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Tolman-Oppenheimer-Volkoff equations in the presence of the Chaplygin gas: Stars and wormholelike solutions

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We study static solutions of the Tolman-Oppenheimer-Volkoff equations for spherically symmetric objects (stars) living in a space filled with the Chaplygin gas. Two cases are considered. In the normal case, all solutions (excluding the de Sitter one) realize a three-dimensional spheroidal geometry because the radial coordinate achieves a maximal value (the ''equator''). After crossing the equator, three scenarios are possible: a closed spheroid having a Schwarzschild-type singularity with infinite blueshift at the ''south pole'', a regular spheroid, and a truncated spheroid having a scalar curvature singularity at a finite value of the radial coordinate. The second case arises when the modulus of the pressure exceeds the energy density (the phantom Chaplygin gas). There is no more equator and all solutions have the geometry of a truncated spheroid with the same type of singularity. We also consider static spherically symmetric configurations existing in a universe filled with only the phantom Chaplygin gas. In this case, two classes of solutions exist: truncated spheroids and solutions of the wormhole type with a throat. However, the latter are not asymptotically flat and possess curvature singularities at finite values of the radial coordinate. Thus, they may not be used as models of observable compact astrophysical objects.

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I. INTRODUCTION

Because of the nowadays accepted existence of cosmic acceleration [1,2], the study of spherically symmetric solutions of the Einstein equations [3,4] in the presence of dark energy is of much interest. This study has already been undertaken for instance in [5–9].

One of the simplest models for dark energy is the Chaplygin gas [10,11]. The model is based on a perfect fluid satisfying the equation of state $p = -A/\rho$, where p is the pressure, ρ is the energy density, and A is a positive constant. Some studies have already appeared where the problem of finding spherically symmetric or wormholelike solutions of Einstein's equations in the presence of the Chaplygin gas have been addressed [6,7].

Here we study static solutions of the Tolman-Oppenheimer-Volkoff (TOV) equations for spherically symmetric objects living in a space filled with the Chaplygin gas. Results obtained appear to be very different from the apparently similar problem of stars in the presence of a cosmological constant [5]. Indeed, in the latter case the exterior solution of the TOV equations is nothing but the well-known Schwarzschild-de Sitter geometry, while the interior problem essentially coincides with the standard TOV case. The only difference is that the pressure does not vanish at the star surface; on the contrary, it is negative and its absolute value is equal to the cosmological constant.

Instead, the Chaplygin gas strongly feels the presence of the star, and consequently the solution acquires quite unusual features. These features are the existence of a maximal value of the radial coordinate, dubbed ''equator,'' and the appearance of curvature singularities at some finite values of the radial coordinate r. We also find that for the case of the phantom Chaplygin gas when the absolute value of the pressure is greater than the energy density ($|p| > \rho$), wormholelike solutions with a throat exist. However, these solutions cannot be identified with the usual Morris-Thorne-Yurtsever wormholes [12] because they are not asymptotically flat. Moreover, they possess curvature singularities at finite values of r as well.

The structure of the paper is as follows. In Sec. II, we write down the TOV equations in the presence of the Chaplygin gas and describe two exact solutions. Section III is devoted to the analysis of the normal case where $|p| < \rho$. In Sec. IV, we study the phantom case with

$|p| > \rho$, and we consider the starlike configurations analogous to those studied in Sec. III. In Sec. V, we consider the solutions of the TOV equations existing in a universe filled exclusively with the phantom Chaplygin gas. Section VI contains conclusions and discussion.

II. TOLMAN-OPPENHEIMER-VOLKOFF EQUATIONS IN THE PRESENCE OF THE CHAPLYGIN GAS

We suppose that the universe is filled with a perfect fluid with energy-momentum $T_{\mu\nu} = (\rho + p)u_\mu u_\nu - g_{\mu\nu}p$ and consider a static spherically symmetric interval

$$ds^2 = e^{\nu(r)}dt^2 - e^{\mu(r)}dr^2 - r^2(d\theta^2 + \sin^2\theta d\phi^2). \quad (1)$$

Then the Einstein system reduces to the following pair of equations:

$$e^{-\mu}\left(\frac{1}{r}\frac{d\mu}{dr} - \frac{1}{r^2}\right) + \frac{1}{r^2} = 8\pi\rho, \quad (2)$$

$$e^{-\mu}\left(\frac{1}{r}\frac{d\nu}{dr} + \frac{1}{r^2}\right) - \frac{1}{r^2} = 8\pi p, \quad (3)$$

plus the energy-momentum conservation equation

$$\frac{dp}{dr} = -\frac{d\nu}{dr}\frac{\rho + p}{2}. \quad (4)$$

Solving Eq. (2) with the boundary condition $e^{-\mu(0)} = 1$ gives

$$e^{-\mu} = \left(1 - \frac{2M}{r}\right) \quad (5)$$

where, as usual, $M(r) = 4\pi \int_0^r drr^2\rho(r)$. This is equivalent to

$$\frac{dM}{dr} = 4\pi r^2\rho, \qquad M(0) = 0. \quad (6)$$

Equation (3)–(5) together give rise to the TOV differential equation [3,4]

$$\frac{dp}{dr} = -\frac{(\rho + p)(M + 4\pi r^3 p)}{r(r - 2M)}. \quad (7)$$

Complementing Eqs. (6) and (7) with an equation of state relating p and ρ one has a closed system of three equations for the three variables p, ρ, and M. In this paper, we investigate the case in which the fluid is the Chaplygin gas whose equation of state is

$$p = -\frac{\Lambda^2}{\rho}. \quad (8)$$

Then Eqs. (6) and (7) give rise to the following system of first-order differential equations for p and M:

$$\frac{dp}{dr} = \frac{(\Lambda^2 - p^2)(M + 4\pi r^3 p)}{pr(r - 2M)}, \quad (9)$$

$$\frac{dM}{dr} = -\frac{4\pi\Lambda^2 r^2}{p}. \quad (10)$$

We denote the radius of the star by r_b. As usual we suppose that the pressure is continuous at the surface of the star. The "exterior" problem amounts to considering a system (10) in the interval $r > r_b$ with some properly chosen boundary conditions $p(r_b)$ and $M(r_b)$ at $r = r_b$. It is easy to see that at $r > r_b$ the system admits two exact solutions with constant pressure. The first solution

$$p = -\rho = -\Lambda, \qquad M = \tfrac{4}{3}\pi\Lambda r^3, \quad (11)$$

describes the geometry of the de Sitter space with

$$e^\nu = e^{-\mu} = 1 - \frac{r^2}{r_{dS}^2}, \qquad r_{dS} = \sqrt{\frac{3}{8\pi\Lambda}}. \quad (12)$$

The second solution is the Einstein static universe

$$p = -\frac{\Lambda}{\sqrt{3}}, \qquad \rho = \sqrt{3}\Lambda, \qquad M = \frac{4\sqrt{3}\pi\Lambda r^3}{3} \quad (13)$$

with the radius

$$r_E = \sqrt{\frac{\sqrt{3}}{8\pi\Lambda}} = \frac{r_{dS}}{3^{1/4}}. \quad (14)$$

III. THE NORMAL CASE: $|p| < \rho$

We now consider solutions with a nonconstant pressure. Some additional constraints have to be imposed on the boundary conditions:

$$-\Lambda < p(r_b) < 0, \quad (15)$$

$$M(r_b) < \frac{r_b}{2}. \quad (16)$$

First of all, note that the pressure p cannot attain the values $p = 0$ and $p = -\Lambda$ in the region where $2M(r) < r$. Indeed, for $2M(r) < r$ the right-hand side of (9) is negative, while in order to approach $p = 0$ starting from negative values of p, it is necessary to have $dp/dr > 0$. In addition, let us rewrite Eq. (9) as follows:

$$d\ln(\Lambda^2 - p^2) = -2dr\frac{(M + 4\pi r^3 p)}{r(r - 2M)}, \quad (17)$$

and suppose that $p \to -\Lambda$ as $r \to r_1$, with $r_1 > 2M(r_1)$. Then, upon integration of Eq. (17), we get a divergence on the left-hand side and a regular expression on the right-hand side, a contradiction.

Thus, as long as the condition $2M(r) < r$ is satisfied we have $-\Lambda < p(r) < 0$ and $\rho(r) > \Lambda$. This means that the mass $M(r)$ is growing at least as fast as r^3, so that at some radius $r = r_0$ the equality

$$M(r_0) = \frac{r_0}{2} \quad (18)$$

is achieved. Then at $r = r_0$ we must have $p(r_0) = p_0 = -\frac{1}{8\pi r_0^2}$. Indeed, let us expand the relevant quantities around r_0:

$$r = r_0 - \varepsilon, \tag{19}$$

$$M(r) = \frac{r_0}{2} - \tilde{M}(\varepsilon), \tag{20}$$

$$p(r) = p_0 + \tilde{p}(\varepsilon), \tag{21}$$

where $\tilde{M}(\varepsilon)$ and $\tilde{p}(\varepsilon)$ tend to zero when $\varepsilon \to 0$. Equation (9) has the following asymptotic form:

$$\frac{d\tilde{p}}{d\varepsilon} = \frac{(\Lambda^2 - p_0^2)(1 + 8\pi r_0^2 p_0)}{2\varepsilon(p_0 + 8\pi r_0^2 \Lambda^2)}, \tag{22}$$

from which it is easily seen that $\tilde{p} \sim \ln\varepsilon$ when $\varepsilon \to 0$, unless $p_0 = -\frac{1}{8\pi r_0^2}$.

We now determine a lower and an upper bound for the radius r_0. First, note that since $p_0 > -\Lambda$ we have

$$r_0 > \sqrt{\frac{1}{8\pi\Lambda}}. \tag{23}$$

On the other hand, since $\rho(r) > \Lambda$, Eq. (6) implies that of $M(r) > \frac{4\pi\Lambda r^3}{3}$ so that

$$r_0 < \sqrt{\frac{3}{8\pi\Lambda}}. \tag{24}$$

The asymptotic equation for $\tilde{p}$ has the form

$$\frac{d\tilde{p}}{d\varepsilon} = \frac{\tilde{p}}{2\varepsilon} + C_0 \tag{25}$$

where

$$C_0 = \frac{1}{8\pi r_0^3}\left(\frac{3}{2} - 32\pi^2\Lambda^2 r_0^4\right), \tag{26}$$

and its solution is

$$\tilde{p} = A\sqrt{\varepsilon} + 2C_0\varepsilon \tag{27}$$

where A is an arbitrary coefficient. Thus, the family of the solutions $p(r)$, $M(r)$ can be characterized by the two parameters r_0 and A which, in turn, are determined by the boundary conditions $M(r_b)$, $p(r_b)$ on the surface of the star.

However, the coordinates which we have used so far are not convenient for the problem under consideration since the metric coefficient $g_{rr} = e^{\mu} = (1 - 2M/r)^{-1}$ has a fictitious (coordinate) singularity at $r = r_0$. Therefore, instead of the coordinate r, we introduce a new coordinate χ defined by $r = r_0 \sin\chi$, so that the corresponding metric becomes

$$ds^2 = e^{\bar{\nu}(\chi)}dt^2 - e^{\bar{\mu}(\chi)}d\chi^2 - r_0^2\sin^2\chi(d\theta^2 + \sin^2\theta d\phi^2). \tag{28}$$

Then the tt component of the Einstein equations has the form

$$e^{-\bar{\mu}}(\bar{\mu}'\cot\chi + 2 - \cot^2\chi) + \frac{1}{r_0^2\sin^2\chi} = 8\pi\rho \tag{29}$$

where prime denotes differentiation with respect to the variable χ. Integration with the boundary condition $e^{-\bar{\mu}(0)} = 1/r_0^2$ gives

$$e^{-\bar{\mu}} = \frac{1}{r_0^2\cos^2\chi}\left(1 - \frac{8\pi r_0^2}{\sin\chi}\int_0^\chi \rho(\chi)\sin^2\chi\cos\chi d\chi\right). \tag{30}$$

From Eqs. (6) and (18) it follows that

$$r_0^2 = \frac{1}{8\pi\int_0^{\pi/2}\rho(\chi)\sin^2\chi\cos\chi d\chi} \tag{31}$$

that implies the positivity and finiteness of the expression (30).

For latter purposes we also write down the $\chi\chi$ component of the Einstein equations:

$$e^{-\bar{\mu}}(\cot^2\chi + \bar{\nu}'\cot\chi) - \frac{1}{r_0^2\sin^2\chi} = 8\pi p. \tag{32}$$

The energy-momentum conservation equation now reads

$$\bar{\nu}' = -\frac{2p'}{p + \rho}, \tag{33}$$

and for the case of the Chaplygin gas it can be easily integrated to give

$$e^{\bar{\nu}} = \frac{A_0}{\Lambda^2 - p^2} \tag{34}$$

where A_0 is some positive constant which fixes the choice of the time scale.

Combining Eqs. (29), (32), and (33) one gets

$$p' = -\frac{\cos\chi(p + \rho)(\bar{M} + 4\pi r_0^3\sin^3\chi p)}{\sin\chi(r_0\sin\chi - 2\bar{M})} \tag{35}$$

where

$$\bar{M}(\chi) = 4\pi r_0^3\int_0^\chi \rho\sin^2\chi\cos\chi d\chi \tag{36}$$

and

$$\bar{M}' = 4\pi r_0^3\rho\sin^2\chi\cos\chi, \qquad \bar{M}(0) = 0. \tag{37}$$

Relation (18) can be rewritten as

$$r_0 = 2\bar{M}(\pi/2), \tag{38}$$

and for the Chaplygin gas Eqs. (35) and (37) acquire the forms

$$p' = \frac{\cos\chi(\Lambda^2 - p^2)(\bar{M} + 4\pi r_0^3 \sin^3\chi p)}{p \sin\chi(r_0 \sin\chi - 2\bar{M})}, \tag{39}$$

$$\bar{M}' = -\frac{4\pi r_0^3 \Lambda^2 \sin^2\chi \cos\chi}{p}, \tag{40}$$

respectively.

We study these equations in the vicinity of the equator ($\chi = \pi/2$) by introducing a small positive variable α such that

$$\chi = \frac{\pi}{2} - \alpha, \tag{41}$$

and the functions $\tilde{p}(\alpha)$ and $\tilde{M}(\alpha)$:

$$p = -\frac{1}{8\pi r_0^2} + \tilde{p}(\alpha), \tag{42}$$

$$\bar{M} = \frac{r_0}{2} - \tilde{M}(\alpha). \tag{43}$$

A simple calculation shows that

$$\tilde{M} = 16\pi^2 r_0^5 \Lambda^2 \alpha^2 + \cdots, \tag{44}$$

while for $\tilde{p}(\alpha)$ one can write down the following asymptotic equation

$$\frac{d\tilde{p}}{d\alpha} = \frac{\tilde{p}}{\alpha} + C_1 \alpha, \tag{45}$$

where

$$C_1 = \frac{\frac{3}{2} - 32\pi^2 r_0^4 \Lambda^2}{8\pi r_0^2}. \tag{46}$$

The solution of Eq. (45) is

$$\tilde{p} = B\alpha + C_1 \alpha^2, \tag{47}$$

with B an arbitrary constant. This solution can be continued to negative values of the parameter α that corresponds to the equator crossing. Thus, all trajectories intersecting the equator $\chi = \pi/2$ can be characterized by two parameters, which could be chosen as r_0 and B. The static Einstein solution corresponds to the values $B = 0$ and $C_1 = 0$, the latter condition being equivalent to $r_0^4 = 3/64\pi^2\Lambda^2$.

In order to investigate the behavior of the trajectories after crossing the equator, we find it convenient to introduce a new variable

$$y \equiv \frac{1}{\sin\chi}, \tag{48}$$

so that $1 \leq y < \infty$. In terms of this variable Eqs. (39) and (40) can be rewritten as

$$\frac{dp}{dy} = -\frac{(\Lambda^2 - p^2)(\bar{M} y^3 + 4\pi r_0^3 p)}{p y^3 (r_0 - 2\bar{M} y)}, \tag{49}$$

$$\frac{d\bar{M}}{dy} = \frac{4\pi\Lambda^2 r_0^3}{p y^4}. \tag{50}$$

Now one can show that the expression $r_0 - 2\bar{M}y$ in the denominator of the right-hand side of Eq. (49) is always positive at $y > 1$. In order to prove this statement, we first show that it is true if $p > -\Lambda$.

To this end we introduce the function

$$f(y) \equiv r_0 - r_0 y + \frac{8\pi\Lambda r_0^3 (y^3 - 1)}{3y^2}. \tag{51}$$

Since $p > -\Lambda$, it satisfies the inequality

$$f(y) \leq r_0 - 2\bar{M}(y) y. \tag{52}$$

We have

$$f(1) = 0 \tag{53}$$

and

$$f'(y) = -r_0 + \frac{8\pi\Lambda r_0^3}{3} + \frac{16\pi\Lambda r_0^3}{3y^3}, \tag{54}$$

so that

$$f'(1) = r_0(8\pi\Lambda r_0^2 - 1) = \left(\frac{\Lambda}{|p(r_0)|} - 1\right) > 0. \tag{55}$$

Now assume that the function $r_0 - 2\bar{M}(y)y$ becomes equal to zero at some $y = y_1 > 1$. This means that at some value $y = y_2 \leq y_1$, the function $f(y)$ vanishes. In turn, this last condition requires the vanishing of the derivative $f'(y)$ at some value $y = y_3 < y_2$. From Eq. (54) one finds

$$y_3 = \left(\frac{16\pi\Lambda r_0^2}{3 - 8\pi\Lambda r_0^2}\right)^{1/3}, \tag{56}$$

and the requirement $y_3 > 1$ is equivalent to

$$1 < 8\pi\Lambda r_0^2 < 3. \tag{57}$$

The vanishing of the function $r_0 - 2\bar{M}y$ at the point y_1 implies also the vanishing of the expression $\bar{M}y^3 + 4\pi r_0^3 p$ at this point, i.e. the vanishing of the numerator of the expression in the right-hand side of Eq. (49). Thus, the values of the functions $\bar{M}$ and p at the point y_1 are given by

$$\bar{M}(y_1) = \frac{r_0}{2y_1}, \qquad p(y_1) = -\frac{y_1^2}{8\pi r_0^2}. \tag{58}$$

It follows from the condition $p > -\Lambda$ that

$$y_1^2 < 8\pi\Lambda r_0^2, \tag{59}$$

while from $y_3 < y_1$ we find

$$y_3^2 < 8\pi\Lambda r_0^2. \tag{60}$$

Substituting the expression (56) into the inequality (60) we get that this inequality is satisfied provided

$$8\pi\Lambda r_0^2 > 4 \tag{61}$$

that contradicts the condition (57).

Finally, looking at Eq. (49) one can see that the pressure can, in principle, achieve the value $p = -\Lambda$ at some value $y = y_\Lambda$ only if $r_0/y_\Lambda = 2\bar{M}(y_\Lambda)$ and $(\frac{8\pi\Lambda r_0^2}{y_\Lambda^2} - 1) > 0$. A simple analysis similar to the one carried out above shows that this is impossible as well. Thus, we have shown that the expression $r_0 - 2\bar{M}(y)y$ cannot vanish at any value of y in the range $1 < y < \infty$.

We now study the behavior of the pressure at $y > 1$. Here we find three families of solutions (geometries). The first one contains trajectories arriving at the south pole of the three-dimensional spatial manifold ($y = \infty, \chi = \pi$) with some value $0 > p(\infty) > -\Lambda$. Looking at Eq. (49) we see that the necessary condition for such solutions to exist is the convergent behavior of the integral

$$\int dy \frac{\bar{M}y^3 + 4\pi r_0^3 p}{y^3(r_0 - 2\bar{M}y)} \tag{62}$$

at $y \to \infty$ that implies the vanishing of the function $\bar{M}$ at $y \to \infty$. Indeed, assume $\bar{M}(\infty) = M_0 \neq 0$. Then $M_0 > 0$ contradicts the positivity of the expression $r_0 - 2\bar{M}y$, while $M_0 < 0$ implies the integral (62) to diverge logarithmically. Thus, the only value $\bar{M}(\infty)$ compatible with $p(\infty) > -\Lambda$ is $\bar{M}(\infty) = 0$. Then it follows from Eq. (50) that the asymptotic behavior of $\bar{M}$ at $y \to \infty$ is

$$\bar{M} = \frac{m}{y^3}, \tag{63}$$

where

$$m = -\frac{4\pi\Lambda^2 r_0^3}{3p(\infty)}. \tag{64}$$

Substituting the value of m into the integrand of the right-hand side of Eq. (49), we see that the sign of the derivative dp/dy at $y \to \infty$ is determined by the sign of the expression $(\Lambda^2 - 3p^2(\infty))$. If $p < -\Lambda/\sqrt{3}$ this derivative is negative, while it is positive for $p > -\Lambda/\sqrt{3}$.

Thus, there exists a two-parameter family of regular space-time geometries for which the spatial manifold represents a three-dimensional spheroid parameterized by the two parameters r_0 and $p(\infty)$, and the metric coefficient g_{tt} given by the formula (34) is always positive.

The second family of geometries includes those where the value of the pressure p becomes equal to zero. Let us describe basic features of such geometries. We suppose that $p(y_0) = 0$ at $y_0 > 1$. The function $\bar{M}(y)$ cannot become negative at $y = y_0$ because in this case the derivative dp/dy would be negative and it would be impossible to reach the value $p(y_0) = 0$. Hence we consider the case when $\bar{M}(y_0) = M_0$ where $0 < M_0 < \frac{r_0}{2y_0}$. We assume that in the neighborhood of the point y_0 the pressure behaves as

$$p(y) = -D(y_0 - y)^\alpha \tag{65}$$

where D and α are some positive constants. Substituting the expression (65) into Eq. (49), one gets

$$\alpha = \frac{1}{2}, D = \sqrt{\frac{2\Lambda^2 M_0}{r_0 - 2M_0 y_0}}. \tag{66}$$

Consider also the case in which $\bar{M}(y_0) = 0$. In this case, we look for the expressions describing the behavior of p and $\bar{M}$ in the vicinity of $y = y_0$ in the form

$$\bar{M}(y) = M_1(y_0 - y)^\beta, \tag{67}$$

$$p(y) = -E(y_0 - y)^\gamma, \tag{68}$$

where M_1 and E are positive and $0 < \beta < \gamma$. Substituting expressions (67) and (68) into Eqs. (49) and (50) one finds the following values for the parameters β, γ, M_1 and E:

$$\beta = \frac{1}{3}, \qquad \gamma = \frac{2}{3}, \qquad E = \left(\frac{18\pi\Lambda^4 r_0^2}{y_0^4}\right)^{1/3}, \qquad M_1 = \frac{2r_0}{3\Lambda^2}E^2. \tag{69}$$

Note that in this case the values of the of the parameters M_1 and E are uniquely fixed by the value of y_0. In the case of $M(y_0) > 0$ considered above one has a one parameter family of geometries parameterized by the value of M_0 or by the value of D. Thus, it seems that one has a three-parameter family of geometries corresponding to $p \to 0$ and these parameters are r_0, y_0 and M_0. However, Eqs. (66) or (69) describe necessary conditions which should be satisfied to provide the existence of the geometry having the maximal radius r_0 and the pressure p vanishing at $y = y_0$. Not all the solutions satisfying the relations (66) or (69) correspond to "initial conditions" at $y = 1$, i.e. $\bar{M}(y = 1) = r_0/2$. Moreover, taking into account the monotonic behavior of the function $M(y)$ one can believe that at least one value of the parameter $\bar{M}(y_0)$ corresponds to a geometry with the desirable initial and final conditions. Thus, the family of solutions (geometries) ending with $p = 0$ is also two parametric and can be parameterized by the two parameters r_0 and y_0.

An interesting feature of the geometries described above consists in the presence of the singularity at $y = y_0$. Indeed, the Chaplygin gas equation of state implies an infinite growth of the energy density when the pressure tends to zero that in turn determines the divergence of the scalar curvature. Thus, the space-time under consideration cannot be continued beyond $y = y_0$ or, in other terms, beyond $\chi = \pi - \arcsin y_0^{-1}$.

The third family of possible geometries includes those for which the pressure p tends to the value $-\Lambda$ when $y \to \infty$ ($\chi \to \pi$). In this case the acceptable behavior of the function $\bar{M}$ is $\bar{M}(\infty) = -M_2$, where $M_2 > 0$. The behavior of the pressure at $y \to \infty$ can be represented as

$$p(y) = -\Lambda + \bar{p}(y) \tag{70}$$

where $\bar{p}(y)$ is a positive function vanishing at $y \to \infty$. Substituting (70) into Eq. (49), one gets an asymptotic equation

$$\frac{d\bar{p}}{dy} = -\frac{\bar{p}}{y}, \tag{71}$$

with the solution

$$\bar{p} = \frac{F}{y} \tag{72}$$

where F is a positive constant. This constant does not depend on the value of the parameter M_2. Thus, the family of the geometries with $p \to -\Lambda$ at $y \to \infty$ appears to be described by the three parameters r_0, F, and M_2. However, as in the case of the family of geometries described above with the pressure vanishing at some of $y = y_0$, we are not free in the choice of the value of M_2 after the values r_0 and F are fixed. Indeed, due to the monotonic behavior of the function $\bar{M}(y)$, at least one value of the parameter M_2 corresponds to a solution $\bar{M}(y)$ satisfying the initial condition $\bar{M}(1) = r_0/2$. Thus, one has a two-parameter family of geometries defined by fixing the values of r_0 and F. These geometries have a singularity of the Schwarzschild-type at $y = \infty$ ($\chi = \pi$, $r = 0$) due to the nonvanishing mass $\bar{M} = -M_2$. They have another curious feature: the metric coefficient $g_{tt} = e^{\bar{\nu}(\chi)}$ given by the formula (34) tends to infinity as $p \to -\Lambda$ and, hence, intervals of the proper time $d\tau = \sqrt{g_{tt}}dt$ tend to infinity. So, one has an infinite blueshift effect in contrast to the well-known redshift effects in the vicinity of the Schwarzschild and de Sitter horizons.

Summarizing, solutions of the Tolman-Oppenheimer-Volkoff equations in the presence of the Chaplygin gas have the following curious features:

(1) All the spatial sections of the space-time manifolds (excluding a special case of the de Sitter space-time) are closed.

(2) Some geometries have a divergent scalar curvature invariant at a finite value of r.

(3) Some geometries manifest an infinite blueshift effect.

Unfortunately, the relations between the boundary conditions p_b, M_b, the parameters characterizing the crossing of the equator r_0, B, and the "final parameters" characterizing the three family of geometries with qualitatively different behaviors at $\chi > \pi/2$ cannot be found analytically and should be studied numerically.

IV. THE PHANTOM CASE: $|p| > \rho$

Now consider the system of Eqs. (9) and (10) with the boundary condition

$$p(r_b) < -\Lambda. \tag{73}$$

In this case $|p| > \rho$ that corresponds to phantom dark energy and, in principle, to a possibility of creation of wormholes. If the condition (73) is satisfied, two cases are possible.

Case A:

$$M(r_b) + 4\pi r_b^3 p(r_b) < 0. \tag{74}$$

In this case, the pressure p is decreasing and its absolute value is growing. Correspondingly, the energy density is also decreasing and hence the mass M grows slower than r^3. Then the left-hand side of the expression (74) is decreasing, too, so the expressions $(M + 4\pi r^3 p)$, $(r - 2M)$, and $(p^2 - \Lambda^2)$ cannot change their signs.

We examine three possible subcases:

(1) p tends to some finite value $-\infty < p_1 < -\Lambda$ when $r \to \infty$;

(2) p tends to $-\infty$ when $r \to \infty$;

(3) p grows indefinitely when r tends to some finite value r_1.

Subcase 1 cannot take place because the left-hand side of Eq. (9) is regular while its right-hand side diverges as r^2 when $r \to \infty$.

Likewise, subcase 2 cannot be realized. Indeed, suppose that $p = -p_1 r^\alpha$, $\alpha > 0$, $p_1 > 0$ when $r \to \infty$. Then Eq. (74) becomes

$$\frac{dp}{dr} = -4\pi p^2 r, \tag{75}$$

which implies

$$\alpha - 1 = 2\alpha + 1 \tag{76}$$

or $\alpha = -2$ which contradicts the positivity of α.

We are left with subcase 3 which can be realized with

$$p = -\frac{p_1}{r_1 - r}, \qquad p_1 > 0, \quad \text{when } r \to r_1. \tag{77}$$

Substituting expression (77) into Eq. (9), we have the following relation:

$$\frac{p_1}{(r_1 - R)^2} = \frac{p_1^2}{(r_1 - r)^2} \cdot \frac{4\pi r_1^2}{r_1 - 2M_1}, \tag{78}$$

from which we find the value of parameter p_1 as a function of the radius r_1 and the mass $M(r_1) = M_1$:

$$p_1 = \frac{r_1 - 2M_1}{4\pi r_1^2}. \tag{79}$$

Thus, we obtain a two-parameter family of solutions in which one encounters a singularity at $r = r_1$ because the scalar curvature R diverges there.

Case B:

$$M(r_b) + 4\pi r_b^3 p(r_b) > 0. \tag{80}$$

In this case, the pressure grows with r. Then, in the right-hand side of Eq. (9) we have three decreasing positive terms $(p^2 - \Lambda^2)$, $(M + 4\pi p r^3)$, and $(r - 2M)$. The problem is which term vanishes before the others, if any.

The above terms cannot simultaneously remain positive as $r \to \infty$ because in this case $|p| > \Lambda > \rho$, and the expression $(M + 4\pi p r^3)$ would unavoidably change sign.

The case in which $p = -\Lambda$ while the other two expressions remain positive is also excluded. Indeed, if $p \to -\Lambda$ as r approaches some finite value, the left-hand side of Eq. (17) has a logarithmic divergence while its right-hand side is regular. On the other hand, if $p \to -\Lambda$ as $r \to \infty$, the expression $(M + 4\pi p r^3)$ will change its sign.

If $(p^2 - \Lambda^2)$ and $(r - 2M)$ vanish at some $r = r_0$, Eq. (17) takes the form

$$d\ln(p^2 - \Lambda^2) = -\frac{dr}{r - r_0} \tag{81}$$

which implies

$$p^2 - \Lambda^2 \sim \frac{1}{r_0 - r} \tag{82}$$

contradicting the hypothesis. Thus, the pressure cannot achieve the value $p = -\Lambda$.

As shown in the preceding section, the denominator $(r - 2M)$ at the right-hand side of Eq. (9) can only vanish at some $r = r_0$ simultaneously with $(M + 4\pi p r^3)$ and the pressure at $r = r_0$ should be equal $p = -\frac{1}{8\pi r_0^2}$ (equator). Let us prove that it is impossible to achieve the equator if $p < -\Lambda$. Indeed, the formula (10) shows that in this case

$$\begin{aligned} r - 2M &= r_0 + (r - r_0) - 2M(r_0) - 2M'(r_0)(r - r_0) \\ &= (r - r_0)(1 - 64\pi^2\Lambda^2 r_0^4) \end{aligned} \tag{83}$$

as $r \to r_0$. The difference $(r - 2M)$ should be positive as $r \to r_0$ from below, so the expression $(1–64\pi^2\Lambda^2 r_0^4)$ should be negative. However, at $|p| > \Lambda$ this expression is positive—a contradiction.

Thus, if (80) is satisfied, p grows until some maximum value $p_{\max} < -\Lambda$ when the expression $(M + 4\pi p r^3)$ changes sign while the terms $(p^2 - \Lambda^2)$ and $(r - 2M)$ are positive, and we are led back to case A.

So, we have proved that only two regimes are possible for a starlike object immersed into the phantom Chaplygin gas. If initial conditions satisfy (74), the pressure is decreasing and diverges at some finite value of r. Then the space-time acquires a scalar curvature singularity there. On the other hand, if initial conditions satisfy (80), then the pressure grows with the r until some maximum value $p = p_{\max}$ where the expression in the right-hand side of Eq. (9) changes sign. After that we come back to case A: the pressure decreases and explodes according to (77) at some finite value of the radial coordinate r. No equator and no horizon are attained in the case of the phantom Chaplygin gas: the quantity $(r - 2M)$ is always positive. As in the case of the nonphantom Chaplygin gas, the relation between the initial values of the parameters functions $p(r_b)$ and $M(r_b)$ and the parameters r_1 and $M(r_1)$ cannot be found analytically.

We can summarize the results of the above considerations in the following theorem: In a static spherically symmetric universe filled with the phantom Chaplygin gas, the scalar curvature becomes singular at some finite value of the radial coordinate and the universe is not asymptotically flat.

V. STATIC SPHERICALLY SYMMETRIC UNIVERSE FILLED EXCLUSIVELY WITH THE PHANTOM CHAPLYGIN GAS

Now we study spherically symmetric static solutions for a universe filled exclusively with the phantom Chaplygin gas. The theorem above is valid in this case, too. This situation is of much interest because when the weak energy condition is violated, $\rho + p < 0$, wormholes may appear (though not necessarily; see [9] in this connection). Suppose that at some finite value of the radial variable $r = r_b$, the factor $r_b - 2M(r_b)$ is positive and $p(r_b) < -\Lambda$. Then, as in the preceding analysis, one can consider the evolution of the functions $M(r)$ and $p(r)$ in accordance with Eqs. (9) and (10) but with a *decreasing* value of the radial variable r. Now only two possibilities may be realized: one can arrive at the value $r = 0$ keeping always a positive value of the factor $r - 2M$, or one can encounter a situation when at some finite value of $r = r_0$ this factor vanishes.

Consider first the case in which $(r - 2M)$ is positive for all values $r > 0$. Here, one can imagine two different regimes as r approaches zero. In the first one the mass in the vicinity of $r = 0$ is positive and behaves as $M \sim r^\alpha$, $\alpha > 0$. In the second regime the mass tends to a negative constant when $r \to 0$. A detailed analysis shows that only the first regime is compatible with the TOV Eqs. (9) and (10) in the presence of the phantom Chaplygin gas. Precisely, in the vicinity of $r = 0$ the pressure and the mass functions have the form

$$p = p_0 - \frac{8\pi^2(3p_0^2 - \Lambda^2)(p_0^2 - \Lambda^2)r^2}{3p_0^2}, \tag{84}$$

$$M = -\frac{4\pi\Lambda^2 r^3}{3p_0} \tag{85}$$

where p_0 is an arbitrary number such that $p_0 < -\Lambda$. Being regular at the center $r = 0$, this static configuration develops a singularity at some finite value of the radius r_1, where the pressure becomes equal to minus infinity. Thus, we have a one-parameter family of static spherically symmetric solutions of the Tolman-Oppenheimer-Volkoff equations in a world filled with the phantom Chaplygin gas. This family is parameterized by the value of the pressure at the center $r = 0$.

Now suppose that the factor $r > 2M(r)$ vanishes at some value $r = r_0$. An analysis similar to the one carried out in Sec. III shows that this is positive only if also the expression $M + 4\pi p r^3$ in the numerator of the right-hand-side of

Eq. (9) vanishes at r_0. In Sec. III, the surface $r = r_0$ was called equator because it corresponded to the maximal value of the radial variable r. Now it corresponds to the minimal value of r, and it is nothing but a throat. Just like in the case of the equator considered in Sec. III, the throat can be achieved only at $p = -\frac{1}{8\pi r_0^2}$. In the phantom case $p < -\Lambda$, hence, there is a restriction on the size of the throat

$$r_0 < \sqrt{\frac{1}{8\pi\Lambda}}. \tag{86}$$

In order to describe the crossing of the throat, it is convenient to introduce the hyperbolic coordinate η instead of the radius r,

$$r = r_0 \cosh\eta, \tag{87}$$

which plays a role similar to that played by the trigonometrical angle χ in the description of the equator.

Now the TOV equations look like

$$\frac{dp}{d\eta} = \frac{(\Lambda^2 - p^2)(M + 4\pi p r_0^3 \cosh^3\eta)\sinh\eta}{p\cosh\eta(r_0\cosh\eta - 2M)}, \tag{88}$$

$$\frac{dM}{d\eta} = -\frac{4\pi\Lambda^2 r_0^3 \cosh^2\eta\sinh\eta}{p}. \tag{89}$$

The solution of (89) at small values of η is

$$M = r_0 + 16\pi^2\Lambda^2 r_0^5\eta^2. \tag{90}$$

Representing the pressure as

$$p(\eta) = -\frac{1}{8\pi r_0^2} + \tilde{p}(\eta), \tag{91}$$

one can rewrite Eq. (88) in the neighborhood of the throat as

$$\frac{d\tilde{p}}{d\eta} = \frac{\tilde{p}}{\eta} + C_T\eta \tag{92}$$

where the negative constant C_T is equal to

$$C_T = \frac{64\pi^2\Lambda^2 r_0^4 - 3}{16\pi r_0^2}. \tag{93}$$

The solution of Eq. (92) is

$$\tilde{p} = D\eta + \tfrac{1}{2}C_T\eta^2 \tag{94}$$

where D is an arbitrary constant.

Thus, geometries with a throat constitute a two-parameter family, characterized by the value of the throat radius r_0 and by the value of the parameter D. From Eq. (94) we see that when the hyperbolic parameter η grows, the negative term $\frac{1}{2}C_T\eta^2$ starts dominating, while the pressure decreases and achieves an infinite negative value at a finite value of the radius r where we encounter a curvature singularity. The peculiarity of this configuration with a throat consists in the fact that these singularities are achieved at both sides of the throat or, in other words, at one positive $\eta_1 > 0$ and one negative $\eta_2 < 0$ values of the hyperbolic parameter. The values η_1 and η_2 correspond to values $r_1 = r_0\cosh\eta_1$ and $r_2 = r_0\cosh\eta_2$ of the radial parameter. If $D = 0$ one has $r_1 = r_2$. The solutions with a throat could be seen as wormholelike, but in contrast with the Morris-Thorne-Yurtsever wormholes [12], they do not connect two asymptotically flat regions.

We summarize the preceding considerations concerning solutions of the Tolman-Oppenheimer-Volkoff equations in the universe filled with the phantom Chaplygin gas as follows. Suppose we start from an initial condition $\bar{r} - 2M(\bar{r}) > 0$ and $p(\bar{r}) < -\Lambda$ for some given value $\bar{r}$ of r. Then letting the functions $M(r)$ and $p(r)$ evolve to values $r > \bar{r}$, we unavoidably arrive to an infinite negative value of the pressure at some finite value r_f of the radius r, thus encountering a singularity. Instead, for $r < \bar{r}$ two qualitatively different situations may arise: either we arrive to $r = 0$ in a regular way, or we may discover a throat at some finite value $r = r_0$, this being the generic situation. Upon passing the throat, an observer finds itself in another patch of the world and then, with the radius increasing, stumbles again upon a curvature singularity at some finite distance from the throat.

Note that in the traditional view of wormholes, one supposes that there is a minimal value of the radial parameter characterizing a throat, and the space-time at both sides of the throat is either asymptotically flat, or has some other traditional structure (for example, wormholes could also connect two expanding asymptotically Friedmann universes). Here we have found a different kind of wormholelike solutions: those connecting two space-time patches which have a scalar curvature singularity at some finite value of r.

We conclude this section with the brief comment on the results of paper [7] where the problem of the existence of wormhole solutions supported by the phantom Chaplygin gas was studied. The main part of this paper is devoted to the consideration of the so called "anisotropic" Chaplygin gas, i.e. a fluid whose radial pressure satisfies the Chaplygin gas equation of state, while the tangential pressure can be arbitrary. Then the system of TOV equations is underdetermined and its solution contains one arbitrary function. Choosing this function in a convenient way, one can construct a lot of solutions, satisfying the desired properties. However, this fluid is not the Chaplygin gas and, moreover, is not a barotropic fluid. The case of the isotropic Chaplygin gas is also considered in Ref. [7]. The author studies numerically the behavior of metric coefficients in the vicinity of the throat without considering the problem of continuation of this solution to larger values of the radial coordinate. However, the general theorem proved at the end of Sec. IV of the present paper states that all spherically symmetric solutions supported by the phantom Chaplygin gas (with or without a throat) have a curvature singularity at some finite value of the radial coordinate and hence cannot be asymptotically flat.

VI. CONCLUSION

In this paper, we have studied the Tolman-Oppenheimer-Volkoff equations for static spherically symmetric objects immersed in the space filled with the Chaplygin gas. Both cases, phantom and nonphantom, were considered. In the nonphantom case all solutions (excluding the de Sitter one) represent a spheroidal geometry, where the radial coordinate achieves a maximal value (equator). After crossing the equator, depending on the boundary conditions, three types of solutions can arise: a closed spheroid having a Schwarzschild-type singularity with infinite blueshift at the ‘‘south pole’’, a regular spheroid, and a truncated spheroid having a scalar curvature singularity at a finite value of the radial coordinate.

For the case of the phantom Chaplygin gas, the equator is absent and all starlike external solutions have the geometry of a truncated spheroid having a scalar curvature singularity at some finite value of the radial coordinate. Besides, we have also considered the static spherically symmetric configurations existing in a universe filled exclusively with the phantom Chaplygin gas. Here two cases are possible: geometries which are regular at the center $r = 0$ and having a scalar curvature singularity at some finite value of r, and geometries containing throats connecting two patches of the world which again have scalar singularities at some finite values of the radius. Because of these singularities, the construction of stable, traversable, and asymptotically flat wormholes using the phantom Chaplygin gas is prohibited, in spite of breaking the weak energy condition in this case.

Finally, we note that many of the solutions of the TOV equations studied above possess singularities arising at some finite values of the radial coordinate, while intuitively it is more habitual to think about singularities arising at the point characterized by the vanishing of this coordinate. Something similar happens in the study of isotropic and homogeneous cosmological models, too. Here, in addition to the traditional big bang and big crunch singularities, an intensive study of the singularities which take place at finite or at infinite values of the cosmological scale factor is under way (see e.g. [13]).

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