Hawking effect in dielectric media and the Hopfield model

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We consider the so-called Hopfield model for the electromagnetic field in a dielectric dispersive medium in a framework in which one allows a space-time dependence of microscopic parameters, aimed at a phenomenological description of a space-time varying dielectric perturbation induced by means of the Kerr effect. We discuss the analogue Hawking effect by introducing a simplified model which avoids some difficulties which characterize in the full Hopfield model, still keeping the same dispersion relation. Our main result is an analytical calculation of the spontaneous thermal emission in the single-branch case, which is provided nonperturbatively for the first time in the framework of dielectric black holes. A universal mechanism for thermality which is shared both by optical black holes and acoustic black holes is also pointed out.

I. INTRODUCTION

In the framework of analogue gravity, a very interesting possibility to check experimentally the existence of the analogue of Hawking radiation is represented by black holes in dielectric media. By taking into account in a very short summary the historical developments, we point out that there are several contributions to this topic, which was reassessed in an interesting framework in [1] and received a further impulse, especially on the experimental side, in [2], grounded on [3,4], and then corroborated also by numerical simulations in [5]. Experimental results in [2] are controversial (see e.g. [6–9]). Subsequent studies involve both numerical and experimental situations. On the former side, an interesting 2D model where there is a sharp step behavior of the dielectric properties of the medium has been discussed in [10–12], with numerical evaluations of the pair-production processes involved. Numerical but also analytical perturbative studies appeared in [13,14], where a smooth behavior of the refractive index was assumed, in the framework of a phenomenological model grounded on the unidirectional pulse propagation equation approximation for the electromagnetic field, and thermality has been explored again in particular in [14]. The aforementioned studies contributed in a very important way to the comprehension of the physics at hand. In the present paper, our main contribution is to complement these studies on the analytical side, providing both a theoretical framework and analytical nonperturbative deduction of thermality which has been lacking up to now for dielectric black holes. Our reference model is the covariant Hopfield model discussed in [15,16]. It represents our basic tool for analyzing conceptual issues which characterize the physics at hand.

We then introduce a simplified model for dispersion, which is still covariant, but has the great advantage to be not involved in a constrained quantization procedure and also to be based on a couple of scalar fields (in place of a couple of vector fields). Its physical content is nontrivial, and can allow for a number of very interesting physical situations, which can be also experimentally tested. We stress that the full Hopfield model can also be exactly solved in the same physical situations, with the difference that it is much more difficult to be handled, both for the vectorial nature of the fields involved, and for their being constrained systems.

The main focus of our paper is in the analytical deduction of thermality for dielectric black holes. This represents our key result, and main motivation of our analysis. Furthermore, we point out a peculiar Fuchsian singularity structure of the field equations near the horizon, and stress its role in determining thermality of the emitted radiation, and is shared with other analytical calculations in the acoustic black hole case [17,18]. As a by-product of our analysis, we provide also a derivation of the so-called generalized Manley-Rowe relations, which play an important role in the presence of pair creation. See Appendix C.

The plan of the paper is the following. In Sec. II we start from a short discussion of the Hopfield model, and describe a simplified model which avoids some formal difficulties of the Hopfield model, still maintaining the same dispersion relation. We develop the scattering picture in order to point out the presence of pair creation from the vacuum. Then in Sec. III we derive thermality in the limit of weak dispersion by matching asymptotic solutions with solutions obtained in the near horizon region, in strict analogy with calculations developed in [17,18]. In Sec. IV we discuss the

problem of the nature of the horizon involved in the pair creation process. In Sec. V we adapt our results to the case of multiple resonances in the dielectric medium. Then in Sec. VI we summarize our main results. For the sake of completeness, we also provide some appendices. In Appendix A we show that our model reproduces exactly the same dispersion relation as in the full electromagnetic case. Moreover, we show that, in line of principle, the model can be solved exactly, being reducible to a Gaussian path integral. In Appendix B we discuss some formal aspects of scattering theory. In Appendix C we derive the generalized Manley-Rowe relations in our scattering framework. In Appendix D, some discussion of geometrical optics aspects is given.

II. HOPFIELD MODEL REVISITED: THE $\varphi\psi$–MODEL AS A THEORETICAL BENCHMARK FOR THE ELECTROMAGNETIC CASE

The Hopfield model is aimed at the description of dielectric dispersive media, considered as transparent [19,20]. The model, as known, introduces mesoscopic fields, representing the polarization fields of mesoscopic electrodynamics, which are coupled with the electromagnetic field. These fields are in the form of a number of harmonic oscillators, each with a proper oscillation frequency characterizing dispersive effects of light in the medium. The medium itself is considered in the first approximation as transparent. The latter requirement can be relaxed, as e.g. in [21–23]. In particular, in [22] an interesting model including space-time variations of the susceptibility is provided. The original model can be described as follows:

$$
\begin{aligned}
\mathcal{L}_{em}: \ =\ & \frac{1}{8\pi}\left(\frac{1}{c}\dot{\mathbf{A}}+\nabla\phi\right)^2-\frac{1}{8\pi}(\nabla\wedge\mathbf{A})^2 \\
& +\frac{1}{2\chi}(\dot{\mathbf{P}}^2-\omega_0^2\mathbf{P}^2)-\frac{1}{2c}(\mathbf{P}\cdot\dot{\mathbf{A}}+\dot{\mathbf{A}}\cdot\mathbf{P}) \\
& -\frac{1}{2}(\mathbf{P}\cdot\nabla\phi+\nabla\phi\cdot\mathbf{P}).
\end{aligned}
$$

In the previous equation, $\mathbf{A}, \phi$ stand for the components of the electromagnetic potential vector field, $\mathbf{P}$ is the polarization field, ω_0 is the proper frequency for the oscillation of the polarization field, χ is the dielectric susceptibility, and c is the speed of light. We introduced in [15,16] a full covariant 4D version of the Hopfield model, taking into account the requirements for a relativistic extension, which is apt for a simple discussion of physics in the comoving frame of a uniformly moving dielectric perturbation induced by means of the Kerr effect. This is only a phenomenological approximation, in the sense that, in place of the nonlinear interactions involved in the Kerr effect, we consider its effects through the space-time

dependence of parameters like the susceptibility. See below.

The covariant Lagrangian density deduced in [16] is

$$
\begin{aligned}
\mathcal{L}=&-\frac{1}{16\pi}F_{\mu\nu}F^{\mu\nu}-\frac{1}{2\chi\omega_0^2}[(v^\rho\partial_\rho P_\mu)(v^\sigma\partial_\sigma P^\mu)]\\
&+\frac{1}{2\chi}P_\mu P^\mu-\frac{g}{2c}(v_\mu P_\nu-v_\nu P_\mu)F^{\mu\nu},
\end{aligned}\tag{1}
$$

where P_μ is the polarization field, $F_{\mu\nu}$ is the field strength tensor, and v^μ is the four velocity associated with the dielectric medium. In this model, we allow the microscopic parameters χ, ω_0, g to depend on space-time variables. We notice that, as e.g. in the (noncovariant) polariton model introduced in [21,24], where a space dependent coupling between the electromagnetic field and the polarization field is introduced, we have as well introduced a coupling g, which is *a priori* space-time dependent.

The covariant quantization of the model has been pursued in [16], and is a nontrivial task because of the gauge constraints implied by the model. See also [15], where further analysis is presented in a perturbative framework. In order to look for our main goal, which is the Hawking effect in dielectric media, we simplify as possible the theoretical framework, as discussed in the following.

The electromagnetic Lagrangian for the full Hopfield model is quite involved, and in order to test quantum effects is not so manageable. In order to gain insights into the real situation, and carry out analytical calculations as far as possible, we introduce a simplified model where the couple *electromagnetic field–polarization field* is simulated by a couple of scalar fields: φ, ψ in place of the electromagnetic field and of the polarization field respectively. The model we propose is constructed in such a way to maintain the same dispersion relation and to simulate the same coupling as in the full case. The model at hand is related to the 2D reduction of the Hopfield model which is adopted in [11].

We introduce

$$
\begin{aligned}
\mathcal{L}_{\varphi\psi}=&\frac{1}{2}(\partial_\mu\varphi)(\partial^\mu\varphi)+\frac{1}{2\chi\omega_0^2}[(v^\alpha\partial_\alpha\psi)^2-\omega_0^2\psi^2]\\
&+\frac{g}{c}(v^\alpha\partial_\alpha\psi)\varphi,
\end{aligned}\tag{2}
$$

where χ plays the role of the dielectrical susceptibility, v^μ is the four-velocity vector, ω_0 stays for the proper frequency of the medium, and g is a constant which plays the role of coupling constant between the fields. *A priori*, in a phenomenological model aimed at describing the electromagnetic field, we can leave room for a space-time dependence of microscopic parameters like χ, ω_0, g. Moreover, we can extend the model in such a way to include also $N > 1$ polarization fields ψ_i, each one characterized by a different ω_{0i}, χ_i, g_i. We shall not use the latter

freedom until the final sections of the paper, and focus our attention on the single-resonance model with just a single polarization field.

The equations of motion are (we omit space-time arguments):

$$\Box\varphi - \frac{g}{c}(v^\alpha\partial_\alpha\psi) = 0, \tag{3}$$

$$(v^\alpha\partial_\alpha)\frac{1}{\chi\omega_0^2}(v^\alpha\partial_\alpha\psi) + \frac{1}{\chi}\psi + \frac{1}{c}v^\alpha\partial_\alpha(g\varphi) = 0. \tag{4}$$

Let G_ψ be the Green function for ψ:

$$\left[(v^\alpha\partial_\alpha)\frac{1}{\chi\omega_0^2}(v^\alpha\partial_\alpha) + \frac{1}{\chi}\right]G_\psi = \delta, \tag{5}$$

where δ stays for the Dirac delta function. Then, defining $\tilde{x} = (t, x, y, z)$, we get the following system:

$$\psi(\tilde{x}) = \frac{1}{c}\int d^4w G_\psi(\tilde{x} - w)(v^\alpha\partial_\alpha g\varphi)(w), \tag{6}$$

$$\Box\varphi(\tilde{x}) - \frac{g}{c^2}(v^\alpha\partial_\alpha)\int d^4w G_\psi(\tilde{x} - w)(v^\beta\partial_\beta g\varphi)(w) = 0, \tag{7}$$

which represent a simplified model equations set simulating dispersive effects in optics.

It is also useful to introduce the Hamiltonian equations for the given model. In particular, we introduce the conjugate momenta

$$\pi_\varphi := \frac{\partial\mathcal{L}}{\partial\partial_0\varphi} = \partial_0\varphi, \tag{8}$$

$$\pi_\psi := \frac{\partial\mathcal{L}}{\partial\partial_0\psi} = \frac{1}{\chi\omega_0^2}v^0v^\alpha\partial_\alpha\psi + \frac{g}{c}v^0\varphi. \tag{9}$$

Then we calculate the Hamiltonian density $\mathcal{H}$:

$$\begin{aligned}\mathcal{H} &= \partial_0\psi\pi_\psi + \partial_0\varphi\pi_\varphi - \mathcal{L}\\ &= \frac{1}{2}\pi_\varphi^2 + \frac{\chi\omega_0^2}{2(v^0)^2}\pi_\psi^2 - \frac{v^k}{v^0}(\partial_k\psi)\pi_\psi - \frac{g\chi\omega_0^2}{c}\frac{v^k}{v^0}\pi_\psi\varphi\\ &\quad + \frac{1}{2c^2}\chi\omega_0^2g^2\varphi^2 + \frac{1}{2\chi}\psi^2 + \frac{1}{2}(\partial_k\varphi)^2.\end{aligned} \tag{10}$$

Let us also define $h := \int dx\mathcal{H}$. Then we obtain

$$\partial_0\varphi = \{\varphi, h\}, \tag{11}$$

$$\partial_0\psi = \{\psi, h\}, \tag{12}$$

$$\partial_0\pi_\varphi = \{\pi_\varphi, h\}, \tag{13}$$

$$\partial_0\pi_\psi = \{\pi_\psi, h\}. \tag{14}$$

Of course, the following nontrivial Poisson brackets hold true:

$$\{\varphi, \pi_\varphi\} = \delta, \tag{15}$$

$$\{\psi, \pi_\psi\} = \delta, \tag{16}$$

which will play a role in the quantization of the fields, as in the Hopfield model [16].

A. Conserved scalar product

By following the same line of thought as in the electromagnetic case, we can show that the following (global) phase transformation,

$$\varphi \mapsto e^{ia}\varphi; \qquad \varphi^* \mapsto e^{-ia}\varphi^*, \tag{17}$$

$$\psi \mapsto e^{ia}\psi; \qquad \psi^* \mapsto e^{-ia}\psi^*, \tag{18}$$

is a symmetry for the complexified Lagrangian,

$$\mathcal{L}_{\varphi\psi}^{\text{complex}} = \frac{1}{2}(\partial_\mu\varphi^*)(\partial^\mu\varphi) + \frac{1}{2\chi\omega_0^2}[(v^\alpha\partial_\alpha\psi^*)(v^\alpha\partial_\alpha\psi) + \omega_0^2\psi^2] + \frac{g}{2c}(v^\alpha\partial_\alpha\psi^*)\varphi + \frac{g}{2c}(v^\alpha\partial_\alpha\psi)\varphi^*; \tag{19}$$

Noether's theorem implies that the following current:

$$J^\mu := \frac{i}{2}\left[\varphi^*\partial^\mu\varphi - (\partial^\mu\varphi^*)\varphi + \frac{1}{\chi\omega_0^2}v^\mu\psi^*v^\alpha\partial_\alpha\psi - \frac{1}{\chi\omega_0^2}v^\mu\psi v^\alpha\partial_\alpha\psi^* + \frac{g}{c}v^\mu(\psi^*\varphi - \psi\varphi^*)\right] \tag{20}$$

is conserved: $\partial_\mu J^\mu = 0$ along the solutions of the equations of motion. Then the charge

$$\int_{\Sigma_t} dx J^0 \tag{21}$$

is conserved, and allows us to define a conserved scalar product. In particular, we obtain

$$((\varphi\psi)|(\tilde{\varphi}\,\tilde{\psi})) = \frac{i}{2}\int_{\Sigma_t}\left[\varphi^*\partial^0\tilde{\varphi} - (\partial^0\varphi^*)\tilde{\varphi} + \frac{1}{\chi\omega_0^2}v^0\psi^*v^\alpha\partial_\alpha\tilde{\psi} - \frac{1}{\chi\omega_0^2}v^0\tilde{\psi}v^\alpha\partial_\alpha\psi^* + \frac{1}{c}gv^0(\psi^*\tilde{\varphi} - \tilde{\psi}\varphi^*)\right]. \qquad (22)$$

This scalar product will be particularly important in the definition of the quantum states for the model at hand.

The same scalar product can be obtained also as associated with the symplectic structure of the classical Hamiltonian equations. The starting point is still the complexified Lagrangian. One first defines

$$\pi_\varphi := \frac{\partial\mathcal{L}_c}{\partial\partial_0\varphi} = \frac{1}{2}\partial_0\varphi, \qquad (23)$$

$$\pi_\varphi^* := \frac{\partial\mathcal{L}_c}{\partial\partial_0\varphi^*} = \frac{1}{2}\partial_0\varphi^*, \qquad (24)$$

$$\pi_\psi := \frac{\partial\mathcal{L}_c}{\partial\partial_0\psi} = \frac{1}{2\chi\omega_0^2}v^0v^\alpha\partial_\alpha\psi + \frac{g}{2c}v^0\varphi, \qquad (25)$$

$$\pi_\psi^* := \frac{\partial\mathcal{L}_c}{\partial\partial_0\psi^*} = \frac{1}{2\chi\omega_0^2}v^0v^\alpha\partial_\alpha\psi^* + \frac{g}{2c}\varphi^*. \qquad (26)$$

Then, introducing

$$\Omega := -i\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \qquad (27)$$

where the square matrix represents the symplectic form of standard Hamiltonian classical mechanics, we can obtain the conserved scalar product as follows:

$$\langle\Psi_1, \Psi_2\rangle := (\Psi_1, \Omega\Psi_2) = : \int dx\Psi_1^* \cdot \Omega\Psi_2, \qquad (28)$$

where $\cdot$ stands for the ordinary scalar product in $\mathbb{R}^4$, and

$$\Psi := \begin{pmatrix} \psi \\ \varphi \\ \pi_\psi \\ \pi_\varphi \end{pmatrix}. \qquad (29)$$

B. Separation of variables in the comoving frame for $v = $ const

From the system (6) and (7), it is not immediately evident that it is possible to separate variables in the comoving frame for $v = $ const. As to the dependence on transversal spatial variables y, z, it is evident that separation of variables occurs. In the case of the time variable t, it is necessary to write the system (6) and (7) as a first order system in t; in other words, we have to rewrite the aforementioned system of equations of motion in Hamiltonian form. Then we obtain the following Hamiltonian form of the system:

$$i\partial_0\begin{pmatrix} \varphi \\ \psi \\ \pi_\varphi \\ \pi_\psi \end{pmatrix} = i\begin{bmatrix} 0 & 0 & 1 & 0 \\ -\frac{g}{c}\chi\omega_0^2 & -\frac{v^k}{v^0}\partial_k & 0 & -\frac{\chi\omega_0^2}{v^0} \\ \left(\nabla^2 - \frac{g^2}{c^2}\chi\omega_0^2\right) & 0 & 0 & \frac{g}{c}\chi\omega_0^2 \\ 0 & -\frac{1}{\chi} & 0 & -\frac{v^k}{v^0}\partial_k \end{bmatrix}\begin{pmatrix} \varphi \\ \psi \\ \pi_\varphi \\ \pi_\psi \end{pmatrix}. \qquad (30)$$

In a more concise form, we can write the previous equation as follows:

$$i\partial_t\Psi = H\Psi, \qquad (31)$$

where $\Psi := (\varphi, \psi, \pi_\varphi, \pi_\psi,)^T$ and where H is the matrix operator displayed in (30). It is to be noted that H is formally self-adjoint in the scalar product $\langle,\rangle$ we have defined in the previous subsection. Indeed, it is not difficult to show that the Hermitian conjugate of H, say $H_c = (\Omega H\Omega)^\dagger$, has the same form as the operator H.

Since H is independent on t, it follows that one can find stationary solutions in the form

$$\Psi = \exp(-i\omega t)F(x, y, z), \qquad (32)$$

and separation of variables in t becomes evident. Notice that ω is a conserved quantity, and that this amounts to the following conservation in the lab frame

$$\omega_{\text{lab}} - vk_{\text{lab},x} = \text{const}, \qquad (33)$$

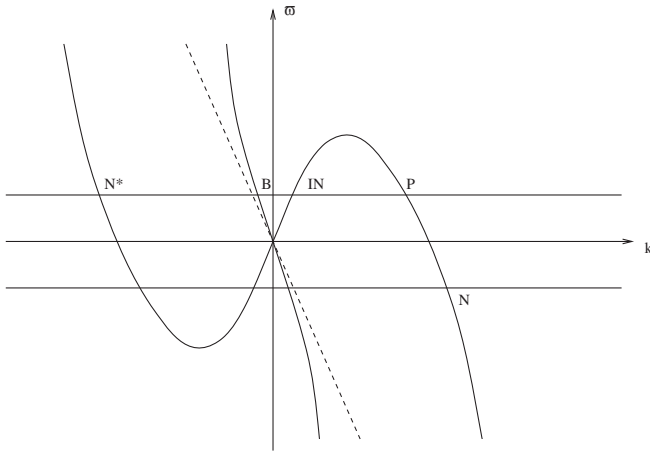which was previously obtained in a perturbative approach and now is confirmed in an exact model. Note that this

FIG. 1. Asymptotic dispersion relation for the Cauchy case in the comoving frame. The monotone branch is G_+, the non-monotone one is G_-. The dashed line divides antiparticle states (below it) and particle ones (above it). Two lines at $\omega = $ const and at $-\omega = $ const are also drawn, and relevant states introduced in the text are explicitly indicated.

conservation is simply associated with the conservation of energy in the comoving frame.

C. Norm of the states and field quantization

The norm of the states is inherited by a conserved scalar product:

$$\|\Psi\|^2 := \langle \Psi, \Psi \rangle$$
$$= \frac{i}{2} \int_{\Sigma_t} \left[\varphi^* \partial^0 \varphi - (\partial^0 \varphi^*) \varphi + \frac{1}{\chi \omega_0^2} v^0 \psi^* v^\alpha \partial_\alpha \psi \right.$$
$$\left. - \frac{1}{\chi \omega_0^2} v^0 \psi v^\alpha \partial_\alpha \psi^* + \frac{g}{c} v^0 (\psi^* \varphi - \psi \varphi^*) \right]. \quad (34)$$

Particle states correspond to positive norm states (which is a notion which remains invariant under Lorentz group), negative norm states correspond to antiparticles. This can

be exemplified in the homogeneous case, i.e. in absence of the perturbation. Then, in the case of plane-wave solutions, it can be shown that

$$\|\Psi\|^2 := \langle \Psi, \Psi \rangle \propto \omega_{\text{lab}}, \quad (35)$$

so that $\text{sgn}\|\Psi\|^2 = \text{sgn}\omega_{\text{lab}}$.

This results match well what happens in the electromagnetic case, and show that particles in the lab are defined by the condition $\omega_{\text{lab}} > 0$. Then, in the comoving frame, we have that particle states are defined by the condition $\omega > -vk$. These conditions should represent a good indication for particle and antiparticle states also for the full problem. See Appendix B for more details. This corroborates the common use of the asymptotic dispersion relation, to be indicated as (DR)-asymptotic in the following, in order to identify particle and antiparticle states. See Fig. 1. For example, in the Cauchy approximation we have

$$n(\omega_{\text{lab}}) = n(0) + B\omega_{\text{lab}}^2 + \delta n(x - vt), \quad (36)$$

where $n(0)$ is $n(\omega_{\text{lab}} = 0)$, B is constant, and the Kerr effect induces the right-moving perturbation $\delta n(x - vt)$; the (DR)-asymptotic displays three states on the same branch G_- [cf. Eq. (48)] with the same ω in the comoving frame: the positive group velocity particle state, to be called *IN*, the negative group velocity particle state, to be called *P*, and a negative group velocity antiparticle state N^*. It is also to be remarked that each state (ω, k) is conjugated to a state with $(-\omega, -k)$ (and then there is e.g. a state *N* which is the conjugate of N^*). See [13,25]. There is also another branch G_+ [cf. Eq. (48)], which for the same ω displays only a single negative group velocity particle state *B*. Only the former branch G_- is involved in the Hawking process.

Quantization follows the general lines we have found for the electromagnetic case [16], with the important simplification associated with the absence of constraints, which makes easier and standard the treatment of the topic. In particular, in the comoving frame (which is static), we have

$$\Psi(t, \mathbf{x}) = \sum \int \frac{d\omega}{2\pi} \frac{d\mathbf{k}}{(2\pi)^3} \frac{1}{N_\Psi} \delta(\text{dispersion relation})(a(\omega, \mathbf{k})U(\omega, \mathbf{k}; t, \mathbf{x}) + a^\dagger(\omega, \mathbf{k})V(\omega, \mathbf{k}; t, \mathbf{x})), \quad (37)$$

where N_Ψ stands for a normalization factor, $\delta(\text{dispersion relation})$ indicates that one considers solutions "on shell," and U, V are positive norm solutions and negative norm solutions of the field equations respectively. To be more specific, we also point out that we are interested in the following fields:

$$\Psi_{(\text{IN})}(t, \mathbf{x}), \quad (38)$$

which represents the field asymptotically in the past: as $t \to -\infty$, and the corresponding asymptotic field in the future:

$$\Psi_{(\text{OUT})}(t, \mathbf{x}). \quad (39)$$

A single (rapid) transition from a region with a given set of parameters χ, g, ω_0 to a region with parameters χ', g', ω_0' is considered, and the above asymptotic fields are interpolated by the field (37).

Notice that the above quantization, in a two-dimensional model, on shell, can give rise both to an ω-representation, where only the integration in $d\omega$ is left, and also to a k-representation, where the dispersion relation is used to

leave only the integral in dk. In a four-dimensional model, the latter choice is to be preferred.

In order to compute amplitudes for pair creation, the strategy is the following. Let us expand in plane waves both the IN state and the P, N, B states emerging from the scattering process. Then, let us consider

$$|N|^2 := \frac{|J_x^N|}{|J_x^{IN}|}, \tag{40}$$

where J_x stands for the conserved current (20), and the indexes N, IN indicate that one is considering the N-particle states and the IN-particle states respectively (and similar for P and B). Indeed, $|N|^2$ above is the ratio between the outgoing flux along x of negative energy particle states and the flux along x of ingoing particle states, where an infinitesimal surface element orthogonal to x has been simplified. $|N|^2$ coincides with the mean number per unit time and unit volume of created particles [26–28]. See also [29]. To this topic is devoted Appendix C, where scattering amplitudes are discussed in detail, and the relation between white hole scattering and black hole scattering is pointed out. Furthermore, a deduction of the generalized Manley-Rowe identities (which are substantially Wronskian relations) is also given. In the following section, we discuss particle creation and in particular its thermal character in our model in the Hawking case.

III. ANALYTICAL CALCULATIONS AND THERMALITY

We wish to set up an analytical approach allowing us to infer thermality for our model. Our reference works will be [17,18]. Indeed, we also try to identify a common mechanism for thermality, including our model, and the general fluid model discussed in [18], which is a generalization and a refinement of the seminal calculations by Corley [17]. The strategy is shared with the aforementioned calculations: Wentzel-Kramers-Brillouin (WKB) approximation is introduced at the level of the calculations for asymptotic states far from the horizon, and this expansion is matched with the near-horizon expansion, which is instead treated in Fourier space, in the approximation up to the linear order in x. It is not difficult to show that, under the hypothesis of not very strong gradient of the refractive index, one of the four asymptotic states (which belongs to the monotone branch of the dispersion relation) decouples and gives rise to a scattering phenomenon which is an almost negligible fraction of the dominant phenomenon represented by the Hawking effect. On the other hand, it is a consequence of the construction of the states in the near-horizon region that the states which match with the asymptotic states P and N (positive and negative norm states emerging from the scattering and lying on the nonmonotone branch of the dispersion relation) are such that the ratio

$$\frac{|N|^2}{|P|^2} = \exp\left(-\beta_h \hbar \omega\right), \tag{41}$$

where β_h is the black hole temperature, i.e. it leads to the standard thermal spectrum as far as the fourth state is negligibly coupled (which means that $|B|^2$ is negligible). We start from the asymptotic expansion.

A. WKB analysis

We consider our equations of motion. Instead of working in full generality, we take into account only the possibility to get $g \mapsto g_0 + \delta g(x)$, where g_0 is constant. We could allow also a dependence on x of the susceptibility $\chi \mapsto \chi_0 + \delta\chi(x)$, as well as $\omega_0^2 \mapsto \omega_0^2 + \delta\omega_0^2(x)$, where χ_0, ω_0^2 are constants, but this choice seems to represent a hindrance to a neat deduction of the Hawking effect, as we show in the following subsection. As a consequence, we avoid such a dependence, and justify this choice by recalling that our model is phenomenological: the traveling perturbation is introduced by means of inhomogeneities in the couplings of the model, which are chosen so that the refractive index displays the expected behavior. What is important is the latter behavior, and not the way the phenomenological model allows us to implement it. As a matter of fact, the model, as it is constructed, is able to reproduce a behavior like $n(x_{\text{lab}} - vt_{\text{lab}}, \omega_{\text{lab}}) = n_0(\omega_{\text{lab}}) + \delta n(x_{\text{lab}} - vt_{\text{lab}})$, with the traveling perturbation described as independent on the dispersion, only in the limit of negligible dispersion [3], when the Cauchy approximation holds true. Otherwise, such phenomenological behavior cannot be approached by means of the Hopfield model, and its extension to the nonlinear regime would be required.

All considerations above in any case do not affect the zeroth order WKB approximation, which is obtained as follows. As in [17,18], we have $\partial_t \mapsto i\omega$. We adopt the following expansion:

$$\begin{pmatrix} \varphi \\ \psi \end{pmatrix} = \sum_{k\geq 0} \left(\frac{\hbar}{i}\right)^k \exp\left(i\frac{\omega}{\hbar}t + i\frac{S(x)}{\hbar}\right) \begin{pmatrix} A_k \\ B_k \end{pmatrix}. \tag{42}$$

Then, defining $p = \partial_x S$, we get the following zeroth order matrix by expanding in powers of $\hbar$:

$$M_{(0)} = \begin{bmatrix} -\frac{\omega^2}{c^2} + p^2 & -i\frac{g}{c}\gamma(\omega + vp) \\ i\frac{g}{c}\gamma(\omega + vp) & \frac{1}{\chi\omega_0^2}(\omega_0^2 - \gamma^2(\omega + vp)^2) \end{bmatrix}. \tag{43}$$

As to the zeroth order, the mandatory vanishing of the determinant of $M_{(0)}$ amounts to

$$c^2 p^2 - \omega^2 = \frac{g^2\gamma^2\chi(\omega + vp)^2}{1 - \frac{\gamma^2(\omega + vp)^2}{\omega_0^2}}, \tag{44}$$

which is equivalent to the usual dispersion relation in the lab, apart for the dependence on space of the

inhomogeneous terms. We obtain a polynomial of degree 4 in p, to which we can associate four solutions, as expected. Due to the intrinsic nature of our model, it makes sense to adopt the Cauchy approximation. As it is easy to understand, at this level there is no substantial difference between solutions one obtains as above and the solutions one can obtain from the two branches of the Cauchy approximation in a phenomenological model where the eikonal approximation is assumed. We assume that

$$p(x) = p_0(x) + \epsilon p_1(x) + \cdots, \qquad (45)$$

i.e. we assume that $p(x)$ is developable in series. Then, we consider the following perturbative solution ansatz, which is equivalent to the one adopted in [17,18]:

$$p(x) = p_0(x) + \epsilon p_1(x), \qquad (46)$$

$$\omega = \epsilon \omega_1, \qquad (47)$$

which is equivalent to the ansatz $\omega \ll pv$. The two branches in the Cauchy approximation are

$$G_{\pm} := \gamma(\omega + vp)(B\gamma^2(\omega + vp)^2 + n(x))$$
$$\pm \frac{c}{v}\left(\gamma(\omega + vp) - \frac{\omega}{\gamma}\right) = 0, \qquad (48)$$

where $G_+ = 0$ corresponds to the monotone branch. From $G_- = 0$, in the above perturbative ansatz, we obtain three solutions:

$$p_{\pm} = \pm \frac{1}{\sqrt{B}}\frac{1}{\gamma v}\sqrt{\frac{c}{v} - n} - \frac{\omega}{v} - \frac{1}{2}\frac{c\omega}{\gamma^2 v^2}\frac{1}{\frac{c}{v} - n}, \qquad (49)$$

$$p_{+s} = -\frac{\omega}{v} + \frac{c\omega}{\gamma^2 v^2}\frac{1}{\frac{c}{v} - n}, \qquad (50)$$

where we have purposefully introduced a notation which resembles the one in [17], and indeed the modes we have found correspond to the modes identified therein. It can be noticed that asymptotically the norm of p_+, p_{+s} is positive, being $\omega + vp > 0$, whereas the norm of p_- is negative. Moreover, $p_{\pm}$ are short wavelength states, and p_{+s} is a long wavelength one. A fourth real solution emerges from $G_+ = 0$, and is the one which decouples from the spectrum in the approximation where a not too strong gradient in the refractive index occurs. The above solutions have to be matched with the ones which will be obtained in the near horizon region. With this aim, we investigate also the amplitude part of the WKB solutions. We can obtain the same results from Eq. (44). We are also interested in the amplitudes associated with the zeroth order equation, which leaves undetermined one solution (due to the fact that $M_{(0)}$, which is a rank 1 matrix, is to be considered on shell, i.e. for p corresponding to a root of $\det M_{(0)} = 0$). We need to look for the first order equation, which is

$$M_{(1)}\begin{pmatrix} A_0 \\ B_0 \end{pmatrix} + M_{(0)}\begin{pmatrix} A_1 \\ B_1 \end{pmatrix} = 0. \qquad (51)$$

We have

$$M_{(1)} = \begin{bmatrix} -i(\partial_x p) - 2ip\partial_x & -\frac{1}{c}\gamma vg\partial_x \\ \frac{1}{c}\gamma v(g\partial_x + (\partial_x g)) & i\frac{\gamma^2 v}{\chi\omega_0^2}((\partial_x s) + 2s\partial_x) \end{bmatrix}, \qquad (52)$$

where p coincides with one of the roots of $\det M_{(0)} = 0$ (at the leading order) and we introduced

$$s = \gamma(\omega + vp). \qquad (53)$$

Following the theory of the multicomponent WKB (see e.g. [30,31]), in order to find out the zeroth order amplitude, we proceed as follows. We first express the zeroth order solution as $r_0 a_0$, where r_0 is a right eigenvector of $M_{(0)}$ relative to the zero eigenvalue, and a_0 is the unknown amplitude. Then we project (51) on the left eigenvector l_0 of $M_{(0)}$ relative to the zero eigenvalue, obtaining then a single equation $l_0 M_{(1)} r_0 a_0 = 0$ for the amplitude. A rather straightforward calculation gives us, in the limit of negligible ω, and for the leading order solutions $p_{\pm}$ ($M_{(1)}|_{p_{\pm}}$ is considered)

$$A_0 \propto \frac{1}{x^{3/4}}, \qquad (54)$$

$$B_0 \propto \frac{1}{x^{1/4}}, \qquad (55)$$

in the near horizon region where the WKB approximation still holds, in the so-called linear region of [18], but not too near the horizon, where it breaks down. This behavior is also expected from the near horizon expansion [see below Eq. (79)].

B. Near horizon region analysis

For the region near the horizon x_+ which is such that $n(x_+) = c/v$, we proceed again by adopting the same ansatz as in [17,18]. Our starting point is to Fourier transform the equations of motion, by keeping into account that $x \mapsto i\partial_p$. The trick is a linearization in the neighborhood of the horizon $x = 0$: $n(x) \sim \frac{c}{v} + \kappa x \mapsto \frac{c}{v} + \kappa i\partial_p$.

In the following analysis, our guide is represented by the presence of a suitable branch cut in Fourier space which should allow us to pinpoint the origin of thermality. At first, we proceed by looking for a Fuchsian singularity as the one occurring at $p = 0$ also in [17,18]. Our ansatz is that, given the universality in the Hawking effect in the nondispersive case, one should be able to find out a common origin of thermally also in the dispersive case. We show in the following that it is indeed possible to implement thermality according to our ansatz. As both our model and the ones in

[17,18] characterize a huge class of analogous systems, we conjecture that such a behavior is a distinctive signal for the Hawking effect in analogous systems. Indeed, provided that a suitable matching exists with WKB solutions in the asymptotic region, the calculation of thermality can be associated with a (hyper)local behavior, in the sense that, once the aforementioned matching allows to find a relation between asymptotic WKB wave functions and local functions near the horizon, thermality can be calculated even locally. It is sufficient to compare $|N|^2/|P|^2$ near the horizon to get the right behavior.

The aforementioned ansatz is interesting, in the sense that it is very selective in the way the phenomenological model should be related to the behavior of microscopical parameters appearing in the Hopfield model. Indeed, in the latter one, we could allow the following dependence on the x variable in the comoving frame of the dielectric perturbation: $\chi \mapsto \chi_0 + \delta\chi(x), \omega_0^2 \mapsto \omega_0^2 + \delta\omega_0^2(x), g \mapsto g_0 + \delta g(x)$. The worst behavior is obtained by the presence of a dependence on $\delta\chi(x)$, as it does not allow any Fuchsian structure and makes as hard as possible the problem of identifying the source of thermality. Even by considering only a first order equation arising from Eq. (57) (see below), it is not possible to identify a mechanism for thermality. So our choice is to factor out this behavior. There is no problem, in our view, in this choice, as in any case the Hopfield model we are trying to adapt in order to match the phenomenological behavior is only an approximate model which should be more correctly described by a nonlinear version of quantum electrodynamics (as the dielectric perturbation arises from nonlinearities of the dielectric medium). Then we consider the correction $\delta\omega_0^2(x)$, which is interesting (indeed, it could be as well considered as arising from a linearization of the nonlinear polarization term P^4 one could add to the Lagrangian). Still, even if a Fuchsian structure can be identified, the roots of the indicial equation depend on ω_0^2 instead of on ω^2, so that again thermality is not extracted from the model. The last possibility, which is to some extent an unexpected variant of the model, because it amounts to a variation of the plasma frequency, but at constant susceptibility χ, is at the very least satisfactory. Indeed, it not only provides us a good Fuchsian behavior with thermality, but it is also a good model at the level of a first order analysis.

We also symmetrize the last term in the Lagrangian (2): $\frac{g}{c}(v^\alpha\partial_\alpha\psi)\varphi \mapsto \frac{g}{2c}(v^\alpha\partial_\alpha\psi)\varphi - \frac{g}{2c}(v^\alpha\partial_\alpha\varphi)\psi$. From the equations of motion we obtain

$$\tilde{\varphi}(p) = \frac{1}{p^2 - \frac{\omega^2}{c^2}}\frac{1}{c}\left[\frac{1}{2}\alpha\gamma v + ig_0\gamma(\omega + vp)\right.$$
$$\left. - \alpha\partial_p\gamma(\omega + vp)\right]\tilde{\psi}(p), \qquad (56)$$

where

$$g(x) = g_0 + \alpha x \mapsto g_0 + i\alpha\partial_p, \qquad (57)$$

and

$$\frac{1}{\chi}\left(1 - \frac{\gamma^2(\omega + vp)^2}{\omega_0^2}\right)\tilde{\psi}(p)$$
$$+ \frac{1}{c}\left(\frac{1}{2}\alpha\gamma v + ig_0\gamma(\omega + vp) - \alpha\partial_p\gamma(\omega + vp)\right)$$
$$\times \tilde{\varphi}(p) = 0. \qquad (58)$$

Equations (56) and (58) lead in a natural way to a second order differential equation, which is better described in terms of the variable s and which displays a Fuchsian singularity for $s = 0$:

$$\left[\partial_s^2 + \frac{2}{s}\partial_s + \left(\frac{1}{4} + \frac{c^2\omega^2}{\gamma^4 v^4 \chi\alpha^2}\right)\frac{1}{s^2}\right]\tilde{\psi}(s) = 0, \qquad (59)$$

whose indicial roots are

$$\alpha_\pm = -\frac{1}{2} \pm i\frac{c\omega}{\gamma^2 v^2\sqrt{\chi}\alpha}. \qquad (60)$$

Then, near $s = 0$, the solutions behave as

$$\tilde{\psi}(s) \sim s^{-\frac{1}{2}\pm i\frac{c\omega}{\gamma^2 v^2\sqrt{\chi}\alpha}}. \qquad (61)$$

We note that a similar Fuchsian singularity appears also in [17,18]. As both the models characterize a huge class of analogous systems, we conjecture that such a behavior is a distinctive signal for the Hawking effect in analogous systems.

By comparing with the phenomenological dispersion relation $G_+G_- = 0$ in the Cauchy approximation [the latter approximation is the only one to be expected to provide a sensible match between the Hopfield model and the zeroth order assumption of nondependence on ω of the dielectric perturbation $\delta n(x)$], we find

$$g_0 = \sqrt{\frac{n(0)^2 - 1}{\chi}}, \qquad (62)$$

$$\delta g = n(0)\sqrt{\frac{1}{(n(0)^2 - 1)\chi}}\delta n, \qquad (63)$$

$$\omega_0^2 = \frac{n(0)^2 - 1}{2n(0)B}. \qquad (64)$$

We assume $\chi > 0$, $g_0 > 0$. In particular, we stress the following relation:

$$\alpha := g' = n(0)\sqrt{\frac{1}{(n(0)^2 - 1)\chi}} n', \qquad (65)$$

where the prime indicates the x-derivative and the above identity is considered at $x = 0$. Note that $n' = \kappa$, which is not yet the surface gravity (because also of the sign to be taken into account).

Even if an almost satisfactory behavior can be identified, there is a problem: it is not possible to find out a solution for (58) in explicit form, so it is not clear how to prescribe the behavior of the solutions. As a consequence, in the spirit of our above considerations, we turn to a first order analysis. This amounts to consider (58), where $\tilde{\varphi}$ is given by (56), and to neglect terms $\propto \alpha^2$.

We assume that $\alpha\gamma v \ll s$ in order to properly deal with the limit as $\omega \to 0$.

The first order equation one obtains admits the following solution:

$$\tilde{\psi}(s) = \sqrt{c^2(s - \gamma\omega)^2 - \gamma^2 v^2 \omega^2} \frac{1}{s} \exp(iq(s)), \qquad (66)$$

where the phase $q(s)$ is

$$q(s) := \frac{c^2 s^3}{6\alpha\chi g_0\gamma^3 v^3\omega_0^2} + \frac{c}{v}\frac{c\omega}{\alpha\chi g_0\gamma^2 v^2}\log(s) + w(s), \qquad (67)$$

where

$$w(s) := \frac{g_0 s}{2\alpha\gamma v} + \frac{c^2\omega^2}{2s\alpha\chi g_0\gamma^3 v^3} - \frac{c^2 s}{2\alpha\chi g_0\gamma^3 v^3} - \frac{c^2 s^2\omega}{2\alpha\chi g_0\gamma^2 v^3\omega_0^2}$$
$$+ \frac{c^2 s\omega^2}{2\alpha\chi g_0\gamma v^3\omega_0^2} - \frac{s\omega^2}{2\alpha\chi g_0\gamma v\omega_0^2}. \qquad (68)$$

We aim to write the field $\psi(t, x)$ as follows:

$$\psi(t, x) = e^{i\omega t}\int_\Gamma dp\frac{1}{\sqrt{2\pi}}\tilde{\psi}(p)e^{ipx}$$
$$= e^{i\omega t}e^{-i\frac{\omega}{v}x}\frac{1}{\gamma v}\int_\Gamma ds\frac{1}{\sqrt{2\pi}}\tilde{\psi}(s)e^{is\frac{1}{\gamma v}x}, \qquad (69)$$

for a suitable path Γ in the complex plane.

Our ansatz is the following: the contribution $w(s)$ is not relevant in the saddle point approximation, in such a way that, keeping into account the contribution of the Fourier transform, the saddle points are determined by

$$\frac{d}{ds}\left(\frac{s}{\gamma v}x + \frac{c^2 s^3}{6\alpha\chi g_0\gamma^3 v^3\omega_0^2}\right) = 0. \qquad (70)$$

In other terms, the relevant saddle points are the ones of the function

$$h(s) := \frac{s}{\gamma v}x + \frac{c^2 s^3}{6\alpha\chi g_0\gamma^3 v^3\omega_0^2}. \qquad (71)$$

According to the saddle point approximation we have

$$\int_\Gamma dsF(s)\exp(ih(s))$$
$$\simeq \sum_k F(s_k)\exp(ih(s_k))\frac{1}{\sqrt{-i\frac{d^2 h}{ds^2}(s_k)}}, \qquad (72)$$

where the sum is extended to the saddle points. From (70) we obtain

$$\frac{c^2 s^2}{2\alpha\chi g_0\gamma^2 v^2\omega_0^2} + x = 0, \qquad (73)$$

i.e.

$$s_\pm = \pm\sqrt{\text{sign}(x)|x|}\sqrt{2\chi|\alpha|g_0\gamma v\omega_0}\frac{1}{c} := \pm\eta|x|^{1/2}, \quad (74)$$

where we have taken into account that $\alpha < 0$. In particular, we get

$$(a)x > 0 \Rightarrow s_\pm = \pm\left(\sqrt{2\chi|\alpha|g_0\gamma v\omega_0}\frac{1}{c}\right)x^{1/2}, \qquad (75)$$

$$(b)x < 0 \Rightarrow s_\pm = \pm i\left(\sqrt{2\chi|\alpha|g_0\gamma v\omega_0}\frac{1}{c}\right)|x|^{1/2}, \qquad (76)$$

where case (a) refers to the outer region and (b) to the inner one. The large parameter in the saddle point approximation is

$$\sqrt{2|\alpha|g_0\chi\omega_0^2} = \sqrt{\frac{n(0)^2 - 1}{B}|\kappa|} \propto \sqrt{\frac{|\kappa|}{B}}; \qquad (77)$$

the above parameter is manifestly large due to the smallness of B.

Due to the strong analogy with the [17,18] case, we can adopt the same choice for circuits in the complex plane (the only difference being in using variable s instead of variable p). See Fig. 2. Forbidden sectors in the complex s-plane are formally the same, albeit s cannot be too high because in our approximation ω_0 acts as an ultraviolet cutoff $s \ll \omega_0$.

Case (b) above provides us modes with exponentially increasing/decreasing behavior. The growing mode is not physically allowed, whereas the decreasing one is assumed to be defined along the curve Γ_0 passing through $-i(\sqrt{2\chi|\alpha|g_0\gamma v\omega_0}\frac{1}{c})x^{1/2}$ and with asymptotes in the allowed regions in the lower s-plane. As to the branch point $s = 0$, we choose the associated branch cut along the positive $\Im(s)$ axis. The decreasing mode, in the saddle point approximation, is
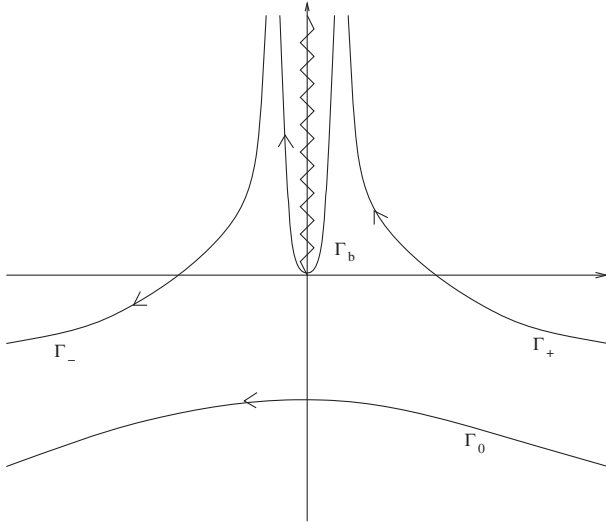
FIG. 2. Circuits in the complex s-plane defining the relevant modes as described in the text.

$$\psi_0(x) \sim C_0 \exp\left(-\frac{4}{3}|x|^{3/2}\frac{\eta}{\gamma v}\right)f(x)\frac{1}{|x|^{1/4}}, \qquad (78)$$

where C_0 is a normalization constant. The function $f(x)$ is associated with the remaining dependence on x (it depends also on η).

Modes with high momentum correspond to curves $\Gamma_\pm$, passing through $\pm(\sqrt{2\chi|\alpha|g_0}\gamma v\omega_0\frac{1}{c})|x|^{1/2}$, ending in the allowed region and flowing asymptotically near the branch cut (see the figure). Γ_- flows from the allowed left region to the allowed region having the positive imaginary $\Im(s)$-axis as asymptote. Γ_+ arises from the left of the aforementioned branch cut (asymptote) and ends in the right allowed region. We get

$$\psi_\pm(x) \sim C_+ \exp(ir_\pm(x))\frac{1}{|x|^{1/4}}\exp\left(\pm\pi\omega\frac{c}{v}\frac{1}{n(0)}\frac{c}{|\kappa|\gamma^2 v^2}\right), \qquad (79)$$

where C_+ is a (common) normalization constant and $r_\pm$ indicates a pure phase function. The real "thermal" characterization arises from the real exponential term, with exponent proportional to ω. In particular, thermality is related to the contribution

$$-\frac{c}{v}\frac{c\omega}{|\alpha|\chi g_0\gamma^2 v^2}\log(s) \qquad (80)$$

in the phase factor, by taking into account that

$$\log(s \pm i\epsilon) \to \log|s| \pm i\pi \quad \text{as } \epsilon \to 0. \qquad (81)$$

One could wonder if this corresponds to a good temperature for our model, matching the one for the nondispersive model. The answer is positive, indeed

$$\beta_H = \frac{2\pi k_b}{\hbar}\frac{c}{v}\frac{c}{|\alpha|\chi g_0\gamma^2 v^2} = \frac{c}{v}\frac{1}{n(0)}\left(\frac{2\pi k_b}{\hbar}\frac{c}{|\kappa|\gamma^2 v^2}\right), \qquad (82)$$

where the term in brackets is immediately realized to have the same form of the inverse of the nondispersive temperature. As it stands, the factor $\frac{c}{v}\frac{1}{n(0)}$ is substantially equal to one, because the horizon condition $n(0) + \delta n(x) = c/v$ implies, due to the smallness of δn, that $n(0) \sim c/v$. Still, it is true that it is not an exact result. The model is imperfect, which is to be expected.

It is interesting to point out that pair creation, in a thermal way, comes out because of the circuits suitably running around the branch point $s = 0$, which is also the threshold for defining particle ($s > 0$) and antiparticle ($s < 0$) states.

There is a fourth circuit Γ_b which is running around the branch cut. Its behavior can be inferred by analogy with [17,18].

Near the horizon, for $\omega \to 0$ (cf. [17,18]), the contribution to the amplitude arising from the saddle points is of the order of $1/\sqrt{s}$, i.e. $x^{-1/4}$, which matches the WKB result in the same order of approximation. Of course, one has to keep into account if $\text{sign}(x)$ is positive or negative (i.e. if the mode is in the inner part or in the outer part of the horizon). In the inner region, there is an evident decreasing exponential factor which tames the modes, as in [17,18].

As to the behavior of the field $\varphi(x)$, by taking into account that $\alpha\gamma v \ll s$, one obtains $\varphi_\pm \propto x^{-3/4}$, again in agreement with the WKB calculations.

In particular, we are interested in the large p modes, which amount to modes $p_\pm$ in the WKB approximation. These modes are involved with $\Gamma_\pm$ which give rise to the well-known thermality factor, with temperature

$$T_H \simeq \frac{\hbar}{2\pi c k_b}|\kappa|\gamma^2 v^2. \qquad (83)$$

We could proceed as in [17] and [18] for getting this result, or we can adopt a simpler ansatz, which is grounded on the nature of modes $p_\pm$ to be associated with positive and negative norm states with an "ultraviolet" momentum. They correspond to P and N modes respectively, and, as far as the fourth mode B is weakly coupled to the scattering, thermality can be inferred simply by the behavior of the ratio

$$\frac{|N|^2}{|P|^2} = \exp\left(-\frac{\hbar\omega}{k_b T_H}\right), \qquad (84)$$

where T_H is given by (83).

IV. EXPANSION POINT: WHICH KIND OF HORIZON DO WE NEED?

We have assumed, up to now, to work with the non-dispersive (geometrical) horizon as an expansion point

where the near horizon analysis is carried out. As a matter of fact, this choice is not the only one. Indeed, we could as well choose two other horizons: the group horizon (GH) and the phase horizon (PH), which are defined respectively as the blocking horizon for propagating waves at a given frequency in the comoving frame (the point where the group velocity of the wave packet vanishes), and the locus where the phase velocity of the waves composing the wave packet vanishes. From a physical point of view, at first sight, there is no doubt that GH is more attractive and meaningful than the PH, and should play the role of nondispersive horizon in the dispersive case, as it results in the optical case in [14], where GH is referred to as "blocking horizon." Pure thermality of the spectrum is supposed to rely on the presence of a GH. Still, we wish to consider both concepts, as they could play a role which is not yet made as evident from our previous calculations.

In our analysis in Sec. III B the horizon surface is at a generic locus simply indicated as "horizon" and shifted to $x = 0$, where only at the end of calculations our microscopic parameters are transformed into the macroscopic ones. Equation (82) remains unaltered, and changes occur only at the level of the subsequent approximation for the factor $\frac{c}{v}\frac{1}{n(0)}$. It is still true that it is substantially equal to one, because $n(0) \sim c/v$. Corrections in the temperature appear, depending on ω in the GH case, and on k_0 in the PH case. In particular, one finds

$$\frac{\frac{c}{v}}{\frac{c}{v} - n_{\text{PH}}} = \frac{1}{1 - \frac{v}{c}zk_0^2} \sim 1 + \frac{v}{c}zk_0^2, \qquad (85)$$

$$\frac{\frac{c}{v}}{\frac{c}{v} - n_{\text{GH}}} = \frac{1}{1 - \frac{v}{c}\zeta_B\omega^{2/3}} \sim 1 + \frac{v}{c}\zeta_B\omega^{2/3}, \qquad (86)$$

provided that in the former case $\frac{v}{c}zk_0^2 \ll 1$ holds true, as well as $\frac{v}{c}\zeta_B\omega^{2/3} \ll 1$ in the latter case and ζ_B is given by Eq. (D7). No modifications in the saddle point approximation and in the choice of the paths appear, as quantities appearing in the calculations, like κ, do not depend on s (they can depend on ω or on k_0, which are fixed parameters).

As a matter of fact, we have also to point out that in fluid models the above distinction between different kinds of horizons, and in particular between nondispersive horizon and group horizon, is *a posteriori* irrelevant, in the sense that conditions can be given such that the wave function of the modes is not able to distinguish between the aforementioned horizons, and also any correction to thermality is washed out. This is the main contribution of Ref. [32]. Due to several analogies between the framework discussed therein and our own, our hypothesis is that a similar conclusion is reasonable also in our case, despite the formal dependence we found above on the expansion point. We do not delve into this question herein.

V. MULTIRESONANCES CASE FOR THE HOPFIELD MODEL

We discuss shortly in this section how to modify the model in order to take into account several resonances $\omega_{01} < \omega_{02} < \cdots < \omega_{0N}$. We introduce

$$\mathcal{L} = \frac{1}{2}(\partial_\mu\varphi)(\partial^\mu\varphi)$$
$$+ \sum_{i=1}^{N}\left\{\frac{1}{2\omega_{0i}^2\chi_i}[(v^\alpha\partial_\alpha\psi_i)^2 - \omega_{0i}^2\psi_i^2] + \frac{g_i}{c}(v^\alpha\partial_\alpha\psi_i)\varphi\right\}, \qquad (87)$$

where N polarization fields ψ_i appear, as well as N *a priori* different ω_{0i}, χ_i, g_i. As a consequence, in the homogeneous case we get

$$n^2 = 1 + \sum_{i=1}^{N}\frac{g_i^2\chi_i}{1 - \frac{s^2}{\omega_{0i}^2}}. \qquad (88)$$

It is rather simple to manage expressions like

$$\frac{1}{1 - \frac{s^2}{\omega_{0i}^2}} = \begin{cases} \sum_{k=0}^{\infty}\left(\frac{s^2}{\omega_{0i}^2}\right)^k & \text{for } s < \omega_{0i}, \\ -\frac{\omega_{0i}^2}{s^2}\sum_{k=0}^{\infty}\left(\frac{\omega_{0i}^2}{s^2}\right)^k & \text{for } s > \omega_{0i}. \end{cases} \qquad (89)$$

This allows one to find expressions to the desired order of approximation for any value of s. The only warning is represented by the regions in a strict neighborhood of the resonances themselves. Such regions cannot be described through the Hopfield model in the present simplified version, because there a large absorption by the dielectric medium, whose dissipative aspects require a special care, takes place. We simply limit ourselves to neglect these regions, deserving their study to more complete models. As to the Hawking effect in dielectric media, we point out that the near horizon analysis does not introduce any new interesting features: indeed, the only Fuchsian term in a second order expansion, or the only logarithmic branch point in a first order expansion, still occurs at $s = 0$. For example with reference to the former expansion, and limiting ourselves to a two-resonance ($N = 2$) model, where the two polarization fields $\tilde{\psi}_1, \tilde{\psi}_2$ are reexpressed in terms of $\tilde{\varphi}$ by using the equations of motion, one finds

$$\left[\partial_s^2 + \frac{2}{s}\partial_s + \left(\frac{1}{4} + \frac{c^2\omega^2}{\gamma^4v^4(\chi_1\alpha_1^2 + \chi_2\alpha_2^2)}\right)\frac{1}{s^2}\right]\tilde{\varphi}(s) = 0, \qquad (90)$$

where α_i amounts to the derivative of g_i, in a straightforward generalization of the single-branch model. Microscopic parameters are related to macroscopic ones as follows in the region $s < \omega_{01}$:

$$g_1^2 \chi_1 + g_2^2 \chi_2 = n(0)^2 - 1, \qquad (91)$$

$$g_1 \chi_1 \delta g_1 + g_2 \chi_2 \delta g_2 = n(0) \delta n, \qquad (92)$$

$$g_1^2 \chi_1 \frac{1}{\omega_{01}^2} + g_2^2 \chi_2 \frac{1}{\omega_{02}^2} = B \sqrt{n(0)^2 - 1}. \qquad (93)$$

We also get the following relation:

$$n' n(0) = g_1 \chi_1 \alpha_1 + g_2 \chi_2 \alpha_2. \qquad (94)$$

A first order expansion in α_1, α_2, allows one to recover a phase term related to thermality of the form

$$i \frac{c^2 \omega}{(g_1 \chi_1 \alpha_1 + g_2 \chi_2 \alpha_2) \gamma^2 v^3} \log(s); \qquad (95)$$

it is easy to show that, thanks to (94), one is able to recover again (82). It is evident that the model is more involved, and it is remarkable that there is the possibility to have more complex scattering processes, as evident also by inspection of the dispersion relation. In particular, there are processes which involve both the lower branch and the upper one, and which are candidated to be Hawking-like. For example, a process $IN' \rightarrow P' + N'^*$ is possible, where N'^* belongs to the lower branch and P', IN' to the upper one (see Fig. 3). We shall consider details in a future publication, and limit ourselves to observe that in the latter case IN' is a higher energy and momentum state than P', and then, whichever nature one would be able to attribute to the aforementioned scattering, what is sure is that vacuum emission still will be peaked at the lower branch emission process we have analyzed in the previous sections, being more energetically favourable.
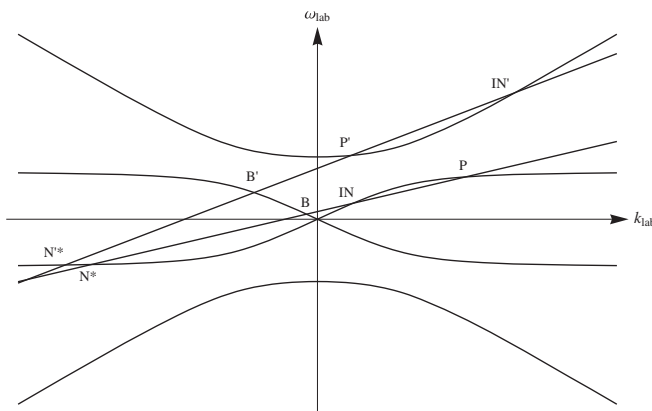

FIG. 3. Asymptotic dispersion relation for the Sellmaier dispersion relation of a diamondlike material (qualitative plot). The lab frame is adopted. Lines of constant ω are represented by straight lines.

VI. CONCLUSIONS

In this paper we focused on the Hawking effect in dispersive dielectric media, and in particular our focus was on the analytical proof of thermally in the spectrum of emitted photons. Our reference framework was the Hopfield model, suitably modified in order to account phenomenologically of some characteristics of the Kerr effect. Several aspects of the model were also taken into account, and interesting results were obtained. We sum up our results as follows.

(1) The microscopical (or rather, mesoscopical) model apt to the phenomenological description of the process involved in thermal pair creation has been previously identified with the Hopfield model. In place of the model with a stepwise behavior of the refractive index [10–12], we have developed a model where microscopic parameters are left free to vary smoothly in space-time coordinates. This model allows extensions to multiresonance situations for the polarization field. We have also introduced a simpler model involving a scalar field doublet in place of the full electromagnetic field and of the polarization field, in order to get a simpler and more manageable model, which still preserves the same dispersion relation as the original one. A conserved scalar product has been identified, in order to provide a norm for identifying particle and antiparticles states. In Appendix C, from Wronskian relations, the generalized Manley-Rowe relations are deduced in a scattering framework (they were deduced by other means in [33–35]).

(2) The analytical calculations for thermality have been based on matching the asymptotic solutions obtained in the WKB approximation and the solutions of the differential equation in Fourier space where an expansion up to the linear order of the refractive index has been performed. In particular, we have shown that calculations can be developed in a nice parallel way with respect to known calculations concerning fluid models [17,18]. We have also tried to identify a common denominator between the two aforementioned classes of models (i.e. fluid models and dielectric models). On one hand, if one pursues a second order expansion in Fourier space near the horizon, a Fuchsian singularity structure of the second order equation one obtains can be associated with Hawking effect. On the other hand, a first order expansion has been shown to be also very interesting (and sufficient for the aim to find out thermality), and in this case a Fuchsian singularity structure can be still identified, and a phase factor with a logarithmic branch point has been identified as the source of thermality. In all cases, thermality is involved with a ternary process (a fourth state is shown to decouple from the spectrum, as in [17,18]).

(3) The aforementioned picture can be extended to the multiresonances situation. Thermality is still preserved, and the possibility to get it in processes involving the lower branch and also the upper one has been indicated. We deserve to come back to this topic in future studies.

(4) A more tricky, and still open, problem concerns what should be meant by horizon in the near horizon expansion. We have discussed this problem in our model, and the nondispersive horizon, the group horizon and also the phase horizon could have chances to be the right places for our model. GH is a strong candidate. Our analysis cannot yet be conclusive. In particular, the expected role of a blocking horizon is not yet emerging in a neat way from the present (as well as in the existing one, albeit on a analytical footing) analysis. Furthermore, a detailed discussion, in comparison with the analysis of the recent paper [32], where it is shown that in fluid models the wave function associated with the modes involved in the scattering process is not able to distinguish between different kinds of horizons, would be required in order to delve into this interesting problem.

Our analysis is completed in the Appendices. In Appendix A we pointed out that the model, in line of principle, can be solved exactly in the general case, without necessarily referring to the physical situations involved in the Hawking effect. In Appendix B, the asymptotic behavior of the solution was considered. In Appendix C the generalized Manley-Rowe identities were discussed, and in Appendix D some relevant aspects of the geometrical optics approximation were discussed too.

Further theoretical studies and further experimental analysis are still required in order to give better and stronger grounds to this fascinating field of investigation.

APPENDIX A: THE GENERAL THEORY: EXACT TREATMENT

In the $\varphi\psi$-model, we can also obtain in line of principle the exact dispersion relation. For an example, we deal with the case where $\chi = \chi(x - vt)$ in the lab frame (see also [15]). In order to delve into the physical content of the above equations, let us introduce $\tilde{G}_\psi$, i.e. the Fourier transform of the Green function G_ψ. Then, for $k = (\frac{\omega}{c}, \vec{k})$, from Eq. (7) we obtain

$$k^\mu k_\mu - \frac{g^2}{c^2}(v^\mu k_\mu)^2 \tilde{G}_\psi(k_\nu) = 0, \qquad (A1)$$

which represents the exact dispersion relation for the theory at hand. This relation holds for a generic space-time dependence of the susceptibility χ, and even of the proper frequency ω_0. It contains all the necessary information in order to explore the problem of pair creation, as we shall see in the following.

Let us consider $\chi = \chi(x)$ in the comoving frame: we obtain in 2D, with $v^\mu = \gamma(c, v)$ and $k^\mu = \gamma(\frac{\omega}{c}, k)$,

$$\frac{\omega^2}{c^2} - k^2 - \frac{g^2}{c^2}\gamma^2(\omega + vk)^2 \tilde{G}_\psi(\omega, k) = 0. \qquad (A2)$$

This equation in the lab becomes

$$k_{\text{lab}}^2 c^2 = \omega_{\text{lab}}^2[1 - g^2 \tilde{G}_\psi^{\text{lab}}(\omega_{\text{lab}}, k_{\text{lab}})], \qquad (A3)$$

where we define

$$\tilde{G}_\psi^{\text{lab}}(\omega_{\text{lab}}, k_{\text{lab}}) \equiv \tilde{G}_\psi\left(\gamma(\omega_{\text{lab}} - vk_{\text{lab}}), \gamma\left(k_{\text{lab}} - \frac{v}{c^2}\omega_{\text{lab}}\right)\right). \qquad (A4)$$

Then, we can define the refractive index as

$$n^2(\omega_{\text{lab}}, k_{\text{lab}}) = 1 - g^2 \tilde{G}_\psi^{\text{lab}}(\omega_{\text{lab}}, k_{\text{lab}}), \qquad (A5)$$

which is an exact relation for the refractive index in the traveling perturbation case with $v = \text{const}$. It is interesting to note that, when no perturbation is present, one obtains

$$n^2(\omega_{\text{lab}}, k_{\text{lab}}) = 1 + g^2 \chi_0 \omega_0^2 \frac{1}{\omega_0^2 - \omega_{\text{lab}}^2}, \qquad (A6)$$

which has exactly the same form as for the standard Hopfield model (apart from a suitable redefinition of g). Indeed, we obtain easily

$$\tilde{G}_\psi^{\text{lab}}(\omega_{\text{lab}}, k_{\text{lab}}) = \frac{-\chi_0 \omega_0^2}{\omega_0^2 - \omega_{\text{lab}}^2} \qquad (A7)$$

in the aforementioned case. It is to be noted that (A2) depends on the explicit form of $\chi(x)$ in the comoving frame, and that in general the refractive index in the lab frame is expected to display a dependence also on k, i.e. also spatial dispersion is to be expected in the general case.

In the 2D case, one can also write formally the exact solutions for the equation of motion as follows:

$$\varphi(t,x) = \int \frac{dk\,d\omega}{(2\pi)^2} \delta\left(\frac{\omega^2}{c^2} - k^2 - \frac{g^2}{c^2}\gamma^2(\omega+vk)^2\tilde{G}_\psi(\omega,k)\right)\tilde{\varphi}(\omega,k)e^{i\omega t+ikx}, \tag{A8}$$

$$\psi(t,x) = g^2 \int \frac{dk\,d\omega}{(2\pi)^2} \delta\left(\frac{\omega^2}{c^2} - k^2 - \frac{g^2}{c^2}\gamma^2(\omega+vk)^2\tilde{G}_\psi(\omega,k)\right)\tilde{G}_\psi(\omega,k)i\gamma(\omega+vk)\tilde{\varphi}(\omega,k)e^{i\omega t+ikx}. \tag{A9}$$

We are of course interested in quantum aspects. It is very useful to point out that the model is, at least in line of principle, exactly soluble. Indeed, if we define

$$\Phi := \begin{pmatrix} \varphi \\ \psi \end{pmatrix}, \tag{A10}$$

we can obtain the following form for the action:

$$S := \frac{1}{2}\int d^4x\, \Phi^T \mathcal{Q}\Phi, \tag{A11}$$

where

$$\mathcal{Q} := \begin{bmatrix} -\Box & gv^\alpha\partial_\alpha \\ -v^\alpha\partial_\alpha g & -v^\alpha\partial_\alpha \frac{1}{\chi\omega_0^2}v^\beta\partial_\beta - \frac{1}{\chi} \end{bmatrix}. \tag{A12}$$

As it stands, the action is quadratic in the field Φ, and this implies that the quantum theory is exactly soluble, as the path-integral formalism immediately reveals. Any amplitude is related to the Green function of the matrix operator $\mathcal{Q}$, and, moreover, any (spontaneous) pair-creation process induced by the presence of a spacetime-dependent χ is in

line of principle exactly calculable, being associated with the imaginary part of the effective action, which could be obtained, as usual, via ζ-function techniques, after having calculated the effective action (which amounts to calculating the functional determinant of $\mathcal{Q}$). This route, even in simple cases can be very involved also for a Gaussian model as the one we are setting up. The approach we have adopted for revealing quantum instabilities, which consists in checking the presence of negative norm states (antiparticles) in stimulated scattering, is simpler. Sometimes, this approach is called transmission coefficient approach [26].

APPENDIX B: ASYMPTOTIC BASES

We consider in what follows the asymptotic behavior (in x) for simplicity in the case of the 2D model (in the 4D case, separation of variables also on transverse variables allows to show that only a little and not substantial modification of the calculations displayed below occurs; see also [16]). We work on stationary solutions, so we get the following second order system of ordinary differential equations:

$$\psi'' = \left(\left(\log\chi\omega_0^2\right)' + 2i\frac{\omega}{v}\right)\psi' + \left[-i\frac{\omega}{v}\left(\log\chi\omega_0^2\right)' - \frac{\omega_0^2}{\gamma^2v^2}\right) + \frac{\omega^2}{v^2}\right]\psi - i\frac{g}{c}\chi\omega_0^2\frac{\omega}{\gamma v^2}\varphi - \frac{1}{\gamma vc}\chi\omega_0^2\partial_x(g\varphi), \tag{B1}$$

$$\varphi'' = \frac{g}{c}\gamma v\psi' - i\frac{g}{c}\gamma\omega\psi - \frac{\omega^2}{c^2}\varphi. \tag{B2}$$

Then, we associate with it a first order system by introducing

$$p := \psi', \qquad q := \varphi'. \tag{B3}$$

Then, if $W(x) := (\psi(x),\varphi(x),p(x),q(x))^T$, we obtain the following first order system:

$$W'(x) = K_4 W(x), \tag{B4}$$

where the 4×4 matrix operator $K_4(x)$ has the following structure:

$$K_4 := \begin{bmatrix} 0_2 & 1_2 \\ A_2 & B_2 \end{bmatrix}. \tag{B5}$$

$0_2, 1_2$ are 2×2 matrices, the first one with all entries equal to zero, and the second one is the identity. As to A_2, B_2, we have

$$A_2 := \begin{bmatrix} \frac{\omega^2}{v^2} - i\frac{\omega}{v}\left(\log\chi\omega_0^2\right)' - \frac{\omega_0^2}{\gamma^2v^2} & -i\frac{g}{c}\frac{\omega}{v}\frac{\chi\omega_0^2}{\gamma v} - \frac{\chi\omega_0^2}{\gamma vc}g' \\ -i\frac{g}{c}\gamma\omega & -\frac{\omega^2}{c^2} \end{bmatrix}, \tag{B6}$$

and

$$B_2 := \begin{bmatrix} 2i\frac{\omega}{v} + \left(\log\chi\omega_0^2\right)' & -\frac{g}{c}\frac{\chi\omega_0^2}{\gamma v} \\ \frac{g}{c}\gamma v & 0 \end{bmatrix}. \tag{B7}$$

Let us write

$$K_4 = \mathcal{C} + \mathcal{R}, \qquad (B8)$$

where $\mathcal{C}$ is a constant matrix and $\mathcal{R} = \mathcal{R}(x)$:

$$\mathcal{C} := \begin{bmatrix} 0_2 & 1_2 \\ A_c & B_c \end{bmatrix}, \qquad (B9)$$

with

$$A_c := \begin{bmatrix} \frac{\omega^2}{v^2} - \frac{\omega_0^2}{\gamma^2 v^2} & 0 \\ 0 & -\frac{\omega^2}{c^2} \end{bmatrix}, \qquad (B10)$$

and

$$B_c := \begin{bmatrix} 2i\frac{\omega}{v} & 0 \\ 0 & 0 \end{bmatrix}; \qquad (B11)$$

moreover,

$$\mathcal{R} := \begin{bmatrix} 0_2 & 0_2 \\ A_r & B_r \end{bmatrix}, \qquad (B12)$$

with

$$A_r := \begin{bmatrix} -i\frac{\omega}{v}(\log\chi\omega_0^2)' & -i\frac{g}{c}\frac{\omega}{v}\frac{\chi\omega_0^2}{\gamma} - \frac{\chi\omega_0^2}{\gamma v c}g' \\ -i\frac{g}{c}\gamma\omega & 0 \end{bmatrix}, \qquad (B13)$$

and

$$B_r := \begin{bmatrix} (\log\chi\omega_0^2)' & -\frac{g}{c}\frac{\chi\omega_0^2}{\gamma v} \\ \frac{g}{c}\gamma v & 0 \end{bmatrix}. \qquad (B14)$$

Let us assume the following condition:

$$\int_a^\infty dx |\mathcal{R}(x)| < \infty, \qquad (B15)$$

which physically can match very well the nature of traveling perturbation of δn, to be implemented by means of a suitable choice of the microscopic parameters g, ω_0, χ. Then, according to theorems in [36], we can infer that, both as $x \to \infty$ and as $x \to -\infty$, the asymptotic behavior of solutions is governed by the eigenvalues of $\mathcal{C}$, which implies that the basis for the homogeneous case with g, ω_0, χ asymptotically constants, is asymptotically a good scattering basis also for the perturbed problem. To be more precise: the asymptotic region solutions are a scattering basis, and, moreover, solutions of the full equations asymptotically behave as the asymptotic region solutions, which then represent a good scattering basis. Furthermore, we are interested in (localized) wave packets, whose support is finite. This is relevant as far as we are concerned with the problem of defining particle and antiparticle states.

APPENDIX C: GENERALIZED MANLEY-ROWE RELATIONS AND PAIR CREATION AMPLITUDES

We give a more systematic account of pair-creation amplitudes, which can be used both for analytical calculations and for numerical ones (see in the latter case results in [5,13,14,37]). At first, we focus on the two-dimensional problem and we fix a scattering basis in the asymptotic regions. At fixed ω we have a number of states as $x \to -\infty$, which is equal to the number of intersections between the horizontal line $\omega = \mathrm{const}$ and the asymptotic dispersion relation $\omega = f(k)$, see Fig. 1. Some of them can have positive group velocity $v_g > 0$ (right moving), and some others can have $v_g < 0$ (left moving). In situations where a blocking horizon is present, only the states belonging to the asymptotic region on the left (with $\delta n = 0$) are involved. In situations where blocking is absent or frequencies involved do not admit blocking, then also (transmitted) scattering states belonging to the asymptotic region on the right (with $\delta n \neq 0$) are involved. See also [11,12].

Some preliminary considerations are in order, concerning the Hawking effect. We shall discuss the scattering process involved in the Hawking effect for a white hole in this section, whereas in Sec. III thermality is discussed for a black hole. There is no contradiction, as white hole is the time reversal of a black hole. Still, at the level of scattering, one has to investigate what happens. Time-reversal implies $(\omega, v) \mapsto (-\omega, -v)$. We notice that the original Hopfield model, and also the one with varying χ, ω_0, g, is invariant under time reversal, as the Lagrangian of the model is. As to the $\varphi\psi$-model we have introduced, equations of motion are invariant under time reversal provided that $g \mapsto -g$. The latter freedom for the scalar model can be assumed without any problem, provided that the correct branch for the relation between microscopic parameters and macroscopic ones is chosen.

In general, we can expect to deal with several branches of the dispersion relation. In that case, it may be more useful to consider the asymptotic dispersion relations in the lab. An analogous reasoning leads to a number of states as $x \to \infty$, for which again $v_g > 0$ or $v_g < 0$. A complete scattering basis is obtained by considering both a scattering with one initial right-moving state, which can give rise to several reflected states and one transmitted state, and a scattering with one initial left-moving state, with an analogous behavior. This is particularly important for the actual computation of the pair creation amplitude in the spontaneous emission case.

Amplitudes can be calculated both in the traditional framework, by means of Bogoliubov transformations, or by means of the conservation of fluxes in the scattering process in the comoving frame. We adopt the latter frame,

which is also more directly related to some previous works in literature [33–35]. We know that, in the comoving frame, there is a current J_x which is conserved, as we have shown in Sec. II. In particular, we can consider J_x as a bilinear form:

$$J_x(\Psi_1, \Psi_2), \qquad (C1)$$

where Ψ_1, Ψ_2 is a couple of asymptotic plane wave solutions of the equations of motion. In the scattering "$\rightarrow$," with a single initial state which is right moving,

$$\Psi_{\rightarrow} = N_{\Psi} e^{-i\omega t} \begin{cases} T_{IN}^{\rightarrow} W_{IN} e^{ik_{IN}x} + T_{P}^{\rightarrow} W_P e^{-ik_P x} + T_{N^*}^{\rightarrow} W_{N^*} e^{-ik_{N^*}x} + T_{B}^{\rightarrow} W_B e^{-ik_B x} & \text{for } x \rightarrow -\infty, \\ T_{T}^{\rightarrow} W_T e^{ik_T x} & \text{for } x \rightarrow \infty, \end{cases} \qquad (C3)$$

where $T_{IN}^{\rightarrow}$, $T_{P}^{\rightarrow}$, $T_{N^*}^{\rightarrow}$, $T_{T}^{\rightarrow}$, $T_{B}^{\rightarrow}$ are the usual scattering coefficients with the additional label $\rightarrow$ indicating that the initial state is right moving, and where W_{IN} etc. are vector Fourier components of the plane wave which is considered. Then, (C2) is of the following form:

$$1 - |P|^2 - |B|^2 - |T|^2 + |N|^2 = 0. \qquad (C4)$$

Interference terms are washed out asymptotically in time, in the sense that separated "photon packets" for the various modes are expected on long time scales (long with respect to the interaction time scale). This emerges from current conservation:

$$J_x^{\text{left}} = J_x^{\text{right}}, \qquad (C5)$$

where "left" and "right" indicate states on the left and on the right of the steplike potential as $x \rightarrow -\infty$ and $x \rightarrow \infty$ respectively. For well-separated packets, we also get

$$J_x^{IN} + J_x^P + J_x^{N^*} + J_x^B = J_x^T. \qquad (C6)$$

The above quantities have a sign which is determined by

$$J_x^U = \text{sign}(v_g^U)|J_x^U|, \qquad (C7)$$

where U is meant as a positive norm asymptotic solution; furthermore, we take into account that

$$J_x^{N^*} = -J_x^N, \qquad (C8)$$

i.e. the antiparticle state (negative norm) current is opposite to the corresponding particle state (positive norm) current. Compare also [26]. Then, we define the following quantities:

$$\frac{J_x^U}{J_x^{IN}} = : \text{sign}\left(\frac{v_g^U}{v_g^{IN}}\right)|U|^2, \qquad U = P, B, T, N. \qquad (C9)$$

scattering solutions are denoted by $\Psi_{\rightarrow}$. An analogous definition is given for the scattering "$\leftarrow$" and $\Psi_{\leftarrow}$. Then we can obtain a number of "Wronskian relations," for example we can calculate

$$J_x(\Psi_{\rightarrow}^*, \Psi_{\rightarrow}). \qquad (C2)$$

For definiteness, let us consider, for the case $IN \rightarrow P + N^* + B + T$, where T stays for a possible transmitted state, the following state:

It is not difficult to show that

$$|U|^2 = F(\omega, k_U)|T_U^{\rightarrow}|^2, \qquad U = P, B, T, N, \qquad (C10)$$

where $F(\omega, k_U)$ is a positive kinematic coefficient which of course depends on the current structure. This analysis can be easily extended to the case of an arbitrary number of states (compatibly with the dispersion relation).

It is also remarkable that, in the spontaneous case, we have to consider

$$\langle J_x \rangle := \langle 0|J_x|0 \rangle$$
$$= \sum_{\rightarrow} \sum_{\leftarrow} \int \frac{d\omega}{(2\pi)} \frac{dk}{(2\pi)} \frac{1}{N_{\Psi}} \delta(\text{dispersion relation}) V^* V, \qquad (C11)$$

where the sum has to be extended both to initial left-moving states and to initial right-moving ones. This leads to the same particle creation amplitudes associated with (C4).

As to the conservation of the fluxes, we recall that we could also define the Poynting vector for the theory at hand. It is easy to conclude that the same amplitudes as above would be obtained.

APPENDIX D: GEOMETRICAL OPTICS

The eikonal approximation is a usual tool for analyzing solutions in the framework of analogue gravity. We explore also this conceptual frame, because it can give useful suggestions and provide us also analytical tools for a better comprehension of the phenomenon at hand. In particular, it is remarkable that thermality of the Hawking radiation arises as associated with the presence, in the comoving frame, of a so-called group horizon (GH), i.e. a turning point (TP) for the waves which reach the perturbation, at least for frequencies in a given interval. For example in the WKB approximation, a turning point is to be handled with care, due to the fact that it violates the requirements of the

approximation itself. In geometrical optics, a TP represents a caustic for rays, so, again, the eikonal approximation fails there. It is necessary to point out immediately the limits of the given approximation, due to the fact that all the phenomenology which we are interested in arises near such a TP, where some other analytical tool has to be assumed.

In what follows, we point out that even in presence of dispersion, the eikonal approximation still gives useful suggestions, and the method of characteristics can be used in order to explore solutions (geometrical optics is a good tool for studying the problem in the nondispersive case, as known). Moreover, the problem of the group horizon, and also the problem of the phase horizon, can be exactly solved in the Cauchy approximation.

We shall limit ourselves mainly to the 2D eikonal equation

$$\omega_{\text{lab}} n(\omega_{\text{lab}}, x - vt) = \pm c k_{\text{lab}}. \tag{D1}$$

Let us consider the Cauchy approximation (36). We have in the comoving frame [3]

$$G = 0 \iff (\omega + vk)(n(x) + B\gamma^2(\omega + vk)^2)$$
$$- ck - \frac{v}{c}\omega = 0. \tag{D2}$$

We are interested in the expression for the group horizon (if any), which is obtained by solving the system [38]

$$G = 0, \tag{D3}$$

$$\partial_k G = 0. \tag{D4}$$

As to the latter equation, we obtain

$$\partial_k G = 0 \iff 3B\gamma^2 v(\omega + vk)^2 - v\left(\frac{c}{v} - n(x)\right) = 0, \tag{D5}$$

which can be solved explicitly:

$$(\omega + vk) = \pm\left(\frac{\frac{c}{v} - n(x)}{3B\gamma^2}\right)^{1/2}. \tag{D6}$$

By substitution of the positive root in $G = 0$, as we mean to get the group horizon for positive norm waves, we obtain an equation for $n(x)$ which allows us to find out explicitly the group horizon:

$$\frac{c}{v} - n(x) = 3B\gamma^2\left(\frac{1}{2B\gamma^4}\frac{c}{v}\right)^{2/3}\omega^{2/3} = :\zeta_B\omega^{2/3}, \tag{D7}$$

where $\zeta_B \propto B^{1/3}$. So we are able to find out $x_{\text{GH}}(\omega)$, which is a function of the frequency ω, as expected.

As to the phase horizon, we have in the comoving frame that it corresponds to $\omega = 0$. By taking into account the dispersion relation (D2), we find that

$$\frac{c}{v} - n(x) = B\gamma^2 v^2 k_0^2, \tag{D8}$$

where k_0 is the value at which the dispersion relation $G = 0$ intersect the k-axis (i.e. $\omega = 0$). In the latter case, $x_{\text{PH}}(k_0)$ is a function of the aforementioned parameter, which is independent from ω.

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