

Exact quantisation of the relativistic Hopfield model

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Abstract

We investigate the quantisation in the Heisenberg representation of a relativistically covariant version of the Hopfield model for dielectric media, which entails the interaction of the quantum electromagnetic field with the matter dipole fields, represented by a mesoscopic polarisation field. A full quantisation of the model is provided in a covariant gauge, with the aim of maintaining explicit relativistic covariance. Breaking of the Lorentz invariance due to the intrinsic presence in the model of a preferred reference frame is also taken into account. Relativistic covariance forces us to deal with the unphysical (scalar and longitudinal) components of the fields, furthermore it introduces, in a more tricky form, the well-known dipole ghost of standard QED in a covariant gauge. In order to correctly dispose of this contribution, we implement a generalised Lautrup trick. Furthermore, causality and the relation of the model with the Wightman axioms are also discussed.

Keywords: Hopfield model, Heisenberg representation, polariton, QED

PACS: 03.70.+k, 11.10.-z, 12.20.-m

1. Introduction

Interaction of the quantum electromagnetic field with quantum matter represents a huge and long-standing field of investigation, which can involve both a microscopic description of the fields and a more phenomenological approach in which some microscopic interactions are described by means of effective fields, which in turn can be quantised and coupled with QED through effective interaction vertices. An example of this kind of approach is provided by models describing interactions of the electromagnetic field with dielectric media. See e.g. [1] and references therein. Interest in this framework has been recently risen up, due to the attempt to reproduce quantum emission by a black hole in the lab by means of analogous systems, i.e. systems displaying the same kinematics which is at the root of the Hawking effect [2], [3], [4], [5]. A possible implementation of the analogue Hawking effect as a quantum effect in dielectric media has been presented and analysed in [6], [7], [8] and further developed in [9]. Improvement of such theoretical analysis required to look for a more fundamental model, able to maintain the main aspects of the phenomenology of the system, and still providing a good mathematical model.

A simple and economic model describing the interaction of the electromagnetic field with a homogeneous and isotropic medium is the Hopfield model [10]. With economic we mean that this model allows to reproduce the observed spectrum of the electromagnetic field into a class of transparent dielectric media (the Sellmeier dispersion relations) without entering the details of a complete physical description of the materials themselves: they are described as a lattice (or a field) of independent oscillators associated with a characteristic frequency ω_0 (the extension to several characteristic frequencies is also allowed). Thus, the model does not

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record any detail about the matter apart from the characteristic frequency of the dipoles.

It is worth mentioning that the Hopfield model represents a cornerstone for many studies which adopt a semi-phenomenological approach to the quantization of the electromagnetic field in dielectric media, by introducing mesoscopic polarization fields. Extensions of the model to include matter fields absorption appear in [11], and then are developed further on by including also spacetime dependence on susceptibility, and anisotropic magnetodielectric media [12, 13, 14, 15]. We also recall that canonical quantization of the electromagnetic field has been performed in [16], and extended further on to the case of a moving dielectric medium in [17]. We don't delve into the analysis of absorption herein, as transparent dielectrics are enough for our present purposes.

Nevertheless, the Hopfield model in its original form is not able to accomplish the preposed targets. Indeed, the analogue Hawking radiation and other perturbative and non perturbative phenomena occur in presence of inhomogeneities propagating through a homogeneous background (see e.g. [5]). These phenomena require to pass from the lab frame to another inertial one, such as the frame comoving with a perturbation or to a carrier signal. For this reason we have developed a relativistic covariant version of the Hopfield model in [18]. In [19] we have considered the quantisation in the lab frame in a simple fixed gauge, in order to study photon production originated by time-dependent perturbations. The covariant and gauge invariant quantisation has been considered in [18], where in the construction of states the linear coupling between the electromagnetic field and the polarisation field was treated perturbatively. The Hawking effect as a non perturbative effect has been also considered in a simplified model in [20].

As the action of the full covariant Hopfield model is quadratic in all fields, we expect that the theory may be quantisable in an exact way in the Heisenberg representation. This is the aim of the present paper. It is worth noting that this is a relevant task for several reasons. First, the fact that the perturbative quantisation seems to be well-defined and causal, does not ensure that the same properties hold true for the full theory, and difficulties may arise in the exact theory in the Heisenberg representation, which instead remains hidden in perturbative construction (but which would appear after adding further perturbative terms). We introduce, for purely exemplificative aims, a simple model where such kind of problems emerge in a straightforward way. Let us consider two Klein-Gordon scalar fields linearly interacting, having the Lagrangian density

$$L = \frac{1}{2}\phi_1\Box\phi_1 - \frac{m_1^2}{2}\phi_1^2 + \frac{1}{2}\phi_2\Box\phi_2 - \frac{m_2^2}{2}\phi_2^2 - \lambda^2\phi_1\phi_2. \quad (1.1)$$

The equations of motion in the Fourier dual space are

$$\begin{pmatrix} -\mathbf{k}^2 + m_1^2 & \lambda^2 \\ \lambda^2 & -\mathbf{k}^2 + m_2^2 \end{pmatrix} \begin{pmatrix} \tilde{\phi}_1 \\ \tilde{\phi}_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad (1.2)$$

where we use the symbol $\mathbf{k} = (k^0, \vec{k})$ for indicating a spacetime vector, whereas \vec{k} indicates a spatial vector. In order to provide nontrivial solutions, the determinant of the matrix above has to vanish, and then one gets the dispersion relation

$$(\mathbf{k}^2)^2 - (m_1^2 + m_2^2)\mathbf{k}^2 + m_1^2 m_2^2 - \lambda^4 = 0. \quad (1.3)$$

Since the discriminant is always non negative, this equation has two real solutions in \mathbf{k}^2 . However, we see that one of these solutions will be negative if the condition

$$\lambda^2 \leq m_1 m_2 \quad (1.4)$$

is not satisfied. This means that, if $\lambda^2 > m_1 m_2$, the exact theory after a linear recombination of the fields, decouples in a pair of fields, one bradionic and the other tachyonic, colored and then outside the light cone. For example, if one of the original masses is zero, say $m_1 = m$, $m_2 = 0$, then the theory is affected by the presence of tachyon states, irrespectively of the smallness of the coupling constant w.r.t. m . It is not evident that this fault of the model could be easily recovered in the perturbative formulation. Presence of the tachyonic modes shows an instability in the theory, due to the fact that the perturbation is not bounded

from below unless it is small enough w.r.t. both the masses.

The covariant version of the Hopfield model is found in a similar situation: we are considering a massless field, the photon, coupled to an oscillator field with a linear coupling which is, a priori, non necessarily small, and, surely not at all small with respect to the zero photon mass. Still, we shall find that the covariant Hopfield model behaves differently and the theory remains well defined for all values of the coupling constant. A second reason justifying the relevance of the exact quantisation is the following. The dipole field is defined as a field of identical oscillators which, in the frame where the matter is at rest, satisfy the harmonic equation

$$\frac{\partial^2 \vec{P}}{\partial t^2} + \omega_0^2 \vec{P} = \vec{0}. \quad (1.5)$$

Now, this equation introduces two conceptual difficulties. The first one is that it is not a priori relativistic: for example, if we interpreted the dipoles as proportional to the displacement of the centre of mass of the electron w.r.t. to the one of the nuclei, then, since the equation of motion is linear, nothing would prevent both the product $\omega_0^2 \vec{P}$ and the speed of the moving clouds of electrons, to be arbitrarily large, also compared to the speed of light. This could give rise to problems with the causality properties of the theory. On the other hand, the fact that in the model the microscopical details of the structure of matter are irrelevant, at least as far as low energy scales are involved, suggests that it is not affected by such microscopical aspects and it might keep satisfying relativistic causality. For this reason it is crucial to check the consistency of the theory by verifying that causality is not violated. At this point enters the second conceptual complication. Indeed, causality can be easily proved as a byproduct of the compatibility of the equal time canonical commutation relations (ETCCR) and relativistic covariance. Since the dipole equations are defined in a specific frame, the one where the dielectric is at rest, in any other frame the velocity \mathbf{v} specifying the frame will enter the equations of motion. Lorentz invariance is broken by the presence of the dielectric medium, which selects a preferred inertial reference frame. This does not implies that also the covariance is broken, but it makes the question more subtle, since it requires for relativistic covariance to be realised correctly in the construction of the representation of the algebra of fields. Again, we shall see that this is the case for the relativistic Hopfield model, so that it preserves both covariance and causality.

Beyond the above mentioned questions, further difficulties arise in the full construction, as the control of gauge invariance, the presence of the dipole ghost and the right choice of the test functions space necessary to define the fields as operator valued distributions.

We mention that some of the results presented here have been anticipated by us in [21], in a simplified form for a more elementary model in which the electromagnetic field and the polarisation field are replaced by scalar fields, so that several technical complications, such as gauge invariance and the presence of constraints, are absent. However, no proves are presented there, only the ideas, and all technical details and mathematical proofs will be found herein. Finally, we add some comments on notations: we shall use the symbols \mathbf{x}, \mathbf{k} or \mathbf{V} for the space-time vectors having components x^μ, k^μ and V^μ , whereas their spatial component will be indicated with $\vec{x}, \vec{k}, \vec{V}$ and so on. We shall use k^2 for the scalar $\mathbf{k}^2 = \mathbf{k} \cdot \mathbf{k}$.

2. The relativistic Hopfield model and its solutions

Let us consider the relativistic Hopfield model with a single polarisation field with resonance frequency ω_0 , as presented in [18]. The Lagrangian density is

$$\begin{aligned} \mathcal{L} = & -\frac{\varepsilon_0 c^2}{4} F_{\mu\nu} F^{\mu\nu} - \frac{1}{2\chi\varepsilon_0\omega_0^2} (v^\rho \partial_\rho P_\mu)(v^\sigma \partial_\sigma P^\mu) + \frac{1}{2\chi\varepsilon_0} P_\mu P^\mu + \frac{g}{2} (v_\mu P_\nu - v_\nu P_\mu) F^{\mu\nu} \\ & + B \partial_\mu A^\mu + \frac{\xi}{2\varepsilon_0 c^2} B^2, \end{aligned} \quad (2.1)$$

¹In equation (1.5) we have purposefully neglected the driving term, as our aim is to point out which kind of troubles could arise in our model with respect to the problem of causality. The driving term would simply make harder the discussion, without substantial improvements.

so that the equation of motion are

$$\varepsilon_0 c^2 (\square \eta_{\mu\nu} - \partial_\mu \partial_\nu) A^\mu - g (\eta_{\mu\nu} v^\rho \partial_\rho - v_\mu \partial_\nu) P^\nu - \partial_\mu B = 0, \quad (2.2)$$

$$g (\eta_{\mu\nu} v^\rho \partial_\rho - v_\nu \partial_\mu) A^\nu + \frac{1}{\chi \varepsilon_0 \omega_0^2} (\omega_0^2 + (v^\rho \partial_\rho)^2) P^\mu = 0, \quad (2.3)$$

$$\partial_\mu A^\mu + \frac{\xi}{\varepsilon_0 c^2} B = 0, \quad (2.4)$$

together with the defining constraint $v^\mu P_\mu = 0$. Here the polarisation field P^μ is related to the polarisation density p^μ by $\mathbf{p} = g\mathbf{P}$. Note that, in order to describe the polarization field in a covariant form, one must introduce also the component P_0 of the field, which is absent in the rest frame. As discussed in [18], this new component is not an independent one, it depends on the spatial components, as can be easily inferred from the constraint $v^\mu P_\mu = 0$, which is required at the level of the classical theory for the polarization vector. Note that in the rest frame, where $v^\mu = (c, 0, 0, 0)$, the above constraint leads to $P_0 = 0$, as expected. It is also remarkable that, with respect to a phenomenological approach, where one introduces by hand an induction tensor $G^{\mu\nu}$, we can deduce such a tensor from our mesoscopic model (cf. e.g. [18]).

After defining the Fourier transforms of the fields

$$\tilde{\mathbf{A}}(\mathbf{k}) = \int_{\mathbb{R}^4} d^4 x e^{i\mathbf{k}\cdot\mathbf{x}} \mathbf{A}(\mathbf{k}), \quad (2.5)$$

$$\tilde{\mathbf{P}}(\mathbf{k}) = \int_{\mathbb{R}^4} d^4 x e^{i\mathbf{k}\cdot\mathbf{x}} \mathbf{P}(\mathbf{k}), \quad (2.6)$$

$$\tilde{B}(\mathbf{k}) = \int_{\mathbb{R}^4} d^4 x e^{i\mathbf{k}\cdot\mathbf{x}} B(\mathbf{k}), \quad (2.7)$$

the system of partial differential equations transform into the algebraic system

$$\mathcal{M}V \equiv \begin{pmatrix} \varepsilon_0 c^2 (k^2 \mathbb{I} - \mathbf{k} \otimes \mathbf{k}) & -ig(\omega \mathbb{I} - \mathbf{v} \otimes \mathbf{k}) & -i\mathbf{k} \\ ig(\omega \mathbb{I} - \mathbf{k} \otimes \mathbf{v}) & \frac{1}{\chi \varepsilon_0} \left(\frac{\omega^2}{\omega_0^2} - 1 \right) \mathbb{I} & 0 \\ i\mathbf{k} & 0 & -\frac{\xi}{\varepsilon_0 c^2} \end{pmatrix} \begin{pmatrix} \tilde{\mathbf{A}} \\ \tilde{\mathbf{P}} \\ \tilde{B} \end{pmatrix} = \begin{pmatrix} \mathbf{0} \\ \mathbf{0} \\ 0 \end{pmatrix}, \quad (2.8)$$

where $\omega := k^\mu v_\mu$. We need to solve this system of equations in a distributional sense. The determinant of the matrix \mathcal{M} defining the system is

$$\det \mathcal{M} = -c^6 \frac{(k^2)^2}{\varepsilon_0 \chi^4} \left(\frac{\omega^2}{\omega_0^2} - 1 \right)^4 \left[c^2 k^2 - \frac{g^2 \chi \omega_0^2 \omega^2}{\omega^2 - \omega_0^2} \right]^2 \left[1 - \frac{g^2 \chi \omega_0^2}{\omega^2 - \omega_0^2} \right]. \quad (2.9)$$

Thus, the solutions will have support in the union of four sets. The first set is defined by $k^2 = 0$, the same spectrum of free photons in vacuum. The associated modes of the photon field do not couple with matter, and are expected to appear as nonphysical states. The second set is defined by $\omega^2 = \omega_0^2$: since ω_0 is the resonance frequency, these solutions correspond to modes where the dipoles of the polarisation field oscillate freely. We shall see that these modes are projected out by the physical constraint $\mathbf{v} \cdot \mathbf{P} = 0$. The next branch is defined by

$$1 - \frac{g^2 \chi \omega_0^2}{\omega^2 - \omega_0^2} = 0. \quad (2.10)$$

This mode looks like a resonance at a shifted frequency

$$\bar{\omega} = \omega_0 \sqrt{1 + g^2 \chi}. \quad (2.11)$$

and it is not particularly significant, as it is shown to correspond to an uncoupled contribution to the polarisation field. Finally, there are the modes satisfying the equation

$$c^2 k^2 - \frac{g^2 \chi \omega_0^2 \omega^2}{\omega^2 - \omega_0^2} = 0. \quad (2.12)$$

These modes reproduce exactly the Sellmeier relation with just one resonance, so that they represent the physical modes we are interested to.

In order to find the solutions we can make use of the identity

$$x\delta^{(n+1)}(x) = -(n+1)\delta^{(n)}(x), \quad (2.13)$$

where $\delta^{(n)}$ is the n -th derivative of the Dirac delta function $\delta = \delta^{(0)}$. We can assume, without loss of generality, that \mathbf{k} and \mathbf{v} are linearly independent, and define the four-vectors \mathbf{e}_1 and \mathbf{e}_2 , satisfying $\mathbf{e}_i \cdot \mathbf{k} = 0$, $\mathbf{e}_i \cdot \mathbf{v} = 0$ and such that $\mathbf{k}, \mathbf{v}, \mathbf{e}_1, \mathbf{e}_2$ is a basis for \mathbb{R}^4 . Then we find for the general solution of the linear system (see Appendix A for a deduction):

$$\begin{aligned} \tilde{\mathbf{A}}(\mathbf{k}) &= (\omega^2 - \omega_0^2)a_0(\mathbf{k})\mathbf{v}\delta(k^2) + \omega [\omega^2 - \omega_0^2 - \xi(\omega^2 - \bar{\omega}^2)] a_0(\mathbf{k})\mathbf{k}\delta^{(1)}(k^2) + a_3(\mathbf{k})\mathbf{k}\delta(k^2) \\ &\quad + \frac{(\omega\mathbf{k} - k^2\mathbf{v})}{\varepsilon_0 c^2 k^2} g b_3(\mathbf{k})\delta(\omega^2 - \bar{\omega}^2) + \sum_{i=1}^2 \delta\left(\varepsilon_0 c^2 k^2 - \varepsilon_0 \frac{g^2 \chi \omega_0^2 \omega^2}{\omega^2 - \omega_0^2}\right) a_i(\mathbf{k})\mathbf{e}_i, \end{aligned} \quad (2.14)$$

$$\begin{aligned} \tilde{\mathbf{P}}(\mathbf{k}) &= -ig\chi\varepsilon_0\omega_0^2(\omega\mathbf{v} - \mathbf{k})a_0(\mathbf{k})\delta(k^2) + i(\omega\mathbf{v} - \mathbf{k})b_3(\mathbf{k})\delta(\omega^2 - \bar{\omega}^2) \\ &\quad - ig \sum_{i=1}^2 \frac{\varepsilon_0\chi\omega_0^2}{\omega^2 - \omega_0^2} \omega \delta\left(\varepsilon_0 c^2 k^2 - \varepsilon_0 \frac{g^2 \chi \omega_0^2 \omega^2}{\omega^2 - \omega_0^2}\right) a_i(\mathbf{k})\mathbf{e}_i, \end{aligned} \quad (2.15)$$

$$\tilde{\mathbf{B}}(\mathbf{k}) = i\varepsilon_0 c^2 \omega (\omega^2 - \bar{\omega}^2) a_0(\mathbf{k}) \delta(k^2). \quad (2.16)$$

There is also an additional mode for $\tilde{\mathbf{P}}$ proportional to $\delta(\omega - \omega_0)$, which is excluded by the condition $\mathbf{v} \cdot \tilde{\mathbf{P}} = 0$. At this point, we can be a little bit more precise on specifying the solutions space. The Fourier transforms are supported on the sets

$$\Sigma_0 := \{\mathbf{k} \in \mathbb{R}^{1,3} | k^2 = 0\}, \quad (2.17)$$

$$\Sigma_1 := \{\mathbf{k} \in \mathbb{R}^{1,3} | \omega^2 = \bar{\omega}^2\}, \quad (2.18)$$

$$\Sigma_2 := \{\mathbf{k} \in \mathbb{R}^{1,3} | c^2 k^2 - \frac{g^2 \chi \omega_0^2 \omega^2}{\omega^2 - \omega_0^2} = 0\}. \quad (2.19)$$

At first sight, we can take the coefficients in the set $\mathcal{S}(\mathbb{R}^4)$ of rapidly decreasing smooth functions. However, a problem arises because these supports have a non vanishing intersection. We shall see in section 5 that Σ_2 intersect Σ_1 and Σ_0 in two isolated points, which give no rise to any singularity. Instead, $\Sigma_0 \cap \Sigma_1$ consists in a two-dimensional ellipsoid. We require for the Fourier coefficient to vanish on this intersection locus at the appropriate degree. The presence of the $\delta^{(1)}(k^2)$ suggests that the order should be two. We shall confirm this in section 5.

Let us discuss shortly the solutions. The presence of the $\delta^{(1)}(k^2)$ distribution is related to the fact that the determinant of \mathcal{M} vanishes at second order in k^2 . This also happens for the Sellmeier modes, but one can check that no $\delta^{(1)}$ contributions are present in that case. The $\delta^{(1)}(k^2)$ term is well known already for the free electromagnetic field, and, indeed, it is not a surprise for it to appear in the sector $k^2 = 0$. This is the dipole ghost, responsible for the infrared divergent terms for the propagators in a covariant gauge different from the Feynman gauge [22, 23]. One practical way to manage it in standard QED consists in choosing the Feynman gauge, $\xi = 1$, where the $\delta^{(1)}$ term disappears and one is dealt with the standard Fock construction where only the spurious harmonic gauge modes are involved. In our case the dipole ghost term is proportional to

$$1 - \xi \frac{\omega^2 - \bar{\omega}^2}{\omega^2 - \omega_0^2}, \quad (2.20)$$

which cannot be eliminated by any choice of the gauge parameter ξ . So, we need a different strategy for managing the dipole ghost. The key observation is to note that the dipole ghost term is proportional to \mathbf{k} , i.e. it is a pure gauge mode, corresponding to a vanishing field strength and thus can be hopefully made harmless in some way. This is essentially the strategy adopted by Lautrup in [24] for constructing the Fock representation for the electromagnetic field in any covariant gauge, and can be easily adapted to our case, so adsorbing the dipole ghost into the pure gauge part. The details are reported in Appendix B.

3. Scalar products and Fourier modes

We can now write down the whole explicit solution. To this end we can eliminate the delta distributions by integrating out the zeroth component of \mathbf{k} . In doing this, we must consider separately each of the supports of the deltas and look at the corresponding spectrum. Let us look shortly at the various spectra:

1. The support $k^2 = 0$ corresponds to the two branches $k^0(\vec{k}) = \pm|\vec{k}|$. We shall use the notation k_{\pm} to indicate the corresponding functions of \vec{k} . More in general, with the notation $f_{\pm}(\mathbf{k})$ we mean that k^0 is evaluated at $k_{\pm}^0(\vec{k})$. The property $k_+^0(\vec{k}) = -k_-^0(\vec{k})$ can be employed in order to get real solutions.
2. The support $(\mathbf{k} \cdot \mathbf{v})^2 - (\bar{\omega})^2 = 0$ has the two branches

$$k^0(\vec{k}) = \frac{\vec{k} \cdot \vec{v} \pm \bar{\omega}}{v^0}. \quad (3.1)$$

We shall reserve the name $k_{>}^0$ for the solution corresponding to the positive sign, which is the positive solution in the lab frame. More in general, given a function $f(\mathbf{k})$ we shall write $f_{>}(\vec{k}) := f(k_{>}^0(\vec{k}), \vec{k})$. If with $k_{<}^0$ we indicate the solution with the minus sign, then $k_{<}^0(-\vec{k}) = -k_{>}^0(\vec{k})$. In particular, $\omega_{>} = \bar{\omega}$.

3. The support

$$c^2 k^2 - \frac{g^2 \chi \omega_0^2 \omega^2}{\omega^2 - \omega_0^2} = 0 \quad (3.2)$$

defines four branches, two with positive ω and two with negative ω . If we call $k_{(a)}^0$, $a = 1, 2$, the two solutions corresponding to positive values of ω , then the solutions with negative ω are $-k_{(a)}^0(-\vec{k})$. Moreover, if we order $k_{(a)}^0$ so that $k_{(1)}^0 > k_{(2)}^0$, then one has $k_{(1)}^0 > \omega_0 > k_{(2)}^0$. See [21] for these properties. Again, we shall use the notation $f_{(a)}(\vec{k}) = f(k_{(a)}^0(\vec{k}), \vec{k})$ for any function $f(\mathbf{k})$. Finally, we introduce the notations

$$DR := \varepsilon_0 \left(c^2 k^2 - \frac{g^2 \chi \omega_0^2 \omega^2}{\omega^2 - \omega_0^2} \right) \quad (3.3)$$

and indicate its derivative w.r.t. k^0 by

$$DR' = \frac{dDR}{dk^0} = 2\varepsilon_0 c^2 k^0 + 2\varepsilon_0 \omega v^0 \frac{g^2 \chi \omega_0^4}{(\omega^2 - \omega_0^2)^2}. \quad (3.4)$$

Thus, the explicit solution of the original system can be written in the form

$$\begin{aligned}
\mathbf{A}(\mathbf{x}) = & \int_{\mathbb{R}^3} \frac{d^3\vec{k}}{(2\pi)^3} \frac{a_0(\vec{k})}{2|\vec{k}|} \left[(\omega_+^2 - \omega_0^2)\mathbf{v} + \frac{1}{2} (\xi(\omega_+^2 - \bar{\omega}^2) - \omega_+^2 + \omega_0^2) (\mathbf{v} - i(\mathbf{v} \cdot \mathbf{x})\mathbf{k}_+) \right] e^{-i\mathbf{k}_+ \cdot \mathbf{x}} \\
& - i \int_{\mathbb{R}^3} \frac{d^3\vec{k}}{(2\pi)^3} \frac{a_0(\vec{k})}{2|\vec{k}|} \frac{\xi(\omega_+^2 - \bar{\omega}^2)^2 + (\omega_+^2 - \omega_0^2)(\omega_+^2 - \bar{\omega}^2) - 2\omega_+^2 g^2 \chi \omega_0^2}{4\omega_+(\omega_+^2 - \bar{\omega}^2)} \mathbf{k}_+ e^{-i\mathbf{k}_+ \cdot \mathbf{x}} \\
& + i \int_{\mathbb{R}^3} \frac{d^3\vec{k}}{(2\pi)^3} \frac{a_3(\vec{k})}{2|\vec{k}|} \mathbf{k}_+ e^{-i\mathbf{k}_+ \cdot \mathbf{x}} \\
& - \frac{g}{\varepsilon_0 c^2} \int_{\mathbb{R}^3} \frac{d^3\vec{k}}{(2\pi)^3} \frac{b_3(\vec{k})}{2\bar{\omega}v^0} \left(\mathbf{v} - \frac{\bar{\omega}}{k_{>}^2} \mathbf{k}_{>} \right) e^{-i\mathbf{k}_{>} \cdot \mathbf{x}} + \sum_{a=1}^2 \sum_{i=1}^2 \int_{\mathbb{R}^3} \frac{d^3\vec{k}}{(2\pi)^3} \frac{a_i^{(a)}(\vec{k})}{DR'_{(a)}} \mathbf{e}_i^{(a)} e^{-i\mathbf{k}_{(a)} \cdot \mathbf{x}} \\
& + \int_{\mathbb{R}^3} \frac{d^3\vec{k}}{(2\pi)^3} \frac{a_0^\dagger(\vec{k})}{2|\vec{k}|} \left[(\omega_+^2 - \omega_0^2)\mathbf{v} + \frac{1}{2} (\xi(\omega_+^2 - \bar{\omega}^2) - \omega_+^2 + \omega_0^2) (\mathbf{v} + i(\mathbf{v} \cdot \mathbf{x})\mathbf{k}_+) \right] e^{i\mathbf{k}_+ \cdot \mathbf{x}} \\
& + i \int_{\mathbb{R}^3} \frac{d^3\vec{k}}{(2\pi)^3} \frac{a_0^\dagger(\vec{k})}{2|\vec{k}|} \frac{\xi(\omega_+^2 - \bar{\omega}^2)^2 + (\omega_+^2 - \omega_0^2)(\omega_+^2 - \bar{\omega}^2) - 2\omega_+^2 g^2 \chi \omega_0^2}{4\omega_+(\omega_+^2 - \bar{\omega}^2)} \mathbf{k}_+ e^{i\mathbf{k}_+ \cdot \mathbf{x}} \\
& - i \int_{\mathbb{R}^3} \frac{d^3\vec{k}}{(2\pi)^3} \frac{a_3^\dagger(\vec{k})}{2|\vec{k}|} \mathbf{k}_+ e^{i\mathbf{k}_+ \cdot \mathbf{x}} \\
& - \frac{g}{\varepsilon_0 c^2} \int_{\mathbb{R}^3} \frac{d^3\vec{k}}{(2\pi)^3} \frac{b_3^\dagger(\vec{k})}{2\bar{\omega}v^0} \left(\mathbf{v} - \frac{\bar{\omega}}{k_{>}^2} \mathbf{k}_{>} \right) e^{i\mathbf{k}_{>} \cdot \mathbf{x}} + \sum_{a=1}^2 \sum_{i=1}^2 \int_{\mathbb{R}^3} \frac{d^3\vec{k}}{(2\pi)^3} \frac{a_i^{(a)\dagger}(\vec{k})}{DR'_{(a)}} \mathbf{e}_i^{(a)} e^{i\mathbf{k}_{(a)} \cdot \mathbf{x}}, \tag{3.5}
\end{aligned}$$

$$\begin{aligned}
\mathbf{P}(\mathbf{x}) = & -ig\varepsilon_0\chi\omega_0^2 \int_{\mathbb{R}^3} \frac{d^3\vec{k}}{(2\pi)^3} \frac{a_0(\vec{k})}{2|\vec{k}|} (\omega_+\mathbf{v} - \mathbf{k}_+) e^{-i\mathbf{k}_+ \cdot \mathbf{x}} + i \int_{\mathbb{R}^3} \frac{d^3\vec{k}}{(2\pi)^3} \frac{b_3(\vec{k})}{2\bar{\omega}v^0} (\omega_+\mathbf{v} - \mathbf{k}_+) e^{-i\mathbf{k}_+ \cdot \mathbf{x}} \\
& - ig\varepsilon_0\chi\omega_0^2 \sum_{a=1}^2 \sum_{i=1}^2 \int_{\mathbb{R}^3} \frac{d^3\vec{k}}{(2\pi)^3} \frac{a_i^{(a)}(\vec{k})}{DR'_{(a)}} \frac{\omega_{(a)}}{\omega_{(a)}^2 - \omega_0^2} \mathbf{e}_i^{(a)} e^{-i\mathbf{k}_{(a)} \cdot \mathbf{x}} \\
& + ig\varepsilon_0\chi\omega_0^2 \int_{\mathbb{R}^3} \frac{d^3\vec{k}}{(2\pi)^3} \frac{a_0^\dagger(\vec{k})}{2|\vec{k}|} (\omega_+\mathbf{v} - \mathbf{k}_+) e^{i\mathbf{k}_+ \cdot \mathbf{x}} - i \int_{\mathbb{R}^3} \frac{d^3\vec{k}}{(2\pi)^3} \frac{b_3^\dagger(\vec{k})}{2\bar{\omega}v^0} (\omega_+\mathbf{v} - \mathbf{k}_+) e^{i\mathbf{k}_+ \cdot \mathbf{x}} \\
& + ig\varepsilon_0\chi\omega_0^2 \sum_{a=1}^2 \sum_{i=1}^2 \int_{\mathbb{R}^3} \frac{d^3\vec{k}}{(2\pi)^3} \frac{a_i^{(a)\dagger}(\vec{k})}{DR'_{(a)}} \frac{\omega_{(a)}}{\omega_{(a)}^2 - \omega_0^2} \mathbf{e}_i^{(a)} e^{i\mathbf{k}_{(a)} \cdot \mathbf{x}}, \tag{3.6}
\end{aligned}$$

$$\mathbf{B}(\mathbf{x}) = i\varepsilon_0 \int_{\mathbb{R}^3} \frac{d^3\vec{k}}{(2\pi)^3} \frac{a_0(\vec{k})}{2|\vec{k}|} \omega_+(\omega_+^2 - \bar{\omega}^2) e^{-i\mathbf{k}_+ \cdot \mathbf{x}} - i\varepsilon_0 \int_{\mathbb{R}^3} \frac{d^3\vec{k}}{(2\pi)^3} \frac{a_0^\dagger(\vec{k})}{2|\vec{k}|} \omega_+(\omega_+^2 - \bar{\omega}^2) e^{i\mathbf{k}_+ \cdot \mathbf{x}}. \tag{3.7}$$

As we see, the expressions for the fields now contain a divergence in $\omega = \omega_0$ intersected with $k^2 = 0$. We shall postpone the choice of the test functions until section 5. However, it is worth noting that the divergences appear exclusively in unphysical components of the \mathbf{A} field. Here the dagger means complex conjugation. In order to represent the algebra of the canonical commutation relations, it is convenient to compute also the conjugate momenta. From [18], and the above construction, we get for the conjugate momenta the following

expressions:

$$\Pi_{\mathbf{A}}^0(\mathbf{x}) = ic\varepsilon_0 \int_{\mathbb{R}^3} \frac{d^3\vec{k}}{(2\pi)^3} \frac{a_0(\vec{k})}{2|\vec{k}|} \omega_+(\omega_+^2 - \bar{\omega}^2) e^{-i\mathbf{k}_+\cdot\mathbf{x}} - ic\varepsilon_0 \int_{\mathbb{R}^3} \frac{d^3\vec{k}}{(2\pi)^3} \frac{a_0^\dagger(\vec{k})}{2|\vec{k}|} \omega_+(\omega_+^2 - \bar{\omega}^2) e^{i\mathbf{k}_+\cdot\mathbf{x}}, \quad (3.8)$$

$$\begin{aligned} \Pi_{\mathbf{A}}^l(\mathbf{x}) &= ic\varepsilon_0 \int_{\mathbb{R}^3} \frac{d^3\vec{k}}{(2\pi)^3} \frac{a_0(\vec{k})}{2|\vec{k}|} (k_+^0 v^l - k_+^l v^0) (\omega_+^2 - \bar{\omega}^2) e^{-i\mathbf{k}_+\cdot\mathbf{x}} \\ &\quad + 2i \frac{\varepsilon_0}{c} \sum_{a=1}^2 \sum_{j=1}^2 \int_{\mathbb{R}^3} \frac{d^3\vec{k}}{(2\pi)^3} \frac{a_j^{(a)}(\vec{k})}{DR'_{(a)}} \mathbf{e}_j^{(a)l} \left[c^2 k_{(a)}^{0l} - g^2 \frac{\chi \omega_0^2 \omega_{(a)}}{\omega_{(a)}^2 - \omega_0^2} v^{0l} \right] e^{-i\mathbf{k}_{(a)}\cdot\mathbf{x}} \\ &\quad - ic\varepsilon_0 \int_{\mathbb{R}^3} \frac{d^3\vec{k}}{(2\pi)^3} \frac{a_0^\dagger(\vec{k})}{2|\vec{k}|} (k_+^0 v^l - k_+^l v^0) (\omega_+^2 - \bar{\omega}^2) e^{i\mathbf{k}_+\cdot\mathbf{x}} \\ &\quad - 2i \frac{\varepsilon_0}{c} \sum_{a=1}^2 \sum_{j=1}^2 \int_{\mathbb{R}^3} \frac{d^3\vec{k}}{(2\pi)^3} \frac{a_j^{(a)\dagger}(\vec{k})}{DR'_{(a)}} \mathbf{e}_j^{(a)l} \left[c^2 k_{(a)}^{0l} - g^2 \frac{\chi \omega_0^2 \omega_{(a)}}{\omega_{(a)}^2 - \omega_0^2} v^{0l} \right] e^{i\mathbf{k}_{(a)}\cdot\mathbf{x}}, \end{aligned} \quad (3.9)$$

$$\begin{aligned} \Pi_{\mathbf{P}}(\mathbf{x}) &= g \frac{v^0}{c} \int_{\mathbb{R}^3} \frac{d^3\vec{k}}{(2\pi)^3} \frac{a_0(\vec{k})}{2|\vec{k}|} \omega_+(\omega_+\mathbf{v} - \mathbf{k}_+) e^{-i\mathbf{k}_+\cdot\mathbf{x}} - \frac{1}{\varepsilon_0 c} \int_{\mathbb{R}^3} \frac{d^3\vec{k}}{(2\pi)^3} \frac{b_3(\vec{k})}{2\chi\omega_0^2} (\omega_>\mathbf{v} - \mathbf{k}_>) e^{-i\mathbf{k}_>\cdot\mathbf{x}} \\ &\quad + g \frac{v^0}{c} \sum_{a=1}^2 \sum_{j=1}^2 \int_{\mathbb{R}^3} \frac{d^3\vec{k}}{(2\pi)^3} \frac{a_j^{(a)}(\vec{k})}{DR'_{(a)}} \frac{\omega_{(a)}^2}{\omega_{(a)}^2 - \omega_0^2} \mathbf{e}_j^{(a)} e^{-i\mathbf{k}_{(a)}\cdot\mathbf{x}} \\ &\quad + g \frac{v^0}{c} \int_{\mathbb{R}^3} \frac{d^3\vec{k}}{(2\pi)^3} \frac{a_0^\dagger(\vec{k})}{2|\vec{k}|} \omega_+(\omega_+\mathbf{v} - \mathbf{k}_+) e^{i\mathbf{k}_+\cdot\mathbf{x}} - \frac{1}{\varepsilon_0 c} \int_{\mathbb{R}^3} \frac{d^3\vec{k}}{(2\pi)^3} \frac{b_3^\dagger(\vec{k})}{2\chi\omega_0^2} (\omega_>\mathbf{v} - \mathbf{k}_>) e^{i\mathbf{k}_>\cdot\mathbf{x}} \\ &\quad + g \frac{v^0}{c} \sum_{a=1}^2 \sum_{j=1}^2 \int_{\mathbb{R}^3} \frac{d^3\vec{k}}{(2\pi)^3} \frac{a_j^{(a)\dagger}(\vec{k})}{DR'_{(a)}} \frac{\omega_{(a)}^2}{\omega_{(a)}^2 - \omega_0^2} \mathbf{e}_j^{(a)} e^{i\mathbf{k}_{(a)}\cdot\mathbf{x}}, \end{aligned} \quad (3.10)$$

$$\Pi_B(\mathbf{x}) = 0, \quad (3.11)$$

where the antisymmetrization $A^{[a}B^{b]}$ means $(A^a B^b - A^b B^a)/2$. From these expressions we can extrapolate a basis of *quasi* plane wave solutions, which we collect into nine dimensional vectors of the form

$$\zeta(\mathbf{x}) = \begin{pmatrix} \mathbf{A}(\mathbf{x}) \\ \mathbf{P}(\mathbf{x}) \\ \mathbf{B}(\mathbf{x}) \end{pmatrix}. \quad (3.12)$$

In particular, we define

$$\zeta_0(\mathbf{x}, \vec{k}) = \begin{pmatrix} (\omega_+^2 - \omega_0^2)\mathbf{v} + \frac{1}{2} (\xi(\omega_+^2 - \bar{\omega}^2) - \omega_+^2 + \omega_0^2) (\mathbf{v} - i(\mathbf{v} \cdot \mathbf{x})\mathbf{k}_+) - \frac{i}{4\omega_+} Z(\vec{k})\mathbf{k}_+ \\ -ig\varepsilon_0\chi\omega_0^2(\omega_+\mathbf{v} - \mathbf{k}_+) \\ i\varepsilon_0 c^2 \omega_+(\omega_+^2 - \bar{\omega}^2) \end{pmatrix} e^{-i\mathbf{k}_+\cdot\mathbf{x}}, \quad (3.13)$$

$$\zeta_3(\mathbf{x}, \vec{k}) = \begin{pmatrix} i\mathbf{k}_+ \\ \mathbf{0} \\ 0 \end{pmatrix} e^{-i\mathbf{k}_+\cdot\mathbf{x}}, \quad (3.14)$$

$$\tilde{\zeta}_3(\mathbf{x}, \vec{k}) = \begin{pmatrix} -\frac{g}{\varepsilon_0 c^2} (\mathbf{v} - \frac{\bar{\omega}}{k_+^2} \mathbf{k}_>) \\ i(\omega_>\mathbf{v} - \mathbf{k}_>) \\ 0 \end{pmatrix} e^{-i\mathbf{k}_>\cdot\mathbf{x}}, \quad (3.15)$$

$$\zeta_{a,i}(\mathbf{x}, \vec{k}) = \begin{pmatrix} \mathbf{e}_i^{(a)} \\ -ig\varepsilon_0 \frac{\chi\omega_0^2\omega_{(a)}}{\omega_{(a)}^2 - \bar{\omega}_0^2} \mathbf{e}_i^{(a)} \\ 0 \end{pmatrix} e^{-i\mathbf{k}^{(a)} \cdot \mathbf{x}}, \quad a = 1, 2, \quad i = 1, 2, \quad (3.16)$$

where

$$Z(\vec{k}) := \xi(\omega_+^2 - \bar{\omega}^2) + \omega_+^2 - \omega_0^2 - \frac{2\omega_+^2 g^2 \chi \omega_0^2}{\omega_+^2 - \bar{\omega}^2}. \quad (3.17)$$

The corresponding momenta are

$$\begin{aligned} \mathbf{\Pi}_{\zeta_0}(\mathbf{x}, \vec{k}) &= \begin{pmatrix} i\varepsilon_0 c(\omega_+^2 - \bar{\omega}^2)\omega_+ \\ i\varepsilon_0 c(\omega_+^2 - \bar{\omega}^2)(k_+^0 \vec{v} - v^0 \vec{k}_+) \\ gv^0 c\omega_+(\omega_+ \mathbf{v} - \mathbf{k}_+) \\ 0 \end{pmatrix} e^{-i\mathbf{k}_+ \cdot \mathbf{x}}, & \mathbf{\Pi}_{\zeta_3}(\mathbf{x}, \vec{k}) &= \begin{pmatrix} [0] \\ [\vec{0}] \\ \mathbf{0} \\ 0 \end{pmatrix}, \\ \mathbf{\Pi}_{\tilde{\zeta}_3}(\mathbf{x}, \vec{k}) &= \begin{pmatrix} [0] \\ [\vec{0}] \\ -\frac{v^0 \bar{\omega}}{\varepsilon_0 c \chi \omega_0^2} (\omega_{>} \mathbf{v} - \mathbf{k}_{>}) \\ 0 \end{pmatrix} e^{-i\mathbf{k}_{>} \cdot \mathbf{x}}, \\ \mathbf{\Pi}_{\zeta_{a,i}}(\mathbf{x}, \vec{k}) &= \begin{pmatrix} 0 \\ i\frac{\varepsilon_0}{c} \mathbf{e}_i^{(a)} \left[c^2 k_{(a)}^0 - v^0 g^2 \frac{\chi\omega_0^2\omega_{(a)}}{\omega_{(a)}^2 - \omega_0^2} \right] - i\frac{\varepsilon_0}{c} \mathbf{e}_i^{(a)0} \left[c^2 \vec{k}_{(a)} - \vec{v} g^2 \frac{\chi\omega_0^2\omega_{(a)}}{\omega_{(a)}^2 - \omega_0^2} \right] \\ g\frac{v^0}{c} \frac{\omega_{(a)}}{\omega_{(a)}^2 - \omega_0^2} \mathbf{e}_i^{(a)} \\ 0 \end{pmatrix} e^{-i\mathbf{k}^{(a)} \cdot \mathbf{x}}, \\ a, 1 &= 1, 2. \end{aligned} \quad (3.18)$$

In particular, let us compute the scalar products among the plane waves and the general solution. The conserved scalar product is [18]

$$(\tilde{\zeta}|\zeta) = i \int_{\Sigma_t} d^3x \left(\tilde{A}^{*\mu}(t; \vec{x}) \Pi_{\mathbf{A}\mu}(t; \vec{x}) + \tilde{P}^{*\mu}(t; \vec{x}) \Pi_{\mathbf{P}\mu}(t; \vec{x}) - \tilde{\Pi}_{\mathbf{A}}^{*\mu}(t; \vec{x}) A_\mu(t; \vec{x}) - \tilde{\Pi}_{\mathbf{P}}^{*\mu}(t; \vec{x}) P_\mu(t; \vec{x}) \right), \quad (3.19)$$

where Σ_t is any spacelike hypersurface. After some algebra we get

$$(\zeta_0(\vec{k})|\zeta) = -i\varepsilon_0 c\omega_+(\omega^2 - \bar{\omega}^2) a_3(\vec{k}), \quad (3.20)$$

$$(\zeta_3(\vec{k})|\zeta) = i\varepsilon_0 c\omega_+(\omega^2 - \bar{\omega}^2) a_0(\vec{k}), \quad (3.21)$$

$$(\tilde{\zeta}_3(\vec{k})|\zeta) = \frac{1}{\varepsilon_0 c \chi \omega_0^2} (\bar{\omega}^2 - k_{>}^2) b_3(\vec{k}), \quad (3.22)$$

$$(\zeta_{a,i}(\vec{k})|\zeta) = \frac{1}{c} a_i^{(a)}(\vec{k}). \quad (3.23)$$

Then, by using the inversions

$$a_0(\vec{k}) = -\frac{i}{\varepsilon_0 c \omega_+ (\omega^2 - \bar{\omega}^2)} (\zeta_3(\vec{k})|\zeta), \quad (3.24)$$

$$b_3(\vec{k}) = \frac{\varepsilon_0 c \chi \omega_0^2}{\bar{\omega}^2 - k_{>}^2} (\tilde{\zeta}_3(\vec{k})|\zeta), \quad (3.25)$$

$$a_i^{(a)}(\vec{k}) = c(\zeta_{a,i}(\vec{k})|\zeta), \quad (3.26)$$

$$a_3(\vec{k}) = \frac{4\pi i}{\varepsilon_0 c \omega_+ (\omega^2 - \bar{\omega}^2)} (\zeta_0(\vec{k})|\zeta), \quad (3.27)$$

we can compute the commutators among the Fourier modes.

3.1. The quantum algebra of Fourier modes

We take into consideration the quantum algebra of fields, by promoting them to distributions taking value in a quantum algebra which realises the equal time canonical commutation relations among the fields $\mathbf{A}(\mathbf{x})$, $\mathbf{P}(\mathbf{x})$, $B(\mathbf{x})$, $\mathbf{\Pi}_A(\mathbf{x})$, $\mathbf{\Pi}_P(\mathbf{x})$, and $\Pi_B(\mathbf{x})$, obtained by applying the correspondence principle to the respective Dirac brackets, see [18]. Since we need to rephrase the algebra in terms of the particle representation (Fock representation), by means of the above formulas we are led to consider the commutators

$$[(\zeta_A(\vec{k})|\zeta), (\zeta_B(\vec{q})|\zeta)], \quad (3.28)$$

and

$$[(\zeta_A(\vec{k})|\zeta), (\zeta_B(\vec{q})|\zeta)^*], \quad (3.29)$$

where A, B take the values $0, 3, \tilde{3}, (a, i)$, and $\zeta_{\tilde{3}} = \tilde{\zeta}_3$. By using the definition of the scalar product in terms of the Poissonian structure, and the canonical commutation relations among the fields, we get

$$[(\zeta_A(\vec{k})|\zeta), (\zeta_B(\vec{q})|\zeta)] = \hbar(\zeta_B^*(\vec{q})|\zeta_A(\vec{k})), \quad (3.30)$$

$$[(\zeta_A(\vec{k})|\zeta), (\zeta_B(\vec{q})|\zeta)^*] = \hbar(\zeta_A(\vec{k})|\zeta_B(\vec{q})). \quad (3.31)$$

The scalar products (3.30) are obviously zero. This means that, if we set α_A so that $\alpha_0 = a_0, \alpha_3 = a_3, \alpha_{\tilde{3}} = b_{\tilde{3}}, \alpha_{a,i} = a_{a,i}$, then

$$[\alpha_A(\vec{k}), \alpha_B(\vec{q})] = [\alpha_A^\dagger(\vec{k}), \alpha_B^\dagger(\vec{q})] = 0 \quad (3.32)$$

as expected. On the other hand, we get

$$(\zeta_0(\vec{k})|\zeta_0(\vec{q})) = 0, \quad (3.33)$$

$$(\zeta_0(\vec{k})|\zeta_3(\vec{q})) = -2i(2\pi)^3 \varepsilon_0 c \omega_+ |\vec{k}| (\omega^2 - \bar{\omega}^2) \delta^3(\vec{k} - \vec{q}), \quad (3.34)$$

$$(\zeta_3(\vec{k})|\zeta_3(\vec{q})) = 0 \quad (3.35)$$

$$(\tilde{\zeta}_3(\vec{k})|\tilde{\zeta}_3(\vec{q})) = 2(2\pi)^3 \frac{v^0 \bar{\omega}}{\varepsilon_0 c \chi \omega_0^2} (\bar{\omega}^2 - k_{>}^2) \delta^3(\vec{k} - \vec{q}), \quad (3.36)$$

$$(\zeta_{a,i}(\vec{k})|\zeta_{b,j}(\vec{q})) = (2\pi)^3 \frac{\varepsilon_0}{c} \left(2k^0 + 2\omega v^0 \frac{g^2 \chi \omega_0^4}{(\omega^2 - \omega_0^2)^2} \right) \delta_{ab} \delta_{ij} \delta^3(\vec{k} - \vec{q}), \quad (3.37)$$

whereas all the remaining terms trivially vanish. Then, the nontrivial commutators are

$$[a_0(\vec{k}), a_0^\dagger(\vec{q})] = [a_3(\vec{k}), a_3^\dagger(\vec{q})] = 0, \quad (3.38)$$

$$[a_0(\vec{k}), a_3^\dagger(\vec{q})] = (2\pi)^3 \hbar \frac{|\vec{k}|}{\varepsilon_0 c \omega_+} \frac{2i}{\omega^2 - \bar{\omega}^2} \delta^3(\vec{k} - \vec{q}), \quad (3.39)$$

$$[b_3(\vec{k}), b_3^\dagger(\vec{q})] = 2(2\pi)^3 \hbar \frac{\varepsilon_0 c \chi \omega_0^2 v^0 \bar{\omega}}{\bar{\omega}^2 - k_{>}^2} \delta^3(\vec{k} - \vec{q}), \quad (3.40)$$

$$[a_i^{(a)}(\vec{k}), a_j^{(b)\dagger}(\vec{q})] = (2\pi)^3 \hbar \frac{\varepsilon_0}{c} \left(2c^2 k^0 + 2\omega_{(a)} v^0 \frac{g^2 \chi \omega_0^4}{(\omega_{(a)}^2 - \omega_0^2)^2} \right) \delta_{ij} \delta_{ab} \delta^3(\vec{k} - \vec{q}). \quad (3.41)$$

Notice that the commutator (3.39) is singular in $\omega = \bar{\omega}$. This is not merely a consequence of the singular redefinition of the fields introduced above. It is easy to see that the singularity would be appeared even with the original definitions.

3.2. The Hamiltonian

We shall discuss later about the generators of the Noether symmetries related to the Poincaré group. However, a privileged role is played by the Hamiltonian, since it also defines the dynamics of the system. For this reason we anticipate here the determination of the Hamiltonian. Even though we have not yet introduced the construction of the Fock representation, being the latter one quite standard (apart from some technical points) we think we can still introduce without any ambiguity the normal ordering of the operators, with respect to what will be the vacuum state of the representation. Then, we shall turn on the discussion of the Hamiltonian after the explicit construction of the representation. Now, we assume the normal ordering to be understood. The computation of the Hamiltonian in terms of the particle modes is quite involved by the presence of the $\delta^{(1)}$ contributions. However, its determination can be notably simplified by means of the following trick. The Hamiltonian can be computed directly, with the warning that the only terms which are problematic are the ones containing $a_0^\dagger a_0$ (to be indicated with H_{00} henceforth). The direct computation gives

$$\begin{aligned} H = & \sum_{j=1}^2 \sum_{a=1}^2 \int_{\mathbb{R}^3} \frac{d^3 \vec{k}}{(2\pi)^3} \frac{1}{DR'_{(a)}} a_j^{(a)\dagger}(\vec{k}) a_j^{(a)}(\vec{k}) k_{(a)}^0 + \int_{\mathbb{R}^3} \frac{d^3 \vec{k}}{(2\pi)^3} \frac{\omega_{>}^2 - k_{>}^2}{2v^0 \varepsilon_0 \chi \omega_0^2 \bar{\omega}} b_3^\dagger(\vec{k}) b_3(\vec{k}) k_{>}^0 \\ & + i \int_{\mathbb{R}^3} \frac{d^3 \vec{k}}{(2\pi)^3} \frac{\omega_+}{2} \varepsilon_0 c^2 (\omega^2 - \bar{\omega}^2) (a_3^\dagger(\vec{k}) a_0(\vec{k}) - a_0^\dagger(\vec{k}) a_3(\vec{k})) + \dots \end{aligned} \quad (3.42)$$

where the dots stay for the $a_0^\dagger a_0$ terms, whose computation requires to handle the derivatives of the delta function and it is quite tricky. In order to determine them, let us consider first the mixed term (to be defined as H_{03}) involving creation-annihilation operators with indices 0, 3:

$$H_{03} := i \int_{\mathbb{R}^3} \frac{d^3 \vec{k}}{(2\pi)^3} \frac{\omega_+}{2} \varepsilon_0 c^2 (\omega^2 - \bar{\omega}^2) (a_3^\dagger(\vec{k}) a_0(\vec{k}) - a_0^\dagger(\vec{k}) a_3(\vec{k})). \quad (3.43)$$

By using the commutation rules we first notice that (since, in particular, $[a_0(\vec{k}), a_0^\dagger(\vec{k})] = 0$)

$$[H, B(t, \vec{x})] = [H_{03}, B(t, \vec{x})] = i\hbar \frac{\partial B(t, \vec{x})}{\partial t}, \quad (3.44)$$

as expected. Now, let us consider the gauge field

$$\mathbf{A}_3(t, \vec{x}) := \int_{\mathbb{R}^3} \frac{d^3 \vec{k}}{(2\pi)^3} \frac{i}{2|\vec{k}|} \mathbf{k}_+ \left[a_3(\vec{k}) e^{-i\mathbf{k}_+ \cdot \mathbf{x}} - a_3^\dagger(\vec{k}) e^{i\mathbf{k}_+ \cdot \mathbf{x}} \right], \quad (3.45)$$

and set

$$H_{00} := \int_{\mathbb{R}^3} \frac{d^3\vec{k}}{(2\pi)^3} f(\vec{k}) a_0^\dagger(\vec{k}) a_0(\vec{k}) \quad (3.46)$$

for the $a_0^\dagger a_0$ terms in the Hamiltonian. Thus, $f(\vec{k})$ is completely fixed by requiring that

$$[H, \mathbf{A}_3(t, \vec{x})] = [H_{03} + H_{00}, \mathbf{A}_3(t, \vec{x})] \equiv i\hbar \frac{\partial \mathbf{A}_3(t, \vec{x})}{\partial t}, \quad (3.47)$$

when \mathbf{A}_3 is a solution, which means when all the particle modes, but a_3 , are set to zero. One could be erroneously tempted to write $H_{00} = 0$. Still, if we look at the first line of (3.5), we see that H_{03} alone does not provide the right temporal evolution of the a_0 components and we must take $f \neq 0$. Indeed, the H_{00} term does not commute with \mathbf{A}_3 , but, being the commutator proportional to a_0 , this does not affect the relation (3.47), which must be true on shell (i.e. when $\mathbf{A} = \mathbf{A}_3$ is a solution, thus corresponding, in particular, to $a_0 = 0$). In order to determine f we must impose the right behaviour for the commutator between H and A . Imposing it to the solutions corresponding to have only a_0 non zero, this provides

$$f(\vec{k}) = -\frac{\varepsilon_0 c^2 \omega_+ v^0}{4|\vec{k}|} (\xi(\omega^2 - \bar{\omega}^2) - \omega^2 + \omega_0^2) (\omega^2 - \bar{\omega}^2). \quad (3.48)$$

Thus, the Hamiltonian takes the form

$$\begin{aligned} H = & \sum_{j=1}^2 \sum_{a=1}^2 \int_{\mathbb{R}^3} \frac{d^3\vec{k}}{(2\pi)^3} \frac{1}{DR'_{(a)}} a_j^{(a)\dagger}(\vec{k}) a_j^{(a)}(\vec{k}) k_{(a)}^0 + \int_{\mathbb{R}^3} \frac{d^3\vec{k}}{(2\pi)^3} \frac{\bar{\omega}^2 - k_{>}^2}{2v^0 \varepsilon_0 \chi \omega_0^2 \bar{\omega}} b_3^\dagger(\vec{k}) b_3(\vec{k}) k_{>}^0 \\ & + i \int_{\mathbb{R}^3} \frac{d^3\vec{k}}{(2\pi)^3} \frac{\omega_+}{2} \varepsilon_0 c^2 (\omega^2 - \bar{\omega}^2) (a_3^\dagger(\vec{k}) a_0(\vec{k}) - a_0^\dagger(\vec{k}) a_3(\vec{k})) \\ & - \int_{\mathbb{R}^3} \frac{d^3\vec{k}}{(2\pi)^3} \frac{\varepsilon_0 c^2 \omega_+ v^0}{4|\vec{k}|} (\xi(\omega^2 - \bar{\omega}^2) - \omega^2 + \omega_0^2) (\omega^2 - \bar{\omega}^2) a_0^\dagger(\vec{k}) a_0(\vec{k}), \end{aligned} \quad (3.49)$$

which, indeed, provides the right time evolution of all the fields. Notice the compatibility with the commutation rules. A direct comparison shows that we can interpret

$$dn_j^{(a)}(\vec{k}) = \frac{1}{\hbar c DR'_{(a)}} a_j^{(a)\dagger}(\vec{k}) a_j^{(a)}(\vec{k}) \frac{d^3\vec{k}}{(2\pi)^3} \quad (3.50)$$

as the number of excitations of the field, in the mode $\zeta_i^{(a)}$ with linear momentum \vec{k} in a volume $d^3\vec{k}$. This should be corroborated by the explicit form of the linear momentum operator. Analogously

$$dn_{b_3}(\vec{k}) = \frac{\bar{\omega}^2 - k_{>}^2}{2\hbar c v^0 \varepsilon_0 \chi \omega_0^2 \bar{\omega}} b_3^\dagger(\vec{k}) b_3(\vec{k}) \frac{d^3\vec{k}}{(2\pi)^3} \quad (3.51)$$

is the number of states for the excitation of the $\omega_{>\mathbf{v} - \mathbf{k}_{>}}$ component of the field, with linear momentum \vec{k} in a volume $d^3\vec{k}$. We have intentionally avoided to speak about excitation of the electromagnetic field rather than the polarisation field, in the light of the fact that the excited modes indeed correspond to an excitation of both \mathbf{A} and \mathbf{P} . This is the effect of the ‘‘interaction’’ in the Heisenberg representation: fields \mathbf{A} and \mathbf{P} are not associated with independent particle modes (of course, this would happen in the interaction representation).

4. Covariance of the theory under the Poincaré group and causality

Let us now discuss the covariance properties of the theory constructed up to now. The classical theory is covariant since it provides a working model in any specified inertial frame, selected by a specification of

the velocity vector \mathbf{v} . Indeed, it is immediate to verify that, consistently, given a Lorentz transformation $\mathbf{\Lambda} \in SO(1,3)$ such that $\mathbf{v}' = \mathbf{\Lambda}\mathbf{v}$, then the theory specified by \mathbf{v}' is obtained by transforming the system specified by \mathbf{v} under $\mathbf{\Lambda}$. Nevertheless, covariance is not the same as invariance: the presence of the dielectric medium, and then of the vector \mathbf{v} , explicitly breaks the Poincaré symmetry so that, in general, boosts are no more expected to be symmetries, and, indeed, they are not. For example, if we consider the free oscillators describing the dielectric medium, thus with coupling $g = 0$, its equations of motion in the frame at rest w.r.t. the medium are

$$\partial_t^2 P^i + \omega_0^2 P^i = 0, \quad (4.1)$$

$$P^0 = 0. \quad (4.2)$$

Their covariant form is

$$(v^\nu \partial_\nu)^2 P^\mu + \omega_0^2 P^\mu = 0, \quad (4.3)$$

$$v^\mu P_\mu = 0. \quad (4.4)$$

The Lorentz symmetry is broken since one can individuate the inertial frame just by looking at the form of the equations of motion. Nevertheless, the equations remain covariant in the sense that the first set of equations can be recovered from the second one simply by setting $\mathbf{v} \equiv (1; \vec{0})$, and in any other frame by specifying the velocity \mathbf{v} .

In order to discuss this point appropriately, as a general fact, let us consider a theory for a field ψ in which a velocity \mathbf{v} , specifying a privileged inertial frame, appears explicitly in the Lagrangian in such a way that the theory in a given frame, boosted by \vec{v}/v^0 with respect to the privileged frame, is described simply by fixing the spacetime vector $\mathbf{v} \equiv (v^0, \vec{v})$. Obviously, our Lagrangian belongs into this class. The point is that such a Lagrangian is not a scalar under Lorentz transformations unless we transform \mathbf{v} also. In other words we have

$$L(\psi(\mathbf{\Lambda}\mathbf{x}), \partial_\mu \psi(\mathbf{\Lambda}\mathbf{x}); \mathbf{\Lambda}\mathbf{v}) = L(\psi(\mathbf{x}), \partial_\mu \psi(\mathbf{x}); \mathbf{v}), \quad (4.5)$$

whereas

$$L(\psi(\mathbf{\Lambda}\mathbf{x}), \partial_\mu \psi(\mathbf{\Lambda}\mathbf{x}); \mathbf{v}) \neq L(\psi(\mathbf{x}), \partial_\mu \psi(\mathbf{x}); \mathbf{v}). \quad (4.6)$$

Thus, such a transformation does not represent a symmetry of a specified model, but it is indeed a covariance transformation and, technically, it looks more as a duality transformation relating two different set of parameters (\mathbf{v} and \mathbf{v}') in a family of models. In order to further clarify this point, let us investigate quickly its role in the Noether theorem. If we consider a Lorentz transformation $\mathbf{\Lambda}$ specified by an infinitesimal transform ϵ such that $\eta\epsilon = -\epsilon^T\eta$, η being the usual Minkowski tensor, then the field will transform as

$$\delta_\epsilon \psi(\mathbf{x}) = \frac{1}{2} \epsilon_{\mu\nu} \sigma^{\mu\nu} \psi(\mathbf{x}) + \epsilon^{\mu\nu} x_\mu \partial_\nu \psi(\mathbf{x}), \quad (4.7)$$

where $\sigma^{\mu\nu}$ are the representation matrices of the Lorentz group determined by the spin of the field. Such a transformation, as discussed above, must be also associated with a Lorentz transformation of the velocity

$$v^\mu \longmapsto v^\mu + \epsilon^\mu{}_\nu v^\nu \quad (4.8)$$

for consistency. Only after including this transformation, the Lagrangian will transform as a scalar, which means

$$\frac{\partial L}{\partial \psi} \delta_\epsilon \psi + \frac{\partial L}{\partial \partial_\mu \psi} \partial_\mu \delta_\epsilon \psi + \frac{\partial L}{\partial v^\mu} \delta_\epsilon v^\mu = \epsilon^{\mu\nu} x_\mu \partial_\nu L(\mathbf{x}), \quad (4.9)$$

where in the r.h.s. we emphasised that the Lagrangian has been seen as a scalar function of \mathbf{x} . Equivalently, this can be recast in the form

$$\left(\frac{\partial L}{\partial \psi} - \partial_\mu \frac{\partial L}{\partial \partial_\mu \psi} \right) \delta_\epsilon \psi + \partial_\mu \left(\frac{\partial L}{\partial \partial_\mu \psi} \delta_\epsilon \psi + \epsilon^{\mu\nu} x_\nu L \right) = - \frac{\partial L}{\partial v^\mu} \delta_\epsilon v^\mu. \quad (4.10)$$

Usually, for theories with scalar couplings, the right hand side vanishes so that on any solution of the equations of motion the quantity

$$j_\epsilon^\mu(\mathbf{x}) = \frac{\partial L}{\partial \partial_\mu \psi} \delta_\epsilon \psi + \epsilon^{\mu\nu} x_\nu L \quad (4.11)$$

defines a conserved current. But in our general case the story is quite different since the r.h.s. of (4.10) is different from zero. One may wonder if $\frac{\partial L}{\partial v^\mu} \delta_\epsilon v^\mu$ is a divergence, but also this cannot be true in general. For example, if L depends polynomially on \mathbf{v} , like in our specific case, then this would imply that also each term in L explicitly containing \mathbf{v} would be a divergence. But in this case such terms would not influence the dynamics and could be completely eliminated from the action from the beginning.

Thus, the Lorentz symmetry is broken down to one of its maximal compact subgroups: the little group associated to \mathbf{v} , which, \mathbf{v} being a timelike vector, is a three dimensional rotations group. Since it leaves \mathbf{v} invariant, we have that, for $\epsilon_{\mathbf{v}}$ a generator of the little group, $\delta_{\epsilon_{\mathbf{v}}} v^\mu = 0$ and the corresponding current $\mathbf{j}_{\epsilon_{\mathbf{v}}}$ is conserved. We recall that, in case of full Lorentz invariance, the Noether charge associated with Lorentz boosts is the so-called ergocenter velocity. In the present case, as discussed above, boosts are not symmetries of our model. In other words, the presence of the dielectric medium defines a sort of external source which does not allow for the conservation of the ergocenter velocity (and maybe the ergocenter is not even well defined since the dielectric medium essentially looks like a system with infinite inertia).

We can rephrase the problem we discussed above also by referring to the ideas developed in [25]. See also [26], pp. 356-360. Our model Lagrangian is associated with equations of motion which are covariant with respect to the Lorentz group. This fact does not correspond to a full Lorentz invariance, due to the fact that the Lorentzian metric is not the only absolute object of the theory, but the velocity \mathbf{v} of the medium is an absolute object too. This implies that our theory, and any covariant theory of a dielectric medium (cf. e.g. [27]), is involved with a preferred frame: the rest frame of the medium. This kind of behaviour is not exceptional, as it occurs e.g. in all the cases where Klein-Gordon equation is studied in presence of an external potential $V(\phi(\mathbf{x}), \mathbf{x})$ as, for example, in presence of a fixed electromagnetic background $A^\mu(\mathbf{x})$ where the Klein-Gordon equation takes the form

$$\hbar^2 (\partial_\mu - iA_\mu(\mathbf{x})) (\partial^\mu - iA^\mu(\mathbf{x})) \phi(x) + m^2 c^2 \phi(\mathbf{x}) = 0. \quad (4.12)$$

Loss of Lorentz invariance is evident, but, at the same time, the field equations are covariant, and solutions are transformed into solutions of the Klein-Gordon equation by the Lorentz group, provided that external potential is transformed too [28].

We shall see soon how to compute the conserved charges. Before doing this, we want to discuss the covariance of the quantum theory on the light of the stated observations.

4.1. Covariance and the quantum theory

How do the above considerations reflect in the quantum theory construction? Since the Lorentz group is no more a symmetry group, but only the little group of \mathbf{v} for any given \mathbf{v} , beyond spacetime translations, we cannot expect to represent the whole Poincaré group on a single Fock space, but over a family $\mathcal{F}_{\mathbf{v}}$ of Fock spaces, one for each \mathbf{v} . Each $\mathcal{F}_{\mathbf{v}}$ will be cyclically generated by the polynomial action of the fields on a vacuum state $|0\rangle_{\mathbf{v}}$, annihilated by the set of operators

$$a_0(\vec{k}; \mathbf{v}), a_3(\vec{k}; \mathbf{v}), b_3(\vec{k}; \mathbf{v}), a_i^{(a)}(\vec{k}; \mathbf{v}), i = 1, 2, a = 1, 2, \vec{k} \in \mathbb{R}^3. \quad (4.13)$$

We shall refer to the Fock space $\mathcal{F}_{\mathbf{v}}$ as the \mathbf{v} sector. Since they correspond to different inertial frames, any two sectors are expected to be equivalent and, then, related by a boost in some way. More precisely, we infer the existence of a map

$$U : SO(1, 3)^\dagger \times V^+ \longrightarrow \mathcal{U}, \quad (4.14)$$

$$(\mathbf{\Lambda}, \mathbf{v}) \longmapsto U(\mathbf{\Lambda}, \mathbf{v}) : \mathcal{F}_{\mathbf{v}} \longrightarrow \mathcal{F}_{\mathbf{\Lambda v}}, \quad \forall \mathbf{v} \text{ timelike}, \quad (4.15)$$

from the orthochronous Lorentz group times the set of all future directed timelike vectors to the set \mathcal{U} of all unitary maps among Fock spaces $\mathcal{F}_{\mathbf{v}}$ and $\mathcal{F}_{\mathbf{w}}$, in such a way that $U(\mathbf{\Lambda}, \mathbf{v})$ must satisfy

$$U(\mathbf{\Lambda}, \mathbf{v})|0\rangle_{\mathbf{v}} = |0\rangle_{\mathbf{\Lambda v}}, \quad (4.16)$$

and

$$U(\mathbf{\Lambda}, \mathbf{v})o(\vec{k}; \mathbf{v})U(\mathbf{\Lambda}, \mathbf{v})^{-1} = o(\mathbf{\Lambda}\vec{k}; \mathbf{\Lambda v}), \quad (4.17)$$

for o any of the operators in (4.13). Here, with $\mathbf{\Lambda}\vec{k}$ we mean $\mathbf{\Lambda}(k^0(\vec{k}), \vec{k})$, which makes perfectly sense since all dispersion relations are Lorentz covariant. The maps $U(\mathbf{\Lambda}, \mathbf{v})$ are indeed well defined and unitary. There is no particular difficulty in proving that and we shall sketch the proof.

Given $\mathbf{\Lambda} \in SO(1, 3)^\uparrow$, $\mathbf{v} \in V^+$, and setting $\mathbf{w} = \mathbf{\Lambda v}$, we can construct the Fock spaces $\mathcal{F}_{\mathbf{v}}$ and $\mathcal{F}_{\mathbf{w}}$, in the usual way, as the Cauchy completion of the spaces generated by the polynomial actions of the operators

$$\int_{\mathbb{R}^3} d^3\nu_o(\vec{k})o^\dagger(\vec{k}, \mathbf{z})f(\vec{k}), \quad (4.18)$$

on $|0\rangle_{\mathbf{z}}$, where $\mathbf{z} = \mathbf{v}, \mathbf{w}$, o is any of the operators (4.13), $d^3\nu_a(\vec{k})$ the corresponding covariant measure,² and f is any test function (wave packet). This way, the two Fock spaces are well defined, and the maps defined by (4.16) and (4.17) are also well defined and isometric on all polynomially generated states (obviously one has to take into account all subtleties relative to the restriction to the set of physical states, projecting out the null norm states, but it is standard so we do not enter into details). Since the maps just defined are bounded operators over dense sets, they remain well defined by Cauchy completion, and provide indeed unitary equivalences among the different representations.

In other words, the representation of the quantum algebra is finally realised on a bundle of Fock spaces over the quotient space $\mathcal{B} = SO(1, 3)^\uparrow/SO(3) \simeq V^+$. The detailed construction of the Fock spaces (and, then, the details of the proof sketched above) is omitted as it is standard. Thus, we have a different quantum field theory for any given velocity field \mathbf{v} of the dielectric medium. The different Fock spaces represent different quantum theories, and the set of $\{\mathcal{F}_{\mathbf{v}}\}_{\mathbf{v}}$ defines a bundle of quantum field theories over V^+ , or, equivalently, a family of quantum field theories parameterized by \mathbf{v} , which plays the role of external parameter.

4.2. On the causality properties

In a quantum field theory, causality imposes that the commutators among fields vanish when the supports of the fields have a spacelike separation, or, in the point-like formulation, if the fields are evaluated at spatially separated points. Having ensured covariance, in our construction such causality relations are simply a consequence of the equal time canonical commutation relations. Notice that in the presence of gauge fields this cannot be realised when also charged fields are involved. However, in our case no charged fields are present.

4.3. Conserved quantities and the little group

For completeness let us shortly discuss the conserved quantities associated to the continuous symmetries of the relativistic Hopfield model. The symmetry group is $G_{\mathbf{v}} \times \mathbb{R}^4$, the semidirect product of the little group $G_{\mathbf{v}}$ of \mathbf{v} and the spacetime translations group \mathbb{R}^4 . The conserved currents associated with translations do not require any particular care, and are computed exactly in the same way as for any relativistic invariant theory. The same happens for the little group, with the only care to appropriately select the correct infinitesimal

²_{so}

$$d^3\nu_o(\vec{k}) = \frac{d^3\vec{k}}{(2\pi)^3 DR'_o(\vec{k})}, \quad (4.19)$$

where $DR_o = 0$ is the dispersion relation describing the \mathbf{k} spectrum relative to the operator o , and the prime means derivative w.r.t. k^0 .

transformations. For a Lorentz invariant theory, the infinitesimal generators are parameterised by any antisymmetric matrix $\epsilon_{\mu\nu}$, and give rise to six conserved quantities $M^{\mu\nu} = -M^{\nu\mu}$, which, for a set of fields Ψ_a , $a = 1, \dots, n$, having spin S_a (with representation matrices $\sigma^{\mu\nu} = -\sigma^{\nu\mu}$) and Lagrangian \mathcal{L} , can be expressed in the form

$$K^j = M^{0j} = \sum_{a=1}^n \int_{\Sigma} [i\sigma^{0j}\Psi_a(\mathbf{x})\pi_a(\mathbf{x}) + (x^0\partial^j - x^j\partial^0)\Psi_a(\mathbf{x})\pi_a(\mathbf{x}) + \frac{1}{n}x^j\mathcal{L}]d^3\vec{x} \quad (4.20)$$

for the boosts, where π_a is the momentum field canonically conjugate to Ψ_a , whereas for rotations we have

$$J_i = \frac{1}{2}\epsilon_{ijk}M^{jk} = \sum_{a=1}^n \int_{\Sigma} [i\mathcal{S}_i\Psi_a(\mathbf{x})\pi_a(\mathbf{x}) + \frac{1}{2}\epsilon_{ijk}(x^j\partial^k - x^k\partial^j)\Psi_a(\mathbf{x})\pi_a(\mathbf{x})]d^3\vec{x}, \quad (4.21)$$

with $2\mathcal{S}_i = \epsilon_{ijk}\sigma^{jk}$. However, we cannot apply directly this formula to our case since we have to restrict the parameters generating symmetries to the set of antisymmetric matrices $\epsilon_{\mu\nu}$ that satisfy

$$\epsilon_{\mu\nu}v^\mu = 0. \quad (4.22)$$

The solutions are given by

$$\epsilon_{\mu\nu} = \varepsilon_{ij}\mathbf{\Lambda}(\mathbf{v})^i{}_{\mu}\mathbf{\Lambda}(\mathbf{v})^j{}_{\nu}, \quad (4.23)$$

where $\varepsilon_{\mu\nu}$ is an arbitrary antisymmetric matrix which vanishes if one of the indices is zero, and $\mathbf{\Lambda}(\mathbf{v})$ is any boost transforming \mathbf{v} into $\mathbf{c} \equiv (c, \vec{0})$. We can choose

$$\mathbf{\Lambda}(\mathbf{v})^0{}_0 = \frac{v^0}{c}, \quad (4.24)$$

$$\mathbf{\Lambda}(\mathbf{v})^i{}_0 = \mathbf{\Lambda}(\mathbf{v})^0{}_i = -\frac{v^i}{c}, \quad (4.25)$$

$$\mathbf{\Lambda}(\mathbf{v})^i{}_j = (\gamma(\mathbf{v}) - 1)\frac{v^iv^j}{v^2} + \delta_j^i. \quad (4.26)$$

In this way we see that the little group rotations are generated by the conserved quantities

$$R^{ij} = \mathbf{\Lambda}(\mathbf{v})^i{}_{\mu}\mathbf{\Lambda}(\mathbf{v})^j{}_{\nu}M^{\mu\nu}, \quad (4.27)$$

where $M^{\mu\nu}$ are the quantities defined above (and computed in our specific case). More precisely

$$R^{ij} = M^{ij} - (\gamma(\mathbf{v}) - 1)\frac{v^k}{v^2}(v^iM^{kj} - v^jM^{ki}) - \frac{v^i}{c}M^{0j} + \frac{v^j}{c}M^{0i}, \quad (4.28)$$

where $v_k = -v^k$, or, equivalently, recalling that $\epsilon_{ijk} = -\epsilon^{ijk}$:

$$J_{\mathbf{v}}^l := \frac{1}{2}\epsilon^{lij}R_{ij} = v^0J^l - \epsilon^{lij}v_iK_j, \quad (4.29)$$

with J^l and K^l as given above. Obviously, the normal ordering w.r.t. the vacuum state $|0\rangle_v$ has to be understood in the quantum expressions. We don't display the explicit formulas giving the operators J^l in terms of the annihilation and creation operators, as they are cumbersome and of little interest.

The physical meaning of the conserved quantities is clear from their construction, which can be understood as follows. Since the Lorentz group is broken, and in particular boosts are not symmetries, in the lab frame, where the dielectric is at rest, the conserved quantities are just the components of the total angular momentum w.r.t. the origin of the coordinate system. Thus, going back to the generic inertial frame characterised by the velocity vector \mathbf{v} , we see that $J_{\mathbf{v}}^l$ are simply the components of the boosted angular momentum.

5. The two point function

Our theory, being quadratic in the fields, is essentially characterised by the propagator, which can be computed directly as the vacuum expectation value of the time ordered product of the fields

$$iG_{\mathbf{v}}^{IJ}(\mathbf{x}, \mathbf{y}) = {}_{\mathbf{v}}\langle 0|T(\Phi^I(\mathbf{x})\Phi^J(\mathbf{y}))|0\rangle_{\mathbf{v}}/\hbar, \quad I, J = 1, \dots, 9, \quad (5.1)$$

where

$$\Phi^I = \begin{cases} A^{I-1} & \text{if } I = 1, 2, 3, 4, \\ P^{I-5} & \text{if } I = 5, 6, 7, 8, \\ B & \text{if } I = 9. \end{cases} \quad (5.2)$$

We define $G_{\mathbf{v}}^{IJ}(\mathbf{x}, \mathbf{y})_{\pm}$ by writing

$$G_{\mathbf{v}}^{IJ}(\mathbf{x}, \mathbf{y}) = G_{\mathbf{v}}^{IJ}(\mathbf{x}, \mathbf{y})_+ \theta(x^0 - y^0) + G_{\mathbf{v}}^{IJ}(\mathbf{x}, \mathbf{y})_- \theta(y^0 - x^0). \quad (5.3)$$

Since $G_{\mathbf{v}}^{IJ}(\mathbf{x}, \mathbf{y})_-$ is easily obtained from $G_{\mathbf{v}}^{IJ}(\mathbf{x}, \mathbf{y})_+$, we shall write down only the latter. This computation does not require any particular shrewdness and we can directly write down the relevant components, which, for $\mu, \nu = 0, 1, 2, 3$, are

$$\begin{aligned} iG_{\mathbf{v}}^{(\mu+1)(\nu+1)}(\mathbf{x}, \mathbf{y})_+ &= {}_{\mathbf{v}}\langle 0|A^\mu(\mathbf{x})A^\nu(\mathbf{y})|0\rangle_{\mathbf{v}}/\hbar \\ &= \int_{\mathbb{R}^3} \frac{d^3\vec{k}}{(2\pi)^3} e^{-i\mathbf{k}_+ \cdot (\mathbf{x}-\mathbf{y})} \frac{v^\mu k_+^\nu + v^\nu k_+^\mu}{4\varepsilon_0 c^2 |\vec{k}| \omega_+} \left(\xi + \frac{\omega^2 - \omega_0^2}{\omega^2 - \bar{\omega}^2} \right) \\ &\quad - i \int_{\mathbb{R}^3} \frac{d^3\vec{k}}{(2\pi)^3} e^{-i\mathbf{k}_+ \cdot (\mathbf{x}-\mathbf{y})} \frac{(\mathbf{x}-\mathbf{y}) \cdot \mathbf{v}}{4\varepsilon_0 c^2 |\vec{k}| \omega_+} \left(\xi - \frac{\omega^2 - \omega_0^2}{(\omega^2 - \bar{\omega}^2)} \right) k^\mu k^\nu \\ &\quad + \int_{\mathbb{R}^3} \frac{d^3\vec{k}}{(2\pi)^3} e^{-i\mathbf{k}_+ \cdot (\mathbf{x}-\mathbf{y})} \frac{\xi(\omega^2 - \bar{\omega}^2)^2 + (\omega^2 - \omega_0^2)(\omega^2 - \bar{\omega}^2) - 2\omega_+^2 g^2 \chi \omega_0^2}{4\varepsilon_0 c^2 \omega_+^2 |\vec{k}| (\omega^2 - \bar{\omega}^2)^2} k^\mu k^\nu \\ &\quad + \frac{g^2 \chi \omega_0^2}{2\varepsilon_0 c^4 \bar{\omega} v^0} \int_{\mathbb{R}^3} \frac{d^3\vec{k}}{(2\pi)^3} e^{-i\mathbf{k}_> \cdot (\mathbf{x}-\mathbf{y})} \frac{1}{\bar{\omega}^2 - k_>^2} \left(v^\mu - \frac{\bar{\omega}}{k_>^2} k_>^\mu \right) \left(v^\nu - \frac{\bar{\omega}}{k_>^2} k_>^\nu \right) \\ &\quad + \sum_{a=1}^2 \sum_{i=1}^2 \int_{\mathbb{R}^3} \frac{d^3\vec{k}}{(2\pi)^3} e^{-i\mathbf{k}_{(a)} \cdot (\mathbf{x}-\mathbf{y})} \frac{e_i^{(a)\mu}(\vec{k}) e_i^{(a)\nu}(\vec{k})}{DR'_{(a)}(\vec{k})}; \end{aligned} \quad (5.4)$$

$$\begin{aligned} iG_{\mathbf{v}}^{(\mu+1)(\nu+5)}(\mathbf{x}, \mathbf{y})_+ &= {}_{\mathbf{v}}\langle 0|A^\mu(\mathbf{x})P^\nu(\mathbf{y})|0\rangle_{\mathbf{v}}/\hbar \\ &= i \int_{\mathbb{R}^3} \frac{d^3\vec{k}}{(2\pi)^3} e^{-i\mathbf{k}_+ \cdot (\mathbf{x}-\mathbf{y})} \frac{g\chi\omega_0^2}{2c^2 |\vec{k}| \omega_+} \frac{\omega_+ k_+^\mu v^\nu - k_+^\mu k_+^\nu}{\omega_+^2 - \bar{\omega}^2} \\ &\quad + i \frac{g\omega_0^2 \chi}{2c^2 \bar{\omega} v^0} \int_{\mathbb{R}^3} \frac{d^3\vec{k}}{(2\pi)^3} e^{-i\mathbf{k}_> \cdot (\mathbf{x}-\mathbf{y})} \frac{1}{\bar{\omega}^2 - k_>^2} \left(v^\mu - \frac{\bar{\omega}}{k_>^2} k_>^\mu \right) (\bar{\omega} v^\nu - k_>^\nu) \\ &\quad + ig\chi\omega_0^2 \sum_{a=1}^2 \sum_{i=1}^2 \int_{\mathbb{R}^3} \frac{d^3\vec{k}}{(2\pi)^3} e^{-i\mathbf{k}_{(a)} \cdot (\mathbf{x}-\mathbf{y})} \frac{\omega_{(a)}}{\omega_{(a)}^2 - \omega_0^2} \frac{e_i^{(a)\mu}(\vec{k}) e_i^{(a)\nu}(\vec{k})}{DR'_{(a)}(\vec{k})}; \end{aligned} \quad (5.5)$$

$$iG_{\mathbf{v}}^{(\mu+1)9}(\mathbf{x}, \mathbf{y})_+ = {}_{\mathbf{v}}\langle 0|A^\mu(\mathbf{x})B(\mathbf{y})|0\rangle_{\mathbf{v}}/\hbar = -\frac{i}{2} \int_{\mathbb{R}^3} \frac{d^3\vec{k}}{(2\pi)^3} e^{-i\mathbf{k}_+ \cdot (\mathbf{x}-\mathbf{y})} \frac{k^\mu}{|\vec{k}|}; \quad (5.6)$$

$$\begin{aligned}
iG_{\mathbf{v}}^{(\mu+5)(\nu+5)}(\mathbf{x}, \mathbf{y})_+ &= {}_{\mathbf{v}}\langle 0|P^\mu(\mathbf{x})P^\nu(\mathbf{y})|0\rangle_{\mathbf{v}}/\hbar \\
&= \frac{\chi\varepsilon_0\omega_0^2}{2\bar{\omega}v^0} \int_{\mathbb{R}^3} \frac{d^3\vec{k}}{(2\pi)^3} e^{-i\mathbf{k}_{>}\cdot(\mathbf{x}-\mathbf{y})} \frac{1}{\bar{\omega}^2 - \omega_0^2} (\bar{\omega}v^\mu - k_{>}^\mu)(\bar{\omega}v^\nu - k_{>}^\nu) \\
&\quad + g^2\chi^2\varepsilon_0^2\omega_0^4 \sum_{a=1}^2 \sum_{i=1}^2 \int_{\mathbb{R}^3} \frac{d^3\vec{k}}{(2\pi)^3} e^{-i\mathbf{k}_{(a)}\cdot(\mathbf{x}-\mathbf{y})} \frac{\omega_{(a)}}{\omega_{(a)}^2 - \omega_0^2} \frac{e_i^{(a)\mu}(\vec{k})e_i^{(a)\nu}(\vec{k})}{DR'_{(a)}(\vec{k})}; \quad (5.7)
\end{aligned}$$

$$iG_{\mathbf{v}}^{(\mu+5)9}(\mathbf{x}, \mathbf{y})_+ = {}_{\mathbf{v}}\langle 0|P^\mu(\mathbf{x})B(\mathbf{y})|0\rangle_{\mathbf{v}}/\hbar = 0; \quad (5.8)$$

$$iG_{\mathbf{v}}^{99}(\mathbf{x}, \mathbf{y})_+ = {}_{\mathbf{v}}\langle 0|B(\mathbf{x})B(\mathbf{y})|0\rangle_{\mathbf{v}}/\hbar = 0. \quad (5.9)$$

At a first sight, these quite involved expressions do not immediately communicate any particular information. However, in the definition of the fields, we have seen that we need to restrict the space of test functions in order to avoid a singular behaviour around $\omega^2 = \bar{\omega}^2$ intersected with the future directed light cone. By looking at the expressions for the propagator, we see precisely what is the degree at which the singularities appear. These new singularities have been originated by the commutator (3.39) between the a_0 and the a_3 modes. In particular, it involves only gauge configuration and is thus expected to be eliminable from any contribution to physical quantities. We can avoid them by a suitable choice of the distribution space, thus making all expressions well defined.

The origin of the extra singularities is due to the fact that the different branches of the spectrum have non-trivial intersections. The three main components of the spectrum are defined respectively by the equations

$$k^2 = 0, \quad (5.10)$$

$$\omega^2 - \bar{\omega}^2 = 0, \quad (5.11)$$

$$c^2k^2 - \frac{g^2\chi_0\omega_0^2\omega^2}{\omega^2 - \omega_0^2} = 0, \quad (5.12)$$

together with positivity conditions, and which we shall indicate with Σ_+ , $\Sigma_{>}$ and $\Sigma_{(a)}$ respectively. The denominators that generate divergences in the propagator have the form $(\omega_{\pm}^2 - \bar{\omega}^2)^l$, $l = 1, 2$. The polar region is thus in the intersection $\Sigma_p = \Sigma_+ \cap \Sigma_{>}$. Since $k^0 = |\vec{k}|$ this is equivalent to write

$$|\vec{k}|v^0 - \vec{v} \cdot \vec{k} = \pm\bar{\omega}. \quad (5.13)$$

Now $v^2 = c^2$ implies $v^0 > |\vec{v}|$ so that we must take the positive sign since v^0 is positive. The equation is thus equivalent to

$$v_0^2\vec{k}^2 = (\bar{\omega} + \vec{v} \cdot \vec{k})^2. \quad (5.14)$$

If we now introduce the shift

$$\vec{k} = \vec{q} + \bar{\omega}\vec{v}, \quad (5.15)$$

the equation becomes

$$\frac{c^2\vec{q}^2}{\bar{\omega}^2} - \frac{c^2(\vec{v} \cdot \vec{q})^2}{\bar{\omega}^2v_0^2} = 1. \quad (5.16)$$

This shows that the singular locus is a prolate ellipsoid, symmetric under rotation around the direction of \vec{v} , centred in

$$\vec{k} = \frac{\bar{\omega}}{c^2}\gamma(\vec{v})\vec{v}, \quad (5.17)$$

where

$$\vec{v} = \gamma(\vec{v})\vec{v}, \quad \gamma(\vec{v}) = \frac{1}{\sqrt{1 - \vec{v}^2/c^2}}, \quad (5.18)$$

with shorter axis of length

$$b = c\bar{\omega} \quad (5.19)$$

whereas the longer one has length

$$b = c\bar{\omega}\gamma(\vec{v}). \quad (5.20)$$

We can cure the singularities by defining the field operators as distributions acting on test rapidly decreasing smooth functions whose Fourier transforms vanish on the singular ellipsoid at least with degree two. It is easily verified that this domain gives rise to well defined distributions.

It is interesting to note that the remaining intersections among spectra are just pointlike and give no new singularities. Indeed, we have that $\Sigma_+ \cap \Sigma_{(a)}$ is given by the point $\vec{k} = 0$, whereas $\Sigma_{>} \cap \Sigma_{(a)}$ corresponds to

$$\vec{k} = \bar{\omega}\vec{v}/c^2. \quad (5.21)$$

To verify these, we first note that $k^2 = 0$ in $\Sigma_{(a)}$ implies $\omega = 0$ that means $\mathbf{v} \cdot \mathbf{k} = 0$. Since \mathbf{k} is null and \mathbf{v} is timelike, we get $\mathbf{k} = 0$. Similarly, for $\Sigma_{>} \cap \Sigma_{(a)}$, if we set $\omega = \omega_{>}$ in $\Sigma_{(a)}$, we get

$$\frac{g^2 \chi \omega_0^2 \omega^2}{\omega^2 - \omega_0^2} = \bar{\omega}^2, \quad (5.22)$$

so that the points in the intersection must satisfy

$$c^2 k^2 - \bar{\omega}^2 = 0, \quad (5.23)$$

where $k^0 v^0 = \bar{\omega} + \vec{k} \cdot \vec{v}$. In the frame where $\vec{v} = 0$ this has solution $\vec{k} = 0$, and boosting at a general \vec{v} gives (5.21). This can also be determined directly. Assuming $\vec{v} \neq \vec{0}$, let \hat{v} the direction of \vec{v} . We can write

$$\vec{k} = \alpha \hat{v} + \beta \hat{v}_\perp, \quad (5.24)$$

where \hat{v}_\perp is a normalised vector perpendicular to \hat{v} . The equation for the intersection thus becomes

$$\left(\frac{\bar{\omega}}{v^0} + \alpha \frac{|\vec{v}|}{v^0} \right)^2 = \bar{\omega}^2 + \alpha^2 + \beta^2. \quad (5.25)$$

After some manipulations it takes the equivalent form

$$0 = \beta^2 c^2 + \left(\bar{\omega} |\vec{v}|/c - \alpha \sqrt{c^2 - \vec{v}^2} \right)^2, \quad (5.26)$$

which gives $\beta = 0$ and (5.21). The intersection points are schematically depicted in figure 1.

5.1. The test functions

In order to avoid the above singularities, and get well defined fields and propagators, we choose as test function space the set

$$\mathcal{S}_{\mathbf{v}}(\mathbb{R}^4) := \{f \in \mathcal{S}(\mathbb{R}^4) | g(\mathbf{k}) = (\omega^2 - \bar{\omega}^2)^{-2} \mathcal{F}[f](\mathbf{k}) \Rightarrow g \in \mathcal{S}(\mathbb{R}^4)\}, \quad (5.27)$$

where \mathcal{F} is the Fourier transform. It is immediate to verify that with this choice all fields and propagators are well defined distributions. Moreover, the set of such spaces transform covariantly under the Lorentz

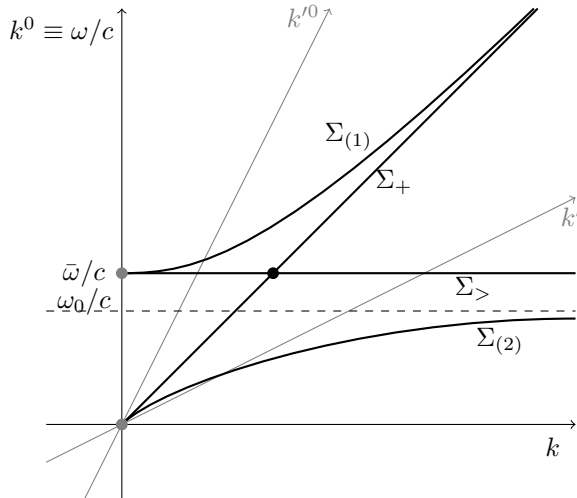


Figure 1: The thick black lines represent the dispersion relations (as seen in the lab frame). The grey lines represent the axes of a boosted frame. The grey points are single point intersections. The black point is a two-dimensional intersection surface.

transformation, and each of them is dense in $L^2(\mathbb{R}^4, d\nu_a)$, see (4.19), as a consequence of the Wiener Tauberian theorems [29].

A comment is in order. Indeed, all singularities are generated by commutators of the kind (3.39), each one giving a singularity of order one. The second order pole arises because the a_0 coefficients in \mathbf{A} have already a first order singularity, and a second one is added by the commutator. The singularity in the field has been introduced by the shift (B.28), which has been introduced as a redefinition of the mode $a_3(\vec{k})$ in order to get the commutators $[a_3(\vec{k}), a_3^\dagger(\vec{q})] = 0$. So, we may wonder if such a singularity is simply due to a bad redefinition of the fields, then questioning if such redefinition was allowable. However, it is easy to check that if we were not performing the shift (B.28), then we would not have a singular expression for the field, but we would get

$$[a_3(\vec{k}), a_3^\dagger(\vec{q})] = \frac{\hbar}{\varepsilon_0 c^2} \left[\xi + \frac{(\omega_+^2 - \omega_0^2)(\omega_+^2 - \bar{\omega}^2) - 8\pi\omega_+^2\omega_0^2 g^2 \chi}{(\omega_+^2 - \bar{\omega}^2)^2} \right] \frac{|\vec{k}|}{\omega_+^2} \delta^3(\vec{k} - \vec{q}), \quad (5.28)$$

in place of zero. This shows that the second order pole appears in the propagator independently from the shift (B.28).

6. Conclusions and outlooks

We have provided a full relativistic quantum theory of the electromagnetic field interacting with an homogeneous and isotropic medium. The fields, which are indicated for simplicity with Φ , are represented as operator valued distributions

$$\mathcal{S}'_{\mathbf{v}}(\mathbb{R}^4) \otimes \mathcal{L}(\mathcal{F}_{\mathbf{v}}), \quad (6.1)$$

where $\mathcal{L}(\mathcal{F}_{\mathbf{v}})$ is the set of linear operators on the separable Hilbert space $\mathcal{F}_{\mathbf{v}}$. Indeed, one gets a family of representations parameterised by $\mathbf{v} \in \mathcal{B} = SO(1,3)^\uparrow/SO(3) \simeq V^+ = \{\mathbf{x} \in \mathbb{R}^{1,3} | \mathbf{x} \cdot \mathbf{x} > 0\}$. In this sense the vacuum state is not the unique state invariant by translations, but we have a family of vacuum states which are however substantially identified by unitary maps. Indeed, on each $\mathcal{F}_{\mathbf{v}}$ it is not represented the whole Poincaré group but only the little group of \mathbf{v} , the set of Poincaré transformations which leave \mathbf{v} invariant. The remaining Poincaré transformations $\mathbf{v} \mapsto \Lambda\mathbf{v}$, the boosts relating different inertial frames, essentially

realise the unitary equivalences $U(\mathbf{\Lambda}) : \mathcal{F}_{\mathbf{v}} \rightarrow \mathcal{F}_{\mathbf{\Lambda v}}$. Each vacuum state $\Omega_{\mathbf{v}} = |0\rangle_{\mathbf{v}}$ is unique in $\mathcal{F}_{\mathbf{v}}$ and cyclic for it. We have realised the theory by means of the correspondence principle, so starting from the equal time canonical commutation relations. This, combined with Lorentz covariance, ensures that microcausality (or locality) is satisfied: $[\Phi(f), \Phi(g)] = 0$ for any $f, g \in \mathcal{S}_{\mathbf{v}}(\mathbb{R}^4)$ whose supports are spatially separated.

It may be interesting to compare these characteristics with the standard Wightman axioms, which we report here for convenience, [30]:

1. The states are described by vectors of a separable Hilbert space \mathcal{H} ;
2. Energy momentum spectral condition:
 - a) The space-time translations are a symmetry of the theory, and consequently they are described by (strongly continuous) unitary operators $U(a)$, $a \in \mathbb{R}^4$, in \mathcal{H} ;
 - b) The spectrum of the generator P_{μ} is contained in the forward closed cone \bar{V}_+ ;
 - c) There is a vacuum state Ω , with the property of being the unique translationally invariant state in \mathcal{H} ;
3. The theory is formulated in terms of fields $\{\Phi\}$ which are operator valued tempered distributions (densely defined) in \mathcal{H} (with Ω in the domain of the polynomials P of the smeared fields $\Phi(f)$) and the vacuum is a cyclic vector with respect to P , namely $\mathcal{H} = \overline{\{P\Omega\}}$;
4. Covariance:
 - a) The Lorentz transformations are described by (strongly continuous) unitary operators $U(\mathbf{\Lambda})$, $\mathbf{\Lambda} \in SO(1, 3)_{\pm}^{\uparrow}$ (the restricted Lorentz group);
 - b) The fields transforms covariantly under the Poincaré transformations $U(\mathbf{a}, \mathbf{\Lambda}) \equiv U(\mathbf{a})U(\mathbf{\Lambda})$,

$$U(\mathbf{a}, \mathbf{\Lambda})\Phi_J(\mathbf{x})U(\mathbf{a}, \mathbf{\Lambda})^{-1} = S_J(\mathbf{\Lambda}^{-1})\Phi_J(\mathbf{\Lambda x} + \mathbf{a}),$$

with S_J a finite dimensional representation of $SL(2, \mathbb{C})$, the universal covering of $SO(1, 3)_{\pm}^{\uparrow}$;

5. Microcausality or locality: The fields either commute or anticommute at spacelike separated points

$$[\Phi_J(f), \Phi_K(g)]_{\pm} = 0, \quad \text{if } f, g \text{ have support in spacelike separated regions,}$$

where the \pm stands for anti- and commutator respectively.

We see that while axioms 1 and 2a are satisfied, the first difference is in axiom 2b: in our model the spectrum is not contained in $\bar{V}^+ = \{\mathbf{x} | x^0 \geq 0, \mathbf{x}^2 \geq 0\}$, since the oneparticle spectrum contains the branches $\Sigma_>$, Σ_2 . Since the branch Σ_2 is well-known in experimental physics, it must be included in the spectrum defining the fields. We shall turn on this point later. Axiom 2c is weakened in our case, since we have not a unique invariant vacuum Ω , but a family of (equivalent) translationally invariant vacua $\Omega_{\mathbf{v}}$, each one unique in $\mathcal{F}_{\bar{\mathbf{v}}}$. But this is not a novelty peculiar of our model, since it must happen each time a preferred frame is selected, e.g. when an external field is present, as previously discussed. Axiom 3 is also slightly modified by restricting the set of test functions to the Schwartzian functions whose Fourier transform vanish on suitable singular loci. Since these loci have zero Lebesgue measure, it follow from the Wiener Tauberian theorems that these spaces remain dense. It is worth noting that the singularities we have chosen to smear out involve only the unphysical gauge components $\mathbf{d}\sigma$ of the fields, and are thus expected not to contribute to physical quantities.

Axiom 4 again is respected in a weaker form: on each $\mathcal{F}_{\mathbf{v}}$ is represented only a subgroup $SO(3)_{\mathbf{v}} \subset SO(1, 3)_{\pm}^{\uparrow}$, isomorphic to $SO(3)$, by strongly continuous unitary operators. The remaining transformations extend these to an action of unitary operators over a family of Hilbert spaces. Finally, microcausality is realised with the minus sign, since we are working with bosonic fields, thanks to the covariance and the implementation of the equal time canonical commutation relations. Microcausality may be not sufficient to grant causality, if axiom 2b is not satisfied. Nevertheless, this is not our case, where causality is respected, but, again, it could not be a surprise, see [31].

This way we reached the goal of realising the exact quantisation of a nontrivial physical system, in the Heisenberg picture. There are several possibilities for extending this work. First of all, the model can also be quantised by means of the path integral methods. Even though it is a less rigorous way from the mathematical point of view, in a sense it sheds some new light on the quantisation of the system, and deserves

investigation. This topic is presented elsewhere [32]. An interesting generalisation would be to improve the model from the point of view of the physical properties of the dielectric medium, which in our minimal realisation are essentially absent. However, at the moment this task is out of our goals. A simpler step would be to include interactions, inhomogeneities and non linearities. For example, in order to improve our analysis of the analogue Hawking effect in dielectrics we need to modify the model enough in order to include at least some nonlinear effects, as the Kerr effect. More in general, it would be interesting to investigate other representations, like, for example, the thermal ones. All these topics are under investigation. Our model can be extended to the case of an arbitrary number of polarisation fields, in a quite straightforward way. Another interesting possibility would be to cure singularities at resonances by including absorption. This may become relevant if one needs to investigate regions not too far from the resonance, so that the absorption becomes important. Far from the resonances the relativistic Hopfield model is enough for describing light in a transparent medium.

Appendix A. Solution of the system (2.8)

A way for solving the algebraic system (2.8) consists in looking at the eigenvalues and eigenvectors of the matrix \mathcal{M} . It has seven eigenvalues, two of which are double degenerate. The two degenerate eigenvalues are

$$\lambda_0^\pm = \frac{1}{2} \left[\varepsilon_0 c^2 k^2 + \frac{\omega^2 - \omega_0^2}{\chi \varepsilon_0 \omega_0^2} \pm \sqrt{\left(\varepsilon_0 c^2 k^2 - \frac{\omega^2 - \omega_0^2}{\chi \varepsilon_0 \omega_0^2} \right)^2 - 4g^2 \omega^2} \right], \quad (\text{A.1})$$

with eigenvectors

$$V_{\lambda_0^\pm, i} = \begin{pmatrix} ig\omega \mathbf{e}_i \\ \frac{1}{2} \left[\varepsilon_0 c^2 k^2 - \frac{\omega^2 - \omega_0^2}{\chi \varepsilon_0 \omega_0^2} \mp \sqrt{\left(\varepsilon_0 c^2 k^2 - \frac{\omega^2 - \omega_0^2}{\chi \varepsilon_0 \omega_0^2} \right)^2 - 4g^2 \omega^2} \right] \mathbf{e}_i \\ 0 \end{pmatrix}, \quad i = 1, 2, \quad (\text{A.2})$$

where \mathbf{e}_i are two spacelike vectors orthogonal to \mathbf{k} and \mathbf{v} , such that $\mathbf{e}_i \cdot \mathbf{e}_j = -\delta_{ij}$. The remaining five eigenvalues are:

$$\lambda_1^\pm = -\frac{\xi}{2\varepsilon_0 c^2} \pm \sqrt{\frac{\xi^2}{4\varepsilon_0^2 c^4} + k^2}, \quad (\text{A.3})$$

with eigenvectors

$$V_{\lambda_1^\pm} = \begin{pmatrix} \mathbf{k} \\ \mathbf{0} \\ -\frac{i\xi}{2\varepsilon_0 c^2} \pm i\sqrt{\frac{\xi^2}{4\varepsilon_0^2 c^4} + k^2} \end{pmatrix}; \quad (\text{A.4})$$

$$\lambda_2^\pm = \frac{1}{2} \left(\varepsilon_0 c^2 k^2 + \frac{\omega^2 - \omega_0^2}{\chi \varepsilon_0 \omega_0^2} \right) \pm \frac{1}{2} \sqrt{\left(\varepsilon_0 c^2 k^2 - \frac{\omega^2 - \omega_0^2}{\chi \varepsilon_0 \omega_0^2} \right)^2 - 4g^2 c^2 k^2} \quad (\text{A.5})$$

with eigenvectors

$$V_{\lambda_2^\pm} = \begin{pmatrix} \mathbf{k} - \frac{k^2}{\omega} \mathbf{v} \\ -\frac{i}{2g\omega} \left[\varepsilon_0 c^2 k^2 - \frac{\omega^2 - \omega_0^2}{\chi \varepsilon_0 \omega_0^2} \mp \sqrt{\left(\varepsilon_0 c^2 k^2 - \frac{\omega^2 - \omega_0^2}{\chi \varepsilon_0 \omega_0^2} \right)^2 - 4g^2 c^2 k^2} \right] \left(\mathbf{k} - \frac{k^2}{\omega} \mathbf{v} \right) \\ 0 \end{pmatrix}; \quad (\text{A.6})$$

$$\lambda_3 = \frac{\omega^2 - \omega_0^2}{\chi \varepsilon_0 \omega_0^2} \quad (\text{A.7})$$

with eigenvector

$$V_{\lambda_3} = \begin{pmatrix} \mathbf{0} \\ \mathbf{v} \\ 0 \end{pmatrix}. \quad (\text{A.8})$$

Because of the condition $\mathbf{v} \cdot \mathbf{P} = 0$, the last eigenvector can be omitted. These eigenvectors, each one defined up to multiplication by an arbitrary function of \mathbf{k} , are solutions of the algebraic system only when the eigenvalues are zero. By direct inspection we see that the only possibilities are:

$$\lambda_0^- = 0, \quad (\text{A.9})$$

which corresponds to the branch (2.19) and gives the solutions

$$f_i(\vec{k})V_{\lambda_0^-,i}\delta\left(c^2k^2 - \frac{g^2\chi\omega_0^2\omega^2}{\omega^2 - \omega_0^2}\right), \quad i = 1, 2, \quad (\text{A.10})$$

where f_i are arbitrary functions;

$$\lambda_1^+ = 0, \quad (\text{A.11})$$

corresponding to the branch (2.17), with eigenvector

$$h(\vec{k})\begin{pmatrix} \mathbf{k} \\ \mathbf{0} \\ 0 \end{pmatrix}\delta(k^2) \quad (\text{A.12})$$

where h is an arbitrary function, and, finally, the condition

$$\lambda_2^- = 0 \quad (\text{A.13})$$

which corresponds to two contributions. The first one is (2.18) with eigenvector

$$l(\vec{k})V_{\lambda_2^-}\delta(\omega^2 - \bar{\omega}^2), \quad (\text{A.14})$$

where l is an arbitrary function, and the other contribution is again $k^2 = 0$ with eigenvector (A.12). The last one suggests that this solution is degenerate and, indeed, we find that the combination

$$W := \frac{2}{\xi}V_{\lambda_2^-} - \left(1 + \frac{g^2\chi\omega_0^2}{\omega^2 - \omega_0^2}\right)V_{\lambda_1^+} \quad (\text{A.15})$$

is such that

$$\mathcal{M}W = O((k^2)^2), \quad (\text{A.16})$$

so that

$$m(\vec{k})W\delta^{(1)}(k^2) \quad (\text{A.17})$$

is a second independent solution of (2.8). Putting all together, with a suitable redefinition of the arbitrary functions, gives the general solution (2.14), (2.15) and (2.16).

Appendix B. The generalised Lautrup strategy

A strategy for getting rid of the dipole ghost term is as follows. First we note that the $\delta^{(1)}$ term is proportional to \mathbf{k} , which means that in the configurations space it is of the form $\partial_\mu f(\vec{x}, t)$ for some f . In an unconstrained theory, this would be a pure gauge configuration (it does not contribute to the electromagnetic field). However, because of the introduction of the auxiliary field B , the whole gauge symmetry is broken and only harmonic gauge transformation, that are transformations of the form $A_\mu \rightarrow A_\mu + \partial_\mu \Lambda$ with Λ harmonic ($\square\Lambda = 0$), are allowed as symmetry transformation. Thus, there is no way to eliminate the undesired term by a symmetry gauge transformation in the constrained theory. Nevertheless, we can always separate this pure gauge term from the rest by decomposing the field \mathbf{A} as follows:

$$\mathbf{A}(\mathbf{x}) = \mathbf{A}^F(\mathbf{x}) + \mathbf{grad}\sigma(\mathbf{x}), \quad (\text{B.1})$$

for some scalar function σ , in such a way that the $\delta^{(1)}$ contribution is contained in $\mathbf{grad}\sigma$. The suffix F stays for Feynman, for analogy with the usual case. To this aim, we need to look at the very origin of the $\delta^{(1)}$

solution. By working in the momentum space, we have $\tilde{\mathbf{A}} = \tilde{\mathbf{A}}^F - i\tilde{\sigma}\mathbf{k}$, where $\tilde{\sigma}$ is the Fourier transform of σ , and the equations of motion take the form

$$\varepsilon_0 c^2 k^2 \tilde{\mathbf{A}}^F - \left[\varepsilon_0 c^2 k^2 \tilde{\sigma} - (\xi - 1) \tilde{B} \right] i\mathbf{k} - ig \left[\omega \tilde{\mathbf{P}} - (\mathbf{k} \cdot \tilde{\mathbf{P}}) \mathbf{v} \right] = \mathbf{0}, \quad (\text{B.2})$$

$$\frac{1}{\chi \varepsilon_0} \left(\frac{\omega^2}{\omega_0^2} - 1 \right) \tilde{\mathbf{P}} + ig \left(\omega \tilde{\mathbf{A}}^F - (\mathbf{v} \cdot \tilde{\mathbf{A}}^F) \mathbf{k} \right) = \mathbf{0}, \quad (\text{B.3})$$

$$\varepsilon_0 c^2 \mathbf{k} \cdot \tilde{\mathbf{A}} = -i\xi \tilde{B}. \quad (\text{B.4})$$

Being $\omega^2 \neq \omega_0^2$, as $\omega^2 = \omega_0^2$ is associated with a mode $\tilde{\mathbf{P}} \propto \mathbf{v}$ that is forbidden by the physical constraint $\mathbf{v} \cdot \mathbf{P} = 0$, we can write

$$\tilde{\mathbf{P}} = -ig \frac{\varepsilon_0 \chi \omega_0^2}{\omega^2 - \omega_0^2} \left(\omega \tilde{\mathbf{A}}^F - (\mathbf{v} \cdot \tilde{\mathbf{A}}^F) \mathbf{k} \right), \quad (\text{B.5})$$

and, then,

$$\varepsilon_0 c^2 k^2 \tilde{\mathbf{A}}^F - \left[\varepsilon_0 c^2 k^2 \tilde{\sigma} - (\xi - 1) \tilde{B} \right] i\mathbf{k} - \frac{g^2 \varepsilon_0 \chi \omega_0^2}{\omega^2 - \omega_0^2} \left[\omega^2 \tilde{\mathbf{A}}^F - \omega (\mathbf{v} \cdot \tilde{\mathbf{A}}^F) \mathbf{k} - \omega (\mathbf{k} \cdot \tilde{\mathbf{A}}^F) \mathbf{v} + k^2 (\mathbf{v} \cdot \tilde{\mathbf{A}}^F) \mathbf{v} \right] = \mathbf{0}. \quad (\text{B.6})$$

It is now convenient to proceed by looking at solution in the different supports determined above, branch by branch. Let us start with the solutions having support in $k^2 = 0$, which is the branch containing the dipole ghost. We assume that $\tilde{\sigma}$ and \tilde{B} are any two given scalar distributions such that

$$\left[\varepsilon_0 c^2 k^2 \tilde{\sigma} - (\xi - 1) \tilde{B} \right] i\mathbf{k} =: U(\mathbf{k}) i\mathbf{k} \delta(k^2) \quad (\text{B.7})$$

so that only the δ appears. Thus, substituting the ansatz

$$\tilde{\mathbf{A}}^F(\mathbf{k}) = a(\mathbf{k}) \delta(k^2) \mathbf{k} + b(\mathbf{k}) \delta^{(1)}(k^2) \mathbf{k} + r(\mathbf{k}) \delta(k^2) \mathbf{v} + \sum_{i=1}^2 d_i(\mathbf{k}) \delta(k^2) \mathbf{e}_{(i)}(\mathbf{k}), \quad (\text{B.8})$$

in (B.6), we get $d_i(\mathbf{k}) = 0$, whereas $a(\mathbf{k})$ and $r(\mathbf{k})$ can be arbitrary, and the coefficient of the $\delta^{(1)}$ is

$$b(\mathbf{k}) = -\frac{i}{\varepsilon_0 c^2} U(\mathbf{k}) + \frac{g^2 \chi \omega_0^2 \omega}{c^2 (\omega^2 - \omega_0^2)} r(\mathbf{k}). \quad (\text{B.9})$$

We can eliminate the $\delta^{(1)}$ from $\tilde{\mathbf{A}}^F$ by choosing $\tilde{\sigma}$ in such a way that the coefficient $b(\mathbf{k})$ vanishes, which is equivalent to say that we have to cancel out all terms linear in \mathbf{k} in (B.6). Thus, the desired decomposition is defined by the condition

$$\varepsilon_0 c^2 k^2 \tilde{\sigma} - (\xi - 1) \tilde{B} + i\varepsilon_0 \frac{g^2 \chi \omega_0^2}{\omega^2 - \omega_0^2} \omega (\mathbf{v} \cdot \tilde{\mathbf{A}}^F) = 0. \quad (\text{B.10})$$

At this point, the equation for $\tilde{\mathbf{A}}^F$ is

$$c^2 k^2 \tilde{\mathbf{A}}^F - \frac{g^2 \chi \omega_0^2}{\omega^2 - \omega_0^2} \left[\omega^2 \tilde{\mathbf{A}}^F - \omega (\mathbf{k} \cdot \tilde{\mathbf{A}}^F) \mathbf{v} + k^2 (\mathbf{v} \cdot \tilde{\mathbf{A}}^F) \mathbf{v} \right] = \mathbf{0}, \quad (\text{B.11})$$

which has general solution

$$\begin{aligned} \tilde{\mathbf{A}}^F(\mathbf{k}) &= (\omega^2 - \omega_0^2) a_0(\mathbf{k}) \delta(k^2) \mathbf{v} + \sum_{i=1}^2 \delta \left(\varepsilon_0 c^2 k^2 - \frac{g^2 \varepsilon_0 \chi \omega_0^2 \omega^2}{\omega^2 - \omega_0^2} \right) a_i(\mathbf{k}) \mathbf{e}_i \\ &+ \delta \left(\varepsilon_0 c^2 k^2 - \frac{g^2 \chi \varepsilon_0 \omega_0^2 \omega^2}{\omega^2 - \omega_0^2} \right) \tilde{a}_3(\mathbf{k}) \mathbf{k} - \frac{g}{\varepsilon_0} b_3(\mathbf{k}) \delta(\omega^2 - \bar{\omega}^2) \mathbf{v}, \end{aligned} \quad (\text{B.12})$$

which corresponds to a sum of plane wave solutions.
For the polarisation vector we get

$$\begin{aligned}\tilde{\mathbf{P}}(\mathbf{k}) = & -ig\varepsilon_0\chi\omega_0^2a_0(\mathbf{k})\delta(k^2)(\omega\mathbf{v} - \mathbf{k}) - ig\frac{\varepsilon_0\chi\omega_0^2\omega}{\omega^2 - \omega_0^2}\sum_{i=1}^2\delta\left(\varepsilon_0c^2k^2 - \varepsilon_0\frac{g^2\chi\omega_0^2\omega^2}{\omega^2 - \omega_0^2}\right)a_i(\mathbf{k})\mathbf{e}_i \\ & + ib_3(\mathbf{k})\delta(\omega^2 - \bar{\omega}^2)(\omega\mathbf{v} - \mathbf{k}).\end{aligned}\tag{B.13}$$

Notice that from

$$\frac{\tilde{B}}{\varepsilon_0c^2} = \frac{i}{\xi}\mathbf{k} \cdot \tilde{\mathbf{A}} = \frac{i}{\xi}\mathbf{k} \cdot \tilde{\mathbf{A}}^F + \frac{i}{\xi}k^2\tilde{\sigma}\tag{B.14}$$

and using (B.10), we get

$$\tilde{B}(\mathbf{k}) = i\varepsilon_0c^2(\mathbf{k} \cdot \tilde{\mathbf{A}}^F) - i\varepsilon_0c^2\frac{g^2\chi\omega_0^2\omega}{\omega^2 - \omega_0^2}(\mathbf{v} \cdot \tilde{\mathbf{A}}^F),\tag{B.15}$$

and so

$$\tilde{B}(\mathbf{k}) = (\omega^2 - \bar{\omega}^2)i\omega\varepsilon_0c^2a_0(\mathbf{k})\delta(k^2).\tag{B.16}$$

In particular, B is harmonic, as expected. Finally, we have to solve the equation for $\tilde{\sigma}$

$$k^2\tilde{\sigma}(\mathbf{k}) = \frac{(\xi - 1)}{\varepsilon_0}\tilde{B}(\mathbf{k}) - i\omega\frac{g^2\chi\omega_0^2}{\omega^2 - \omega_0^2}(\mathbf{v} \cdot \tilde{\mathbf{A}}^F),\tag{B.17}$$

that is

$$\begin{aligned}k^2\tilde{\sigma}(\mathbf{k}) = & (\xi(\omega^2 - \bar{\omega}^2) - \omega^2 + \omega_0^2)i\omega a_0(\mathbf{k})\delta(k^2) - ik^2\tilde{a}_3(\mathbf{k})\delta\left(\varepsilon_0c^2k^2 - \frac{g^2\chi\varepsilon_0\omega_0^2\omega^2}{\omega^2 - \omega_0^2}\right) \\ & + i\frac{g}{\varepsilon_0c^2}\omega b_3(\mathbf{k})\delta(\omega^2 - \bar{\omega}^2).\end{aligned}\tag{B.18}$$

In order to find the right representation, let us simplify the expressions by setting

$$V(\mathbf{k}) := (\xi(\omega^2 - \bar{\omega}^2) - \omega^2 + \omega_0^2)i\omega a_0(\mathbf{k})\tag{B.19}$$

and look for a particular solution of the equation

$$k^2\tilde{\sigma}(\mathbf{k}) = V(\mathbf{k})\delta(k^2).\tag{B.20}$$

Since $(k^2)^2\tilde{\sigma}(\mathbf{k}) = 0$, if we are working in the space of tempered distributions, the solution must be of the form

$$\tilde{\sigma}(\mathbf{k}) = \alpha(\mathbf{k})\delta(k^2) + \beta(\mathbf{k})\delta^{(1)}(k^2),\tag{B.21}$$

for some scalar functions α and β . Here is exactly where the $\delta^{(1)}$ enters the game. Note that the α part is harmonic, so it is not determined by the equation, which, indeed, simply gives

$$\beta(\mathbf{k}) = -V(\mathbf{k}).\tag{B.22}$$

In the configuration space, let us give a look at the particular solution

$$\sigma(\mathbf{x}) \equiv -\int_{\mathbb{R}^4}\frac{d^4k}{(2\pi)^4}V(\mathbf{k})e^{-i\mathbf{k}\cdot\mathbf{x}}\delta^{(1)}(k^2),\tag{B.23}$$

where with the symbol \equiv we mean modulo harmonic terms, which can be reabsorbed in the pure harmonic gauge terms, corresponding to the a_3 term in (2.14). If we consider the identity

$$\delta^{(1)}(k^2) = \frac{1}{2\omega} v^\mu \frac{\partial}{\partial k^\mu} \delta(k^2) \quad (\text{B.24})$$

we can write

$$\sigma(\mathbf{x}) \equiv -\frac{i}{2} \int_{\mathbb{R}^4} \frac{d^4 k}{(2\pi)^4} V(\mathbf{k}) \frac{\mathbf{v} \cdot \mathbf{x}}{\omega} e^{-i\mathbf{k} \cdot \mathbf{x}} \delta(k^2). \quad (\text{B.25})$$

After inserting back the expression of V we get

$$\sigma(\mathbf{x}) \equiv \frac{1}{2} \int_{\mathbb{R}^4} \frac{d^4 k}{(2\pi)^4} (\xi(\omega^2 - \bar{\omega}^2) - \omega^2 + \omega_0^2) a_0(\mathbf{k}) (\mathbf{v} \cdot \mathbf{x}) e^{-i\mathbf{k} \cdot \mathbf{x}} \delta(k^2), \quad (\text{B.26})$$

which is a covariant version of the function Λ introduced by Lautrup in [24]. The corrections due to the remaining terms in (B.18) are³

$$\begin{aligned} \sigma(\mathbf{x}) &\equiv \frac{1}{2} \int_{\mathbb{R}^4} \frac{d^4 k}{(2\pi)^4} (\xi(\omega^2 - \bar{\omega}^2) - \omega^2 + \omega_0^2) a_0(\mathbf{k}) (\mathbf{v} \cdot \mathbf{x}) e^{-i\mathbf{k} \cdot \mathbf{x}} \delta(k^2) \\ &\quad - i \int_{\mathbb{R}^4} \frac{d^4 k}{(2\pi)^4} \tilde{a}_3(\mathbf{k}) e^{-i\mathbf{k} \cdot \mathbf{x}} \delta \left(\varepsilon_0 c^2 k^2 - \frac{g^2 \chi \varepsilon_0 \omega_0^2 \omega^2}{\omega^2 - \omega_0^2} \right) \\ &\quad + i \frac{g}{\varepsilon_0 c^2} \int_{\mathbb{R}^4} \frac{d^4 k}{(2\pi)^4} \frac{\omega}{k^2} b_3(\mathbf{k}) e^{-i\mathbf{k} \cdot \mathbf{x}} \delta(\omega^2 - \bar{\omega}^2). \end{aligned} \quad (\text{B.27})$$

In adding $\mathbf{grad}\sigma$ to \mathbf{A}^F , the terms with \tilde{a}_3 in (B.12) cancel out. Thus, we apparently lose a degree of freedom. However, σ is defined modulo additive harmonic terms, as we said, which can be equally included in \mathbf{A}^F without changing its property of excluding the dipole ghost. For this reason we prefer to drop the \tilde{a}_3 term out from σ , and substitute the corresponding term in (B.12) with the harmonic term $a_3(\mathbf{k})\delta(k^2)\mathbf{k}$, whereas we include, for later convenience, a pure harmonic gauge term in $\mathbf{grad}\sigma$, whose Fourier transform is

$$-\frac{i}{4\omega_+} \frac{\xi(\omega^2 - \bar{\omega}^2)^2 + (\omega^2 - \omega_0^2)(\omega^2 - \bar{\omega}^2) - 2\omega_+^2 g^2 \chi \omega_0^2}{(\omega^2 - \bar{\omega}^2)(\omega^2 - \omega_0^2)} a_0(\mathbf{k}) \mathbf{k} \delta(k^2). \quad (\text{B.28})$$

In conclusion, we write

$$\mathbf{A}(\mathbf{x}) = \mathbf{A}^F(\mathbf{x}) + \mathbf{grad}\sigma, \quad (\text{B.29})$$

with

$$\begin{aligned} \mathbf{A}^F(\mathbf{x}) &= \int_{\mathbb{R}^4} \frac{d^4 k}{(2\pi)^4} \left[(\omega^2 - \omega_0^2) a_0(\mathbf{k}) \delta(k^2) \mathbf{v} + \sum_{i=1}^2 \delta \left(\varepsilon_0 c^2 k^2 - \frac{g^2 \chi \varepsilon_0 \omega_0^2 \omega^2}{\omega^2 - \omega_0^2} \right) a_i(\mathbf{k}) \mathbf{e}_i + \delta(k^2) a_3(\mathbf{k}) \mathbf{k} \right. \\ &\quad \left. - \frac{g}{\varepsilon_0 c^2} b_3(\mathbf{k}) \delta(\omega^2 - \bar{\omega}^2) \mathbf{v} \right] e^{-i\mathbf{k} \cdot \mathbf{x}}, \end{aligned} \quad (\text{B.30})$$

and

$$\begin{aligned} \mathbf{grad}\sigma(\mathbf{x}) &= \int_{\mathbb{R}^4} \frac{d^4 k}{(2\pi)^4} \left[\frac{1}{2} (\xi(\omega^2 - \bar{\omega}^2) - \omega^2 + \omega_0^2) (\mathbf{v} - i(\mathbf{v} \cdot \mathbf{x}) \mathbf{k}) + \right. \\ &\quad \left. - i \frac{\xi(\omega^2 - \bar{\omega}^2)^2 + (\omega^2 - \omega_0^2)(\omega^2 - \bar{\omega}^2) - 2\omega_+^2 g^2 \chi \omega_0^2}{4\omega(\omega^2 - \bar{\omega}^2)} \mathbf{k} \right] \delta(k^2) a_0(\vec{k}) e^{-i\mathbf{k} \cdot \mathbf{x}} \\ &\quad + \frac{g}{\varepsilon_0 c^2} \int_{\mathbb{R}^4} \frac{d^4 k}{(2\pi)^4} \frac{\omega}{k^2} \delta(\omega^2 - \bar{\omega}^2) \mathbf{k} b_3(\mathbf{k}) e^{-i\mathbf{k} \cdot \mathbf{x}}. \end{aligned} \quad (\text{B.31})$$

³Notice that, in the limit $g \rightarrow 0$, the last row disappears and the second one becomes harmonic.

In particular, we note that \mathbf{A}^F contains a basis of plane waves. Let us notice that the above redefinition of the fields introduces a singularity in $\omega^2 = \bar{\omega}^2$, so one could be prevented in doing it. However, as we have discussed in section 2, such a singular locus is expected to appear in any case, as we shall see, because of the non vanishing intersection of the different branches.

Appendix C. Deduction of (B.12)

We need to solve the equation

$$\mathcal{M}\tilde{\mathbf{A}}^F = \mathbf{0}, \quad (\text{C.1})$$

where

$$\mathcal{M} = \left[c^2 k^2 - \frac{g^2 \chi \omega_0^2 \omega^2}{\omega^2 - \omega_0^2} \right] \mathbb{I}_4 + \frac{g^2 \chi \omega_0^2}{\omega^2 - \omega_0^2} [\omega \mathbf{v} \otimes \mathbf{k} - k^2 \mathbf{v} \otimes \mathbf{v}]. \quad (\text{C.2})$$

Since $\mathbf{k} \cdot \mathbf{v} = \omega$, \mathbf{k} is in the kernel of the last parenthesis. Also, being $\mathbf{e}_i \cdot \mathbf{k} = \mathbf{e}_i \cdot \mathbf{v} = 0$, \mathbf{e}_i are also in the kernel of the same matrix. Thus, \mathcal{M} acts on these terms only by means of the first part which is proportional to the identity. The coefficient of proportionality is thus an eigenvalue with multiplicity 3. The fourth eigenvalue can be computed by observing that the image of the second matrix is proportional to \mathbf{v} . Since the first part is proportional to the identity, this means that \mathbf{v} is the fourth eigenvector:

$$\mathcal{M}\mathbf{v} = k^2 c^2 \left[1 - \frac{g^2 \chi \omega_0^2}{\omega^2 - \omega_0^2} \right] \mathbf{v}. \quad (\text{C.3})$$

The determinant of the matrix is the product of the four eigenvalues:

$$\det \mathcal{M} = \left[c^2 k^2 - \frac{g^2 \chi \omega_0^2 \omega^2}{\omega^2 - \omega_0^2} \right]^3 c^2 k^2 \frac{\omega^2 - \bar{\omega}^2}{\omega^2 - \omega_0^2}. \quad (\text{C.4})$$

The supports of the equation are thus defined by $\det \mathcal{M} = 0$.

The component

$$\left[c^2 k^2 - \frac{g^2 \chi \omega_0^2 \omega^2}{\omega^2 - \omega_0^2} \right]^3 = 0 \quad (\text{C.5})$$

corresponds to 3 different eigenvectors, $\mathbf{e}_1, \mathbf{e}_2, \mathbf{k}$, so it defines the solution

$$\mathbf{f}(\mathbf{k}) = \delta \left(c^2 k^2 - \frac{g^2 \chi \omega_0^2 \omega^2}{\omega^2 - \omega_0^2} \right) [a_1(\mathbf{k})\mathbf{e}_1 + a_2(\mathbf{k})\mathbf{e}_2 + a_1(\mathbf{k})\mathbf{k}]. \quad (\text{C.6})$$

The other support pieces, $k^2 = 0$ and $\omega^2 - \bar{\omega}^2 = 0$, correspond to the eigenvector \mathbf{v} , so they define the solution

$$\mathbf{g}(\mathbf{k}) = (\omega^2 - \omega_0^2) a_0(\mathbf{k}) \delta(k^2) \mathbf{v} + \tilde{b}_3(\mathbf{k}) \delta(\omega^2 - \bar{\omega}^2) \mathbf{v}. \quad (\text{C.7})$$

Putting all together and defining $\tilde{b}_3(\mathbf{k}) =: 4\pi g b_3(\mathbf{k})$, we get (B.12).

Remark: the reason for defining b_3 is to get the right behaviour in the limit $g = 0$.

Remark: the same result can be obtained directly from (2.14), by considering that

$$\begin{aligned} & \int \frac{d^4 k}{(2\pi^3)} \omega [\omega^2 - \omega_0^2 - \xi(\omega^2 - \bar{\omega}^2)] a_0(\mathbf{k}) \mathbf{k} \delta^{(1)}(k^2) e^{-i\mathbf{k} \cdot \mathbf{x}} \\ &= \int \frac{d^4 k}{(2\pi^3)} [\omega^2 - \omega_0^2 - \xi(\omega^2 - \bar{\omega}^2)] a_0(\mathbf{k}) \frac{1}{2} \mathbf{k} v^\mu \frac{\partial}{\partial k^\mu} \delta(k^2) e^{-i\mathbf{k} \cdot \mathbf{x}} \\ &= \frac{i}{2} \int d^4 k (2\pi^3) [\omega^2 - \omega_0^2 - \xi(\omega^2 - \bar{\omega}^2)] a_0(\mathbf{k}) (\mathbf{v} \cdot \mathbf{x}) \mathbf{k} \delta(k^2) e^{-i\mathbf{k} \cdot \mathbf{x}} \\ &\quad - \frac{1}{2} \int d^4 k (2\pi^3) [\omega^2 - \omega_0^2 - \xi(\omega^2 - \bar{\omega}^2)] a_0(\mathbf{k}) \mathbf{v} \delta(k^2) e^{-i\mathbf{k} \cdot \mathbf{x}} + \dots, \end{aligned} \quad (\text{C.8})$$

where we have integrated by parts, and the first line of the last expression comes out from the derivative of the phase, the second line from the derivative of the vector \mathbf{k} , and the ellipses, which are the derivative of the remaining factor, stay for harmonic pure gauge terms.

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