \tilde a_j(t)\omega_j \qquad \text{and} \qquad \xi^t(s)=\sum_{j=1}^m \tilde b_j(t)\zeta_j(s)$$

where $\{\tilde a_j\}_{j=1}^m$ and $\{\tilde b_j\}_{j=1}^m$ are given functions in $\mathcal{D}((0,T))$. Then (2.5) holds with $(\sigma(t),\xi^t)$ in place of (ω_k,ζ_j). Denoting by $\langle\!\langle\cdot,\cdot\rangle\!\rangle$ the duality map between $H^1_\mu(\mathbb{R}^+,H_0^1)$ and its dual space, it is straightforward to see that

$$\lim_{n\to\infty}\langle\!\langle\eta_n',\xi\rangle\!\rangle=\langle\!\langle\eta',\xi\rangle\!\rangle.$$

Indeed, for every $\psi\in L^2_\mu(\mathbb{R}^+,H_0^1)$,

$$\langle\!\langle\psi',\xi\rangle\!\rangle=-\int_0^\infty \mu'(s)\langle\nabla\psi(s),\nabla\xi(s)\rangle\,ds-\int_0^\infty \mu(s)\langle\nabla\psi(s),\nabla\xi'(s)\rangle\,ds$$

and

$$\xi\in H^1_\mu(\mathbb{R}^+,H_0^1))\cap L^2_{\frac{(\mu')^2}{\mu}}(\mathbb{R}^+,H_0^1)).$$

Integrating over $(0,T)$ and passing to the limit, in view of (2.15) and of the fact that

$$\partial_t z_n\to z_t \qquad \text{in} \qquad \mathcal{D}'((0,T),H_0^1\cap L^{2p})\times\mathcal{D}'((0,T),\mathcal{D}(\mathbb{R}^+,H_0^1))$$

we get

$$\begin{aligned}\int_0^T\langle z,u_t\rangle_{\mathcal{H}}\,dt &= \int_0^T\Big[k_0\langle\nabla\theta,\nabla\sigma\rangle+\langle\eta,\sigma\rangle_{1,\mu}-\langle\theta,\xi\rangle_{1,\mu}+\langle\!\langle\eta',\xi\rangle\!\rangle-\langle h,\sigma\rangle\Big]\,dt\\ &\quad+\lim_{n\to\infty}\int_0^T\int_\Omega g(\theta_n)\sigma\,dxdt.\end{aligned} \tag{2.16}$$

We next claim that

$$\lim_{n\to\infty}\int_0^T\int_\Omega g(\theta_n)\sigma\,dxdt=\int_0^T\int_\Omega g(\theta)\sigma\,dxdt. \tag{2.17}$$

In order to prove the claim we will show that

$$g(\theta_n(t,x))\to g(\theta(t,x)) \qquad \text{for a.e. } (t,x)\in[0,T]\times\Omega$$

and

$$\|g(\theta_n)\|_{L^q([0,T]\times\Omega)}\le C<\infty$$

where $q = 2p/(2p-1)$ is the conjugate exponent of p. Then a classical result of measure theory (see, e.g., [16], p.12) implies that $g(\theta_n) \rightharpoonup g(\theta)$ weakly in $L^q([0,T]\times\Omega)$, and (2.17) holds. Indeed, from (2.5),

$$\begin{aligned}
&\|\partial_t\theta_n\|_{L^2([0,T],H^{-1})+L^q([0,T],L^q)} \\
&\le \|\nabla\theta_n\|_{L^2([0,T],H^{-1})} + \left\| \int_0^\infty \mu(s)\Delta\eta_n(s)\,ds \right\|_{L^2([0,T],H^{-1})} \qquad (2.18)\\
&\quad + \|h\|_{L^2([0,T],H^{-1})} + \|g(\theta_n)\|_{L^q([0,T],L^q)}.
\end{aligned}$$

From (1.4) we know that

$$|g(\theta_n)|^q \le K(1+|\theta_n|^{2p}) \qquad (2.19)$$

for some $K>0$. Using (2.14) and (2.19) it is easy to show that the second term of inequality (2.18) is bounded. Thus, up to a subsequence, we get that

$$\partial_t\theta_n \rightharpoonup \psi \qquad \text{weakly in} \qquad L^2([0,T],H^{-1}) + L^q([0,T],L^q). \qquad (2.20)$$

By standard arguments we can infer that $\psi = \theta_t$. Since

$$L^2([0,T],H^{-1}) + L^q([0,T],L^q) \subset L^q([0,T],H^{-1}+L^q)$$

and

$$L^2([0,T],H_0^1) \subset L^q([0,T],H_0^1)$$

(notice that $1 < q \le 2$), by (2.15) and (2.20) we argue that

$$\theta_n \rightharpoonup \theta \qquad \text{weakly in} \qquad H^{1,q}([0,T],H^{-1}+L^q) \cap L^q([0,T],H_0^1). \qquad (2.21)$$

Applying a compactness argument (see, e.g., [16], p.57) we know that the injection

$$H^{1,q}([0,T],H^{-1}+L^q) \cap L^q([0,T],H_0^1) \hookrightarrow L^q([0,T],L^q)$$

is compact, and therefore (2.21) implies that

$$\theta_n \to \theta \qquad \text{strongly in} \qquad L^q([0,T],L^q).$$

By the continuity of g we get that (up to a subsequence)

$$g(\theta_n(t,x)) \to g(\theta(t,x)) \qquad \text{for a.e.}\,(t,x) \in [0,T]\times\Omega.$$

In virtue of (2.19),

$$\|g(\theta_n)\|^q_{L^q([0,T]\times\Omega)} = \int_0^T \int_\Omega |g(\theta_n)|^q\,dx\,dt \le K|\Omega|T + K\int_0^T \|\theta_n\|_{2p}^{2p}\,dt$$

which is bounded uniformly in n. Therefore the claim is proved, and (2.16) becomes

$$\begin{aligned}\int_0^T \langle z, u_t\rangle_{\mathcal{H}}\,dt &= \int_0^T \Big[k_0\langle\nabla\theta,\nabla\sigma\rangle + \langle\eta,\sigma\rangle_{1,\mu} - \langle\theta,\xi\rangle_{1,\mu} \\ &\qquad + \langle\!\langle\eta',\xi\rangle\!\rangle - \langle h,\sigma\rangle + \int_\Omega g(\theta)\sigma\,dx\Big]\,dt.\end{aligned}$$

which in turn implies (2.4), using a density argument.

Step 4 (Continuity of solution). From equation (2.4) it is immediate to see that $z_t = (\theta_t, \eta_t)$ fulfills

$$\begin{aligned}&\theta_t \in L^2([0,T], H^{-1}) + L^q([0,T], L^q)\\ &\eta_t \in L^2([0,T], H^{-1}_\mu(\mathbb{R}^+, H^1_0)).\end{aligned}$$

The space $L^2([0,T],H^{-1})+L^q([0,T],L^q)$ is the dual of $L^2([0,T],H^1_0)\cap L^{2p}([0,T],L^{2p})$ Recalling (2.15), and using a slightly modified version of Lemma III.1.2 in [26], we see that $\theta \in C([0,T], L^2)$. Concerning η we get at once

$$\eta \in C([0,T], H^{-1}_\mu(\mathbb{R}^+, H^1_0)).$$

Thus $z(0)$ makes sense, and the equality $z(0) = z_0$ follows from the fact that $(P_n\theta_0, Q_n\eta_0)$ converges to z_0 strongly. To get the required further continuity in η, let us consider the linear equation

$$\begin{aligned}&\tilde\eta_t = \mathcal{A}\tilde\eta + \theta(t)\\ &\tilde\eta|_{t=0} = \eta_0.\end{aligned} \tag{2.22}$$

where θ is the first component of the solution z of (1.6) and $\mathcal{A}$ is the linear operator defined by

$$D(\mathcal{A}) = \big\{\varrho \in H^1_\mu(\mathbb{R}^+, H^1_0) : \varrho(0) = 0\big\} \subset L^2_\mu(\mathbb{R}^+, H^1_0)$$

$$\mathcal{A}\varrho(s) = -\frac{d}{ds}\varrho(s).$$

It is easy to see that $\mathcal{A}$ is dissipative, i.e.,

$$\langle\mathcal{A}\varrho,\varrho\rangle \le 0 \qquad \forall\,\varrho \in D(\mathcal{A})$$

and

$$\mathrm{Range}(\mathbb{1} - \mathcal{A}) = L^2_\mu(\mathbb{R}^+, H^1_0).$$

Therefore by Lumer-Phillips theorem (see [21], Theorem 4.3), $\mathcal{A}$ generates a strongly continuous semigroup of contractions $U(t)$ on $L^2_\mu(\mathbb{R}^+, H^1_0)$. The solution of (2.22) is then given by the Duhamel integral

$$\tilde\eta(t) = U(t)\eta_0 + \Phi(t)$$

where

$$\Phi(t) = \int_0^t U(t-\tau)\theta(\tau)\,d\tau.$$

Select $t_0 > 0$, and let $0 \le t < t_0$. We have

$$\begin{aligned}
&\|\Phi(t_0) - \Phi(t)\|_{1,\mu} \\
&\le \int_0^{t_0} \|U(t_0-\tau)\theta(\tau) - U(t-\tau)\theta(\tau)\|_{1,\mu} d\tau + \int_t^{t_0} \|U(t-\tau)\theta(\tau)\|_{1,\mu} d\tau \\
&\le \int_0^{t_0} \|U(t_0-\tau)\theta(\tau) - U(t-\tau)\theta(\tau)\|_{1,\mu} d\tau + \int_t^{t_0} \|\theta(\tau)\|_{1,\mu} d\tau.
\end{aligned}$$

Notice that, by (2.3),

$$\|\theta(\tau)\|_{1,\mu} = \left[\int_0^\infty \mu(s)\|\nabla\theta(\tau)\|^2\,ds\right]^{\frac{1}{2}} = \|\mu\|_{L^1(\mathbb{R}^+)}\|\nabla\theta(\tau)\| \in L^1([0,t_0]).$$

On the other hand, the strong continuity of the semigroup yields

$$\lim_{t\uparrow t_0} \|U(t_0-\tau)\theta(\tau) - U(t-\tau)\theta(\tau)\|_{1,\mu} = 0$$

for a.e. $\tau \in [0, t_0]$, and

$$\|U(t_0-\tau)\theta(\tau) - U(t-\tau)\theta(\tau)\|_{1,\mu} \le 2\|\mu\|_{L^1(\mathbb{R}^+)}\|\nabla\theta(\tau)\| \in L^1([0,t_0]).$$

Hence, in force of the dominated convergence theorem,

$$\lim_{t\uparrow t_0} \|\Phi(t_0) - \Phi(t)\|_{1,\mu} = 0$$

which implies the left-continuity of $\tilde{\eta}^t$. The same argument, with obvious modifications, entails the right-continuity, and therefore $\tilde{\eta} \in C([0,T], L^2_\mu(\mathbb{R}^+, H^1_0))$. To gain the required continuity on η we are left to show that $\tilde{\eta} = \eta$, and this will be a consequence of the uniqueness of the solution of (1.6).

Step 5 (Uniqueness). Suppose that $z_1 = (\theta_1, \eta_1)$ and $z_2 = (\theta_2, \eta_2)$ are two solutions of (1.6) with initial data z_{10} and z_{20}, respectively, and set $\tilde{z} = (\tilde{\theta}, \tilde{\eta}) = z_1 - z_2$ and $\tilde{z}_0 = z_{10} - z_{20}$. Then

$$\frac{d}{dt}\|\tilde{z}\|^2_{\mathcal{H}} \le -2\langle g(\theta_1) - g(\theta_2), \tilde{\theta}\rangle \tag{2.23}$$

where, with abuse of notation, $\langle\cdot,\cdot\rangle$ is the duality between L^p and L^q. The above calculation is obtained formally taking product in $\mathcal{H}$ between $\tilde{z}$ and the difference of (1.6) with z_1 and z_2 in place of z, and it can be made rigorous with the use of mollifiers. In particular, $\tilde{\eta}(s)$ is to be replaced with

$$\int_0^s \tilde{\eta}(y)\rho_\epsilon(s-y)\,dy$$

where $\rho_\epsilon(y)$ is a positive C^∞ function supported in $(0,\epsilon)$ of L^1-norm equal to one. In this case, since both $\mu\|\nabla\tilde\eta\|^2$ and $\mu\|\nabla\tilde\eta'\|^2$ belong to $L^1(\mathbb{R}^+)$, and $\tilde\eta(0)=0$, it is easy to see that the boundary term of the integration by parts of $\langle\tilde\eta,\tilde\eta'\rangle_{1,\mu}$ vanishes. Indeed, using Hölder inequality we get

$$\begin{aligned}\lim_{s\to 0}\mu(s)\|\nabla\tilde\eta(s)\|^2 &= \lim_{s\to 0}\mu(s)\left\|\int_0^s \nabla\tilde\eta'(\tau)\,d\tau\right\|^2\\ &\leq \limsup_{s\to 0}\left(\int_0^s \mu(s)^{1/2}\|\nabla\tilde\eta'(\tau)\|\,d\tau\right)^2\\ &\leq \limsup_{s\to 0} s\int_0^s \mu(\tau)\|\nabla\tilde\eta'(\tau)\|^2\,d\tau = 0.\end{aligned}$$

Integration by parts then bears

$$2\langle\tilde\eta',\tilde\eta\rangle_{1,\mu} = \lim_{s\to\infty}\mu(s)\|\nabla\tilde\eta(s)\|^2 - \int_0^\infty \mu'(s)\|\nabla\tilde\eta(s)\|^2\,ds.$$

But the left-hand side of the equation is bounded, and since from (h2) both terms of the right-hand side are positive, we conclude that the above limit exists and is finite, and therefore equals zero.

Notice that $g(y)$ is increasing for $|y|\geq M$ for some $M>0$. Fix $t\in[0,T]$, and let

$$\Omega_1 = \{x\in\Omega : |\theta_1(x,t)|\leq M \text{ and } |\theta_2(x,t)|\leq M\}.$$

If $x\in\Omega_1$, calling

$$N = 2\sup_{|y|\leq M}|g'(y)|$$

we have that

$$2|g(\theta_1(x)) - g(\theta_2(x))| \leq N|\tilde\theta(x)|.$$

Then, by the monotonicity of $g(y)$ for $|y|\geq M$, and by Poincaré inequality (2.1),

$$\begin{aligned}-2\langle g(\theta_1)-g(\theta_2),\tilde\theta\rangle &\leq -2\int_{\Omega_1}\big(g(\theta_1(x))-g(\theta_2(x))\big)\tilde\theta(x)\,dx\\ &\leq \int_{\Omega_1} N|\tilde\theta(x)|^2\,dx\\ &\leq B\|\tilde z\|^2_{\mathcal{H}}\end{aligned}$$

where we put $B=N/\lambda_0$. Hence (2.23) leads to

$$\frac{d}{dt}\|\tilde z\|^2_{\mathcal{H}} \leq B\|\tilde z\|^2_{\mathcal{H}}$$

and Gronwall lemma yields

$$\|\tilde z(t)\|^2_{\mathcal{H}} \leq \|\tilde z_0\|^2_{\mathcal{H}} e^{Bt}$$

which implies uniqueness and continuous dependence on initial data.

Step 6 (Further regularity). Multiply $(1.3)_1$ by $-\Delta\theta$ with respect to the inner product of L^2, and the laplacian of $(1.3)_2$ by $\Delta\eta$ with respect to the inner product of $L^2_\mu(\mathbb{R}^+, L^2)$. Adding the two terms,

$$\frac{d}{dt}\|z\|^2_{\mathcal{V}} + 2k_0\|\Delta\theta\|^2 + 2\langle\eta', \eta\rangle_{2,\mu} = 2\langle g(\theta) - h, \Delta\theta\rangle. \tag{2.24}$$

Since g is a polynomial of odd degree, there exists $d_0 > 0$ such that

$$g'(y) \geq -\frac{d_0}{2} \qquad \forall\, y \in \mathbb{R}.$$

Thus (1.4), $(1.6)_2$, Young inequality, and the Green formula yield

$$\begin{aligned} 2\langle g(\theta), \Delta\theta\rangle &= 2\int_\Omega g_{2p-1}\Delta\theta\, dx - 2\int_\Omega g'(\theta)\nabla\theta\cdot\nabla\theta\, dx \\ &\leq 2k_0 g^2_{2p-1}|\Omega|^2 + \frac{k_0}{2}\|\Delta\theta\|^2 + d_0\|\nabla\theta\|^2. \end{aligned}$$

Young inequality gives also

$$2|\langle h, \Delta\theta\rangle| \leq \frac{k_0}{2}\|\Delta\theta\|^2 + \frac{2}{k_0}\|h\|^2.$$

Then, setting

$$f = 2k_0 g^2_{2p-1}|\Omega|^2 + d_0\|\nabla\theta\|^2 + \frac{2}{k_0}\|h\|^2 \tag{2.25}$$

(2.24) becomes

$$\frac{d}{dt}\|z\|^2_{\mathcal{V}} + k_0\|\Delta\theta\|^2 + 2\langle\eta', \eta\rangle_{2,\mu} = f. \tag{2.26}$$

Notice that $f \in L^1([0,T])$. Under suitable spatial regularity assumptions on η, integrating by parts in time, and using (h2), we get

$$\langle\eta', \eta\rangle_{2,\mu} = -\int_0^\infty \mu'(s)\|\Delta\eta(s)\|^2\, ds \geq 0. \tag{2.27}$$

So the term $2\langle\eta', \eta\rangle_{2,\mu}$ in (2.26) can be neglected, and integration on $(0,t)$, $t \in (0,T)$ leads to

$$\|z(t)\|^2_{\mathcal{V}} + \int_0^t k_0\|\Delta\theta(\tau)\|^2\, d\tau \leq \int_0^T f(\tau)\, d\tau. \tag{2.28}$$

From the above equation (2.28) we conclude that

$$\theta \in L^\infty([0,T], H^1_0) \cap L^2([0,T], H^2 \cap H^1_0)$$

$$\eta \in L^\infty([0,T], L^2_\mu(\mathbb{R}^+, H^2 \cap H^1_0)).$$

Concerning the last assertion of the theorem, continuity of θ follows again using a slightly modified version of Lemma III.1.2 in [26] (see also [27], p.91) Continuity of η is obtained mimicking the above Step 4, with $H^2 \cap H^1_0$ in place of H^1_0. □

3 Existence of absorbing sets in $\mathcal{H}$ and in $\mathcal{V}$

Let H be the Hilbert space into which move all orbits of problem (1.6), namely

$$z : [0,T] \to H \qquad \text{where} \qquad H = \mathcal{H} \text{ or } H = \mathcal{V}$$

depending on the regularity of the initial data z_0. In this section we shall prove in both cases the existence of an absorbing set, that is, a bounded set $\mathcal{B}_0 \subset H$ into which every orbit eventually enters. Such a set is defined as follows:

Definition Let $B(0,R)$ be the open ball with center 0 and radius $R > 0$ in H. A bounded set $\mathcal{B}_0 \subset H$ is called an *absorbing set* for the problem (1.6) if for any initial value $z_0 \in B(0,R) \subset H$ there exists $t_H = t_H(R)$ such that

$$z(t) \in \mathcal{B}_0 \qquad \forall t \geq t_H$$

where $z(t)$ is the solution starting from z_0.

We now state and prove the result of this section.

Theorem *Under assumptions* (h1)-(h3)*, there exist absorbing sets in* $\mathcal{H}$ *and* $\mathcal{V}$ *for problem* (1.6).

Proof

We begin deriving a uniform estimate in $\mathcal{H}$. To this purpose we consider (2.11)–(2.12) for z. From (2.12) and (h3) we obtain

$$2\langle \eta', \eta \rangle_{1,\mu} \geq \delta \|\eta\|^2_{1,\mu}.$$

Thus, defining $\epsilon_0 = \min\{\alpha_0, \delta\}$, (2.11) entails

$$\frac{d}{dt}\|z\|^2_{\mathcal{H}} + \epsilon_0 \|z\|^2_{\mathcal{H}} + k_0 \|\nabla\theta\|^2 \leq \Lambda. \tag{3.1}$$

By Gronwall lemma we get the uniform estimate

$$\|z(t)\|^2_{\mathcal{H}} \leq \|z(t_0)\|^2_{\mathcal{H}} e^{-\epsilon_0(t-t_0)} + \frac{\Lambda}{\epsilon_0}\left[1 - e^{-\epsilon_0(t-t_0)}\right] \qquad \forall\, t \geq t_0. \tag{3.2}$$

In particular,

$$\|z(t)\|^2_{\mathcal{H}} \leq \|z_0\|^2_{\mathcal{H}} e^{-\epsilon_0 t} + \frac{\Lambda}{\epsilon_0}$$

from which it follows at once that

$$\limsup_{t\to+\infty} \|z(t)\|^2_{\mathcal{H}} \leq \rho^2_{\mathcal{H}} = \frac{\Lambda}{\epsilon_0}.$$

Therefore every ball $B(0,\rho) \subset \mathcal{H}$, with radius $\rho > \rho_{\mathcal{H}}$, is an absorbing set in $\mathcal{H}$. Indeed, for any open ball $B(0,R) \subset \mathcal{H}$, we have

$$S(t)B(0,R) \subset B(0,\rho) \qquad \forall t \geq t_{\mathcal{H}}$$

where

$$t_{\mathcal{H}} = t_{\mathcal{H}}(R,\rho) = \frac{1}{\epsilon_0} \log \left[\frac{R^2}{\rho^2 - \rho_{\mathcal{H}}^2} \right]. \tag{3.3}$$

In order to achieve uniform estimates involving the existence of a bounded absorbing set in $\mathcal{V}$ we consider (2.26). In force of (2.27) and (h3),

$$2\langle \eta', \eta \rangle_{2,\mu} \geq \delta \|\eta\|^2_{2,\mu}.$$

Thus, recalling (2.2) and defining $\epsilon_1 = \min\{\gamma_0 k_0, \delta\}$, (2.26) turns into

$$\frac{d}{dt}\|z\|^2_{\mathcal{V}} + \epsilon_1 \|z\|^2_{\mathcal{V}} \leq f \tag{3.4}$$

with f given in (2.25). Integration of (3.1) over $(t, t+1)$, for $t \geq t_0$, and (3.2) provide also the estimate

$$\int_t^{t+1} k_0 \|\nabla\theta(\tau)\|^2 \, d\tau \leq \|z(t)\|^2_{\mathcal{H}} + \Lambda \leq \|z(t_0)\|^2_{\mathcal{H}} + \Lambda \left[\frac{1+\epsilon_0}{\epsilon_0} \right]. \tag{3.5}$$

Defining the positive constants

$$K_1 = \frac{d_0}{k_0} \qquad \text{and} \qquad K_2 = \Lambda \, d_0 \left[\frac{1+\epsilon_0}{\epsilon_0 k_0} \right] + \frac{1}{k_0}\|h\|^2$$

from (3.5) it is clear that, for $t \geq t_0$,

$$\int_t^{t+1} f(\tau)\, d\tau \leq K_1 \|z(t_0)\|^2_{\mathcal{H}} + K_2. \tag{3.6}$$

By Gronwall lemma applied to (3.4) we obtain the uniform estimate

$$\|z(t)\|^2_{\mathcal{V}} \leq \|z(t_0)\|^2_{\mathcal{V}} e^{-\epsilon_1(t-t_0)} + \int_{t_0}^t e^{-\epsilon_1(t-\tau)} f(\tau)\, d\tau \qquad \forall\, t \geq t_0.$$

Let $m \in \mathbb{N}$ such that $t_0 + m - 1 < t \leq t_0 + m$. Since

$$\begin{aligned} \int_{t_0}^t e^{-\epsilon_1(t-\tau)} f(\tau)\, d\tau &\leq e^{-\epsilon_1(t-t_0)} \sum_{j=0}^{m-1} e^{\epsilon_1(j+1)} \int_{t_0+j}^{t_0+j+1} f(\tau)\, d\tau \\ &\leq (K_1\|z(t_0)\|^2_{\mathcal{V}} + K_2)\, e^{-\epsilon_1(t-t_0)} \sum_{j=0}^{m-1} e^{\epsilon_1(j+1)} \\ &\leq \frac{e^{2\epsilon_1}}{e^{\epsilon_1} - 1} (K_1\|z(t_0)\|^2_{\mathcal{H}} + K_2) \end{aligned}$$

we conclude that

$$\|z(t)\|_{\mathcal{V}}^2 \le \|z(t_0)\|_{\mathcal{V}}^2 e^{-\epsilon_1(t-t_0)} + \frac{e^{2\epsilon_1}}{e^{\epsilon_1}-1}\left(K_1\|z(t_0)\|_{\mathcal{H}}^2 + K_2\right) \qquad \forall\, t \ge t_0. \tag{3.7}$$

Let now $z_0 \in B(0,R)$ in $\mathcal{V}$. Recalling (2.1)–(2.2),

$$\|z_0\|_{\mathcal{H}} \le R_1 = \max\left\{\frac{1}{\lambda_0}, \frac{1}{\gamma_0}\right\} R.$$

Then, using (3.7) with $t_0 = 0$, we get

$$\|z(t)\|_{\mathcal{V}}^2 \le C(R) = R^2 + \frac{e^{2\epsilon_1}}{e^{\epsilon_1}-1}(K_1 R_1^2 + K_2). \tag{3.8}$$

Select $\rho > \rho_{\mathcal{H}}$, and let $t_0 = t_{\mathcal{H}}(R_1, \rho)$ as in (3.2). By (3.7) and (3.8),

$$\|z(t)\|_{\mathcal{V}}^2 \le C(R) e^{-\epsilon_1(t-t_0)} + \frac{e^{2\epsilon_1}}{e^{\epsilon_1}-1}(K_1\rho^2 + K_2) \qquad t \ge t_0.$$

Defining

$$\rho_{\mathcal{V}}^2 = \frac{e^{2\epsilon_1}}{e^{\epsilon_1}-1}(K_1\rho_0^2 + K_2)$$

every ball $B(0,\rho) \subset \mathcal{V}$, with $\rho > \rho_{\mathcal{V}}$, is an absorbing set in $\mathcal{V}$. Indeed, defining

$$t_{\mathcal{V}} = t_{\mathcal{V}}(R,\rho) = t_{\mathcal{H}}(R,\rho) + \frac{1}{\epsilon_1}\log\left[\frac{C(R)}{\rho^2 - \rho_{\mathcal{V}}^2}\right]$$

for any open ball $B(0,R) \subset \mathcal{V}$, we have

$$S(t)B(0,R) \subset B(0,\rho) \qquad \forall t \ge t_{\mathcal{V}}$$

as desired. □

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