Nonlocal Singular Problems and Applications to MEMS

D. Cassani, L. Fattorusso, A. Tarsia

***Abstract*—We consider fourth order nonlinear problems which describe electrostatic actuation in MicroElectroMechanicalSystems (MEMS) both in the stationary case and in the evolution case; we prove existence, uniqueness and regularity theorems by exploiting the Near Operators Theory.**

***Index Terms*—Singular nonlinearities; Integro-differential equations; Higher order elliptic and hyperbolic PDE; Regularity results; Steklov boundary conditions; Near operators theory; Implicit function theorem; MEMS and NEMS; Electrostatic actuation.**

I. Introduction

Recently a lot of attention has been devoted to the study of mathematical models which describe, with different levels of accuracy, the so-called electrostatic actuation in *MicroElectroMechanicalSystems* (MEMS), see e.g. [11] and references therein. These models are studied by considering nonlinear problems involving nonlinearities which develop singularities.

As an example we consider a plate set on a micro scale which is suitably *fixed* at boundary of a region $\Omega \subset \mathbb{R}^N$. Once that a drop voltage is applied between the deflecting plate and a ground plate, the micro-plate leaves the steady state $u = 0$ moving towards the ground plate set at height $u = 1$.

The deformation profile u of the MEMS is then governed, in the stationary case, by the following model:

$$\begin{cases} \alpha\Delta^2 u = \left(\beta \displaystyle\int_\Omega |\nabla u|^2\,dx + \gamma\right)\Delta u \\ \quad + \dfrac{\lambda f(x)}{(1-u)^\sigma \left(1 + \chi \displaystyle\int_\Omega \dfrac{dx}{(1-u)^{\sigma-1}}\right)}, \quad x \in \Omega \\ u = \Delta u - d u_\nu = 0, \quad x \in \partial\Omega,\ d \geq 0 \\ 0 \leq u < 1, \quad x \in \Omega \end{cases} \tag{1}$$

Here:

- $\Omega \subset \mathbb{R}^N$ is a smooth bounded domain;
- $u : \Omega \longrightarrow \mathbb{R}$ is the unknown profile of the deflecting MEMS plate;
- $f : \Omega \longrightarrow \mathbb{R}^+$ is a bounded function which carries dielectric properties of the material (permittivity profile);
- $\lambda \geq 0$ is the drop voltage between the ground plate and the deflecting plate;
- the positive parameters $\alpha, \beta, \gamma, \chi$ which are respectively related to the thickness (rigidity) of the deflecting plate, material deformation (self-stretching), tangential tension forces (stretching), and nonlocal dependence of the electrostatic potential on the solution itself (non uniform electric charge distribution) and for $\sigma \geq 2$ which takes into account more general potential than Coulomb's.

This is an extension of the nonlocal MEMS problem studied by Cassani-do Ó-Ghoussoub [8] where the case $\sigma = 2$ (Coulomb potential) was considered, as well as $\alpha = 1$, $\beta = 0$, $\gamma = 0$ and $\chi = 0$, namely:

$$\begin{cases} \Delta^2 u(x) = \dfrac{\lambda\ f(x)}{[1-u(x)]^2}; \\ 0 \leq u(x) < 1, \quad \text{in } \Omega, \\ u = \Delta u - d u_\nu = 0, \quad \text{on } \partial\Omega,\ d \geq 0 \end{cases} \tag{2}$$

It is well known that the role of boundary conditions in higher order problems is very delicate, as pointed out in [12].

Here we deal with a rather general physical situation in which Steklov boundary conditions are considered and given by

$$u = \Delta u - d u_\nu = 0, \quad x \in \partial\Omega,\ d \geq 0$$

and from which we obtain Dirichlet ($u = u_\nu = 0$) and Navier ($u = \Delta u = 0$) boundary conditions by setting respectively $d = \infty$ and $d = 0$.

In the stationary case we consider: $u(t,x) = u(x)$; $\lambda(t) = \lambda \geq 0$; $0 \leq f(x) \leq 1$

$$(S_\lambda) \begin{cases} \Delta^2 u = \lambda \dfrac{f(x)}{(1-u)^2}, \quad \text{in } \Omega \subset \mathbb{R}^N \\ 0 \leq u < 1, \quad \text{in } \Omega \\ \begin{cases} u = \Delta u = 0 \text{ (Navier)} \\ u = u_\nu = 0 \text{ (Dirichlet)} \\ u = \Delta u - d u_\nu = 0 \text{ (Steklov),} \end{cases} \quad \text{on } \partial\Omega \end{cases} \tag{3}$$

where f is the permittivity profile of the material. Solutions have to be understood in the following sense:

<u>Weak solutions</u>: $u_\lambda, 1/(1-u_\lambda)^2 \in L^1(\Omega)$, $0 \leq u_\lambda \leq 1$ such that

$$\int_\Omega u_\lambda \Delta^2 \varphi\, dx = \lambda \int_\Omega \frac{h(x)}{(1-u_\lambda)^2}\varphi\, dx, \quad \varphi\text{-test}$$

<u>Energy solutions</u>: weak-solutions such that $u_\lambda \in \mathcal{H}$, the Sobolev space H_0^2 or $H^2 \cap H_0^1$, accordingly to boundary conditions.

The first result in the case of Dirichlet boundary conditions was obtained in [8] and can be summarized as follows:

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- There exists a minimal (pointwise) classical (smooth) solution $\underline{u}_\lambda$ for $0 < \lambda < \lambda_*(\Omega, f, N)$;
- $\lambda_* = \lambda^* =: \sup\{\lambda \,|\, \text{there exists a weak solution}\}$;
- $u^*(x) = \lim_{\lambda \nearrow \lambda^*} u_\lambda(x)$ is an energy solution and it is unique,

which was further developed in [10, Cowan-Esposito-Ghoussoub-Moradifam]. Navier boundary conditions were considered in [16, Lin-Yang'07], [15, Guo-Wei'08] and [9, Cowan-Esposito-Ghoussoub'10] whereas Steklov boundary conditions in [2, Berchio-Cassani-Gazzola'10].

II. Main results

The following result was proved in [6, Cassani-Fattorusso-Tarsia '11].

Theorem 1: Let the dimension $N < 8$, $\Omega \subset \mathbb{R}^N$ be a bounded domain, $f \in L^\infty(\Omega)$ and $\alpha, \beta, \gamma, \chi > 0$. Then, there exist $\lambda^*, d_0 \in (0, \infty)$ such that for $\lambda \in (0, \lambda^*)$ problem (1) *possesses a solution $u \in H^4(\Omega)$ provided one of the following holds:*

(SN) $0 \le d < d_0$
or
(D) $d = \infty$ *and Ω is a ball*

and the diameter of Ω is sufficiently small.

A. Remarks

1) It is worth to emphasize that the solution provided by Theorem 1, $u = u_\lambda$ for a fixed $0 < \lambda < \lambda^*$, is such that $\|u_\lambda\|_\infty \le C < 1$; as a consequence, by elliptic regularity u_λ turns out to be smooth.
2) Let us mention that in applications, the domain Ω represents the region occupied by the undeflected MEMS plate which is set on a micro-scale basis; therefore, in this respect, restrictions from above on the diameter of the domain do not seem too stringent.
3) The restriction in Theorem 1 on the dimension $N < 8$ is somehow expected as a consequence of [10], [2] as for $N \ge 9$ the solution in the semilinear case and avoiding nonlocal effects, is singular approaching λ^*, in the sense that for $\lambda = \lambda^*$ one has $\|u_0\|_\infty = 1$; this clearly prevents any existence result to (1).

B. The abstract setting: a version of the Implicit Function Theorem

The key-ingredient in the study of the stationary case is provided by the following version of the implicit function theorem proved in [20, Tarsia '98] and which remarkably extends the *Near Operator Theory* introduced in [4, Campanato '94].

Theorem 2: Let X be a topological space, Y a set, Z a Banach space and the following mappings $\boldsymbol{F} : X \times Y \to Z$, $\boldsymbol{B} : Y \to Z$. Suppose that:

(i) *there exists $(\boldsymbol{x}_0, \boldsymbol{y}_0) \in X \times Y$ such that $\boldsymbol{F}(\boldsymbol{x}_0, \boldsymbol{y}_0) = 0$;*
(ii) *the map $\boldsymbol{x} \to \boldsymbol{F}(\boldsymbol{x}_0, \boldsymbol{y}_0)$ is continuous at $\boldsymbol{x}_0$;*
(iii) *there exist $k_1 > 0, k_2 \in (0,1)$ and a neighborhood of $\boldsymbol{x}_0$, $U(\boldsymbol{x}_0) \subset X$, such that, for all $\boldsymbol{y}_1$, $\boldsymbol{y}_2 \in Y$ and for all $\boldsymbol{x} \in U(\boldsymbol{x}_0)$, we have*

$$\begin{aligned}\|\boldsymbol{B}(\boldsymbol{y}_1) - \boldsymbol{B}(\boldsymbol{y}_2) - k_1[\boldsymbol{F}(\boldsymbol{x}, \boldsymbol{y}_1) - \boldsymbol{F}(\boldsymbol{x}, \boldsymbol{y}_2)]\|_Z \\ \le k_2 \|\boldsymbol{B}(y_1) - \boldsymbol{B}(y_2)\|_Z\end{aligned}$$

(iv) *$\boldsymbol{B}$ is injective;*
(v) *$\boldsymbol{B}(Y)$ is a neighborhood of $\boldsymbol{z}_0 = \boldsymbol{B}(\boldsymbol{y}_0)$.*

Then, there exists a ball $S(\boldsymbol{z}_0, r) \subset \boldsymbol{B}(Y)$ and a neighborhood of $\boldsymbol{x}_0$, $V(\boldsymbol{x}_0) \subset U(\boldsymbol{x}_0)$, such that the following problem:

$$\begin{cases} \boldsymbol{F}(\boldsymbol{x}, \boldsymbol{y}(\boldsymbol{x})) = 0, \ \forall \boldsymbol{x} \in V(\boldsymbol{x}_0), \\ \\ \boldsymbol{y}(\boldsymbol{x}_0) = \boldsymbol{y}_0 \end{cases} \tag{4}$$

posseses an unique solution $\boldsymbol{y} : V(\boldsymbol{x}_0) \to \boldsymbol{B}^{-1}(S(\boldsymbol{z}_0, r))$. Moreover, if condition (iii) holds for all $\boldsymbol{x} \in X$, then the solution $\boldsymbol{y} = \boldsymbol{y}(\boldsymbol{x})$ turns out to be defined in the whole X.

The second main tool which is actually the starting point to set up the strategy outlined in Theorem 2, can be obtained by joining some results of [8] and [2] which we recall in the following

Theorem 3: Let $\beta = \gamma = \chi = 0$ in (1). *Then there exists $\lambda^* \in (0, \infty)$ such that for all $\lambda \in (0, \lambda^*)$ problem* (1) *possesses a classical solution provided one of the following holds:*

(S) *$0 \le d < d_0$, where d_0 is the first simple boundary eigenvalue of the biharmonic operator Δ^2 under Steklov boundary conditions;*
(D) *Ω is a ball and $d = \infty$, which corresponds to the Dirichlet boundary conditions $u = u_\nu = 0$ on $\partial\Omega$.*

C. Sketch of the proof of Theorem 1

First we apply implicit function theorem to show that the problem is well posed and for this purpose we assume

$$\boldsymbol{x} \,=\, (\alpha,\, \beta,\, \gamma,\, \chi,\, f,\, \lambda), \quad \boldsymbol{y} \,=\, y(x);$$

$$\begin{aligned} &\boldsymbol{F}(\boldsymbol{x}, \boldsymbol{y}) = F(\alpha,\, \beta,\, \gamma,\, \chi,\, f,\, \lambda,\, y) \\ &= \alpha\, \Delta^2 y(x) - \left[\beta \int_\Omega |\nabla y(x)|^2\, dx\; \gamma\right] \Delta y(x) \\ &\quad + \frac{\lambda\, f(x)}{[1 - y(x)]^\sigma \left\{1 + \chi \displaystyle\int_\Omega \frac{1}{[1 - y(x)]^{\sigma - 1}}\right\} dx}; \end{aligned} \tag{5}$$

$$\begin{aligned} X = {} & \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+ \\ & \times \{f \in L^\infty(\Omega) : |x : f(x) > 0| \ne 0\} \\ & \times \{\lambda : 0 \le \lambda < \lambda^* < +\infty\}; \end{aligned}$$

$$\boldsymbol{B}(\boldsymbol{y}) \,=\, \Delta^2 y(x);$$

$$\begin{aligned} Y = Y_d = \Big\{ & u \in H^4(\Omega) : \, u = \Delta u \,-\, d\, \frac{\partial u}{\partial \nu} = 0, \\ & \text{a.e. in } \partial\Omega, \;\; d \ge 0,\, 0 < u < 1, \\ & \int_\Omega \frac{1}{[1 - u(x)]^{8\sigma}}\, dx \,<\, M_1,\, \int_\Omega |\Delta u(x)|^2 dx < M_2 \Big\}; \end{aligned} \tag{6}$$

$$Z = L^2(\Omega).$$

We consider $(\boldsymbol{x}_0, \boldsymbol{y}_0)$, belonging to $X \times Y$, which enjoys $\boldsymbol{F}(\boldsymbol{x}_0, \boldsymbol{y}_0) = 0$, where $\boldsymbol{x}_0 = (1, 0, 0, 0, f_0, \lambda)$ e $\boldsymbol{y}_0 = \boldsymbol{y}_0(\boldsymbol{x}_0) = u_0(x)$ is the solution of

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$$\begin{cases} \Delta^2 u_0(x) = \dfrac{\lambda\, f_0(x)}{[1-u_0(x)]^2} \\ 0 < u_0(x) < 1, \quad \text{in } \Omega \\ u_0(x) = \Delta u_0 - d\,\dfrac{\partial u_0(x)}{\partial \nu} = 0, \quad \text{on } \partial\Omega \end{cases} \tag{7}$$

as provided by Theorem 3. Next we sketch how assumptions of Theorem 2 turn out to be verified.

Assumption (i):

$$\boldsymbol{F}(\boldsymbol{x}_0, \boldsymbol{y}_0) = \Delta^2 u_0(x) - \frac{\lambda\, f_0(x)}{[1-u_0(x)]^2} = 0$$

follows directly from the existence results for problem (S_λ) subject to Steklov boundary conditions as proved in [2].

Condition (ii) is verified since the dependence of F through parameters is continuous.

The assumptions (iv) e (v) follow by the properties of the operator Δ^2.

To verify the (iii) we show that there exists $k_1 \in (0,1)$ such that for every $(\alpha, \beta, \gamma, \chi, f, \lambda) \in X$ one has

$$\int_\Omega |\alpha\Delta^2 y_1(x) - \alpha\Delta^2 y_2(x)$$

$$-\left[F(\alpha, \beta, \gamma, \chi, f, \lambda, y_1) - F(\alpha, \beta, \gamma, \chi, f, \lambda, y_2)\right]|^2\, dx$$

$$\leq k_1 \int_\Omega |\alpha\Delta^2 y_1(x) - \alpha\Delta^2 y_2(x)|^2\, dx \tag{8}$$

and in turn we have

$$\int_\Omega |[G(\beta, \gamma, y_1(x)) + H(\lambda, \chi, f(x), y_1(x))]$$

$$- [G(\beta, \gamma, y_2(x)) + H(\lambda, \chi, f(x), y_2(x))]|^2\, dx$$

$$\leq 2\int_\Omega |G(\beta, \gamma, y_1(x)) - G(\beta, \gamma, y_2(x))|^2\, dx$$

$$+2\int_\Omega |\, H(\lambda, \chi, f(x), y_1(x))] - H(\lambda, \chi, f(x), y_2(x))|^2\, dx$$

$$\leq k_1 \int_\Omega |\alpha\Delta^2 y_1(x) - \Delta^2 y_2(x)|^2\, dx \tag{9}$$

where we have set

$$G(\beta, \gamma, u(x)) = \left[\beta \int_\Omega |\nabla u(x)|^2\, dx + \gamma\right] \Delta u(x) \tag{10}$$

and

$$H(\lambda, \chi, f(x), u(x)) = \frac{\lambda\, f(x)}{[1-u(x)]^2 \left[1 + \chi \displaystyle\int_\Omega \frac{1}{[1-u(x,t)]^2}\right]}\, dx$$

Observe that we obtain the existence result globally in the positive parameters $\alpha, \beta, \gamma, \chi$ as well as for any $\sigma \geq 2$, $f \in L^\infty$, $\lambda \in (0, \lambda^*)$ as consequence of the last claim in Theorem 2.

III. Nonlocal time dependent problems

Recently in [7] the authors obtain existence, uniqueness and regularity results for a model which takes into account the dynamic of the problem as follows:

$$\begin{cases} \alpha\Delta^2 u(x,t) + c\, u'(x,t) + p\, u''(x,t) \\ = \left[\beta \int_\Omega |\nabla u(x,t)|^2\, dx + \gamma\right] \Delta u(x,t) \\ \quad + \dfrac{\lambda(t) f(x)}{[1-u(x,t)]^\sigma \left[1 + \chi \displaystyle\int_\Omega \frac{1}{[1-u(x,t)]^{\sigma-1}}\, dx\right]}, \\ 0 \leq u(x,t) < 1, \quad \text{in } \Omega \times [0,T] \\ u(x,0) = u_0, \quad \text{in } \Omega \\ u'(x,0) = 0, \quad \text{in } \Omega \\ u(x,t) = \Delta u(x,t) - d\,\dfrac{\partial u(x,t)}{\partial \nu} = 0, \quad \text{on } \partial\Omega \times [0,T] \end{cases} \tag{11}$$

Theorem 4: Let $\Omega \subset \mathbb{R}^N$, $1 \leq N \leq 3$, be a bounded domain with sufficiently small diameter, $\sigma \geq 2$, non negative constants β, γ, χ and $0 \leq d < d_0$, where d_0 is the first boundary eigenvalue of the biharmonic operator subject to Steklov boundary conditions. Let also p, c be bounded functions and $\lambda \in C^1((0,T); L^2(\Omega))$ such that $\|\lambda\|_\infty < \lambda^*$, $u_0 \in H^2 \cap H_0^1(\Omega)$ (satisfying suitable compatibility conditions) and $u_1 \in L^2(\Omega)$. Then, problem (11) possesses a unique solution $u \in C^0([0,T]; H^2(\Omega)) \cap C^1([0,T]; L^2(\Omega))$. The same conclusion holds if $d = \infty$ and Ω is a ball.

Theorem 5: Let

$$u \in C^0([0,T]; H_0^2(\Omega)) \cap C^1([0,T]; L^2(\Omega))$$

be the solution to problem (11) given by Theorem 4. Assume $u_0, u_1 \in H^2 \cap H_0^1(\Omega)$ and $c \in W^{1,\infty}((0,T); L^2(\Omega))$. Then, the solution enjoys the following regularity:

$$u \in C^0([0,T]; H^4(\Omega)) \cap C^1([0,T]; H^2 \cap H_0^1(\Omega)) \\ \cap C^2([0,T]; L^2(\Omega))$$

Theorems (4) and (5) follows by means of a non straightforward extension of the technique used in the stationary case to the dynamic setting and again this approach enables us to prove existence and uniqueness of the solution locally in time but globally in the physical parameters involved in the problem.

Moreover, differently from the stationary case, here the problem of regularity is somehow delicate as the equation manifests itself through an hyperbolic nature. Then we are concerned with proving regularity of solutions by adapting and further developing abstract results of [1] and [13]. In this respect, it is worth to mention that standard interpolation theory does not suite optimal regularity results even with the aid of higher order (operator) perturbations in a penalized framework.

Finally, let us mention that the inverse identification problem of identifying the pull-in voltage $\lambda(t)$ in (11), under a suitable (accessible and measurable) supplementary information on the solution, has been studied in [5, Cassani-Kaltenbacher-Lorenzi '09].

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