Steering Bose-Einstein Condensates despite Time Symmetry

Dario Poletti,[1] Giuliano Benenti,[2,3] Giulio Casati,[2,3] Peter Hänggi,[4,1] and Baowen Li[1,5]

[1]*Department of Physics and Centre for Computational Science and Engineering, National University of Singapore, Singapore 117542, Republic of Singapore*

[2]*CNISM, CNR-INFM, and Center for Nonlinear and Complex Systems, Università degli Studi dell'Insubria, Via Valleggio 11, 22100 Como, Italy*

[3]*Istituto Nazionale di Fisica Nucleare, Sezione di Milano, Via Celoria 16, 20133 Milano, Italy*

[4]*Institut für Physik, Universität Augsburg, Universitätsstr. 1, D-86135 Augsburg, Germany*

[5]*NUS Graduate School for Integrative Sciences and Engineering, 117597, Republic of Singapore*

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A Bose-Einstein condensate in an oscillating spatially asymmetric potential is shown to exhibit a directed current for unbiased initial conditions despite time symmetry. This phenomenon occurs only if the interaction between atoms, treated in mean-field approximation, exceeds a critical value. Our findings can be described with a three-mode model. These three-mode model results corroborate well with a many-body study over a time scale which increases with increasing atom number. The duration of this time scale probes the validity of the used mean-field approximation.

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The realization of Bose-Einstein condensates (BECs) [1] of dilute atomic gases has opened new possibilities for the investigation of interesting physical phenomena induced by atom-atom interactions [2]. For example, for a BEC in a double-well potential, it was found that the tunneling can be suppressed due to interaction-induced self-trapping [3–5], thus maintaining population imbalance between the two wells. A similar phenomenon has been predicted also for a periodically driven BEC in a double well [6–10] and shows the importance of interaction in the coherent control of quantum tunneling between wells.

In recent years, ample interest has arisen in the field of directed transport in classical and quantum systems in absence of a dc bias (see [11,12] and references therein). In particular it has been shown that by breaking space- and time-inversion symmetries it is possible to achieve directed (ratchet) transport [13,14]. The study of the role of interactions on the quantum ratchet transport, however, truly remains *terra incognita*, with only a very few preliminary studies available [15–17]. For instance, in [15] it has been shown that atom-atom interactions in a BEC can cause a finite current while the noninteracting system would not exhibit directed transport; however, all these systems possess a Hamiltonian that is both space and time asymmetric.

With this Letter, instead, we consider a nondissipative, i.e., Hamiltonian, system in which the time-reversal symmetry is not broken and the system is driven smoothly. We show that, starting out from initial conditions that are symmetric both in momentum and space, interactions, treated in the limits of a mean-field approximation, induce directed current in a BEC. In a full many-body analysis of the system, symmetry considerations imply that this directed current is asymptotically decaying. Despite this fact we show that a finite current persists over a time scale which increases with increasing atom number in the condensate.

To start out, we consider a BEC composed of N atoms confined in a toroidal trap of radius R and cross section πr^2, subjected to the condition $r \ll R$, so that the motion is essentially one dimensional. The condensate is driven by a time-periodic external potential, reading

$$V_{\rm Ext}(x,t) = [V_1\cos(x) + V_2\cos(2x+\phi)]\cos[\omega(t-t_0)], \tag{1}$$

where V_1 and V_2 denote the potential depths of the two lattice harmonics, ϕ their relative phase, and ω the angular frequency of the oscillations of the perturbation. For ^{6}Li and a trap of radius $R = 2.5\ \mu$m our used unit of time is $mR^2/\hbar = 6.93\times10^{-4}$ s. This potential can be readily reproduced in experiments [18]. The space symmetry ($x \to -x+\gamma$, with γ constant [13]), is broken when $\phi \neq 0, \pi$. The driving is symmetric under $t \to -t + 2t_0$ inversion and, in particular, for $t_0 = 0$, under $t \to -t$ time symmetry.

At zero temperature and as long as the number N_0 of condensate particles is much larger than the number δN of noncondensate ones, the evolution of a BEC is well described by the Gross-Pitaevskii (GP) equation [1]. Thus, for our system the evolution of the condensate wave function $\psi(x,t)$, normalized to 1, is described by (taking $\hbar = m = 1$)

$$i\frac{\partial\psi(x,t)}{\partial t} = \left[-\frac{1}{2}\frac{\partial^2}{\partial x^2} + g|\psi(x,t)|^2 + V_{\rm Ext}(x,t)\right]\psi(x,t), \tag{2}$$

where $x \in [0, 2\pi)$ is the azimuthal angle of the torus, $g = 8NaR/r^2$ the scaled strength of the nonlinear interaction (we consider the repulsive case, i.e., $g > 0$), and a is the s-wave scattering length for elastic atom-atom collisions.

As the initial condition we use a wave function which is symmetric both in space, x, and in momentum p; i.e., it is of the form $\psi(x, 0) = \sum_n a_n e^{i\alpha_n} \cos(nx)$, where $n \in \mathbb{N}$ and a_n, $\alpha_n \in \mathbb{R}$. For simplicity, and without loss of generality, we choose the initial condition to be

$$\psi(x, 0) = \frac{1}{\sqrt{2}}\left[\frac{1}{\sqrt{2\pi}} + \frac{e^{i\alpha}}{\sqrt{\pi}} \cos(x)\right], \tag{3}$$

where α is the relative phase between the homogeneous part and the last term which corresponds to a cosinusoidal modulation in the initial density of particles. While in the noninteracting ($g = 0$) case the asymptotic current is zero, for any initial condition that is symmetric in x and p, we show that this is not the case if $g \neq 0$.

As depicted with Fig. 1 (top panel), our model may exhibit a nonzero asymptotic current, $\langle p\rangle_{\mathrm{asym}} = \lim_{t\to\infty} \frac{1}{t} \int_0^t \langle p\rangle(t')dt'$, when $g \neq 0$ and $\alpha \neq 0$ with $\langle p\rangle \times (t') = -i \int_0^{2\pi} \psi^*(x, t') \frac{\partial}{\partial x} \psi(x, t')dx$. The asymptotic time-averaged momentum $\langle p\rangle_{\mathrm{asym}}$ vs g [Fig. 1 (bottom panel)] clearly shows that (i) there is a finite threshold value $g^\star$ such that $\langle p\rangle_{\mathrm{asym}} \neq 0$ when $g > g^\star$ and (ii) there exists an optimal value g_{opt} that maximizes the current.

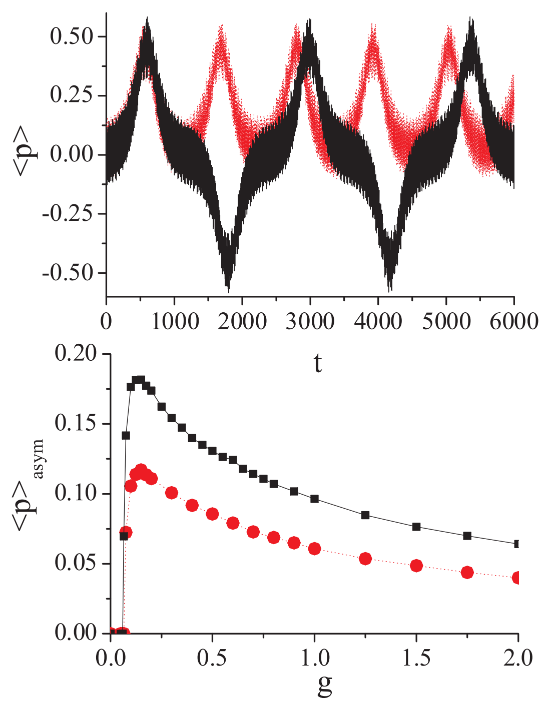

FIG. 1 (color online). Upper panel: Momentum expectation $\langle p\rangle$ vs time t for $\alpha = 0$ (black curve) and $\alpha = \pi/2$ [gray (red) curve]. Used parameter values are $g = 0.2$, $V_1 = V_2 = 2$, $\phi = \pi/2$, $\omega = 10$, $t_0 = 0$. Lower panel: Asymptotic time-average momentum $\langle p\rangle_{\mathrm{asym}}$ vs the interaction strength g for $\alpha = \pi/2$. Data are obtained from the GP equation (2) (black squares) or from the three-mode model ansatz (5) (red circles). In the first case $g^\star \approx 0.065$ and $g_{\mathrm{opt}} \approx 0.15$; in the latter $g^\star \approx 0.070$ and $g_{\mathrm{opt}} \approx 0.15$.

The asymptotic time-averaged current is shown as a function of the initial phase ωt_0 of the driving field in Fig. 2. We can deduce that there is a strong asymmetry around $\omega t_0 = \pi$, in particular, for the coupling strength $g = 0.075$. As a result, the directed current survives even after averaging over t_0 (cf. the inset of Fig. 2).

The asymptotic time-averaged momentum is depicted as a function of the relative phase α in Fig. 3 (top panel). We note that for certain intervals of the values of α it is possible to obtain a nonzero current, and these regions widen as g increases. There is a symmetry for $\alpha \to 2\pi - \alpha$, and hence the average over all α yields zero current.

This $\alpha \to (2\pi - \alpha)$ symmetry can be reasoned as follows: With $t_0 = 0$, let $t \to \tau = -t$ and perform complex conjugation in the GP equation (2), to obtain

$$i\frac{\partial \tilde{\psi}(x, \tau)}{\partial \tau} = \left[-\frac{1}{2}\frac{\partial^2}{\partial x^2} + g|\tilde{\psi}(x, \tau)|^2 + V_{\mathrm{Ext}}(x, \tau)\right]\tilde{\psi}(x, \tau), \tag{4}$$

where $\tilde{\psi}(x, \tau) \equiv \psi^\star(x, -\tau)$. Therefore, if $\psi(x, t)$ is a solution of the GP equation, so is $\tilde{\psi}(x, \tau)$. It follows that $\langle \tilde{p}(\tau)\rangle$ of the wave function $\tilde{\psi}(x, \tau)$ is such that $\langle \tilde{p}(\tau)\rangle = -\langle p(t)\rangle$, where $\langle p(t)\rangle$ is the momentum expectation of $\psi(x, t)$. For the initial condition (3), $\psi(x, 0) \to \psi^\star(x, 0) = \tilde{\psi}(x, 0)$ when $\alpha \to (2\pi - \alpha)$. We thus conclude that $\langle p\rangle_{\mathrm{asym}}(\alpha) = -\langle p\rangle_{\mathrm{asym}}(2\pi - \alpha)$. Clearly, if $\psi(0) \in \mathbb{R}$ [$\alpha = 0$, π in Eq. (3)], then $\langle p\rangle_{\mathrm{asym}} = 0$. While in the noninteracting case the current is asymptotically vanishing for any initial condition, complex or real [14], in the $g \neq 0$ case a directed current can arise when $\alpha \neq 0$, π.

In Fig. 3 (bottom panel) we show that a directed current can be generated only when the potential is spatially asymmetric, that is, $\phi \neq 0$, π. Note that the nonzero current is obtained and remains stable in two ϕ regions around the maximum symmetry violation values $\phi = \pi/2, 3\pi/2$.

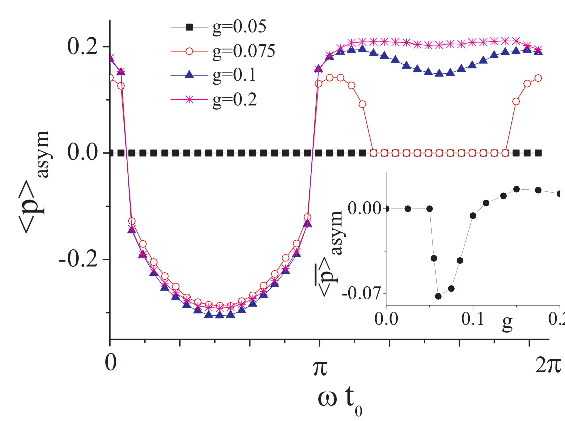

FIG. 2 (color online). Asymptotic time-averaged momentum $\langle p\rangle_{\mathrm{asym}}$ vs ωt_0 for $g = 0.05$ (filled black squares), $g = 0.075$ (empty red circles), $g = 0.1$ (filled blue triangles), and $g = 0.2$ (pink asterisks). Inset: Asymptotic current averaged over t_0, $\overline{\langle p\rangle}_{\mathrm{asym}}$, as a function of the interaction strength g. Parameter values: $V_1 = V_2 = 2$, $\phi = \pi/2$, $\omega = 10$, $\alpha = \pi/2$.

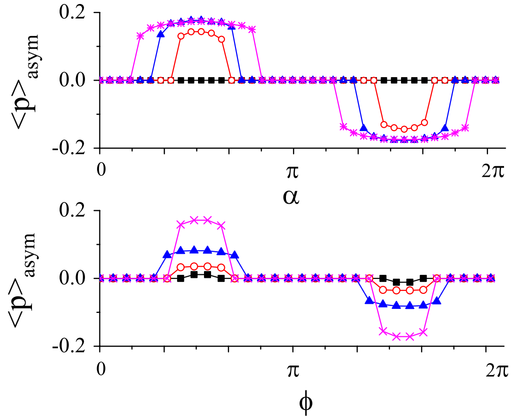

FIG. 3 (color online). Top panel: Asymptotic time-averaged momentum $\langle p\rangle_{\rm asym}$ vs α for $g = 0.05$ (filled black squares), $g = 0.075$ (empty red circles), $g = 0.15$ (filled blue triangles), and $g = 0.2$ (pink asterisks). The other parameters' values are $V_1 = V_2 = 2$, $\phi = \pi/2$, $\omega = 10$, and $t_0 = 0$. Bottom panel: Asymptotic time-averaged momentum $\langle p\rangle_{\rm asym}$ vs ϕ for $V_1 = V_2 = 0.2$ (filled black squares), $V_1 = V_2 = 0.5$ (empty red circles), $V_1 = V_2 = 1$ (filled blue triangles), and $V_1 = V_2 = 2$ (pink crosses). Other parameters' values are $g = 0.2$, $\alpha = \pi/2$, $\omega = 10$, and $t_0 = 0$.

From our numerical studies we notice that the only states that are significantly excited during the dynamical evolution of the system assume the momenta 0, +1, or −1. We hence attempt to reproduce our present results, at least on a qualitative level, with the use of a mean-field three-mode model (TMM). We use the ansatz

$$\psi(x, t) = \frac{1}{\sqrt{2\pi}}[A(t) + P(t)e^{ix} + M(t)e^{-ix}], \tag{5}$$

where $A(t)$, $P(t)$, and $M(t)$ are time-dependent, complex coefficients such that $|A|^2 + |P|^2 + |M|^2 = 1$. Using (5) we find the effective mean-field Hamiltonian of the TMM,

$$H_{\rm eff} = \int_0^{2\pi} dx\,\psi^\star(x, t)\left[-\frac{1}{2}\frac{\partial^2}{\partial x^2} + \frac{g}{2}|\psi(x, t)|^2 + V_{\rm Ext}(x, t)\right]\psi(x, t), \tag{6}$$

together with the equations of motion

$$\begin{aligned} i\dot{A} &= \frac{g}{2\pi}[(|A|^2 + 2|P|^2 + 2|M|^2)A + 2A^*PM] + \frac{K}{2}(P + M)\cos(\omega t),\\ i\dot{P} &= \frac{P}{2} + \frac{g}{2\pi}[(|P|^2 + 2|A|^2 + 2|M|^2]P + A^2M^*) + \frac{K}{2}(A + iM)\cos(\omega t),\\ i\dot{M} &= \frac{M}{2} + \frac{g}{2\pi}[(|M|^2 + 2|A|^2 + 2|P|^2)M + A^2P^*] + \frac{K}{2}(A - iP)\cos(\omega t). \end{aligned} \tag{7}$$

This TMM qualitatively reproduces the behavior of the asymptotic time-averaged current [see Fig. 1 (bottom panel)]. In Fig. 4 we depict the evolution of the populations $|A|^2$, $|P|^2$, and $|M|^2$ of the three modes. For $\alpha = 0$ ($\alpha = \pi/2$) the time-averaged current is zero (nonzero).

The obtained results may appear surprising if one observes that the full second-quantized quantum many-body operator is linear, obeying time-reversal symmetry, and symmetry arguments necessarily imply a vanishing asymptotic current [14]. Actually there occurs no contradiction because the two limits $t \to \infty$ and $N \to \infty$ do not commute: a directed current is obtained only in the mean-field limit in which we let first $N \to \infty$ and then $t \to \infty$. The issue for a physical system in which both the number of particles and the time of the experimental run are finite constitutes the vindication of the regime of validity of the used nonlinear GP equation. For this purpose, we next study the second-quantized many-body problem within the three-mode approximation, being also accessible to numerical investigation.

The second-quantized quantum many-body Hamiltonian reads

$$\hat{H}_{\rm eff} = \int_0^{2\pi} dx\,\hat{\psi}^\dagger\left[-\frac{1}{2}\frac{\partial^2}{\partial x^2} + \frac{g}{2N}\hat{\psi}^\dagger\hat{\psi} + V_{\rm Ext}(x, t)\right]\hat{\psi}, \tag{8}$$

where, within the three-mode approximation,

$$\hat{\psi}^\dagger = \frac{1}{\sqrt{2\pi}}(\hat{a}^\dagger + \hat{p}^\dagger e^{ix} + \hat{m}^\dagger e^{-ix}), \tag{9}$$

$\hat{a}^\dagger$, $\hat{p}^\dagger$, and $\hat{m}^\dagger$ denote the boson creation operators for the states with momenta 0, +1, and −1. These operators satisfy the commutation relations $[\hat{a}, \hat{a}^\dagger] = 1$, $[\hat{p}, \hat{p}^\dagger] = 1$, $[\hat{m}, \hat{m}^\dagger] = 1$, and operators corresponding to different modes commute. The particle conservation reads $\langle \hat{a}^\dagger\hat{a} + \hat{p}^\dagger\hat{p} + \hat{m}^\dagger\hat{m}\rangle = N$, where N is the total number of particles. As an initial condition, we consider N atoms in the

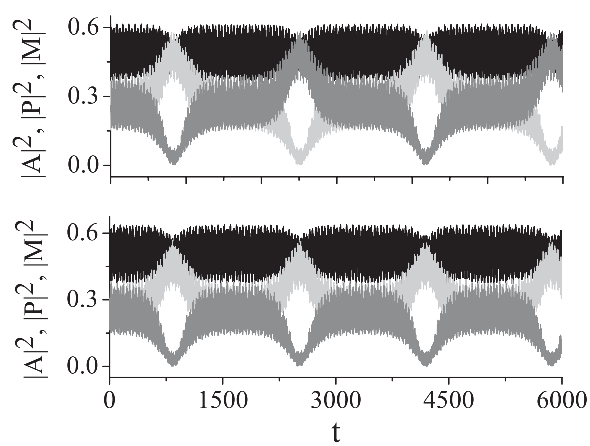

FIG. 4. Population of the three levels, $|A|^2$ (black), $|P|^2$(light gray), and $|M|^2$ (dark gray), which carry momentum 0, +1, and −1, respectively, for $\alpha = 0$ (top) and $\pi/2$ (bottom), at $g = 0.2$, $V_1 = V_2 = 2$, $\phi = \pi/2$, $\omega = 10$, $t_0 = 0$.

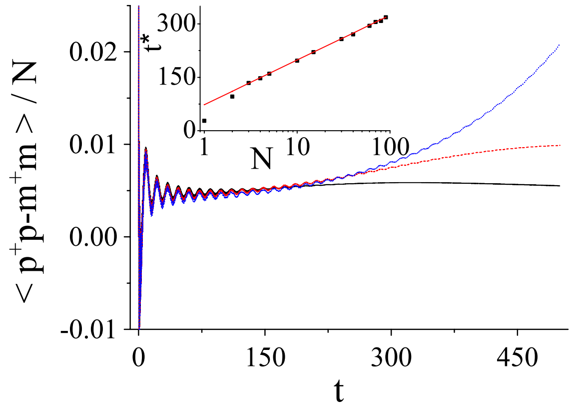

FIG. 5 (color online). Population imbalance divided by N between levels with $p = +1$ and $p = -1$ vs time t. From bottom to top: $N = 10$, 80, and mean-field TMM, for $g = 0.2$, $V_1 = V_2 = 2$, $\phi = \pi/2$, $\omega = 10$, $t_0 = 0$, $\alpha = \pi/2$. Inset: t^* vs N (squares) and numerical fit $t^\star = A + B\ln N$, with $A \approx 73$ and $B \approx 54$.

state

$$\frac{1}{\sqrt{N!}}\left[\frac{\hat{a}^\dagger}{\sqrt{2}} + \frac{e^{i\alpha}(\hat{p}^\dagger + \hat{m}^\dagger)}{2}\right]^N |0\rangle. \tag{10}$$

We studied numerically the evolution of the N-body system governed by the second-quantized three-mode Hamiltonian (8). The normalized population imbalance between levels with momenta $+1$ and -1 is depicted in Fig. 5. The TMM holds up to increasingly longer times $t^\star$ as N increases. We estimate $t^\star(N)$ as the time instant at which the N-body population imbalance deviates by more than 1% from the mean-field treatment value: Our numerical data are well fitted by a logarithmic dependence, $t^\star \propto \ln N$ (cf. inset of Fig. 5).

To check the stability of the condensate, we have diagonalized the single-particle reduced density matrix, whose matrix elements read $\langle \hat{x}^\dagger \hat{y} \rangle$, with $\hat{x}$, $\hat{y}$ being $\hat{a}$, $\hat{p}$ or $\hat{m}$. According to [19], we have a BEC if this matrix possesses one eigenvalue of order N. Defining $\tilde{t}$ as the time for which the condensed state is highly populated, we have found numerically that $\tilde{t} \propto N^{0.45}$. This means that for large N and times t obeying $\tilde{t} > t > t^*$, the mean-field approach no longer provides an accurate description of the many-body system despite the presence of a condensate.

In conclusion, we have shown that in BEC the interaction plays a prominent role in inducing long lasting currents, when otherwise impossible. We note that BECs in toroidal traps have recently been realized in different experiments [20–23]. The initial condition (3) can be implemented by properly exciting the ground-state of a condensate in a toroidal trap, and the phase α can be tuned by letting the excited condensate evolve over a given time span. The interaction strength can be changed by tuning an external magnetic field [24]. Extrapolating the dependence given in Fig. 5 of $t^\star$ on the number N of condensate particles, we obtain $t^\star \approx 800$ (in an *in situ* experiment this corresponds to a time scale 0.1 s) for $N \approx 5 \times 10^5$ particles [22]. Changing the number of condensed particles [25] opens the possibility to explore *in situ* the range of validity of the GP description via the measurement of our predicted, intriguing directed transport.

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