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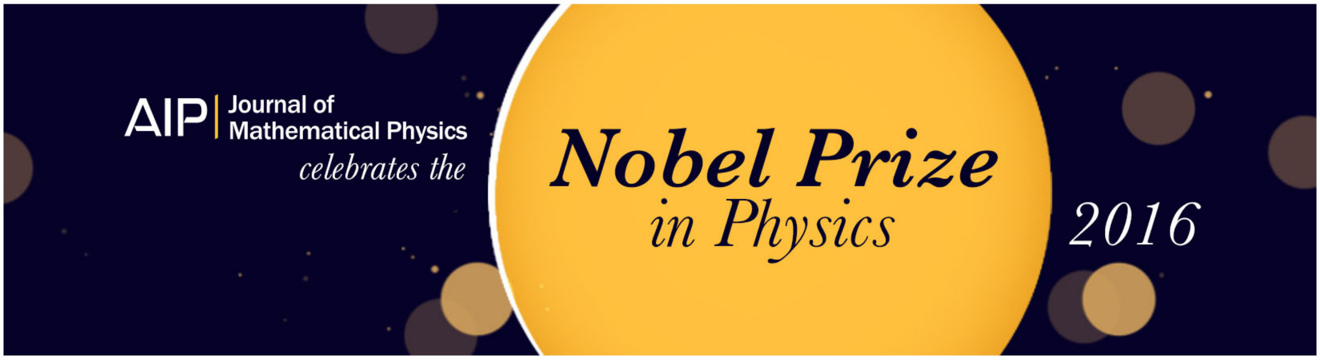

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Point interactions in acoustics: One-dimensional models

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A one-dimensional system made up of a compressible fluid and several mechanical oscillators, coupled to the acoustic field in the fluid, is analyzed for different settings of the oscillator array. The dynamical models are formulated in terms of singular perturbations of the decoupled dynamics of the acoustic field and the mechanical oscillators. Detailed spectral properties of the generators of the dynamics are given for each model we consider. In the case of a periodic array of mechanical oscillators it is shown that the energy spectrum presents a band structure. © *2006 American Institute of Physics.* [DOI: 10.1063/1.2209553]

INTRODUCTION

In this paper we study the dynamics of a system consisting of one or more mechanical oscillators (the sources) coupled with the acoustic field they produce in the compressible fluid surrounding them.

Classical electromagnetism is perhaps the most well-known case in theoretical physics where all attempts to construct a complete, covariant, causal, divergence-free theory for the evolution of the fields together with their sources were unsuccessful up to now (in fact, it is hard to say that there is a single case in classical or in quantum physics in which this problem was completely solved).

Whereas theories with extended rigid charges are quite well understood both at the classical and the quantum level [see, e.g., a recent book (Spohn, 2004) for a systematic introduction to the subject and for a long list of references], there is no mathematically consistent theory of point charges interacting with their own electromagnetic field. Indeed, Newton equations with a Lorentz force require the fields to be evaluated at the particle positions, and this produces infinities due to the presence of the pointlike sources. These difficulties directly lead to the need of mass renormalization. In his seminal paper, Dirac (1938) [see also Infeld and Wallace (1940), Kijowski (1994), and Marino (2002)], without using a Lorentz force but exploiting the conservation of energy and momentum and considering their flow through a thin tube of radius r, derived an equation for the motion of a charged point particle (the Lorentz-Dirac equation). As Dirac himself pointed out, the equation obtained in the limit $r\downarrow 0$, together with the mass renormalization, leads to the presence of runaway solutions, i.e., solutions for which the acceleration increases beyond any bound even in the absence of external fields.

An approach based on the theory of singular perturbations of the free dynamics was initiated in Noja and Posilicano (1998, 1999) for the case of classical electrodynamics of a point particle in the dipole (or linearized) case. Here the generator of the limit dynamics of both the field and the particle appears to be a singular perturbation of the generator of the free dynamics. The phenom-

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enological mass plays the role of the parameter describing a suitable family of self-adjoint extensions and the boundary condition naturally appearing in the domain of the generator results to be nothing else that a regularized (and linearized) version of the usual velocity-momentum relation in the presence of an electromagnetic field. In this framework runaway solutions are unavoidable because a negative eigenvalue appears in the spectrum of the generator after mass renormalization.

Our interest in a similar problem in acoustics was prompted by the appearance in 1999 of a paper by Templin (1999). In that paper the author analyzed the dynamics of a simple model of a spherical oscillator interacting with the acoustic field it generates. The existence of a spherically symmetric radiation field (the *acoustic monopole*) makes the acoustic case significantly different from the electromagnetic one. Moreover, the pressure field at the surface of the sphere completely characterizes the contact forces responsible for the interaction between the source and field in the acoustic case.

Templin performed a detailed analysis of the field emitted by the acoustic monopole, explicitly computing both its radiation and near-field components. He noticed that a deduction of the *reaction field* obtained from the emitted radiation power, therefore neglecting the near field component, brings to an equation for the radius of the oscillating sphere showing *runaway solutions*.

In analogy with what was done for the electromagnetic case in Noja and Posilicano (1998), we want to provide a formalization of the problem of a finite or infinite number of oscillators coupled with their acoustic field in terms of singular perturbations of the generator of the free dynamics.

In this paper we will consider only the one-dimensional case. In the abstract setting we will work in the physical model of interaction between the sources and the field will appear as the only possible extension of the free dynamics. The generalization to three dimensions is not straightforward. On the one hand, a model of a physically relevant, symmetric, mechanical oscillator with finite degrees of freedom is lacking. On the other hand, point perturbations of the free dynamics are much more singular in higher dimensions. We plan to discuss the three-dimensional case in future work.

We want to stress an aspect of the dynamical system we analyze here that was extensively studied in different contexts. As an immediate consequence of the third Newton's law and of the assumption of persistent contact between the fluid and the surface of the oscillators, the total energy, sum of the (positive) energy of the acoustic field E_{ac} and the (positive) energy of the oscillators E_{osc}, is a constant of motion. As an immediate consequence, one can exclude the existence of runaway solutions in this case. Moreover, lacking a mechanism of reflection of the acoustic waves at some exterior boundary, the motion of the oscillators should be damped and the energy should finally diffuse over the field degrees of freedom for almost every initial condition. The situation is reminiscent of the one investigated in Soffer and Weinstein (1998a, 1998b, 1999) that concerned the diffusion of energy from bound states to continuous states triggered by time-dependent perturbations in quantum and classical systems. In our system there is no external potential, the interaction being given by internal forces.

This paper is organized as follows. In Sec. I we introduce a list of notations and we briefly recall the equations for the acoustic field. Afterwards we exemplify the problem of the interaction between the field and a source in the completely solvable case of a single wall attracted toward the origin by a linear restoring force.

In Sec. II we analyze the case of a finite number of sources in the framework of the possible extensions of the free dynamics outside the points where the sources are placed. In Sec. III we generalize the construction to the case of infinitely many sources and study the case of sources periodically placed on the real line. We give detailed results on the characteristic band structure of the spectrum of the generator of the dynamics.

To the best of our knowledge, these kinds of systems of oscillators coupled with the acoustic field was never proposed and solved. A remark on the band structure of a similar model is in Griffiths and Steinke (2001).

I. THE ACOUSTIC MONOPOLE IN ONE DIMENSION

We give a detailed description of our model in the simplest case of one oscillator coupled with the acoustic field.

Consider an infinite pipe filled with a nonviscous, compressible fluid. We suppose that there is no friction between the fluid and the pipe, and we choose a coordinate system with the x axis parallel to the axis of the pipe. The mechanical oscillator is made up of a very thin wall of mass M positioned in the pipe perpendicularly to the axis in $x=0$. The thin wall is connected to a spring of elastic constant K. We analyze only one-dimensional cases, hence the acoustic field is described by the pressure field $p(x,t)$ and the velocity field $v(x,t)$. The motion of the mechanical oscillator is described through the position and the velocity of the thin wall.

The field $p(x,t)$ represents deviations of the pressure in point x at time t with respect to an equilibrium pressure P_0. In the linearized acoustics regime the continuity equation, the Newton's second law, and the adiabatic equation of state read

$$\frac{\partial \rho}{\partial t} + \rho_0 \frac{\partial v}{\partial x} = 0, \quad \rho_0 \frac{\partial v}{\partial t} = -\frac{\partial p}{\partial x}, \quad p = a^2 \rho, \tag{1.1}$$

where $\rho(x,t)$ is the deviation of the density in point x at time t with respect to the equilibrium density ρ_0 and a is the velocity of sound in the fluid.

Then we have for $p(x,t)$ and $v(x,t)$ the following coupled differential equations:

$$\frac{\partial p}{\partial t} = -a^2 \rho_0 \frac{\partial v}{\partial x}, \quad \frac{\partial v}{\partial t} = -\frac{1}{\rho_0}\frac{\partial p}{\partial x}. \tag{1.2}$$

We consider only small oscillations of the thin wall around its equilibrium position $x=0$, we indicate with $y(t)$ the displacement of the wall from its equilibrium position at time t, and we suppose that the wall always remains in contact with the fluid:

$$v(y(t),t) = \frac{dy(t)}{dt} \quad \forall\, t \geqslant 0. \tag{1.3}$$

Notice that we consider a wall of zero thickness. We make the approximation $v(y(t),t) \simeq v(0,t)$ and condition (1.3) becomes

$$v(0,t) = \frac{dy(t)}{dt} \quad \forall\, t \geqslant 0. \tag{1.4}$$

The equation of motion for the position of the thin wall $y(t)$ is

$$M\ddot{y}(t) = -Ky(t) - S(p(0^+,t) - p(0^-,t)), \tag{1.5}$$

where S is the area of the transverse section of the pipe and we made the approximation $p(y^{\pm}(t),t) \simeq p(0^{\pm},t)$.

The total energy of the system is given by

$$E_{\text{tot}} = E_{\text{ac}} + E_{\text{osc}}, \tag{1.6}$$

with

$$E_{\text{ac}} = \frac{S}{2a^2\rho_0}\int_{-\infty}^{\infty} p(x)^2 dx + \frac{S\rho_0}{2}\int_{-\infty}^{\infty} v(x)^2 dx, \tag{1.7}$$

$$E_{\text{osc}} = \frac{K}{2}y^2 + \frac{M}{2}\dot{y}^2. \tag{1.8}$$

E_{ac} is the energy stored in the acoustic field while E_{osc} is the energy of the mechanical oscillator.

As the system is isolated, the energy is constant. The motion of the wall produces acoustic waves, thus transferring continuously energy from the oscillator to the acoustic field. One then expects that $y(t)$ decreases to zero when $t \to \infty$.

In spite of being a simple exercise, the exact computation of the solution of problems (1.2), (1.4), and (1.5) and, in turn, of the damping rate of the oscillations rarely appears in textbooks.

In the following we give the solution of the Cauchy problem of coupled ordinary and partial differential equations with time-dependent boundary conditions

$$\begin{gathered}\frac{\partial p}{\partial t} = -a^2\rho_0\frac{\partial v}{\partial x} \quad \forall\, t \geqslant 0 \;\; \forall\, x \in \mathbb{R}\setminus\{0\},\\ \frac{\partial v}{\partial t} = -\frac{1}{\rho_0}\frac{\partial p}{\partial x} \quad \forall\, t \geqslant 0 \;\; \forall\, x \in \mathbb{R}\setminus\{0\},\\ \ddot{y}(t) = -\omega_0^2 y(t) - \frac{S}{M}(p(0^+,t) - p(0^-,t)) \quad \forall\, t \geqslant 0,\\ p(x,0) = f(x) \quad \forall\, x \in \mathbb{R}\setminus\{0\},\\ v(x,0) = g(x) \quad \forall\, x \in \mathbb{R}\setminus\{0\},\\ y(0) = y_0,\\ \dot{y}(0) = \dot{y}_0,\\ v(0,t) = \dot{y}(t) \quad \forall\, t \geqslant 0,\end{gathered} \tag{1.9}$$

where $f(x)$ and $g(x)$ are two real functions and $\omega_0^2 = K/M$.

Suppose that

$$f(x) \in C_0^2(\mathbb{R}); \quad g(x) \in C_0^2(\mathbb{R}) \quad \text{and} \quad y_0 = \frac{1}{\omega_0^2\rho_0}f'(0); \quad \dot{y}_0 = g(0), \tag{1.10}$$

then the solution of problem (1.9) reads

$$p(x,t) = p_f(x,t) + a\rho_0\,\mathrm{sgn}(x)\,Y\left(t - \frac{|x|}{a}\right), \tag{1.11}$$

$$v(x,t) = v_f(x,t) + Y\left(t - \frac{|x|}{a}\right), \tag{1.12}$$

$$y(t) = -\frac{\dot{v}_f(0,t)}{\omega_0^2} + \frac{\displaystyle\int_0^t \frac{F(t')}{\beta_+}e^{\beta_+(t-t')}dt' - \int_0^t \frac{F(t')}{\beta_-}e^{\beta_-(t-t')}dt'}{\beta_+ - \beta_-}, \tag{1.13}$$

where $p_f(x,t)$ and $v_f(x,t)$ are solution of the wave equation in $(-\infty, +\infty)$ with initial conditions $f(x)$ and $g(x)$:

$$p_f(x,t) = \frac{f(x-at) + f(x+at)}{2} + \frac{a\rho_0}{2}(g(x-at) - g(x+at)), \tag{1.14}$$

$$v_f(x,t)=\frac{g(x-at)+g(x+at)}{2}+\frac{1}{2a\rho_0}(f(x-at)-f(x+at)),$$
$$F(t)=-\ddot{v}_f(0,t)-\omega_0^2 v_f(0,t), \tag{1.15}$$

and

$$Y(t)=\frac{\int_0^t F(t')e^{\beta_+(t-t')}dt'-\int_0^t F(t')e^{\beta_-(t-t')}dt'}{\beta_+-\beta_-}, \tag{1.16}$$

with $\beta_\pm=(-\gamma\pm\sqrt{\gamma^2-4\omega_0^2})/2$ with $\gamma=2a\rho_0 S/M$.

With conditions (1.10) one can easily obtain that $y(t)$ and $\dot{y}(t)$ are both continuous and decrease exponentially to zero with a decay constant $\tau=\gamma/2$.

II. SINGULAR PERTURBATIONS OF THE FREE DYNAMICS

In this section we present a generalization of problem (1.9) formulated in terms of a unitary flow on a space of finite energy.

Let us consider a system of n thin walls positioned in a pipe, perpendicular to its axis. Let $\mathcal{S}=\{s_1,\ldots,s_n\}\subset\mathbb{R}$ be the set of equilibrium positions of the thin walls. The ith thin wall, placed in s_i has mass M_i and is connected to a spring of elastic constant K_i. The acoustic field is described by the pressure field p and the velocity field v. The motion of the walls is described by the displacements y_j from their equilibrium positions and by the corresponding velocities z_j.

The generator of the dynamics, $\hat{A}$, will be defined as a singular perturbation of the skew-adjoint operator A generating the uncoupled evolution of the acoustic field and of the oscillators.

The system of first-order differential equations,

$$\frac{\partial p}{\partial t}=-a^2\rho_0\frac{\partial v}{\partial x}\quad \forall\, x\in\mathbb{R}, \tag{2.1}$$

$$\frac{\partial v}{\partial t}=-\frac{1}{\rho_0}\frac{\partial p}{\partial x}\quad \forall\, x\in\mathbb{R}, \tag{2.2}$$

$$\frac{dy_j}{dt}=z_j\quad 1\leqslant j\leqslant n, \tag{2.3}$$

$$\frac{dz_j}{dt}=-\frac{K_j}{M_j}y_j\quad 1\leqslant j\leqslant n, \tag{2.4}$$

describes, in the linear approximation, the independent evolution of n mechanical oscillators and of the acoustic field.

We want to show first how Eqs. (2.1)–(2.4) define an unitary flow in a complex Hilbert space. To this aim let us consider Eqs. (2.1)–(2.4) for complex functions v, p, y_j, z_j of position and time.

The set of all the displacements and velocities will be represented respectively by the vectors in $\mathbb{C}^n$:

$$\underline{y}=y_1\underline{e}_1+\cdots+y_n\underline{e}_n,\quad \underline{z}=z_1\underline{e}_1+\cdots+z_n\underline{e}_n, \tag{2.5}$$

where $\underline{e}_1,\cdots,\underline{e}_n$ is the canonical orthonormal basis in $\mathbb{C}^n$.

Let us denote by $L^2(\mathbb{R})$ the space of square-integrable functions on the real line. $\bar{H}^1(\mathbb{R})$ indicates the homogeneous Sobolev space of locally square-integrable functions with a square-integrable (distributional) derivative, and $H^1(\mathbb{R})$ is the usual Sobolev space $H^1(\mathbb{R}) := \bar{H}^1(\mathbb{R}) \cap L^2(\mathbb{R})$.

Therefore the linear operator A in $L^2(\mathbb{R}) \oplus L^2(\mathbb{R}) \oplus \mathbb{C}^n \oplus \mathbb{C}^n$ generating the dynamics (2.1)–(2.4) is defined by

$$A: H^1(\mathbb{R}) \oplus H^1(\mathbb{R}) \oplus \mathbb{C}^n \oplus \mathbb{C}^n \to L^2(\mathbb{R}) \oplus L^2(\mathbb{R}) \oplus \mathbb{C}^n \oplus \mathbb{C}^n, \tag{2.6}$$

$$A(p,v,\underline{y},\underline{z}) := \left(-a^2\rho_0 \frac{dv}{dx}, -\frac{1}{\rho_0}\frac{dp}{dx}, \underline{z}, -\sum_{1\le j\le n} \frac{K_j}{M_j} y_j \underline{e}_j\right), \tag{2.7}$$

where a, ρ_0, K_j, M_j, $1 \le j \le n$, are the positive real constants representing the physical parameters.

In the following, a capital Greek letter will indicate a generic vector $(p,v,\underline{y},\underline{z})$ in $L^2(\mathbb{R}) \oplus L^2(\mathbb{R}) \oplus \mathbb{C}^n \oplus \mathbb{C}^n$.

A is a real operator, i.e., it preserves the (physical) linear subspace of real elements

$$\{(p,v,\underline{y},\underline{z}): p(x) \in \mathbb{R}, v(x) \in \mathbb{R}, \underline{y} \in \mathbb{R}^n, \underline{z} \in \mathbb{R}^n\}. \tag{2.8}$$

A is skew-symmetric with respect to the scalar product

$$\langle\langle \Psi_1, \Psi_2 \rangle\rangle \equiv \frac{1}{a^2\rho_0}\langle p_1, p_2\rangle + \rho_0\langle v_1, v_2\rangle + \frac{1}{S}\sum_{1\le j\le n} K_j \bar{y}_{1j} y_{2j} + M_j \bar{z}_{1j} z_{2j}, \tag{2.9}$$

where $\langle\cdot,\cdot\rangle$ indicates the standard scalar product in $L^2(\mathbb{R})$, $\mathcal{S}$ is the area of the transverse section of the pipe, and denotes complex conjugation. $L^2(\mathbb{R}) \oplus L^2(\mathbb{R}) \oplus \mathbb{C}^n \oplus \mathbb{C}^n$ is a Hilbert space with the scalar product (2.9).

The square norm of a vector Ψ, $\|\Psi\|^2 = \langle\langle \Psi, \Psi \rangle\rangle$, defines the total energy of the system in the state Ψ,

$$E_{\mathrm{tot}} = \frac{S}{2}\|\Psi\|^2 = E_{\mathrm{ac}} + E_{\mathrm{osc}}, \tag{2.10}$$

where E_{ac} is the energy stored in the acoustic field while E_{osc} is the energy of the oscillators:

$$E_{\mathrm{ac}} = \frac{\mathcal{S}}{2a^2\rho_0}\langle p,p\rangle + \frac{\rho_0\mathcal{S}}{2}\langle v,v\rangle; \quad E_{\mathrm{osc}} = \frac{1}{2}\sum_{1\le j\le n}(K_j|y_j|^2 + M_j|z_j|^2). \tag{2.11}$$

For any $\zeta \in \mathbb{C}\backslash i\mathbb{R}$ the resolvent of A is

$$\begin{aligned}(-A+\zeta)^{-1}(p,v,\underline{y},\underline{z}) = \Bigg(&\rho_0\left(-\frac{d^2}{dx^2} + \frac{\zeta^2}{a^2}\right)^{-1}\left(-\frac{dv}{dx} + \frac{\zeta}{a^2\rho_0}p\right),\\ &\frac{1}{a^2\rho_0}\left(-\frac{d^2}{dx^2} + \frac{\zeta^2}{a^2}\right)^{-1}\left(-\frac{dp}{dx} + \zeta\rho_0 v\right), \quad \sum_{1\le j\le n}\frac{M_j z_j + \zeta M_j y_j}{K_j + \zeta^2 M_j}\underline{e}_j,\\ &\sum_{1\le j\le n}\frac{-K_j y_j + \zeta M_j z_j}{K_j + \zeta^2 M_j}\underline{e}_j\Bigg).\end{aligned} \tag{2.12}$$

Since

$$\mathrm{Ran}(-A \pm 1) = L^2(\mathbb{R}) \oplus L^2(\mathbb{R}) \oplus \mathbb{C}^n \oplus \mathbb{C}^n, \tag{2.13}$$

A is skew-adjoint. Moreover, the essential spectrum of A is purely absolutely continuous and

$$\sigma_{\rm ess}(A) = \sigma_{\rm ac}(A) = i\mathbb{R}, \quad \sigma_{\rm pp}(A) = \left\{ \pm i\sqrt{\frac{K_j}{M_j}}, 1 \leqslant j \leqslant n \right\}. \tag{2.14}$$

Being skew-adjoint, the operator A describes, by the Stone theorem, the uncoupled evolution of the acoustic field and of the oscillators through the unitary flow $\exp tA$ corresponding to the Cauchy problem for the first-order differential equation

$$\frac{d}{dt}\Psi(t) = A\Psi(t), \tag{2.15}$$

which is equivalent to the system written at the beginning of the section.

Now we consider the linear operator A_0 obtained by restricting A on the set of vectors in its domain satisfying

$$\{v(s_j) = z_j, \ 1 \leqslant j \leqslant n\}, \tag{2.16}$$

which represents the kinematic constraint (1.4) at each thin wall. A_0 is a closed, densely defined, skew-symmetric linear operator with defect indices (n,n). We want to characterize the skew-adjoint extensions of A_0. The family of extensions of A_0 can be parameterized by relations $K \subset \mathbb{C}^n \oplus \mathbb{C}^n$, which are skew-symmetric, i.e., such that $K = (IK)^{\perp}$, where $I(\underline{z}_1, \underline{z}_2) := (\underline{z}_2, \underline{z}_1)$ [see, e.g., Gorbachuk and Gorbachuk (1991), Theorem 1.6, Chap. 3, for the analogous self-adjoint case]. A skew-symmetric relation in $\mathbb{C}^n \oplus \mathbb{C}^n$ extends the notion of the graph of a skew-symmetric operator $\Theta : \mathbb{C}^n \to \mathbb{C}^n$ through the relation $K = \{(\underline{z}, \Theta\underline{z}), \underline{z} \in \mathbb{C}^n\}$. In order to be a candidate to describe the interacting dynamics of the system under analysis, a skew-adjoint extension of A_0 must be local and real, i.e., it must generate a coupling between the fields evaluated in s_j and the jth oscillator, $1 \leqslant j \leqslant n$, and it must preserve the linear space of physical data defined in (2.8). The only admissible extension different from A itself will be the one corresponding to the graph of the zero operator, $\Theta = 0$. The next theorem completely characterizes such an extension.

Theorem 2.1: *The only local, real, and skew-adjoint extension of A_0 is given by*

$$\hat{A}: D(\hat{A}) \subset L^2(\mathbb{R}) \oplus L^2(\mathbb{R}) \oplus \mathbb{C}^n \oplus \mathbb{C}^n \to L^2(\mathbb{R}) \oplus L^2(\mathbb{R}) \oplus \mathbb{C}^n \oplus \mathbb{C}^n, \tag{2.17}$$

$$D(\hat{A}) = \{\Psi \equiv (p, v, \underline{y}, \underline{z}) : p \in L^2(\mathbb{R}) \cap H^1(\mathbb{R} \setminus \mathcal{S}), v \in H^1(\mathbb{R}), \underline{y} \in \mathbb{C}^n, \underline{z} \in \mathbb{C}^n, p(s_i^+) - p(s_i^-) = \sigma_i, v(s_j) = z_j, \underline{\sigma} \in \mathbb{C}^n\}, \tag{2.18}$$

$$\hat{A}(p, v, \underline{y}, \underline{z}) := \left(-a^2\rho_0 \frac{dv}{dx}, -\frac{1}{\rho_0}\frac{dp_0}{dx}, \underline{z}, -\sum_{1 \leqslant j \leqslant n} \left(\frac{K_j}{M_j} y_j + \frac{S}{M_j}\sigma_j \right) \underline{e}_j \right). \tag{2.19}$$

Here $p_0 \in \bar{H}^1(\mathbb{R})$,

$$p_0(x) := p(x) - \frac{1}{2}\sum_{1 \leqslant j \leqslant n} \sigma_j \operatorname{sgn}(x - s_j), \tag{2.20}$$

denotes the regular part of p. The resolvent of $\hat{A}$ is given by

$$(-\hat{A} + \zeta)^{-1} = (-A + \zeta)^{-1} + \sum_{1 \leqslant i,j \leqslant n} (\Gamma(\zeta)^{-1})_{ij} G_\zeta^i \otimes \breve{G}_{\bar{\zeta}}^j, \quad \zeta \in \mathbb{C} \setminus i\mathbb{R}, \tag{2.21}$$

where

$$\Gamma(\zeta)_{ij} := -\zeta\left(\pm \frac{e^{\mp\zeta|s_i - s_j|/a}}{2a\rho_0\zeta} + \frac{S\delta_{ij}}{K_j + \zeta^2 M_j} \right), \quad \pm \operatorname{Re}\zeta > 0 \tag{2.22}$$

and

$$\breve{G}^j_\zeta(x) = \left(\mathcal{G}'_\zeta(x-s_j), \frac{\zeta}{a^2\rho_0}\mathcal{G}_\zeta(x-s_j), \frac{S}{K_j+\zeta^2 M_j}\underline{e}_j, \frac{-\zeta S}{K_j+\zeta^2 M_j}\underline{e}_j \right), \tag{2.23}$$

$$G^j_\zeta(x) = \left(-\mathcal{G}'_\zeta(x-s_j), \frac{\zeta}{a^2\rho_0}\mathcal{G}_\zeta(x-s_j), \frac{-S}{K_j+\zeta^2 M_j}\underline{e}_j, \frac{-\zeta S}{K_j+\zeta^2 M_j}\underline{e}_j \right), \tag{2.24}$$

$$\mathcal{G}_\zeta(x) = \pm \frac{a}{2\zeta} e^{\mp\zeta|x|/a}, \quad \mathcal{G}'_\zeta(x) = -\frac{1}{2}\,\mathrm{sgn}(x) e^{\mp\zeta|x|/a}, \quad \pm \mathrm{Re}\,\zeta > 0. \tag{2.25}$$

Proof: Since iA_0 is a closed, densely defined, symmetric operator with defect indices (n,n), all its self-adjoint extensions can be obtained by the famed von Neumann theory on self-adjoint extensions [see, e.g., Theorem X.2 in Reed and Simon (1975)]. However, since A_0 is obtained by restricting the skew-adjoint operator A to the kernel of the continuous, surjective linear operator

$$\tau : H^1(\mathbb{R}) \oplus H^1(\mathbb{R}) \oplus \mathbb{C}^n \oplus \mathbb{C}^n \to \mathbb{C}^n, \tag{2.26}$$

$$\tau(p,v,\underline{y},\underline{z}) := \sum_{1\le j\le n} (v(s_j) - z_j)\underline{e}_j, \tag{2.27}$$

it is easier to make use of the (equivalent) procedure developed in Posilicano (2001) [also see the Appendix in Posilicano (in press) for a compact review]. Here below we provide the (almost) self-contained construction of the skew-adjoint extensions of A_0 by using such a procedure.

Given the map τ we can define the bounded linear operator

$$\breve{G}(\zeta) := \tau(-A+\zeta)^{-1} : L^2(\mathbb{R}) \oplus L^2(\mathbb{R}) \oplus \mathbb{C}^n \oplus \mathbb{C}^n \to \mathbb{C}^n. \tag{2.28}$$

By the relation

$$\breve{G}(\zeta)(p,v,\underline{y},\underline{z}) = \sum_{1\le j\le n} \langle\langle \breve{G}^j_{\bar\zeta}, (p,v,\underline{y},\underline{z}) \rangle\rangle \underline{e}_j, \quad 1 \le j \le n, \tag{2.29}$$

$\breve{G}(\zeta)$ is represented by the vector $\breve{G}^j_\zeta$. By $\breve{G}(\zeta)$ we define the bounded linear operator

$$G(\zeta) := -\breve{G}(-\bar\zeta)^* : \mathbb{C}^n \to L^2(\mathbb{R}) \oplus L^2(\mathbb{R}) \oplus \mathbb{C}^n \oplus \mathbb{C}^n, \tag{2.30}$$

where $\breve{G}(\zeta)^*$ indicates the adjoint of $\breve{G}(\zeta)$. The action of $G(\zeta)$ on $\mathbb{C}^n$ is given by

$$G(\zeta)\underline{e}_j = G^j_\zeta, \quad 1 \le j \le n. \tag{2.31}$$

Let us notice that

$$\mathrm{Ran}(G(\zeta)) \cap H^1(\mathbb{R}) \oplus H^1(\mathbb{R}) \oplus \mathbb{C}^n \oplus \mathbb{C}^n = \{0\}. \tag{2.32}$$

Now we consider the linear operator $\Gamma_\Theta(\zeta) : \mathbb{C}^n \to \mathbb{C}^n$ represented by the matrix $\Theta_{ij} + \Gamma(\zeta)_{ij}$, where $\Theta : \mathbb{C}^n \to \mathbb{C}^n$ is skew-Hermitian. By noticing that

$$\Gamma_\Theta(\zeta) - \Gamma_\Theta(\xi) = \tau(G(\xi) - G(\zeta)) \tag{2.33}$$

and that, by the definition of $G(\zeta)$ and by the first resolvent identity,

$$(\zeta - \xi)(-A+\xi)^{-1}G(\zeta) = G(\xi) - G(\zeta), \tag{2.34}$$

one has that $\Gamma_\Theta(\zeta)$ satisfies the identity

$$\Gamma_\Theta(\zeta)-\Gamma_\Theta(\xi)=(\zeta-\xi)\breve{G}(\xi)G(\zeta). \tag{2.35}$$

By the definitions of $\breve{G}(\zeta)$ and $G(\zeta)$, by (2.35) and by $\Gamma_\Theta(\bar{\zeta})^*=-\Gamma_\Theta(-\zeta)$, it follows that $\det \Gamma_\Theta(\zeta)\neq 0$ for any $\zeta\in\mathbb{C}\backslash i\mathbb{R}$ and that

$$\hat{R}(\zeta):=(-A+\zeta)^{-1}+G(\zeta)\Gamma_\Theta(\zeta)^{-1}\breve{G}(\zeta) \tag{2.36}$$

satisfies the first resolvent identity

$$(\zeta-\xi)\hat{R}(\xi)\hat{R}(\zeta)=\hat{R}(\xi)-\hat{R}(\zeta) \tag{2.37}$$

and

$$\hat{R}(\bar{\zeta})^*=-\hat{R}(-\zeta) \tag{2.38}$$

[for details see Posilicano (2001)]. Moreover, $\hat{R}(\zeta)$ is injective by (2.32). Therefore

$$\hat{A}:=-\hat{R}(\zeta)^{-1}+\zeta \tag{2.39}$$

is well defined on

$$D(\hat{A}):=\mathrm{Ran}(\hat{R}(\zeta)). \tag{2.40}$$

By (2.37) such a definition of $\hat{A}$ is ζ independent. $\hat{A}$ is skew-symmetric by (2.38) and is skew-adjoint since

$$\mathrm{Ran}(-\hat{A}\pm 1)=L^2(\mathbb{R})\oplus L^2(\mathbb{R})\oplus\mathbb{C}^n\oplus\mathbb{C}^n \tag{2.41}$$

by construction.

Since we require $\hat{A}$ to be real, i.e., to preserve the linear space (2.8), we have to restrict the choice of Θ to real, skew-symmetric matrices. Off-diagonal elements in the matrix Θ would correspond to nonlocal couplings between the pressure field and the oscillators. Since we are looking for local interactions the only admissible choice for the skew-symmetric matrix Θ is $\Theta=0$.

By (2.40) $(p,v,\underline{y},\underline{z})\in D(\hat{A})$ if and only if

$$p(x)=p_\zeta(x)-\sum_{1\leqslant i,j\leqslant n}(\Gamma(\zeta)^{-1})_{ij}(v_\zeta(s_j)-z_{\zeta j})\mathcal{G}'_\zeta(x-s_i), \tag{2.42}$$

$$v(x)=v_\zeta(x)+\frac{\zeta}{a^2\rho_0}\sum_{1\leqslant i,j\leqslant n}(\Gamma(\zeta)^{-1})_{ij}(v_\zeta(s_j)-z_{\zeta j})\mathcal{G}'_\zeta(x-s_i), \tag{2.43}$$

$$\underline{y}=\underline{y}_\zeta-S\sum_{1\leqslant i,j\leqslant n}(\Gamma(\zeta)^{-1})_{ij}\frac{v_\zeta(s_j)-z_{\zeta j}}{K_i+\zeta^2 M_i}\underline{e}_i, \tag{2.44}$$

$$\underline{z}=\underline{z}_\zeta-\zeta S\sum_{1\leqslant i,j\leqslant n}(\Gamma(\zeta)^{-1})_{ij}\frac{v_\zeta(s_j)-z_{\zeta j}}{K_i+\zeta^2 M_i}\underline{e}_i, \tag{2.45}$$

with $(p_\zeta(x),v_\zeta(x),\underline{y}_\zeta,\underline{z}_\zeta)\in D(A)$. Posing

$$\hat{A}(p,v,\underline{y},\underline{z})\equiv(\hat{A}_1(p,v,\underline{y},\underline{z}),\hat{A}_2(p,v,\underline{y},\underline{z}),\hat{A}_3(p,v,\underline{y},\underline{z}),\hat{A}_4(p,v,\underline{y},\underline{z})), \tag{2.46}$$

the action of $\hat{A}$ on $(p,v,\underline{y},\underline{z})$ is given by

$$[\hat{A}_1(p,v,\underline{y},\underline{z})](x) = -a^2\rho_0 \frac{dv_\zeta}{dx}(x) - \zeta \sum_{1\leq i,j\leq n} (\Gamma(\zeta)^{-1})_{ij}(v_\zeta(s_j) - z_{\zeta j})\mathcal{G}'_\zeta(x-s_i), \tag{2.47}$$

$$[\hat{A}_2(p,v,\underline{y},\underline{z})](x) = -\frac{1}{\rho_0}\frac{dp_\zeta}{dx}(x) + \frac{\zeta^2}{a^2\rho_0}\sum_{1\leq i,j\leq n} (\Gamma(\zeta)^{-1})_{ij}(v_\zeta(s_j) - z_{\zeta j})\mathcal{G}'_\zeta(x-s_i), \tag{2.48}$$

$$\hat{A}_3(p,v,\underline{y},\underline{z}) = \underline{z}_\zeta - \zeta S \sum_{1\leq i,j\leq n} (\Gamma(\zeta)^{-1})_{ij}\frac{v_\zeta(s_j) - z_{\zeta j}}{K_i + \zeta^2 M_i}\underline{e}_i, \tag{2.49}$$

$$\hat{A}_4(p,v,\underline{y},\underline{z}) = -\sum_{1\leq j\leq n}\frac{K_j}{M_j}y_{\zeta j}\underline{e}_j - \zeta^2 S \sum_{1\leq i,j\leq n} (\Gamma(\zeta)^{-1})_{ij}\frac{v_\zeta(s_j) - z_{\zeta j}}{K_i + \zeta^2 M_i}\underline{e}_i. \tag{2.50}$$

By the definitions of $D(\hat{A})$ and $\Gamma(\zeta)$ one has

$$\hat{A}_1(p,v,\underline{y},\underline{z}) = -a^2\rho_0\frac{dv}{dx}, \tag{2.51}$$

$$\hat{A}_3(p,v,\underline{y},\underline{z}) = \underline{z}, \tag{2.52}$$

and, defining

$$\sigma_i \coloneqq p(s_i^+) - p(s_i^-) = \sum_{1\leq j\leq n} (\Gamma(\zeta)^{-1})_{ij}(v_\zeta(s_j) - z_{\zeta j}), \tag{2.53}$$

formula (2.50) becomes

$$\hat{A}_4(p,v,\underline{y},\underline{z}) = -\sum_{1\leq j\leq n}\frac{K_j}{M_j}y_{\zeta j}\underline{e}_j - \zeta^2 S\sum_{1\leq i\leq n}\frac{\sigma_i}{K_i + \zeta^2 M_i}\underline{e}_i \tag{2.54}$$

$$= -\sum_{1\leq j\leq n}\left(\frac{K_j}{M_j}y_j + \frac{S}{M_j}\sigma_j\right)\underline{e}_j. \tag{2.55}$$

Then, posing

$$p(x) = p_\zeta(x) - \sum_{1\leq j\leq n}\sigma_j\mathcal{G}'_\zeta(x-s_j) = p_0(x) + \frac{1}{2}\sum_{1\leq j\leq n}\sigma_j \operatorname{sgn}(x-s_j), \tag{2.56}$$

one obtains

$$[\hat{A}_2(p,v,\underline{y},\underline{z})](x) = -\frac{1}{\rho_0}\frac{dp_0}{dx}(x) - \sum_{1\leq j\leq n}\frac{\sigma_j}{\rho_0}\left(\frac{d}{dx}\frac{|x-s_j|}{2(x-s_j)} - \left(-\frac{d^2}{dx^2} + \frac{\zeta^2}{a^2}\right)\mathcal{G}_\zeta(x-y_j)\right) \tag{2.57}$$

$$= -\frac{1}{\rho_0}\frac{dp_0}{dx}. \tag{2.58}$$

Finally

$$v(s_k) = v_\zeta(s_k) + \frac{\zeta}{a^2\rho_0}\sum_{1\leqslant i,j\leqslant n}(\Gamma(\zeta)^{-1})_{ij}(v_\zeta(s_j) - z_{\zeta j})\frac{a}{\pm 2\zeta}e^{\mp\zeta|s_k - s_i|/a} \tag{2.59}$$

$$= v_\zeta(s_k) - \sum_{1\leqslant i,j\leqslant n}(\Gamma(\zeta)^{-1})_{ij}(v_\zeta(s_j) - z_{\zeta j})\left((\Gamma(\zeta))_{ki} + \frac{\zeta S\delta_{ki}}{K_i + \zeta^2 M_i}\right) \tag{2.60}$$

$$= z_{\zeta k} - \zeta S\sum_{1\leqslant j\leqslant n}(\Gamma(\zeta)^{-1})_{kj}\frac{v_\zeta(s_k) - z_{\zeta k}}{K_k + \zeta^2 M_k} = z_k. \tag{2.61}$$

□

By the previous theorem the differential equation

$$\frac{d}{dt}\Psi(t) = \hat{A}\Psi(t) \tag{2.62}$$

is equivalent to the system of equations

$$\frac{\partial p}{\partial t} = -a^2\rho_0\frac{\partial v}{\partial x}, \tag{2.63}$$

$$\frac{\partial v}{\partial t} = -\frac{1}{\rho_0}\frac{\partial p_0}{\partial x} \equiv -\frac{1}{\rho_0}\left(\frac{\partial p}{\partial x} - \sum_{1\leqslant j\leqslant n}\sigma_j\delta_{s_j}\right), \tag{2.64}$$

$$\frac{d\underline{y}}{dt} = \underline{z}, \tag{2.65}$$

$$\frac{d\underline{z}}{dt} = -\sum_{1\leqslant j\leqslant n}\left(\frac{K_j}{M_j}y_j + \frac{S}{M_j}\sigma_j\right)\underline{e}_j, \tag{2.66}$$

and the corresponding Cauchy problem generates the strongly continuous unitary group of evolution $\exp t\hat{A}$ on $L^2(\mathbb{R}) \oplus L^2(\mathbb{R}) \oplus \mathbb{C}^n \oplus \mathbb{C}^n$, which preserves $D(\hat{A})$. Here δ_{s_j} denotes the Dirac mass at the point s_j and σ_j [see (2.18)] is the pressure jump at s_j.

It is worth noting that the only real, skew-adjoint extension of the free operator A restricted to the space of the vectors $(p, v, \underline{y}, \underline{z})$ such that $v(s_i,t) = z_i$ corresponds to the relevant physical coupling between the pressure field and the oscillators.

The next result will be useful in the spectral analysis of $\hat{A}$.

Lemma 2.2: The matrix

$$\Gamma_\pm(\lambda)^{-1} := \lim_{\varepsilon\downarrow 0}\Gamma(\lambda \pm \varepsilon)^{-1} \tag{2.67}$$

is well defined for any $\lambda \in i\mathbb{R}\backslash\{0\}$.

Proof: We give the proof only for the matrix $\Gamma_+(\lambda)$. The proof for $\Gamma_-(\lambda)$ is analogous.

Let the matrix $\Gamma_+(\zeta)$ be the analytic continuation to $\mathbb{C}\backslash\cup_{j=1}^n\{\pm i\sqrt{K_j/M_j}\}$ of $\Gamma(\zeta)$ defined for $\mathrm{Re}\,\zeta > 0$ in (2.22). Suppose that $s_i > s_j$ if $i > j$, then

$$\Gamma_+(\zeta) = -\Pi(\zeta) - T(\zeta), \tag{2.68}$$

where Π is the operator

$$\Pi = (\underline{\phi}^{-}(\zeta) \otimes \underline{\phi}^{+}(\bar{\zeta})) \tag{2.69}$$

with $\underline{\phi}^{\pm}(\zeta)=\Sigma_i e^{\pm\zeta s_i/a}/\sqrt{2a\rho_0}\underline{e}_i$. While $T(\zeta)$ is the upper triangular matrix

$$T(\zeta)_{ij} = \begin{cases} \dfrac{\zeta S\delta_{ij}}{K_i+\zeta^2 M_i} + \dfrac{\sinh \zeta(s_i - s_j)}{a\rho_0} & i \leqslant j \\ 0 & i > j, \end{cases} \tag{2.70}$$

we use the formula

$$\Gamma_+(\zeta)^{-1} = -\frac{1}{\Pi(\zeta)+T(\zeta)} = -\frac{1}{T(\zeta)} + \frac{1}{T(\zeta)}\Pi(\zeta)\frac{1}{\Pi(\zeta)+T(\zeta)} \tag{2.71}$$

$$= -\sum_{n=0}^{\infty}\frac{(-1)^n}{T(\zeta)}\left(\Pi(\zeta)\frac{1}{T(\zeta)}\right)^n, \tag{2.72}$$

valid for all ζ for which the series converges.

Matrix $T(\zeta)$ is invertible and its inverse $T(\zeta)^{-1}$ is a lower triangular matrix with $(T(\zeta)^{-1})_{ii} = 1/(T(\zeta))_{ii}$. The eigenvalues of $T(\zeta)^{-1}$ are $1/(T(\zeta))_{ii}$ and we can write

$$T(\zeta)^{-1} = D(\zeta)\tilde{T}(\zeta)^{-1}D(\zeta)^{-1}, \tag{2.73}$$

where $D(\zeta)$ is a unitary matrix, analytic for $\zeta \in \mathbb{C}\backslash\{0\}$, and

$$(\tilde{T}(\zeta)^{-1})_{ij} = \frac{1}{(T(\zeta))_{ii}}\delta_{ij} = \frac{K_i+\zeta^2 M_i}{\zeta S}\delta_{ij}. \tag{2.74}$$

We obtain for $\Gamma_+(\zeta)^{-1}$ the expression

$$\Gamma_+(\zeta)^{-1} = -D(\zeta)\sum_{n=0}^{\infty}(-1)^n(\underline{\psi}(\zeta)\otimes\underline{\chi}(\zeta))^n\tilde{T}(\zeta)^{-1}D(\zeta)^{-1}, \tag{2.75}$$

with

$$(\underline{\psi}(\zeta))_i = \frac{K_i+\zeta^2 M_i}{\zeta S}(D(\zeta)^{-1}\underline{\phi}^{-}(\zeta))_i, \tag{2.76}$$

$$(\underline{\chi}(\zeta))_i = (D(\zeta)^{-1}\underline{\phi}^{+}(\bar{\zeta}))_i. \tag{2.77}$$

Then

$$\Gamma_+(\zeta)^{-1} = -\frac{1}{T(\zeta)} + \sum_{n=0}^{\infty}(-1)^n(\langle\underline{\chi}(\zeta),\underline{\psi}(\zeta)\rangle_{\mathbb{C}^n})^n D(\zeta)\underline{\psi}(\zeta)\otimes\underline{\chi}(\zeta)\tilde{T}(\zeta)^{-1}D(\zeta)^{-1}. \tag{2.78}$$

For all ζ for which the series converges, one has

$$\Gamma_+(\zeta)^{-1} = -\frac{1}{T(\zeta)} + \frac{D(\zeta)\underline{\psi}(\zeta)\otimes\underline{\chi}(\zeta)\tilde{T}(\zeta)^{-1}D(\zeta)^{-1}}{1+\langle\underline{\chi}(\zeta),\underline{\psi}(\zeta)\rangle_{\mathbb{C}^n}}. \tag{2.79}$$

Consider the scalar product in $\mathbb{C}^n$:

$$\langle \underline{\chi}(\zeta), \underline{\psi}(\zeta)\rangle_{\mathbb{C}^n} = \sum_{i=1}^{n} \overline{(D(\zeta)^{-1}\underline{\phi}^{+}(\bar{\zeta}))_i} \frac{K_i + \zeta^2 M_i}{\zeta S}(D(\zeta)^{-1}\underline{\phi}^{-}(\zeta))_i. \tag{2.80}$$

Notice that, for $\lambda \in i\mathbb{R}\setminus\{0\}$, $\langle \underline{\chi}(\lambda), \underline{\psi}(\lambda)\rangle_{\mathbb{C}^n} \in i\mathbb{R}$ and

$$-i\langle \underline{\chi}(\lambda), \underline{\psi}(\lambda)\rangle_{\mathbb{C}^n} \to +\infty \quad \text{for } \lambda \to +i\infty, \tag{2.81}$$

$$-i\langle \underline{\chi}(\lambda), \underline{\psi}(\lambda)\rangle_{\mathbb{C}^n} \to -\infty \quad \text{for } \lambda \to i0^{+}. \tag{2.82}$$

Then there exists at least one point $\lambda \in i\mathbb{R}$ in which $\langle \underline{\chi}(\lambda), \underline{\psi}(\lambda)\rangle_{\mathbb{C}^n}=0$. In a neighborhood of this point the series converges and defines an analytic function. By (2.79) and (2.80) it is clear that $\Gamma_+(\zeta)^{-1}$ exists for any $\zeta \in \mathbb{C}\setminus\{0\}$. The same relations show that one can put $\Gamma_+(\zeta)^{-1} := 0$ if $\zeta = i\sqrt{K_j/M_j}$, $j=1,\ldots,n$. □

The following theorem completely characterizes the spectrum of $\hat{A}$.

Theorem 2.3: *The essential spectrum of $\hat{A}$ is purely absolutely continuous and*

$$\sigma_{\text{ess}}(\hat{A}) = \sigma_{\text{ac}}(\hat{A}) = i\mathbb{R}, \quad \sigma_{pp}(\hat{A}) = \{0\}. \tag{2.83}$$

Any vector of the kind

$$\left(\frac{1}{2}\sum_{1\le j\le n} \sigma_j \operatorname{sgn}(x - s_j), 0, -\sum_{1\le j\le n} \frac{S}{K_j}\sigma_j \underline{e}_j, \underline{0}\right), \tag{2.84}$$

with

$$\sum_{1\le j\le n} \sigma_j = 0, \tag{2.85}$$

is an eigenvector corresponding to the $(n-1)$-fold degenerate eigenvalue $\lambda=0$.

The generalized eigenfunctions $\hat{\Phi}^{\pm}(\lambda)$ corresponding to the point of the absolutely continuous spectrum relative to right $(+)$ and left $(-)$ incidence are given by

$$\hat{\Phi}^{\pm}(\lambda,x) = (\hat{\phi}_p^{\pm}(\lambda,x), \hat{\phi}_v^{\pm}(\lambda,x), \hat{\phi}_{\underline{y}}^{\pm}(\lambda), \hat{\phi}_{\underline{z}}^{\pm}(\lambda)), \quad \lambda \in i\mathbb{R}, \tag{2.86}$$

$$\hat{\phi}_p^{\pm}(\lambda,x) = Ce^{\pm\lambda x/a} \mp \frac{C}{2a\rho_0}\sum_{1\le i,j\le n} (\Gamma_+(\lambda)^{-1})_{ij} e^{\pm\lambda s_j/a} \operatorname{sgn}(x - s_i) e^{-\lambda|x-s_i|/a}, \tag{2.87}$$

$$\hat{\phi}_v^{\pm}(\lambda,x) = \mp C\frac{e^{\pm\lambda x/a}}{a\rho_0} \mp \frac{C}{2a^2\rho_0^2}\sum_{1\le i,j\le n} (\Gamma_+(\lambda)^{-1})_{ij} e^{\pm\lambda s_j/a} e^{-\lambda|x-s_i|/a}, \tag{2.88}$$

$$\hat{\phi}_{\underline{y}}^{\pm}(\lambda) = \pm\frac{SC}{a\rho_0}\sum_{1\le i,j\le n} (\Gamma_+(\lambda)^{-1})_{ij}\frac{e^{\pm\lambda s_j/a}}{K_i + \lambda^2 M_i}\underline{e}_i, \tag{2.89}$$

$$\hat{\phi}_{\underline{z}}^{\pm}(\lambda) = \pm\frac{\lambda SC}{a\rho_0}\sum_{1\le i,j\le n} (\Gamma_+(\lambda)^{-1})_{ij}\frac{e^{\pm\lambda s_j/a}}{K_i + \lambda^2 M_i}\underline{e}_i, \tag{2.90}$$

with $C=\sqrt{a\rho_0/(4\pi)}$.

Proof: For $\zeta \in \rho(A) \cap \rho(\hat{A})$, $(-\hat{A}+\zeta)^{-1} - (-A+\zeta)^{-1}$ is of finite rank; then from Weyl's criterion [see, e.g., Reed and Simon (1978) Theorem XIII.14] one has $\sigma_{\text{ess}}(\hat{A}) = \sigma_{\text{ess}}(A) = i\mathbb{R}$. Moreover, by the Birman-Kato invariance principle, the wave operators $\Omega_{\pm}(\hat{A},A)$ exist and are complete [see, e.g., Reed and Simon (1979), Corollary 2 to Theorem XI.11]. Thus $\sigma_{ac}(\hat{A})=\sigma_{ac}(A)$.

Let $\hat{\mu}_{\Psi}^{sc}$ be the singular continuous part of the spectral measure on $i\mathbb{R}$ corresponding to $\hat{A}$ and Ψ. Since $\|\breve{G}(\zeta)\Psi\|<\infty$ for all $\zeta\in\mathbb{C}\backslash\sigma_{pp}(A)$ and for all $\Psi\in D$,

$$D := \{\Psi \equiv (p,v,\underline{y},\underline{z}) : p \in L^1(\mathbb{R}) \cap L^2(\mathbb{R}),\ v \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})\}, \tag{2.91}$$

by Lemma 2 and Reed and Simon (1978), Theorem XIII.19, one has supp $\hat{\mu}_{\Psi}^{sc}\subseteq\{0\}\cup\sigma_{pp}(A)$, i.e., supp $\hat{\mu}_{\Psi}^{sc}=\emptyset$, since $\hat{\mu}_{\Psi}^{sc}$ has no atoms by its definition. Since D is dense this gives $\sigma_{sc}(\hat{A})=\emptyset$.

One can check that any vector Ψ of the kind (2.84) is in the domain of $\hat{A}$ and solves the equation $\hat{A}\Psi=0$. The degeneration of eigenvalue $\{0\}$ follows from condition (101).

Suppose now $\lambda\in i\mathbb{R}\backslash\{0\}$ and consider the equation $\hat{A}\Psi=\lambda\Psi$. This produces, if $\Psi\equiv(p,v,\underline{y},\underline{z})$, the equation

$$v'' - \frac{\lambda^2}{a^2}v = -\frac{\lambda}{a^2\rho_0}\sum_{1\leqslant j\leqslant n}\sigma_j\delta_{s_j}, \tag{2.92}$$

with $\sigma_i\in\mathbb{C}$, $i=1,\ldots,n$, which has no square integrable solution.

The expression for the generalized eigenfunctions is a consequence of the Stone's formula [see, e.g., Reed and Simon (1972), Theorem VII.13], which gives the generalized expansion formula

$$\Psi = \mathrm{s}-\lim_{a\downarrow-\infty,b\uparrow\infty}\ \mathrm{s}-\lim_{\varepsilon\downarrow 0}\frac{1}{2\pi}\int_a^b[\hat{R}(\lambda+\varepsilon)-\hat{R}(\lambda-\varepsilon)]\Psi d\lambda. \tag{2.93}$$

□

In the following lemma the asymptotic behavior of the oscillations of the thin walls is characterized. It is proved that the oscillators relax (as $|t|\to\infty$) toward their equilibrium positions for any initial data orthogonal to the eigenspace relative to eigenvalue zero. For example, this is true for any initial datum of the kind $\Psi_0\equiv(p,v,\underline{0},\underline{z})$, where the support of p is outside the interval containing the points $s_1,\cdots,s_n$ which denote the equilibrium position of the walls.

Lemma 2.4: Given Ψ_0 orthogonal to the eigenspace relative to eigenvalue zero, let us denote by $(\underline{y}(t),\underline{z}(t))$ the projection onto $\mathbb{C}^n\oplus\mathbb{C}^n$ of $e^{t\hat{A}}\Psi_0$. Then

$$\lim_{|t|\to\infty}\|\underline{y}(t)\|_{\mathbb{C}^n}=0 \quad \text{and} \lim_{|t|\to\infty}\|\underline{z}(t)\|_{\mathbb{C}^n}=0.$$

Proof: Let $\hat{P}(dk)$ be the projection-valued measure corresponding to the self-adjoint operator $-i\hat{A}$. Since Ψ_0 is in the absolutely continuous subspace, for any Ψ the bounded complex measure $\langle\langle\Psi,\hat{P}(dk)\Psi_0\rangle\rangle$ is absolutely continuous with respect to the Lebesgue measure and hence its density belongs to $L^1(\mathbb{R})$. Thus by the spectral theorem and Riemann-Lebesgue lemma,

$$\lim_{|t|\to\infty}\langle\langle\Psi,e^{t\hat{A}}\Psi_0\rangle\rangle=\lim_{|t|\to\infty}\int_{\mathbb{R}}e^{-itk}\langle\langle\Psi,\hat{P}(dk)\Psi_0\rangle\rangle=0. \tag{2.94}$$

By taking $\Psi=(0,0,\underline{e}_i,\underline{0})$ and $\Psi=(0,0,\underline{0},\underline{e}_i)$, $i=1,\ldots,n$, one then obtains

$$\lim_{|t|\to\infty}y_i(t)=0 \quad \text{and} \lim_{|t|\to\infty}z_i(t)=0. \tag{2.95}$$

□

In order to obtain a more precise estimate on the asymptotic behavior of solutions of Eq. (2.62), for particular initial conditions, a detailed analysis of $\Gamma(\lambda)^{-1}$ is required. For example, in specific cases one can prove the existence of frequencies that are totally transmitted by the array of oscillators.

III. KRONIG-PENNEY MODEL IN ACOUSTICS

It is possible to extend the previous construction to the case of an array of infinitely many oscillators. We prove that in the case of a periodic array of identical oscillators the energy spectrum shows a band structure.

As a first step we define the operator $\hat{A}$ introduced in Sec. II when $\mathcal{S}=\{s_1,s_2,\ldots\}$ is a denumerable set such that

$$d \coloneqq \inf_{i\neq j}|s_i - s_j| > 0 \ i,j \in \mathbb{N}. \tag{3.1}$$

Defining the linear map

$$\tau(p,v,\underline{y},\underline{z}) \coloneqq \sum_{j=1}^{\infty} (v(s_j) - z_j)\underline{e}_j, \tag{3.2}$$

where $\{\underline{e}_j\}_1^\infty$ is the usual complete orthonormal system for ℓ^2, one has the following.

Lemma 3.1: τ is bounded as a map on $H^1(\mathbb{R})\oplus H^1(\mathbb{R})\oplus \ell^2 \oplus \ell^2$ to ℓ^2.

Proof: We will follow closely Albeverio *et al.* (2005). Let $\{I_j\}_1^\infty$ be a partition of $\mathbb{R}$ and let $K(x-y)$ be the kernel of $(-\Delta+1)^{-1/2}$. Since

$$v(x) = \sum_{j=1}^{\infty} \int_{I_j} K(x-y)[(-\Delta+1)^{1/2}v](y)dy, \tag{3.3}$$

to prove the lemma amounts to show that the infinite matrix

$$M_{ij} \coloneqq \left(\int_{I_j} K(x-s_i)^2 dx\right)^{1/2} \tag{3.4}$$

corresponds to a bounded linear operator M on ℓ^2. By Lemma C.3 in Albeverio *et al.* (2005), one has

$$\|M\|^2_{\ell^2,\ell^2} \leq \sup_i \sum_{j=1}^{\infty} \left(\int_{I_j} K(x-s_i)^2 dx\right)^{1/2} \sup_j \sum_{i=1}^{\infty} \left(\int_{I_j} K(x-s_i)^2 dx\right)^{1/2}. \tag{3.5}$$

Since

$$\frac{1}{\sqrt{x}} = \frac{2}{\pi}\int_0^\infty \frac{d\mu}{x+\mu^2}, \quad x>0,$$

by functional calculus one has

$$K(x-y) = \frac{1}{\pi}\int_0^\infty \frac{e^{-\sqrt{1+\mu^2}|x-y}}{\sqrt{1+\mu^2}} d\mu. \tag{3.6}$$

By taking $I_j=[s_j-\varepsilon_j, s_j+\delta_j)$, where ε_j is one half the distance between s_j and the preceding point and δ_j is one half the distance between s_j and the successive point, a straightforward calculation leads to

$$\int_{I_j} K(x-s_i)^2 dx \leq \frac{2}{d^2\pi^2} e^{-|s_i-s_j|/\sqrt{2}}, \tag{3.7}$$

from where the estimate $\|M\|_{\ell^2,\ell^2} < +\infty$ follows immediately.

The construction proceeds now along the same lines as in the case of a finite set of points. We state the final result:

Theorem 3.2: *Let* $\{K_j\}_1^\infty$ $\{M_j\}_1^\infty$, $K_j>0$, $M_j>0$ *be in* ℓ^∞ *and suppose that* $\{K_j/M_j\}_1^\infty$ *and* $\{1/M_j\}_1^\infty$ *are in* ℓ^∞ *too. The linear operator*

$$\hat{A}:D(\hat{A}) \subset L^2(\mathbb{R}) \oplus L^2(\mathbb{R}) \oplus \ell^2 \oplus \ell^2 \to L^2(\mathbb{R}) \oplus L^2(\mathbb{R}) \oplus \ell^2 \oplus \ell^2, \tag{3.8}$$

$$D(\hat{A}) = \{(p,v,\underline{y},\underline{z}):p \in L^2(\mathbb{R}) \cap H^1(\mathbb{R}\setminus\mathcal{S}),v \in H^1(\mathbb{R}),\underline{y} \in \ell^2,\underline{z} \in \ell^2,p(s_i^+) - p(s_i^-) = \sigma_i,v(s_j) = z_j,\underline{\sigma} \in \ell^2\}, \tag{3.9}$$

$$\hat{A}(p,v,\underline{y},\underline{z}) \coloneqq \left(-a^2\rho_0\frac{dv}{dx},-\frac{1}{\rho_0}\frac{dp_0}{dx},\underline{z},-\sum_{j=1}^{\infty}\left(\frac{K_j}{M_j}y_j + \frac{S}{M_j}\sigma_j\right)\underline{e}_j\right) \tag{3.10}$$

is real and skew-adjoint. Here $p_0 \in \bar{H}^1(\mathbb{R})$,

$$p_0(x) \coloneqq p(x) - \frac{1}{2}\sum_{j=1}^{\infty}\sigma_j \,\mathrm{sgn}(x-s_j), \tag{3.11}$$

denotes the regular part of p. *The resolvent of* $\hat{A}$ *is given by*

$$(-\hat{A}+\zeta)^{-1} = (-A+\zeta)^{-1} + \sum_{i,j=1}^{\infty}(\Gamma(\zeta)^{-1})_{ij}G_\zeta^i \otimes \breve{G}_{\bar{\zeta}}^j, \quad \zeta \in C\setminus i\mathbb{R}. \tag{3.12}$$

Now we can proceed to the study of a periodic system. We use the same notation of Albeverio *et al.* (2005).

In this case $\mathcal{S}$ will be the "Bravais" lattice,

$$\mathcal{S} = \{nL:n \in \mathbb{Z}\}, \quad L>0, \tag{3.13}$$

and $\hat{\mathcal{S}}$ the "Brillouin" zone,

$$\hat{\mathcal{S}} = \left[-\frac{b}{2},\frac{b}{2}\right), \quad b = \frac{2\pi}{L}. \tag{3.14}$$

We consider a Hilbert space $\mathcal{H}$ on $L^2 \oplus L^2 \oplus \ell^2 \oplus \ell^2$ in which the scalar product is defined by

$$\frac{1}{a^2\rho_0}\langle p_1,p_2\rangle + \rho_0\langle v_1,v_2\rangle + \frac{K}{S}\langle \underline{y}_1,\underline{y}_2\rangle + \frac{M}{S}\langle \underline{z}_1,\underline{z}_2\rangle \tag{3.15}$$

where $\langle\cdot,\cdot\rangle$ represents either the usual scalar product in L^2, when concerning pressure and velocity fields, or the usual scalar product in ℓ^2, for $\underline{y}$ and $\underline{z}$.

M, K, and S are positive constants representing the mass of oscillating walls, the elastic constant of the springs, and the area of the transverse section of the pipe.

The Hilbert space $\mathcal{H}$ can be decomposed as

$$\mathcal{H} = \widetilde{W}^{-1}\hat{\mathcal{H}}(\hat{\mathcal{S}},b^{-1}d\theta;L^2([-L/2,L/2)) \oplus L^2([-L/2,L/2)) \oplus \mathbb{C} \oplus \mathbb{C}) \tag{3.16}$$

$$= \widetilde{W}^{-1}\int_{[-b/2,b/2)}^{\oplus}\frac{d\theta}{b}(L^2([-L/2,L/2)) \oplus L^2([-L/2,L/2)) \oplus \mathbb{C} \oplus \mathbb{C}), \tag{3.17}$$

where

$$\widetilde{W}:\mathcal{H} \to \hat{\mathcal{H}}(\hat{\mathcal{S}},b^{-1}d\theta;L^2([-L/2,L/2)) \oplus L^2([-L/2,L/2)) \oplus \mathbb{C} \oplus \mathbb{C}), \tag{3.18}$$

$$\widetilde{W}(p,v,\underline{y},\underline{z}) \equiv ((\widetilde{W}p)(\theta,\nu),(\widetilde{W}v)(\theta,\nu),(\widetilde{W}\underline{y})(\theta),(\widetilde{W}\underline{z})(\theta)), \tag{3.19}$$

$$(\widetilde{W}p)(\theta,\nu) \equiv \widetilde{p}(\theta,\nu) = \sum_{n\in Z} e^{in\theta L} p(\nu+nL), \tag{3.20}$$

$$(\widetilde{W}v)(\theta,\nu) \equiv \widetilde{v}(\theta,\nu) = \sum_{n\in Z} e^{in\theta L} v(\nu+nL), \tag{3.21}$$

$$(\widetilde{W}\underline{y})(\theta) \equiv \widetilde{y}(\theta) = \sum_{n\in Z} e^{in\theta L} y_n, \tag{3.22}$$

$$(\widetilde{W}\underline{z})(\theta) \equiv \widetilde{z}(\theta) = \sum_{n\in Z} e^{in\theta L} z_n \quad \nu \in [-L/2,L/2), \quad \theta \in [-b/2,b/2), \tag{3.23}$$

and

$$\widetilde{W}^{-1}:\hat{\mathcal{H}}(\hat{\mathcal{S}},b^{-1}d\theta;L^2([-L/2,L/2)) \oplus L^2([-L/2,L/2)) \oplus \mathbb{C} \oplus \mathbb{C}) \to \mathcal{H}, \tag{3.24}$$

$$\widetilde{W}^{-1}(\widetilde{p},\widetilde{v},\underline{y},\underline{z}) \equiv ((\widetilde{W}^{-1}\widetilde{p})(\nu+nL),(\widetilde{W}^{-1}\widetilde{v})(\nu+nL),\{(\widetilde{W}^{-1}\widetilde{y})_n\},\{(\widetilde{W}^{-1}\widetilde{z})_n\}), \tag{3.25}$$

$$(\widetilde{W}^{-1}\widetilde{p})(\nu+nL) = b^{-1}\int_{-b/2}^{b/2} d\theta e^{-in\theta L}\widetilde{p}(\theta,\nu), \tag{3.26}$$

$$(\widetilde{W}^{-1}\widetilde{v})(\nu+nL) = b^{-1}\int_{-b/2}^{b/2} d\theta e^{-in\theta L}\widetilde{v}(\theta,\nu), \tag{3.27}$$

$$(\widetilde{W}^{-1}\widetilde{y})_n = b^{-1}\int_{-b/2}^{b/2} d\theta e^{-in\theta L}\widetilde{y}(\theta), \tag{3.28}$$

$$(\widetilde{W}^{-1}\widetilde{z})_n = b^{-1}\int_{-b/2}^{b/2} d\theta e^{-in\theta L}\widetilde{z}(\theta) \quad \nu \in [-L/2, \quad L/2), \quad n \in \mathbb{Z}. \tag{3.29}$$

The scalar product in $L^2([-L/2,L/2)) \oplus L^2([-L/2,L/2)) \oplus \mathbb{C} \oplus \mathbb{C}$ is defined by

$$\frac{1}{a^2\rho_0}\langle \widetilde{p}_1,\widetilde{p}_2\rangle_{L/2} + \rho_0\langle \widetilde{v}_1,\widetilde{v}_2\rangle_{L/2} + \frac{K}{S}\bar{\widetilde{y}}_1\widetilde{y}_2 + \frac{M}{S}\bar{\widetilde{z}}_1\widetilde{z}_2, \tag{3.30}$$

where $\langle \cdot,\cdot\rangle_{L/2}$ indicates the usual scalar product in $L^2([-L/2,L/2))$.

From Theorem 3.2 we obtain the following.

Corollary 3.3: The linear operator

$$\hat{A}:D(\hat{A}) \subset \mathcal{H} \to \mathcal{H}, \tag{3.31}$$

$$D(\hat{A}) = \{(p,v,\underline{y},\underline{z}):p \in L^2(\mathbb{R}) \cap H^1(\mathbb{R}\setminus\mathcal{S}),v \in H^1(\mathbb{R}),\underline{y} \in \ell^2,\underline{z} \in \ell^2,p(nL^+) - p(nL^-) = \sigma_n,v(nL) = z_n \ \forall\, n \in \mathbb{Z},\underline{\sigma} \in \ell^2\}, \tag{3.32}$$

$$\hat{A}(p,v,\underline{y},\underline{z}) := \left(-a^2\rho_0\frac{dv}{dx},-\frac{1}{\rho_0}\frac{dp_0}{dx},\underline{z},-\frac{K}{M}\underline{y}-\frac{S}{M}\underline{\sigma}\right), \tag{3.33}$$

where the regular part of $p(x)$, *denoted with* $p_0 \in \bar{H}^1(\mathbb{R})$, is

$$p_0(x) = p(x) - \frac{1}{2}\sum_{n\in Z}\sigma_n \operatorname{sgn}(x-nL), \tag{3.34}$$

is real and skew-adjoint.

We want to study the spectral structure of $\hat{A}$. To this aim we introduce the family of operators $\hat{A}(\theta)$:

$$\hat{A}(\theta):D(\hat{A}(\theta)) \subset L^2((-L/2,L/2)) \oplus L^2((-L/2,L/2)) \oplus \mathbb{C} \oplus \mathbb{C} \to L^2((-L/2,L/2)) \oplus L^2((-L/2,L/2)) \oplus \mathbb{C} \oplus \mathbb{C}, \tag{3.35}$$

$$D(\hat{A}(\theta)) = \Big\{ (\tilde{p}(\theta),\tilde{v}(\theta),\tilde{y}(\theta),\tilde{z}(\theta)):\tilde{p}(\theta) \in H^1((-L/2,L/2)\setminus\{0\}),\tilde{v}(\theta) \in H^1((-L/2,L/2)),\tilde{y}(\theta) \in \mathbb{C},\tilde{z}(\theta) \in \mathbb{C}\tilde{p}(\theta,0^+)-\tilde{p}(\theta,0^-)=\tilde{\sigma}(\theta),\tilde{v}(\theta,0)=\tilde{z}(\theta),\tilde{\sigma}(\theta) \in \mathbb{C},\tilde{p}\left(\theta,-\frac{L}{2}^+\right) = e^{i\theta L}\tilde{p}\left(\theta,\frac{L}{2}^-\right),\tilde{v}\left(\theta,-\frac{L}{2}^+\right) = e^{i\theta L}\tilde{v}\left(\theta,\frac{L}{2}^-\right)\Big\}; \quad \forall\, \theta \in \left[-\frac{b}{2},\frac{b}{2}\right) \tag{3.36}$$

$$\hat{A}(\theta)(\tilde{p}(\theta),\tilde{v}(\theta),\tilde{y}(\theta),\tilde{z}(\theta)) := \left(-a^2\rho_0\frac{d\tilde{v}(\theta)}{d\nu},-\frac{1}{\rho_0}\frac{d\tilde{p}_0(\theta)}{d\nu},\tilde{z}(\theta),-\frac{K}{M}\tilde{y}(\theta)-\frac{S}{M}\tilde{\sigma}(\theta)\right), \tag{3.37}$$

where $\tilde{p}_0(\theta) \in H^1(\mathbb{R})$ is the regular part of $\tilde{p}(\theta)$,

$$\tilde{p}_0(\theta,\nu) = \tilde{p}(\theta,\nu) - \frac{1}{2}\tilde{\sigma}(\theta)\operatorname{sgn}(\nu). \tag{3.38}$$

Boundary conditions for $\tilde{p}(\theta,\nu)$ and $\tilde{v}(\theta,\nu)$ in $\nu=0$ and $\nu=\pm L/2$ are such that all operators in this family are skew-adjoint with respect to the scalar product (3.30).

The operator $\hat{A}$ is related to $\hat{A}(\theta)$ by the relation (see Albeverio *et al.* (2005))

$$\tilde{W}\hat{A}\tilde{W}^{-1} = \int_{[-b/2,b/2)}^{\oplus} \frac{d\theta}{b}\hat{A}(\theta). \tag{3.39}$$

The spectrum of $\hat{A}(\theta)$ is described by the following

Theorem 3.4: *Let* $\theta \in [-b/2,b/2)$, *then the spectrum of* $\hat{A}(\theta)$ *is purely discrete, in particular its eigenvalues* $E_n(\theta)$ are given by

$$E_n(\theta) = \lambda_n(\theta) = 2i\xi_n(\theta)\frac{a}{L}; \quad n \in \mathbb{Z}, \quad \xi_n(\theta) \in \mathbb{R}, \tag{3.40}$$

where $\xi_n(\theta)$ are the real solutions of

$$\sin\xi[\sin\xi - F(\xi)\cos\xi]\cos^2\frac{\theta L}{2} = \cos\xi[\cos\xi + F(\xi)\sin\xi]\sin^2\frac{\theta L}{2}, \tag{3.41}$$

$$F(\xi)=\frac{M}{M_g}\left(\pi^2\frac{\omega_o^2}{\omega_g^2}\frac{1}{\xi}-\xi\right); \quad M_g=\rho_0 SL, \quad \omega_o^2=\frac{K}{M}, \quad \omega_g=2\pi\frac{a}{L}. \tag{3.42}$$

The corresponding eigenfunctions are

$$\Phi_n(\theta,x)=(\tilde{p}_n(\theta,\nu),\tilde{v}_n(\theta,\nu),\tilde{y}_n(\theta),\tilde{z}_n(\theta)); \quad n\in\mathbb{Z}, \quad \theta\in[-b/2,b/2), \tag{3.43}$$

$$\tilde{p}_n(\theta,\nu)=C_n\left[\left(\sin\left(\xi_n-\frac{\theta L}{2}\right)-F(\xi_n)\cos\left(\xi_n-\frac{\theta L}{2}\right)\right)\cos\frac{2\xi_n}{L}\nu+ -i\sin\left(\xi_n-\frac{\theta L}{2}\right)\left(\sin\frac{2\xi_n}{L}\nu-F(\xi_n)\frac{|\nu|}{\nu}\cos\frac{2\xi_n}{L}\nu\right)\right], \tag{3.44}$$

$$\tilde{v}_n(\theta,\nu)=-\frac{iC_n}{a\rho_0}\left[\left(\sin\left(\xi_n-\frac{\theta L}{2}\right)-F(\xi_n)\cos\left(\xi_n-\frac{\theta L}{2}\right)\right)\sin\frac{2\xi_n}{L}\nu +i\sin\left(\xi_n-\frac{\theta L}{2}\right)\left(\cos\frac{2\xi_n}{L}\nu+F(\xi_n)\sin\frac{2\xi_n}{L}|\nu|\right)\right], \tag{3.45}$$

$$\tilde{y}_n(\theta)=-i\frac{C_nL}{a^2\rho_0\xi_n}\sin\left(\xi_n-\frac{\theta L}{2}\right), \tag{3.46}$$

$$\tilde{z}_n(\theta)=\frac{C_n}{a\rho_0}\sin\left(\xi_n-\frac{\theta L}{2}\right). \tag{3.47}$$

For $\theta\in[-b/2,b/2)$ *zero is an eigenvalue with eigenfunction*

$$\Psi_0=\left(C_0\left(\cos\frac{\theta L}{2}-i\sin\frac{\theta L}{2}\,\mathrm{sgn}(\nu)\right),0,2iC_0\frac{S}{K}\sin\frac{\theta L}{2},0\right). \tag{3.48}$$

Moreover, the following chain of inequalities holds:

$$0<E_1(0)<E_1(-b/2)\leqslant E_2(-b/2)<E_2(0)\leqslant E_3(0)<E_3(-b/2)\leqslant E_4(-b/2)<E_4(0)\leqslant E_5(0) <E_4(-b/2)\leqslant E_5(-b/2)<\cdots . \tag{3.49}$$

In general, the eigenvalues $E_n(\theta)$ *are all distinct and nondegenerate. If* $\omega_o/\omega_g=n/2$ *with* $n\in\mathbb{N}$, *there is just one twofold degenerate eigenvalue equal to* $n\pi/2$; *such an eigenvalue corresponds to* $\theta=0$ *for n even and to* $|\theta|=b/2$ *for n odd.*

If $E(\theta)$ *is an eigenvalue then* $-E(\theta)$ *is an eigenvalue.*

Given $\theta\in[-b/2,b/2)$ *the following relation holds*:

$$E_n(-\theta)=E_n(\theta). \tag{3.50}$$

Proof: Eigenvalues and eigenfunctions (3.40)–(3.48) are given by direct computation. We solve the system of equations

$$\hat{A}(\theta)(\tilde{p}(\theta),\tilde{v}(\theta),\tilde{y}(\theta),\tilde{z}(\theta))=\lambda(\tilde{p}(\theta),\tilde{v}(\theta),\tilde{y}(\theta),\tilde{z}(\theta)) \quad \lambda\in i\mathbb{R}; \tag{3.51}$$

with the condition $\tilde{v}(\theta,0)=\tilde{z}(\theta)$, the solution reads

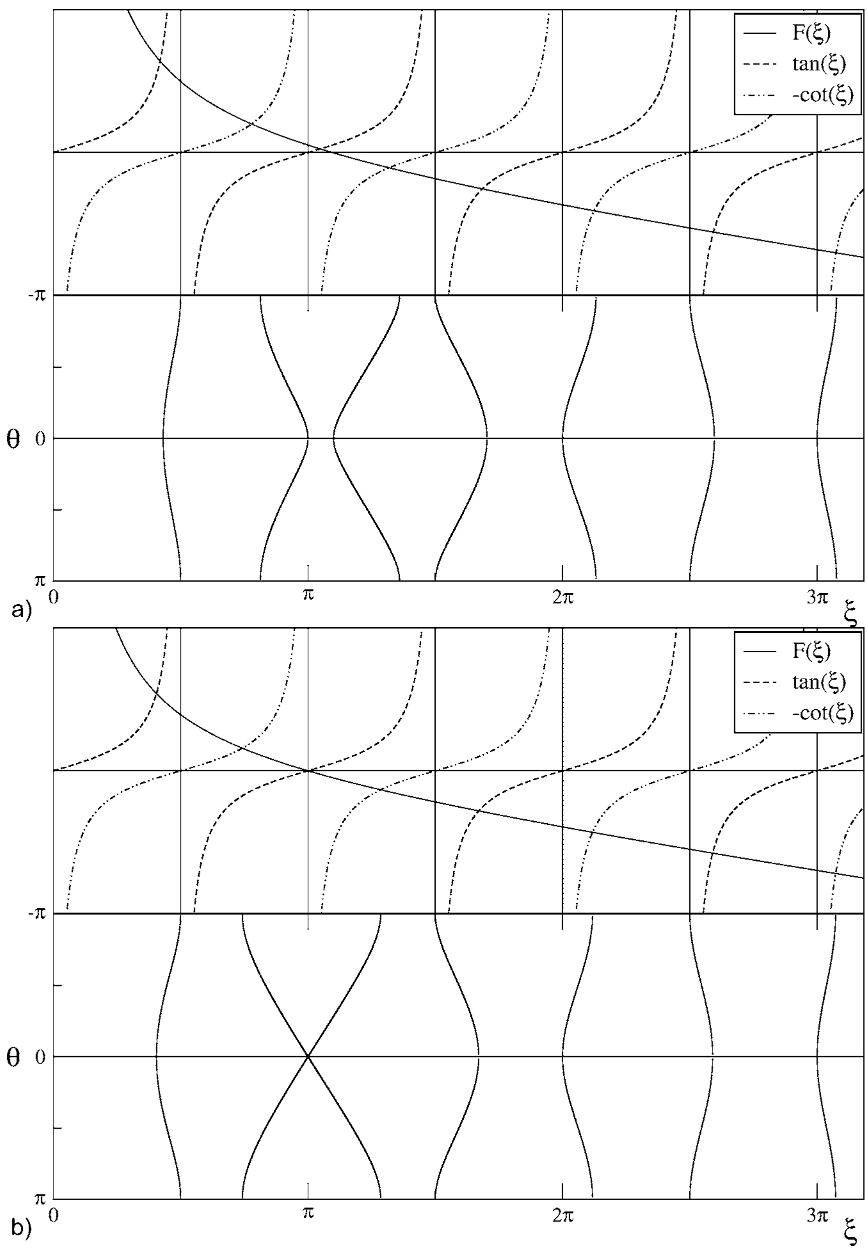

FIG. 1. Graphical solution of Eqs. (3.55) and (3.56). (b) and corresponding band structure for (a) a non-degenerate case (b) a degenerate case.

$$\tilde{p}(\theta,\nu) = C(\xi)\cos\frac{2\xi\nu}{L} + D(\xi)\left[\sin\frac{2\xi\nu}{L} - F(\xi)\mathrm{sgn}(\nu)\cos\frac{2\xi\nu}{L}\right], \tag{3.52}$$

$$\tilde{v}(\theta,\nu) = \frac{C(\xi)}{ia\rho_0}\sin\frac{2\xi\nu}{L} - \frac{D(\xi)}{ia\rho_0}\left[\cos\frac{2\xi\nu}{L} + F(\xi)\sin\frac{2\xi|\nu|}{L}\right], \tag{3.53}$$

where $\xi = -iL\lambda/(2a) \in \mathbb{R}$, and $C(\xi)$ and $D(\xi)$ are two unknown functions of ξ. To determine $C(\xi)$ and $D(\xi)$ we have to take into account the boundary conditions

$$\tilde{p}\left(\theta,-\frac{L_+}{2}\right)=e^{i\theta L}\tilde{p}\left(\theta,\frac{L_-}{2}\right),$$
$$\tilde{v}\left(\theta,-\frac{L_+}{2}\right)=e^{i\theta L}\tilde{v}\left(\theta,\frac{L_-}{2}\right). \tag{3.54}$$

This system has only the trivial solution $C(\xi)=0$ and $D(\xi)=0$ for the values of ξ for which the determinant of the matrix of the coefficients of the system is zero. The condition that the determinant is zero implies Eq. (3.41) for the eigenvalues. For ξ satisfying condition (3.41) the solutions of the system of dependent equations (3.54) give the eigenfunctions.

For $\theta=0$ and $\theta=-b/2$ relation (3.41) becomes

$$\tan\xi=0 \quad \text{or} \quad \tan\xi=F(\xi)=\frac{M}{M_g}\left(\pi^2\frac{\omega_o^2}{\omega_g^2}\frac{1}{\xi}-\xi\right); \quad \theta=0, \tag{3.55}$$

$$\cot\xi=0 \quad \text{or} \quad -\cot\xi=F(\xi)=\frac{M}{M_g}\left(\pi^2\frac{\omega_o^2}{\omega_g^2}\frac{1}{\xi}-\xi\right); \quad \theta=-b/2. \tag{3.56}$$

Graphic solutions of the transcendental equations (3.55) and (3.56) are given in the upper part of Figs. 1(a) and 1(b). The chain of inequalities (3.49) follows by the monotone behavior of $F(\xi)$.

Degeneration of eigenvalues for $\omega_o/\omega_g=n/2$, the fact that $-E(\theta)$ is an eigenvalue if $E(\theta)$ is an eigenvalue and relation (3.50) follow directly by Eq. (3.41) and by $F(\xi)=-F(-\xi)$. □

One can show that there is a band structure writing equation (3.41) as

$$\tan^2\frac{\theta L}{2}=\tan\xi\left[\frac{\tan\xi-F(\xi)}{1+F(\xi)\tan\xi}\right]. \tag{3.57}$$

It is possible to find solutions of Eq. (3.57) only for values of ξ such that the right-hand side is positive. In the lower part of Figs. 1(a) and 1(b) the resulting band structure is shown. The figures clearly show that the width of the gaps is connected to the structure of the spectrum. In particular, Fig. 1(b) shows that when there is a degenerate eigenvalue, $\omega_o/\omega_g=n\pi/2$ with $n\in\mathbb{N}$, a gap disappears because of the overlapping of two bands.

The bandwidth increases when the ratio M/M_g decreases.

[1] Albeverio, S., Gesztesy, F., Høegh-Krohn, R., and Holden, H., *Solvable Models in Quantum Mechanics*, 2nd ed. (AMS Chelsea, Providence, RI, 2005).
[2] Dirac, P. A. M., "Classical theory of radiating electrons," Proc. R. Soc. London, Ser. A **167**, 148–169 (1938).
[3] Gorbachuk, V. I. and Gorbachuk, M. L., *Boundary Value Problems for Operator Differential Equations* (Kluwer Academic, Amsterdam, 1991).
[4] Griffiths, D. J. and Steinke, C. A., "Waves in locally periodic media," Am. J. Phys. **69**, 137–154 (2001).
[5] Infeld, L. and Wallace, P. R., "The equation of motion in electrodynamics," Phys. Rev. **57**, 797–806 (1940).
[6] Kijowski, J., "Electrodynamics of moving particles," Gen. Relativ. Gravit. **26**, 167–201 (1994).
[7] Marino, M., "Classical electrodynamics of point charges," Ann. Phys. **301**, 85–127 (2002).
[8] Noja, D. and Posilicano, A., "The wave equation with one point interaction and the (linearized) classical electrodynamics of a point particle," Ann. Inst. Henri Poincare, Sect. A **68**, 351–377 (1998).
[9] Noja, D. and Posilicano, A. "On the point limit of the Pauli-Fierz model," Ann. Inst. Henri Poincare, Sect. A **71**, 425–457 (1999).
[10] Posilicano, A., "A Kreĭn-like formula for singular perturbations of self-adjoint operators and applications," J. Funct. Anal. **183**, 109–147 (2001).
[11] Posilicano, A., "Singular perturbations of abstract wave equations," J. Funct. Anal. (in press).
[12] Reed, M. and Simon, B., *Methods of Modern Mathematical Physics. Vol 1: Functional Analysis* (Academic, New York, 1972).
[13] Reed, M. and Simon, B., *Methods of Modern Mathematical Physics. Vol 2: Fourier Analysis, Self-Adjointness* (Academic, New York, 1975).
[14] Reed, M. and Simon, B., *Methods of Modern Mathematical Physics. Vol 3: Scattering Theory* (Academic, New York, 1979).
[15] Reed, M. and Simon, B., *Methods of Modern Mathematical Physics. Vol 4: Analysis of Operators* (Academic, New York, 1978).

[16] Soffer, A. and Weinstein, M. I., "Nonautonomous Hamiltonians," J. Stat. Phys. **93**, 359–391 (1998a).
[17] Soffer, A. and Weinstein, M. I., "Time dependent resonances theory," Geom. Funct. Anal. **8**, 1–43 (1998b).
[18] Soffer, A. and Weinstein, M. I., "Resonances, radiation damping and instability in Hamiltonian nonlinear wave equations," Invent. Math. **136**, 9–74 (1999).
[19] Spohn, H., *Dynamics of Charged Particles and their Radiation Field* (Cambridge University Press, Cambridge, 2004).
[20] Templin, J. D., "Radiation reaction and runaway solutions in acoustics," Am. J. Phys. **67**, 407–413 (1999).