PROCEEDINGS

SEVENTH IEEE DIGITAL SIGNAL PROCESSING WORKSHOP

1996 IEEE DIGITAL SIGNAL PROCESSING WORKSHOP

LOEN, NORWAY · SEPTEMBER 1-4, 1996

IEEE ABB Technology NERA NORSIG

New Results on the Evaluation of Equalizers Performance

M. Prandini, M. Campi and R. Leonardi

Dipartimento di Elettronica per l'Automazione, Università di Brescia, via Branze, 38 I-25123 Brescia, Italy
{prandini, campi, leon}@bsing.ing.unibs.it

ABSTRACT

In this paper, we present a new upper bound on the probability of error in PAM systems. The main feature of this bound is that it depends on the variance and fourth-order cumulant of the measurable equalizer output and on the a-priori known statistics of the channel input. It can therefore be useful for an a-posteriori evaluation of the equalization accuracy in a blind equalization context.

1. INTRODUCTION

An important feature in the analysis of a digital linear transmission system is the evaluation of its intersymbol interference phenomenon. A suitable measure of performance is represented by the probability of error ([1]), i.e. the probability that an individual recovered symbol is different from the corresponding transmitted symbol.

Most of the literature on the estimation of the probability of error requires that the coefficients of the channel-equalizer cascade are known ([2]-[5]). In most real applications, unfortunately, such coefficients are not available so that the approach presented in the above mentioned references is not directly applicable. An alternative approach based on correlation measures on the output of the equalizer has been introduced in [6]. However this approach is applicable only when the channel has a finite impulse response of known length.

In this paper, we consider the case of PAM transmission systems and derive an upper bound on the probability of error which can be estimated from the equalizer output and only the known channel input statistics. Such a bound can be used as an a-posteriori test to decide if an equalization algorithm has converged to a satisfactory solution.

The paper is organized as follows. In Section 2 basic concepts on PAM transmission systems are introduced. Then, a preliminary expression for the probability of error for such systems is derived in Section 3. An upper bound on the probability of error is then determined in Section 4, while some simulations are presented in Section 5.

2. SYSTEM DESCRIPTION

Consider a transmission system where the distortion caused by the channel between the information source and the receiver is represented by a linear, causal time-invariant and (possibly) non-minimum phase system with transfer function:

$$\mathrm{H}(z)=\sum_{k=0}^{\infty} h(k) z^{-k}.$$

The signal d(t) transmitted through the channel is assumed to be an i.i.d. sequence of symbols belonging to a finite alphabet *A* with an even number M of equiprobable levels. Precisely:

$\mathrm{d}(t) \in A=\{-(M-1), -(M-3), ..., M-3, M-1\}$

$\Pr(\mathrm{d}(t)=2i-M-1)=1/M, \qquad i=1,2,...,M.$

This is a typical communication system known under the acronym PAM (pulse-amplitude-modulation) (see e.g. [1]). The received signal $\mathrm{v}(t)=\sum_{k=0}^{\infty} h(k)\mathrm{d}(t-k)$ is processed by a linear equalizer. The linear filter usually adopted for the equalization purpose is a tapped-delay line with n parameters: $E(z)=\sum_{k=0}^{n-1} e(k) z^{-k}$.

Denote by x(·) its output.

The equalizer is used to cancel the channel distortion, so as to reconstruct the input d(·). In PAM systems, optimal detection is often achieved by adding at the output of the equalizer E(z) a nearest-neighbour M-ary quantizer (Fig.1). This threshold device is introduced to cancel the residual distortion to which the equalized signal x(t) can still be subject.

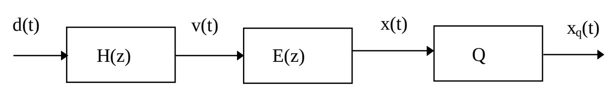

Figure 1. Complete block scheme for the PAM system

Specifically, the quantizer output $x_q(t)$ corresponds to the nearest value in the alphabet *A* of the equalized signal x(t), i.e.

$$x_q(x)=\sum_{j=1-M/2}^{M/2-1} \mathrm{sgn}(x+2j),$$

where

$$\mathrm{sgn}(x)=\begin{cases} 1, & x>0 \\ 0, & x=0 \\ -1, & x<0 \end{cases}.$$

3. PROBABILITY OF ERROR IN PAM SYSTEMS

Let us define the equalization delay $\bar{k}$ as the index of the dominant coefficient of the channel-equalizer cascade impulse response $s(k)=h(k)*e(k)$,

$$\bar{k}=\arg\max_k\{|s(k)|\}.$$

We say that an error occurs when $x_q(t)\neq \mathrm{sgn}(s(\bar{k}))\,\mathrm{d}(t-\bar{k})$, and consequently a meaningful measure of the overall PAM system performance is provided by the probability P_e of the event $x_q(t) \neq \mathrm{sgn}(s(\bar{k}))d(t-\bar{k})$.

Applying the theorem of total probability, P_e can be computed as follows:

$$P_e = \sum_{i=1}^{M} \Pr\{x_q(t) \neq \mathrm{sgn}(s(\bar{k}))d(t-\bar{k}) / d(t-\bar{k}) = 2i - M - 1\} \Pr\{d(t-\bar{k}) = 2i - M - 1\}$$
$$= \frac{1}{M}\sum_{i=1}^{M} \Pr\{x_q(t) \neq \mathrm{sgn}(s(\bar{k}))d(t-\bar{k}) / d(t-\bar{k}) = 2i - M - 1\}$$

The expression of P_e can be simplified by observing that x(t) is given by:

$$x(t) = s(\bar{k})d(t-\bar{k}) + \sum_{k\neq\bar{k}} s(k)d(t-k)$$

where $\xi(t) := \sum_{k\neq\bar{k}} s(k)d(t-k)$ is the *intersymbol interference*. $\xi(t)$ is independent of $s(\bar{k})d(t-\bar{k})$ and it is easily shown that it is symmetrically distributed. This implies that:

$$\Pr\{x_q(t) \neq \mathrm{sgn}(s(\bar{k}))d(t-\bar{k}) / d(t-\bar{k}) = 2i - M - 1\} =$$
$$= \Pr\{x_q(t) \neq \mathrm{sgn}(s(\bar{k}))d(t-\bar{k}) / d(t-\bar{k}) = -(2i - M - 1)\}$$
$$i = 1,2,\ldots,\frac{M}{2}$$

from which it follows:

$$P_e = \frac{2}{M}\sum_{i=M/2+1}^{M} \Pr\{x_q(t) \neq \mathrm{sgn}(s(\bar{k}))d(t-\bar{k}) / d(t-\bar{k}) = 2i - M - 1\} \quad (1)$$

We now introduce the following two technical assumptions:

A.1: $(M-3)|s(\bar{k})| < M-2$

A.2: $(M-1)|s(\bar{k})| > M-2$,

which are both satisfied if $|s(\bar{k})|$ is close enough to 1. Under these hypotheses, an error occurs if $\xi(t)$ makes x(t) overcome the boundaries of the quantization interval in which $s(\bar{k})d(t-\bar{k})$ falls. P_e can then be calculated as

$$P_e = \frac{1}{M}\Big[\sum_{i=M/2+1}^{M-1}\Big[\Pr\{|\xi(t)| \geq (2i-M-1)|s(\bar{k})| - (2i-M-2)\} + \Pr\{|\xi(t)| \geq (2i-M) - (2i-M-1)|s(\bar{k})|\}\Big] + \Pr\{|\xi(t)| \geq (M-1)|s(\bar{k})| - (M-2)\}\Big]. \quad (2)$$

The rigorous evaluation of the different terms in equation (2) would require the knowledge of $s(\bar{k})$ and of the statistical properties of $\xi(t)$, but these quantities cannot solely be estimated from the output measurements. However, it is possible to see that one can first determine an upper bound on P_e depending only on the coefficient $s(\bar{k})$, and then exploit a theoretical result so as to establish the possible values of $s(\bar{k})$ in order to compute the final bound for P_e. This task is accomplished in the next section.

4. THE UPPER BOUND ON THE PROBABILITY OF ERROR

Let us denote by α the normalized fourth-order cumulant of the equalizer output x(t), i.e.

$$\alpha := \frac{c_4^{xx}(0,0,0)}{c_4^{dd}(0,0,0)}.$$

It can be shown, [7], that α can be expressed in terms of the coefficients of the cascade channel-equalizer as follows: $\alpha = \sum_k s^4(k)$.

α can be computed from the output measurements and can be used to define an admissible range for $s(\bar{k})$. In fact, under the condition: $\sum_k s^2(k) = 1$ directly imposed by most equalization techniques ([8]-[11]), we have that if $\alpha = 1$, $s(\bar{k})$ can only have unitary modulus (and $s(k) = 0$ $\forall\, k \neq \bar{k}$), while if $\alpha < 1$ the following proposition holds.

Proposition

Let $\{s(k)\}$ be subject to the conditions

$$\sum_k s^2(k) = 1 \qquad \sum_k s^4(k) = \alpha \cdot \quad (3)$$

Then, for α>1/2 $s^2(\bar{k}) := \max_k \{s^2(k)\}$ is unique and satisfies the inequality (for a proof see the appendix):

$$\frac{1}{2}\left(1+\sqrt{2\alpha-1}\right) \leq s^2(\bar{k}) \leq \sqrt{\alpha}\,. \quad (4)$$

Remark It is easy to show that the lower and upper bounds in (4) cannot be improved. □

Thanks to the above proposition, once we have determined the upper bound on P_e as a function of $s(\bar{k})$, it can be successively converted into a function of α only, by means of (4).

We see from (2) that P_e is the sum of terms all of the form: $\Pr(|\xi(t)| \geq A)$, where A is a function of $s(\bar{k})$.

In order to compute the upper bound for such a probability, one can resort to one of the following three well-known inequalities (the convenience of using one of the three inequalities depends on the actual value of $s(\bar{k})$ and α, see below):

$$\Pr(|\xi(t)| \geq A) \leq \frac{\mathrm{var}\,\xi(t)}{A^2} \qquad \text{(Tchebycheff, [12]),} \quad (5)$$

$$\Pr(|\xi(t)| \geq A) \leq 2\exp\left(-\frac{1}{2}\frac{A^2}{\mathrm{var}\,\xi(t)}\right) \qquad \text{(Chernoff, [3]),} \quad (6)$$

$$\Pr(|\xi(t)| \geq A) \leq \frac{E\left[\xi^4(t)\right]}{A^4} \qquad \text{(Markov, [12]).} \quad (7)$$

The right-hand sides of inequalities (5)-(7) can be expressed in terms of $s(\bar{k})$ and α. This is obvious for (5) and (6), since $\mathrm{var}\xi(t) = (1 - s^2(\bar{k}))\sigma_d^2$, while, as far as equation (7) is concerned, we have

$$E\left[\xi^4(t)\right] = \sum_{k \neq \bar{k}} s^4(k)\, c_4^{dd}(0,0,0) + 3\left(\mathrm{var}\,\xi(t)\right)^2 \text{ and}$$

$$\sum_{k \neq \bar{k}} s^4(k) = \alpha - s^4(\bar{k}) \cdot \qquad (8)$$

The upper bound for the probability (6) is then determined as:

$$\Pr\left(|\xi(t)| \geq A\right) = \min\left\{1, \frac{\left(1 - s^2(\bar{k})\right)\sigma_d^2}{A^2},\ 2\exp\left(-\frac{1}{2}\frac{A^2}{\left(1 - s^2(\bar{k})\right)\sigma_d^2}\right), \frac{\left(\alpha - s^4(\bar{k})\right)c_4^{dd}(0,0,0) + 3\left(1 - s^2(\bar{k})\right)^2\sigma_d^4}{A^4}\right\}$$

By plugging (8) in (2) and using bounds (4), a simple but cumbersome computation leads to the following final upper bound on P_e:

$$P_e \leq \frac{1}{M}\sum_{i=M/2-1}^{M} L_i \,, \qquad (9)$$

where

$$L_i = \min\left\{1, \frac{\left(1 - f(\alpha)\right)\sigma_d^2}{a_i},\ 2\exp\left(-\frac{1}{2}\frac{a_i}{\left(1 - f(\alpha)\right)\sigma_d^2}\right), \frac{\left(\alpha - f^2(\alpha)\right)c_4^{dd}(0,0,0) + 3\left(1 - f(\alpha)\right)^2\sigma_d^4}{a_i^2}\right\} +$$

$$+ \min\left\{1, \frac{\left(1 - f(\alpha)\right)\sigma_d^2}{b_i},\ 2\exp\left(-\frac{1}{2}\frac{b_i}{\left(1 - f(\alpha)\right)\sigma_d^2}\right), \frac{\left(\alpha - f^2(\alpha)\right)c_4^{dd}(0,0,0) + 3\left(1 - f(\alpha)\right)^2\sigma_d^4}{b_i^2}\right\}$$

for $i = \frac{M}{2} + 1, \ldots, M - 1$ and

$$L_M = \min\left\{1, \frac{\left(1 - f(\alpha)\right)\sigma_d^2}{a_M},\ 2\exp\left(-\frac{1}{2}\frac{a_M}{\left(1 - f(\alpha)\right)\sigma_d^2}\right), \frac{\left(\alpha - f^2(\alpha)\right)c_4^{dd}(0,0,0) + 3\left(1 - f(\alpha)\right)^2\sigma_d^4}{a_M^2}\right\}$$

with

$$a_i = \left((2i - M - 1)\sqrt{f(\alpha)} - (2i - M - 2)\right)^2,$$

$$b_i = \left((2i - M) - (2i - M - 1)\sqrt{g(\alpha)}\right)^2,$$

$$f(\alpha) = \frac{1}{2}\left(1 + \sqrt{2\alpha - 1}\right) \text{ and } g(\alpha) = \sqrt{\alpha}\,.$$

The upper bound for P_e is displayed for different values of M in Fig.2. It can be shown that for α close to 1 the tighter bound is obtained by means of the Chernoff inequality which corresponds to the second element under the sign of min in the expression for L_i. The Markov and Tchebycheff inequalities give better bounds for lower values of α. The joint use of the three inequalities provides a tight bound for a wide range of values of α.

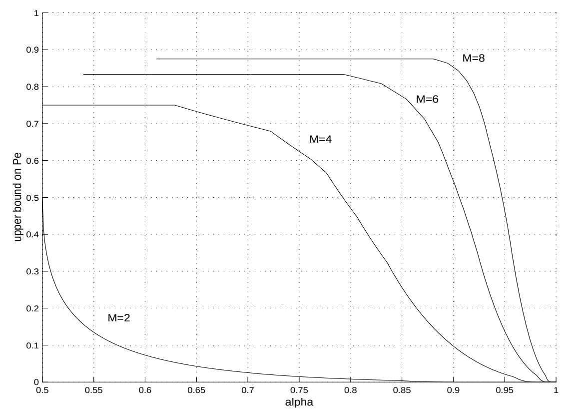

Figure 2. Upper bound on the probability of error P_e

5. SIMULATIONS

In our simulations, the fourth-order cumulant of the equalizer and channel cascade output has been computed from the theoretical probability density function of the source and the channel-equalizer impulse response, rather than from data samples. The obtained value of α is used to determine the upper bound on the probability of error.

The tightness of the bound has been tested by simulation. 10 records of 10000 samples each have been considered in each example.

Example 1

Consider a channel with transfer function

$H(z) = z^{-2}+3,5z^{-1}+1,5$

fed by an i.i.d. equiprobable 4-ary input with values in {-3,-1,1,3}, and the FIR equalizer with 6 coefficients identified with the blind equalization procedure described in [13]. The impulse response of the channel-equalizer cascade is represented in Fig.3.

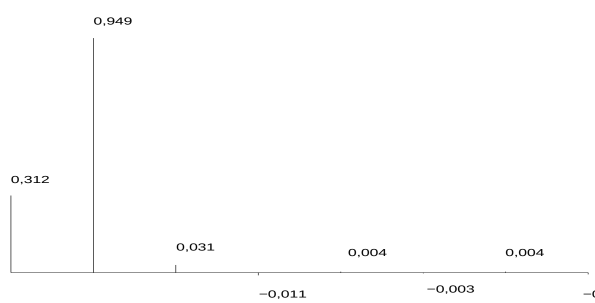

Figure 3. Impulse response of the channel-equalizer cascade

The normalized fourth-order cumulant of the equalizer output is: α=0,8216. The corresponding upper bound for the probability of error computed by means of eq.(9) is: 0,3793.

Example 2

Consider the channel with transfer function

$H(z) = 0,227\, z^{-4} + 0,460\, z^{-3} + 0,688\, z^{-2} + 0,460\, z^{-1} + 0,227$

described in [1], fed by an i.i.d. equiprobable 4-ary input with values in {-3,-1,1,3}, and the FIR equalizer with 31 coefficients identified with the blind equalization procedure in [13]. The impulse response of the channel-equalizer cascade is represented in Fig.4.

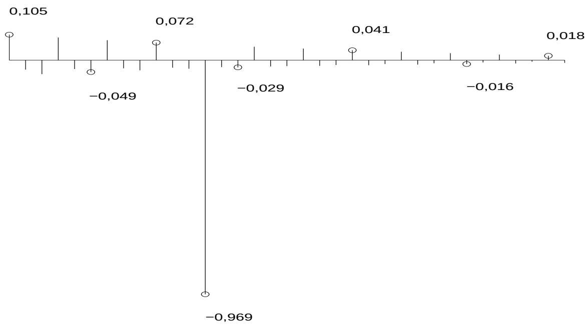

Figure 4. Impulse response of the channel-equalizer cascade

The normalized fourth-order cumulant of the equalizer output is: α=0,8809. The corresponding upper bound for the probability of error is: 0,1463.

The table below displays the number of errors for each simulation.

	Number of errors	
Simulation	***Example 1***	***Example 2***
1	1690	734
2	1751	597
3	1635	533
4	1706	705
5	1658	550
6	1729	645
7	1736	644
8	1693	521
9	1719	553
10	1707	686

In these examples, bound (9) gives results which are conservative by approximately a factor 2 with respect to simulations. Discrepancy between the theoretical bound and practical experiments is expected since the bound in eq.(9) must obviously hold true for any channel-equalizer cascade and not only for the actual one considered in the simulations.

APPENDIX:

Proof of the Proposition in Section 4

The following equation is easily derived from (3): $\sum_k s^2(k)\sum_{j\neq k} s^2(j) = 1-\alpha$.

By extracting the term $s^2(\bar{k})$, it can be rewritten as:

$$2s^2(\bar{k})\sum_{j\neq\bar{k}} s^2(j) + \sum_{k\neq\bar{k}} s^2(k) \sum_{j\neq k,\bar{k}} s^2(j) = 1-\alpha \qquad (10)$$

Obviously $\sum_{j\neq k,\bar{k}} s^2(j) \geq 1-2s^2(\bar{k}), \quad \forall\, k$. (11)

So (10) and (11) entail the inequality:

$$2s^2(\bar{k})\sum_{j\neq\bar{k}} s^2(j) + \sum_{k\neq\bar{k}} s^2(k)\left(1-2s^2(\bar{k})\right) \leq 1-\alpha$$

which can be reduced to

$$s^2(\bar{k}) \geq \alpha\,. \qquad (12)$$

If α>1/2, it is then evident that no other coefficients than $s(\bar{k})$ have square greater than α and consequently $s^2(\bar{k})$ is the only maximum value of the sequence $\left\{s^2(k)\right\}$. Moreover

$$\sum_{j\neq k,\bar{k}} s^2(j) \geq 0 > 1-2s^2(\bar{k}), \quad \forall\, k\,, \qquad (13)$$

therefore (10) and (13) lead to the condition:

$$s^2(\bar{k}) \leq \frac{1}{2}\left(1-\sqrt{2\alpha-1}\right) \cup s^2(\bar{k}) \geq \frac{1}{2}\left(1+\sqrt{2\alpha-1}\right). \qquad (14)$$

Both (12) and (14) have to be satisfied, consequently we get the lower bound: $s^2(\bar{k}) \geq \frac{1}{2}\left(1+\sqrt{2\alpha-1}\right)$, while the upper bound simply follows from condition (3).

REFERENCES

[1] J.G. Proakis: *Digital Communications*, 3rd ed. McGraw-Hill, New York, 1995.

[2] B.R. Saltzberg: "Intersymbol Interference Error Bounds with Application to Ideal Bandlimited Signaling", *IEEE Trans. on Inf. Theory*, vol. IT-14, n. 4, pp. 563-568, 1968.

[3] R. Lugannani: "Intersymbol Interference and Probability of Error in Digital Systems", *IEEE Trans. on Inf. Theory*, vol. IT-15, n. 6, pp. 682-688, 1969.

[4] Y.C. Jenq, B. Liu and J.B. Thomas: "Probability of Error in PAM Systems with Intersymbol Interference and Additive Noise", *IEEE Trans. on Inf. Theory*, vol. IT-23, n. 5, pp. 575-582, 1977.

[5] K. Yao and E.M. Biglieri: "Multidimensional Moment Error Bounds on Digital Communication Systems", *IEEE Trans. on Inf. Theory*, vol. IT-26, n. 4, 1980.

[6] R.A. Kennedy, G. Pulford, B.D.O. Anderson and R.R. Bitmead: "When Has a Decision-Directed Equalizer Converged?", *IEEE Trans. on Commun.*, vol. 37, n. 8, pp. 879-884, 1989.

[7] J.M. Mendel: "Tutorial on Higher-Order Statistics (Spectra) in Signal Processing and System Theory: Theoretical Results and Some Applications", *IEEE Proc.* , vol. 79, n. 3, pp. 278-305, March 1991.

[8] S. Bellini: "Blind Equalization", *ALTA FREQUENZA*, vol. LVII, n. 7, pp. 445-450, 1988.

[9] O. Shalvi and E. Weinstein: "New Criteria for Blind Deconvolution of Nonminimum Phase Systems (Channels)" *IEEE Trans. on Inf. Theory*, vol. IT-36, pp. 312-321, 1990.

[1] A. Benveniste, M. Goursat and G. Ruget: "Robust Identification of a Nonminimum Phase System: Blind Adjustment of a Linear Equalizer in Data Communications", *IEEE Trans. on Automatic Control*, vol. AC-25, pp. 385-399, 1980.

[2] B. Jellonek and K.D. Kammeyer: "A Closed Form Solution to Blind Equalization", *Signal Processing*, vol. 36, issue 3, pp. 251-259, 1994.

[3] A. Papoulis: *Probability, Random variables and Stochastic Processes.* N.Y.: McGraw-Hill, 1984.

[4] B. Jellonek and K.D. Kammeyer: "A Unified Approach to Blind Equalization", private communication, 1995.