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All timelike supersymmetric solutions of $\mathcal{N} = 2$, $D = 4$ gauged supergravity coupled to abelian vector multiplets

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ABSTRACT: The timelike supersymmetric solutions of $\mathcal{N} = 2$, $D = 4$ gauged supergravity coupled to an arbitrary number of abelian vector multiplets are classified using spinorial geometry techniques. We show that the generalized holonomy group for vacua preserving N supersymmetries is $\mathrm{GL}(\frac{8-N}{2}, \mathbb{C}) \ltimes \frac{N}{2}\mathbb{C}^{\frac{8-N}{2}} \subseteq \mathrm{GL}(8, \mathbb{R})$, where $N = 0, 2, 4, 6, 8$. The spacetime turns out to be a fibration over a three-dimensional base manifold with $\mathrm{U}(1)$ holonomy and nontrivial torsion. Our results can be used to construct new supersymmetric AdS black holes with nontrivial scalar fields turned on.

KEYWORDS: Superstring Vacua, Black Holes, Supergravity Models.

Contents

1. Introduction

Supersymmetric solutions to supergravity theories have played, and continue to play, an important role in string- and M-theory developments. This makes it desirable to obtain a complete classification of BPS solutions to various supergravities in diverse dimensions. Progress in this direction has been made in the last years using the mathematical concept of G-structures [1]. The basic strategy is to assume the existence of at least one Killing spinor ϵ obeying $\mathcal{D}_\mu \epsilon = 0$, and to construct differential forms as bilinears from this spinor. These forms, which define a preferred G-structure, obey several algebraic and differential equations that can be used to deduce the metric and the other bosonic supergravity fields. Using this framework, a number of complete classifications [2–4] and many partial results (see e.g. [5–17] for an incomplete list) have been obtained. By complete we mean that the most general solutions for all possible fractions of supersymmetry have been obtained, while for partial classifications this is only available for some fractions. Note that the complete classifications mentioned above involve theories with eight supercharges and holonomy $H = \mathrm{SL}(2, \mathbb{H})$ of the supercurvature $R_{\mu\nu} = \mathcal{D}_{[\mu}\mathcal{D}_{\nu]}$, and allow for either half- or maximally supersymmetric solutions.

An approach which exploits the linearity of the Killing spinors has been proposed [18] under the name of spinorial geometry. Its basic ingredients are an explicit oscillator basis for the spinors in terms of forms and the use of the gauge symmetry to transform them to a preferred representative of their orbit. In this way one can construct a linear system for the background fields from any (set of) Killing spinor(s) [19]. This method has proven fruitful in e.g. the challenging case of IIB supergravity [20 – 22]. In addition, it has been adjusted to impose 'near-maximal' supersymmetry and thus has been used to rule out certain large fractions of supersymmetry [23 – 27]. Finally, a complete classification for type I supergravity in ten dimensions has been obtained in [28], and all half-supersymmetric backgrounds of $\mathcal{N} = 2$, $D = 5$ gauged supergravity coupled to abelian vector multiplets were determined in [29, 30].

In the present paper we would like to address the classification of supersymmetric solutions in four-dimensional $\mathcal{N} = 2$ matter-coupled U(1)-gauged supergravity, generalizing thus the simpler cases of $\mathcal{N} = 1$, considered recently in [31, 32], and minimal $\mathcal{N} = 2$, where a full classification is available both in the ungauged [33] and gauged theories [34]. We shall thereby focus on the class where the Killing vector constructed from the Killing spinor is timelike, deferring the lightlike case to a forthcoming publication. Moreover, only coupling to abelian vector multiplets and gauging of a U(1) subgroup of the SU(2) R-symmetry will be considered, while the inclusion of hypermultiplets and nonabelian vectors, as well as a general gauging, are left for future work [35].

The outline of this paper is as follows. In section 2, we briefly review $\mathcal{N} = 2$ supergravity in four dimensions and its matter couplings. In 3.1 we discuss the orbits of Killing spinors and analyze the holonomy of the supercovariant connection. In section 4 we determine the conditions coming from a single timelike Killing spinor, and obtain all supersymmetric solutions in this class. Finally, in section 5 we present our conclusions and outlook. Appendices A and B contain our notation and conventions for spinors, while in appendix C we show that the Killing spinor equations, together with the Maxwell equations and the Bianchi identities, imply the equations of motion in the timelike case. Finally, in appendix D we discuss the reduced holonomy of the three-dimensional manifold over which the spacetime is fibered.

2. Matter-coupled $\mathcal{N} = 2$, $D = 4$ gauged supergravity

In this section we shall give a short summary of the main ingredients of $\mathcal{N} = 2$, $D = 4$ gauged supergravity coupled to vector- and hypermultiplets [36]. Throughout this paper, we will use the notations and conventions of [37], to which we refer for more details.

Apart from the vierbein e^a_μ and the chiral gravitinos ψ^i_μ, $i = 1, 2$, the field content includes n_H hypermultiplets and n_V vector multiplets enumerated by $I = 0, \ldots, n_V$. The latter contain the graviphoton and have fundamental vectors A^I_μ, with field strengths

$$F^I_{\mu\nu} = \partial_\mu A^I_\nu - \partial_\nu A^I_\mu + g A^K_\nu A^J_\mu f_{JK}{}^I \, .$$

The fermions of the vector multiplets are denoted as $\lambda^{\alpha i}$ and the complex scalars as z^α where $\alpha = 1, \ldots, n_V$. These scalars parametrize a special Kähler manifold, i. e., an n_V-

dimensional Hodge-Kähler manifold that is the base of a symplectic bundle, with the covariantly holomorphic sections

$$\mathcal{V} = \begin{pmatrix} X^I \\ F_I \end{pmatrix}, \qquad \mathcal{D}_{\bar{\alpha}}\mathcal{V} = \partial_{\bar{\alpha}}\mathcal{V} - \frac{1}{2}(\partial_{\bar{\alpha}}\mathcal{K})\mathcal{V} = 0, \tag{2.1}$$

where $\mathcal{K}$ is the Kähler potential and $\mathcal{D}$ denotes the Kähler-covariant derivative.[1] $\mathcal{V}$ obeys the symplectic constraint

$$\langle \mathcal{V}, \bar{\mathcal{V}} \rangle = X^I \bar{F}_I - F_I \bar{X}^I = i. \tag{2.2}$$

To solve this condition, one defines

$$\mathcal{V} = e^{\mathcal{K}(z,\bar{z})/2} v(z), \tag{2.3}$$

where $v(z)$ is a holomorphic symplectic vector,

$$v(z) = \begin{pmatrix} Z^I(z) \\ \frac{\partial}{\partial Z^I} F(Z) \end{pmatrix}. \tag{2.4}$$

F is a homogeneous function of degree two, called the prepotential, whose existence is assumed to obtain the last expression. This is not restrictive because it can be shown that it is always possible to go in a gauge where the prepotential exists via a local symplectic transformation [37, 38].[2] The Kähler potential is then

$$e^{-\mathcal{K}(z,\bar{z})} = -i\langle v, \bar{v} \rangle. \tag{2.5}$$

The matrix $\mathcal{N}_{IJ}$ determining the coupling between the scalars z^α and the vectors A_μ^I is defined by the relations

$$F_I = \mathcal{N}_{IJ} X^J, \qquad \mathcal{D}_{\bar{\alpha}}\bar{F}_I = \mathcal{N}_{IJ}\mathcal{D}_{\bar{\alpha}}\bar{X}^J. \tag{2.6}$$

Given

$$U_\alpha \equiv \mathcal{D}_\alpha \mathcal{V} = \partial_\alpha \mathcal{V} + \frac{1}{2}(\partial_\alpha \mathcal{K})\mathcal{V}, \tag{2.7}$$

the following differential constraints hold:

$$\begin{aligned} \mathcal{D}_\alpha U_\beta &= C_{\alpha\beta\gamma} g^{\gamma\bar{\delta}} \bar{U}_{\bar{\delta}}, \\ \mathcal{D}_{\bar{\beta}} U_\alpha &= g_{\alpha\bar{\beta}} \mathcal{V}, \\ \langle U_\alpha, \mathcal{V} \rangle &= 0. \end{aligned} \tag{2.8}$$

Here, $C_{\alpha\beta\gamma}$ is a completely symmetric tensor which determines also the curvature of the special Kähler manifold.

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[1] For a generic field ϕ^α that transforms under a Kähler transformation $\mathcal{K}(z,\bar{z}) \to \mathcal{K}(z,\bar{z}) + \Lambda(z) + \bar{\Lambda}(\bar{z})$ as $\phi^\alpha \to e^{-(p\Lambda + q\bar{\Lambda})/2}\phi^\alpha$, one has $\mathcal{D}_\alpha \phi^\beta = \partial_\alpha \phi^\beta + \Gamma^\beta{}_{\alpha\gamma}\phi^\gamma + \frac{p}{2}(\partial_\alpha \mathcal{K})\phi^\beta$. $\mathcal{D}_{\bar{\alpha}}$ is defined in the same way. X^I transforms as $X^I \to e^{-(\Lambda - \bar{\Lambda})/2}X^I$ and thus has Kähler weights $(p,q) = (1,-1)$.

[2] This need not be true for gauged supergravity, where symplectic covariance is broken [36]. However, in our analysis we do not really use that the F_I can be obtained from a prepotential, so our conclusions go through also without assuming that $F_I = \partial F(X)/\partial X^I$ for some $F(X)$. We would like to thank Patrick Meessen for discussions on this point.

We now come to the hypermultiplets. These contain scalars q^X and spinors ζ^A, where $X = 1, \ldots, 4n_H$ and $A = 1, \ldots, 2n_H$. The $4n_H$ hyperscalars parametrize a quaternionic Kähler manifold, with vielbein f_X^{iA} and inverse f_{iA}^X (i. e. the tangent space is labelled by indices (iA)). From these one can construct the three complex structures

$$\vec{J}_X{}^Y = -i f_X^{iA} \vec{\sigma}_i{}^j f_{jA}^Y \,, \tag{2.9}$$

with the Pauli matrices $\vec{\sigma}_i{}^j$ (cf. appendix A). Furthermore, one defines SU(2) connections $\vec{\omega}_X$ by requiring the covariant constancy of the complex structures:

$$0 = \mathfrak{D}_X \vec{J}_Y{}^Z \equiv \partial_X \vec{J}_Y{}^Z - \Gamma^W{}_{XY} \vec{J}_W{}^Z + \Gamma^Z{}_{XW} \vec{J}_Y{}^W + 2 \vec{\omega}_X \times \vec{J}_Y{}^Z \,, \tag{2.10}$$

where the Levi-Civita connection of the metric g_{XY} is used. The curvature of this SU(2) connection is related to the complex structure by

$$\vec{R}_{XY} \equiv 2 \, \partial_{[X} \vec{\omega}_{Y]} + 2 \vec{\omega}_X \times \vec{\omega}_Y = -\frac{1}{2} \kappa^2 \vec{J}_{XY} \,. \tag{2.11}$$

Depending on whether $\kappa = 0$ or $\kappa \neq 0$ the manifold is hyper-Kähler or quaternionic Kähler respectively. In what follows, we take $\kappa = 1$.

The bosonic action of $\mathcal{N} = 2$, $D = 4$ supergravity is

$$\begin{aligned}
e^{-1}\mathcal{L}_{\text{bos}} = {}& \frac{1}{16\pi G} R + \frac{1}{4}(\text{Im}\,\mathcal{N})_{IJ} F_{\mu\nu}^I F^{J\mu\nu} - \frac{1}{8}(\text{Re}\,\mathcal{N})_{IJ}\, e^{-1}\epsilon^{\mu\nu\rho\sigma} F_{\mu\nu}^I F_{\rho\sigma}^J \,, \\
& - g_{\alpha\bar{\beta}}\mathcal{D}_\mu z^\alpha \mathcal{D}^\mu \bar{z}^{\bar{\beta}} - \frac{1}{2} g_{XY}\mathcal{D}_\mu q^X \mathcal{D}^\mu q^Y - V \,, \\
& - \frac{g}{6} C_{I,JK} e^{-1}\epsilon^{\mu\nu\rho\sigma} A_\mu^I A_\nu^J \left(\partial_\rho A_\sigma^K - \frac{3}{8} g f_{LM}{}^K A_\rho^L A_\sigma^M \right) \,,
\end{aligned} \tag{2.12}$$

where $C_{I,JK}$ are real coefficients, symmetric in the last two indices, with $Z^I Z^J Z^K C_{I,JK} = 0$, and the covariant derivatives acting on the scalars read

$$\mathcal{D}_\mu z^\alpha = \partial_\mu z^\alpha + g A_\mu^I k_I^\alpha(z) \,, \qquad \mathcal{D}_\mu q^X = \partial_\mu q^X + g A_\mu^I k_I^X \,. \tag{2.13}$$

Here $k_I^\alpha(z)$ and k_I^X are Killing vectors of the special Kähler and quaternionic Kähler manifolds respectively. The potential V in (2.12) is the sum of three distinct contributions:

$$\begin{aligned}
V &= g^2(V_1 + V_2 + V_3) \,, \\
V_1 &= g_{\alpha\bar{\beta}} k_I^\alpha k_J^{\bar{\beta}} e^{\mathcal{K}} \bar{Z}^I Z^J \,, \\
V_2 &= 2 \, g_{XY} k_I^X k_J^Y e^{\mathcal{K}} \bar{Z}^I Z^J \,, \\
V_3 &= 4(U^{IJ} - 3 \, e^{\mathcal{K}} \bar{Z}^I Z^J) \vec{P}_I \cdot \vec{P}_J \,,
\end{aligned} \tag{2.14}$$

with

$$U^{IJ} \equiv g^{\alpha\bar{\beta}} e^{\mathcal{K}} \mathcal{D}_\alpha Z^I \mathcal{D}_{\bar{\beta}} \bar{Z}^J = -\frac{1}{2}(\text{Im}\,\mathcal{N})^{-1|IJ} - e^{\mathcal{K}} \bar{Z}^I Z^J \,, \tag{2.15}$$

and the triple moment maps $\vec{P}_I(q)$. The latter have to satisfy the equivariance condition

$$\vec{P}_I \times \vec{P}_J + \frac{1}{2} \vec{J}_{XY} k_I^X k_J^Y - f_{IJ}{}^K \vec{P}_K = 0 \,, \tag{2.16}$$

which is implied by the algebra of symmetries. The metric for the vectors is given by

$$\mathcal{N}_{IJ}(z, \bar{z}) = \bar{F}_{IJ} + i \frac{N_{IN} N_{JK} Z^N Z^K}{N_{LM} Z^L Z^M}, \qquad N_{IJ} \equiv 2 \operatorname{Im} F_{IJ}, \qquad (2.17)$$

where $F_{IJ} = \partial_I \partial_J F$, and F denotes the prepotential.

Finally, the supersymmetry transformations of the fermions to bosons are

$$\delta \psi_\mu^i = D_\mu(\omega) \epsilon^i - g \Gamma_\mu S^{ij} \epsilon_j + \frac{1}{4} \Gamma^{ab} F_{ab}^{-I} \epsilon^{ij} \Gamma_\mu \epsilon_j (\operatorname{Im} \mathcal{N})_{IJ} Z^J e^{\mathcal{K}/2}, \qquad (2.18)$$

$$D_\mu(\omega) \epsilon^i = \left(\partial_\mu + \frac{1}{4} \omega_\mu^{ab} \Gamma_{ab} \right) \epsilon^i + \frac{i}{2} A_\mu \epsilon^i + \partial_\mu q^X \omega_{X \, j}{}^i \epsilon^j + g A_\mu^I P_{I \, j}{}^i \epsilon^j, \qquad (2.19)$$

$$\delta \lambda_i^\alpha = -\frac{1}{2} e^{\mathcal{K}/2} g^{\alpha \bar{\beta}} \mathcal{D}_{\bar{\beta}} \bar{Z}^I (\operatorname{Im} \mathcal{N})_{IJ} F_{\mu\nu}^{-J} \Gamma^{\mu\nu} \epsilon_{ij} \epsilon^j + \Gamma^\mu \mathcal{D}_\mu z^\alpha \epsilon_i + g N_{ij}^\alpha \epsilon^j,$$

$$\delta \zeta^A = \frac{i}{2} f_X^{Ai} \Gamma^\mu \mathcal{D}_\mu q^X \epsilon_i + g \mathcal{N}^{iA} \epsilon_{ij} \epsilon^j,$$

where we defined

$$S^{ij} \equiv -P_I^{ij} e^{\mathcal{K}/2} Z^I,$$

$$N_{ij}^\alpha \equiv e^{\mathcal{K}/2} \left[\epsilon_{ij} k_I^\alpha \bar{Z}^I - 2 P_{Iij} \mathcal{D}_{\bar{\beta}} \bar{Z}^I g^{\alpha \bar{\beta}} \right], \qquad \mathcal{N}^{iA} \equiv -i f_X^{iA} k_I^X e^{\mathcal{K}/2} \bar{Z}^I.$$

In (2.19), A_μ is the gauge field of the Kähler U(1),

$$A_\mu = -\frac{i}{2} (\partial_\alpha \mathcal{K} \partial_\mu z^\alpha - \partial_{\bar{\alpha}} \mathcal{K} \partial_\mu \bar{z}^{\bar{\alpha}}) - g A_\mu^I P_I^0, \qquad (2.20)$$

with the moment map function

$$P_I^0 = \langle T_I \mathcal{V}, \bar{\mathcal{V}} \rangle, \qquad (2.21)$$

and

$$T_I \mathcal{V} \equiv \begin{pmatrix} -f_{IJ}{}^K & 0 \\ C_{I,KJ} & f_{IK}{}^J \end{pmatrix} \begin{pmatrix} X^J \\ F_J \end{pmatrix}. \qquad (2.22)$$

The major part of this paper will deal with the case of vector multiplets only, i. e., $n_H = 0$. Then there are still two possible solutions of (2.16) for the moment maps $\vec{P}_I$, which are called SU(2) and U(1) Fayet-Iliopoulos (FI) terms respectively [37]. Here we are interested in the latter. In this case

$$\vec{P}_I = \vec{e} \xi_I, \qquad (2.23)$$

where $\vec{e}$ is an arbitrary vector in SU(2) space and ξ_I are constants for the I corresponding to U(1) factors in the gauge group. If, moreover, we assume $f_{IJ}{}^K = 0$ (abelian gauge group), and $k_I^\alpha = 0$ (no gauging of special Kähler isometries), then only the V_3 part survives in the scalar potential (2.14), and one can also choose $C_{I,JK} = 0$. Note that this case corresponds to a gauging of a U(1) subgroup of the SU(2) R-symmetry, with gauge field $\xi_I A_\mu^I$.

3. *G*-invariant Killing spinors in 4D

3.1 Orbits of spinors under the gauge group

A Killing spinor[3] can be viewed as an SU(2) doublet (ϵ^1, ϵ^2), where an upper index means that a spinor has positive chirality. ϵ^i is related to the negative chirality spinor ϵ_i by charge conjugation, $\epsilon_i^C = \epsilon^i$, with

$$\epsilon_i^C = \Gamma_0 C^{-1} \epsilon_i^* \,. \tag{3.1}$$

Here C is the charge conjugation matrix defined in appendix B. As ϵ^1 has positive chirality, we can write $\epsilon^1 = c1 + de_{12}$ for some complex functions c, d. Notice that $c1 + de_{12}$ is in the same orbit as 1 under $\mathrm{Spin}(3,1)$, which can be seen from

$$e^{\gamma \Gamma_{13}} e^{\psi \Gamma_{12}} e^{\delta \Gamma_{13}} e^{h \Gamma_{02}} \, 1 = e^{i(\delta + \gamma)} e^h \cos \psi \, 1 + e^{i(\delta - \gamma)} e^h \sin \psi \, e_{12} \,.$$

This means that we can set $c = 1$, $d = 0$ without loss of generality. In order to determine the stability subgroup of ϵ^1, one has to solve the infinitesimal equation

$$\alpha^{cd} \Gamma_{cd} 1 = 0 \,, \tag{3.2}$$

which implies $\alpha^{02} = \alpha^{13} = 0$, $\alpha^{01} = -\alpha^{12}$, $\alpha^{03} = \alpha^{23}$. The stability subgroup of 1 is thus generated by

$$X = \Gamma_{01} - \Gamma_{12} \,, \qquad Y = \Gamma_{03} + \Gamma_{23} \,. \tag{3.3}$$

One easily verifies that $X^2 = Y^2 = XY = 0$, and thus $\exp(\mu X + \nu Y) = 1 + \mu X + \nu Y$, so that X, Y generate $\mathbb{R}^2$.

Having fixed $\epsilon^1 = 1$, also ϵ_1 is determined by $\epsilon_1 = \epsilon^{1C} = e_1$. A negative chirality spinor independent of ϵ^1 is ϵ_2, which can be written as a linear combination of odd forms, $\epsilon_2 = ae_1 + be_2$, where a and b are again complex valued functions. We can now act with the stability subgroup of ϵ^1 to bring ϵ_2 to a special form:

$$(1 + \mu X + \nu Y)(ae_1 + be_2) = be_2 + [a - 2b(\mu + i\nu)]e_1 \,.$$

In the case $b = 0$ this spinor is invariant, so the representative is $\epsilon^1 = 1$, $\epsilon_2 = ae_1$ (so that $\epsilon^2 = \bar{a}1$), with isotropy group $\mathbb{R}^2$. If $b \neq 0$, one can bring the spinor to the form be_2 (which implies $\epsilon^2 = -\bar{b}e_{12}$), with isotropy group $\mathbb{I}$. The representatives[4] together with the stability subgroups are summarized in table 1. Given a Killing spinor ϵ^i, one can construct the bilinear

$$V_A = A(\epsilon^i, \Gamma_A \epsilon_i) \,, \tag{3.4}$$

with the Majorana inner product A defined in (B.4), and the sum over i is understood. For $\epsilon_2 = ae_1$, V_A is lightlike, whereas for $\epsilon_2 = be_2$ it is timelike, see table 1. The existence of a globally defined Killing spinor ϵ^i, with isotropy group $G \in \mathrm{Spin}(3,1)$, gives rise to a

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[3]Our conventions for spinors and their description in terms of forms can be found in appendix B.

[4]Note the difference in form compared to the Killing spinors of the corresponding theories in five and six dimensions: in six dimensions these can be chosen constant [3] while in five dimensions they are constant up to an overall function [24]. In four dimensions such a choice is generically not possible.

(ϵ^1, ϵ_2)	$G \subset \mathrm{Spin}(3,1)$	$G \subset \mathrm{Spin}(3,1) \times \mathrm{U}(1)$	$V_A E^A = A(\epsilon^i, \Gamma_A \epsilon_i) E^A$
$(1,0)$	$\mathbb{R}^2$	$\mathrm{U}(1) \ltimes \mathbb{R}^2$	$-\sqrt{2} E^-$
$(1, ae_1)$	$\mathbb{R}^2$	$\mathbb{R}^2 \quad (a \in \mathbb{R})$	$-\sqrt{2}(1 + a^2) E^-$
$(1, be_2)$	$\mathbb{I}$	$\mathrm{U}(1)$	$\sqrt{2}(|b|^2 E^+ - E^-)$

Table 1: The representatives (ϵ^1, ϵ_2) of the orbits of Weyl spinors and their stability subgroups G under the gauge groups $\mathrm{Spin}(3,1)$ and $\mathrm{Spin}(3,1) \times \mathrm{U}(1)$ in the ungauged and $\mathrm{U}(1)$-gauged theories, respectively. The number of orbits is the same in both theories, the only difference lies in the stability subgroups and the fact that a is real in the gauged theory. In the last column we give the vectors constructed from the spinors.

G-structure. This means that we have an $\mathbb{R}^2$-structure in the null case and an identity structure in the timelike case.

In $\mathrm{U}(1)$ gauged supergravity, the local $\mathrm{Spin}(3,1)$ invariance is actually enhanced to $\mathrm{Spin}(3,1) \times \mathrm{U}(1)$. For $\mathrm{U}(1)$ Fayet-Iliopoulos terms, the moment maps satisfy (2.23), where we can choose $e^x = \delta_3^x$ without loss of generality. Then, under a gauge transformation

$$A_\mu^I \rightarrow A_\mu^I + \partial_\mu \alpha^I \,, \tag{3.5}$$

the Killing spinor ϵ^i transforms as

$$\epsilon^1 \rightarrow e^{-ig\xi_I \alpha^I} \epsilon^1 \,, \qquad \epsilon^2 \rightarrow e^{ig\xi_I \alpha^I} \epsilon^2 \,, \tag{3.6}$$

which can be easily seen from the supercovariant derivative (cf. eq. (2.19)). Note that ϵ^1 and ϵ^2 have opposite charges under the $\mathrm{U}(1)$. In order to obtain the stability subgroup, one determines the Lorentz transformations that leave the spinors ϵ^1 and ϵ^2 invariant up to a arbitrary phase factors $e^{i\psi}$ and $e^{-i\psi}$ respectively, which can then be gauged away using the additional $\mathrm{U}(1)$ symmetry. If $\epsilon_2 = 0$, one gets in this way an isotropy group generated by X, Y and Γ_{13} obeying

$$[\Gamma_{13}, X] = -2Y \,, \qquad [\Gamma_{13}, Y] = 2X \,, \qquad [X, Y] = 0 \,,$$

i. e. $G \cong \mathrm{U}(1) \ltimes \mathbb{R}^2$. For $\epsilon_2 = ae_1$ with $a \neq 0$, the stability subgroup $\mathbb{R}^2$ is not enhanced, whereas the $\mathbb{I}$ of the representative $(\epsilon^1, \epsilon_2) = (1, be_2)$ is promoted to $\mathrm{U}(1)$ generated by $\Gamma_{13} = i\Gamma_{\bullet\bullet}$. The Lorentz transformation matrix a_{AB} corresponding to $\Lambda = \exp(i\psi \Gamma_{\bullet\bullet}) \in \mathrm{U}(1)$, with $\Lambda \Gamma_B \Lambda^{-1} = a^A{}_B \Gamma_A$, has nonvanishing components

$$a_{+-} = a_{-+} = 1 \,, \qquad a_{\bullet\bar{\bullet}} = e^{2i\psi} \,, \qquad a_{\bar{\bullet}\bullet} = e^{-2i\psi} \,. \tag{3.7}$$

Finally, notice that in $\mathrm{U}(1)$ gauged supergravity one can choose the function a in $\epsilon_2 = ae_1$ real and positive: Write $a = R \exp(2i\delta)$, use

$$e^{\delta \Gamma_{13}} 1 = e^{i\delta} 1 \,, \qquad e^{\delta \Gamma_{13}} ae_1 = e^{-i\delta} ae_1 = e^{i\delta} Re_1 \,,$$

and gauge away the phase factor $\exp(i\delta)$ using the electromagnetic $\mathrm{U}(1)$.

Note that in the gauged theory the presence of G-invariant Killing spinors will in general not lead to a G-structure on the manifold but to stronger conditions. The structure group is in fact reduced to the intersection of G with $\text{Spin}(3,1)$, and hence is equal to the stability subgroup in the ungauged theory.

The representatives, stability subgroups and vectors constructed from the Killing spinors are summarized in table 1 both for the ungauged and the $\text{U}(1)$-gauged cases.

3.2 Generalized holonomy

The variation of the chiral gravitini under supersymmetry transformations is given by (2.18). This can be rewritten in terms of Majorana spinors $\psi_\mu^{\hat{i}} = \psi_\mu^i + \psi_{i\mu}$ and $\epsilon^{\hat{i}} = \epsilon^i + \epsilon_i$, where $\psi_{i\mu}$ and ϵ_i denote the charge conjugate of ψ_μ^i and ϵ^i respectively. One has then

$$\delta\psi_\mu^{\hat{i}} = \hat{\mathcal{D}}_\mu \epsilon^{\hat{i}} = \left(\partial_\mu + \frac{1}{4}\omega_\mu^{ab}\Gamma_{ab}\right)\epsilon^{\hat{i}} + \frac{i}{2}A_\mu\Gamma_5\epsilon^{\hat{i}} + \partial_\mu q^X \left[\text{Re}\,\omega_{X\,j}{}^i + i\Gamma_5\text{Im}\,\omega_{X\,j}{}^i\right]\epsilon^{\hat{j}}$$
$$+gA_\mu^I\left[\text{Re}P_{I\,j}{}^i + i\Gamma_5\text{Im}P_{I\,j}{}^i\right]\epsilon^{\hat{j}} + g\Gamma_\mu e^{\mathcal{K}/2}\left[\text{Re}\left(P_I^{ij}Z^I\right) - i\Gamma_5\text{Im}\left(P_I^{ij}Z^I\right)\right]\epsilon^{\hat{j}}$$
$$+\frac{1}{4}\Gamma\cdot\left[\text{Re}\left(F^{-I}Z^J\right) + i\Gamma_5\text{Im}\left(F^{-I}Z^J\right)\right]\epsilon^{ij}\Gamma_\mu\epsilon^{\hat{j}}(\text{Im}\mathcal{N})_{IJ}e^{\mathcal{K}/2}. \tag{3.8}$$

From this it is evident that the holonomy of the supercovariant derivative $\hat{\mathcal{D}}_\mu$ is contained in $\text{GL}(8,\mathbb{R})$, so that in principle one can have vacua that preserve any number N of supersymmetries with $N = 0, 1, \ldots 8$. In the case without hypermultiplets, and for $\text{U}(1)$ FI terms with $\vec{P}_I = \vec{e}\,\xi_I$ and $e^x = \delta_3^x$, it is instructive to rewrite everything using complex (Dirac) spinors $\psi_\mu = \psi_\mu^1 + \psi_{2\mu}$, $\epsilon = \epsilon^1 + \epsilon_2$.[5] This yields

$$\delta\psi_\mu = \left(\partial_\mu + \frac{1}{4}\omega_\mu^{ab}\Gamma_{ab}\right)\epsilon + \frac{i}{2}A_\mu\Gamma_5\epsilon + ig\xi_I A_\mu^I\epsilon + g\Gamma_\mu\xi_I\left[\text{Im}X^I + i\Gamma_5\text{Re}X^I\right]\epsilon$$
$$+\frac{i}{4}\Gamma\cdot\left[\text{Im}\left(F^{-I}X^J\right) - i\Gamma_5\text{Re}\left(F^{-I}X^J\right)\right](\text{Im}\mathcal{N})_{IJ}\Gamma_\mu\epsilon \tag{3.9}$$

as well as (introducing $\lambda^\alpha = \lambda_2^\alpha + \lambda_1^{\alpha C}$)

$$\delta\lambda^\alpha = \frac{i}{2}e^{\mathcal{K}/2}(\text{Im}\mathcal{N})_{IJ}\Gamma\cdot\left[\text{Im}\left(F^{-J}\mathcal{D}_{\bar\beta}\bar{Z}^I g^{\alpha\bar\beta}\right) - i\Gamma_5\text{Re}\left(F^{-J}\mathcal{D}_{\bar\beta}\bar{Z}^I g^{\alpha\bar\beta}\right)\right]\epsilon$$
$$+\Gamma^\mu\partial_\mu\left[\text{Re}z^\alpha - i\Gamma_5\text{Im}z^\alpha\right]\epsilon + 2ge^{\mathcal{K}/2}\xi_I\left[\text{Im}\left(\mathcal{D}_{\bar\beta}\bar{Z}^I g^{\alpha\bar\beta}\right) - i\Gamma_5\text{Re}\left(\mathcal{D}_{\bar\beta}\bar{Z}^I g^{\alpha\bar\beta}\right)\right]\epsilon.$$

We see that in this case the complex conjugate spinor ϵ^* does not appear in the variation of the fermions, so that the supercovariant derivative has smaller holonomy $\text{GL}(4,\mathbb{C}) \subseteq \text{GL}(8,\mathbb{R})$, and the number of preserved supercharges is necessarily even, $N = 0, 2, 4, 6, 8$. The generalized holonomy group for vacua preserving N supersymmetries is then $\text{GL}(\frac{8-N}{2},\mathbb{C}) \ltimes \frac{N}{2}\mathbb{C}^{\frac{8-N}{2}}$, like in minimal gauged supergravity [39, 34]. To see this, assume that there exists a Killing spinor ϵ_1.[6] By a local $\text{GL}(4,\mathbb{C})$ transformation, ϵ_1 can be brought to the form $\epsilon_1 = (1, 0, 0, 0)^T$. This is annihilated by matrices of the form

$$\mathcal{A} = \begin{pmatrix} 0 & \underline{a}^T \\ \underline{0} & A \end{pmatrix},$$

---

[5]Note that one can reconstuct ψ_μ^1 and $\psi_{2\mu}$ from ψ_μ by projecting on the two chiralities.

[6]The index of ϵ_1 here should not be confused with an $\text{SU}(2)$ index for chiral spinors.

that generate the affine group $A(3, \mathbb{C}) \cong GL(3, \mathbb{C}) \ltimes \mathbb{C}^3$. Now impose a second Killing spinor $\epsilon_2 = (\epsilon_2^0, \underline{\epsilon}_2)^T$. Acting with the stability subgroup of ϵ_1 yields

$$e^{\mathcal{A}} \epsilon_2 = \begin{pmatrix} \epsilon_2^0 + \underline{b}^T \underline{\epsilon}_2 \\ e^A \underline{\epsilon}_2 \end{pmatrix}, \qquad \text{where} \quad \underline{b}^T = \underline{a}^T A^{-1} \left(e^A - 1 \right).$$

We can choose $A \in gl(3, \mathbb{C})$ such that $e^A \underline{\epsilon}_2 = (1, 0, 0)^T$, and $\underline{b}$ such that $\epsilon_2^0 + \underline{b}^T \underline{\epsilon}_2 = 0$. This means that the stability subgroup of ϵ_1 can be used to bring ϵ_2 to the form $\epsilon_2 = (0, 1, 0, 0)$. The subgroup of $A(3, \mathbb{C})$ that stabilizes also ϵ_2 consists of the matrices

$$\begin{pmatrix} 1 & 0 & b_2 & b_3 \\ 0 & 1 & B_{12} & B_{13} \\ 0 & 0 & B_{22} & B_{23} \\ 0 & 0 & B_{32} & B_{33} \end{pmatrix} \quad \in GL(2, \mathbb{C}) \ltimes 2\mathbb{C}^2.$$

Finally, imposing a third Killing spinor yields $GL(1, \mathbb{C}) \ltimes 3\mathbb{C}$ as maximal generalized holonomy group, which is however not realized in $\mathcal{N} = 2$, $D = 4$ minimal gauged supergravity [11, 25].[7] It would be interesting to see whether genuine preons (i.e., 3/4 supersymmetric backgrounds that are not locally AdS) exist in matter-coupled supergravity.

4. Timelike representative $(\epsilon^1, \epsilon_2) = (1, be_2)$

In this section we will analyze the conditions coming from a single timelike Killing spinor, and determine all supersymmetric solutions in this class. We shall first keep things general, i. e., including hypermultiplets and a general gauging, and write down the linear system following from the Killing spinor equations. This system will then be solved for the case of $U(1)$ Fayet-Iliopoulos terms and without hypers, while the solution in the general case will be left for a future publication [35].

4.1 Conditions from the Killing spinor equations

From the vanishing of the hyperini variation one obtains

$$\frac{i}{\sqrt{2}} f_X^{A1} \mathcal{D}_\bullet q^X + \frac{ib}{\sqrt{2}} f_X^{A2} \mathcal{D}_- q^X - g \mathcal{N}^{2A} = 0, \tag{4.1}$$

$$-\frac{i}{\sqrt{2}} f_X^{A1} \mathcal{D}_+ q^X + \frac{ib}{\sqrt{2}} f_X^{A2} \mathcal{D}_{\bar{\bullet}} q^X - g\bar{b} \mathcal{N}^{1A} = 0, \tag{4.2}$$

whereas the gaugino variation yields

$$\bar{b} e^{\mathcal{K}/2} g^{\alpha\bar{\beta}} \mathcal{D}_{\bar{\beta}} \bar{Z}^I (\text{Im} \mathcal{N})_{IJ} (F^{-J\bullet\bar{\bullet}} - F^{-J+-}) - \sqrt{2} \mathcal{D}_+ z^\alpha - g\bar{b} N_{12}^\alpha = 0, \tag{4.3}$$

$$2\bar{b} e^{\mathcal{K}/2} g^{\alpha\bar{\beta}} \mathcal{D}_{\bar{\beta}} \bar{Z}^I (\text{Im} \mathcal{N})_{IJ} F^{-J+\bar{\bullet}} + \sqrt{2} \mathcal{D}_\bullet z^\alpha + g N_{11}^\alpha = 0, \tag{4.4}$$

$$e^{\mathcal{K}/2} g^{\alpha\bar{\beta}} \mathcal{D}_{\bar{\beta}} \bar{Z}^I (\text{Im} \mathcal{N})_{IJ} (F^{-J+-} - F^{-J\bullet\bar{\bullet}}) + b\sqrt{2} \mathcal{D}_- z^\alpha + g N_{21}^\alpha = 0, \tag{4.5}$$

---

[7]3/4 supersymmetric solutions of minimal gauged supergravity are necessarily quotients of AdS_4, which have been constructed in [40].

$$-2e^{\mathcal{K}/2}g^{\alpha\bar{\beta}}\mathcal{D}_{\bar{\beta}}\bar{Z}^I(\text{Im}\,\mathcal{N})_{IJ}F^{-J-\bullet} + b\sqrt{2}\mathcal{D}_{\bar{\bullet}}z^\alpha - g\bar{b}N_{22}^\alpha = 0\,. \qquad (4.6)$$

Finally, from the gravitini we get

$$\frac{1}{2}(\omega_+^{+-} - \omega_+^{\bullet\bar{\bullet}}) + \frac{i}{2}A_+ + \partial_+q^X\omega_{X\,1}{}^1 + gA_+^I P_{I\,1}{}^1$$
$$-\sqrt{2}gbS^{12} + \frac{b}{\sqrt{2}}(F^{-I+-} - F^{-I\bullet\bar{\bullet}})(\text{Im}\,\mathcal{N})_{IJ}Z^J e^{\mathcal{K}/2} = 0\,, \quad (4.7)$$

$$-\omega_+^{-\bullet} - \bar{b}\partial_+q^X\omega_{X\,2}{}^1 - g\bar{b}A_+^I P_{I\,2}{}^1 - \sqrt{2}bF^{-I-\bullet}(\text{Im}\,\mathcal{N})_{IJ}Z^J e^{\mathcal{K}/2} = 0\,, \quad (4.8)$$

$$-\bar{b}\omega_+^{+\bar{\bullet}} + \partial_+q^X\omega_{X\,1}{}^2 + gA_+^I P_{I\,1}{}^2 - \sqrt{2}gbS^{22} = 0\,, \quad (4.9)$$

$$-\partial_+\bar{b} - \frac{\bar{b}}{2}(\omega_+^{\bullet\bar{\bullet}} - \omega_+^{+-}) - \frac{i\bar{b}}{2}A_+ - \bar{b}\partial_+q^X\omega_{X\,2}{}^2 - g\bar{b}A_+^I P_{I\,2}{}^2 = 0\,, \quad (4.10)$$

$$\frac{1}{2}(\omega_-^{+-} - \omega_-^{\bullet\bar{\bullet}}) + \frac{i}{2}A_- + \partial_-q^X\omega_{X\,1}{}^1 + gA_-^I P_{I\,1}{}^1 = 0\,, \quad (4.11)$$

$$-\omega_-^{-\bullet} - \bar{b}\partial_-q^X\omega_{X\,2}{}^1 - \bar{b}gA_-^I P_{I\,2}{}^1 + \sqrt{2}gS^{11} = 0\,, \quad (4.12)$$

$$-\partial_-\bar{b} - \frac{\bar{b}}{2}(\omega_-^{\bullet\bar{\bullet}} - \omega_-^{+-}) - \frac{i\bar{b}}{2}A_- - \bar{b}\partial_-q^X\omega_{X\,2}{}^2 - \bar{b}gA_-^I P_{I\,2}{}^2$$
$$+\sqrt{2}gS^{21} + \frac{1}{\sqrt{2}}(F^{-I\bullet\bar{\bullet}} - F^{-I+-})(\text{Im}\,\mathcal{N})_{IJ}Z^J e^{\mathcal{K}/2} = 0\,, \quad (4.13)$$

$$-\bar{b}\omega_-^{+\bar{\bullet}} + \partial_-q^X\omega_{X\,1}{}^2 + gA_-^I P_{I\,1}{}^2 + \sqrt{2}F^{-I+\bar{\bullet}}(\text{Im}\,\mathcal{N})_{IJ}Z^J e^{\mathcal{K}/2} = 0\,, \quad (4.14)$$

$$\frac{1}{2}(\omega_\bullet^{+-} - \omega_\bullet^{\bullet\bar{\bullet}}) + \frac{i}{2}A_\bullet + \partial_\bullet q^X\omega_{X\,1}{}^1 + gA_\bullet^I P_{I\,1}{}^1 + \sqrt{2}bF^{-I+\bar{\bullet}}(\text{Im}\,\mathcal{N})_{IJ}Z^J e^{\mathcal{K}/2} = 0\,, \quad (4.15)$$

$$-\omega_\bullet^{-\bullet} - \bar{b}\partial_\bullet q^X\omega_{X\,2}{}^1 - \bar{b}gA_\bullet^I P_{I\,2}{}^1 - \sqrt{2}gbS^{12}$$
$$+\frac{b}{\sqrt{2}}(F^{-I\bullet\bar{\bullet}} - F^{-I+-})(\text{Im}\,\mathcal{N})_{IJ}Z^J e^{\mathcal{K}/2} = 0\,, \quad (4.16)$$

$$-\bar{b}\omega_\bullet^{+\bar{\bullet}} + \partial_\bullet q^X\omega_{X\,1}{}^2 + gA_\bullet^I P_{I\,1}{}^2 = 0\,, \quad (4.17)$$

$$-\partial_\bullet\bar{b} - \frac{\bar{b}}{2}(\omega_\bullet^{\bullet\bar{\bullet}} - \omega_\bullet^{+-}) - \frac{i\bar{b}}{2}A_\bullet - \bar{b}\partial_\bullet q^X\omega_{X\,2}{}^2 - g\bar{b}A_\bullet^I P_{I\,2}{}^2 - \sqrt{2}gbS^{22} = 0\,, \quad (4.18)$$

$$\frac{1}{2}(\omega_{\bar{\bullet}}^{+-} - \omega_{\bar{\bullet}}^{\bullet\bar{\bullet}}) + \frac{i}{2}A_{\bar{\bullet}} + \partial_{\bar{\bullet}}q^X\omega_{X\,1}{}^1 + gA_{\bar{\bullet}}^I P_{I\,1}{}^1 - \sqrt{2}gS^{11} = 0\,, \quad (4.19)$$

$$-\omega_{\bar{\bullet}}^{-\bullet} - \bar{b}\partial_{\bar{\bullet}}q^X\omega_{X\,2}{}^1 - \bar{b}gA_{\bar{\bullet}}^I P_{I\,2}{}^1 = 0\,, \quad (4.20)$$

$$-\bar{b}\omega_{\bar{\bullet}}^{+\bar{\bullet}} + \partial_{\bar{\bullet}}q^X\omega_{X\,1}{}^2 + gA_{\bar{\bullet}}^I P_{I\,1}{}^2 - \sqrt{2}gS^{21}$$
$$-\frac{1}{\sqrt{2}}(F^{-I+-} - F^{-I\bullet\bar{\bullet}})(\text{Im}\,\mathcal{N})_{IJ}Z^J e^{\mathcal{K}/2} = 0\,, \quad (4.21)$$

$$-\partial_{\bar{\bullet}}\bar{b} - \frac{\bar{b}}{2}(\omega_{\bar{\bullet}}^{\bullet\bar{\bullet}} - \omega_{\bar{\bullet}}^{+-}) - \frac{i\bar{b}}{2}A_{\bar{\bullet}} - \bar{b}\partial_{\bar{\bullet}}q^X\omega_{X\,2}{}^2$$
$$-\bar{b}gA_{\bar{\bullet}}^I P_{I\,2}{}^2 + \sqrt{2}F^{-I-\bullet}(\text{Im}\,\mathcal{N})_{IJ}Z^J e^{\mathcal{K}/2} = 0\,. \quad (4.22)$$

4.2 Geometry of spacetime

In order to obtain the spacetime geometry, we consider the spinor bilinears

$$V_\mu{}^i{}_j = A(\epsilon^i, \Gamma_\mu\epsilon_j)\,, \qquad (4.23)$$

where the Majorana inner product A is defined in (B.4). The nonvanishing components are

$$V_-{}^1{}_1 = -\sqrt{2}\,, \qquad V_+{}^2{}_2 = \sqrt{2}\bar{b}b\,, \qquad V_\bullet{}^1{}_2 = \sqrt{2}b\,, \qquad V_\bullet{}^2{}_1 = \sqrt{2}\bar{b}\,. \qquad (4.24)$$

Note that $V_\mu{}^i{}_j = V_\mu{}^j{}_i{}^*$, so that we can expand into a basis of hermitian matrices,

$$V_\mu{}^i{}_j = \frac{1}{2}V_\mu\delta^i{}_j + \vec{V}_\mu\cdot\vec{\sigma}_j{}^i\,. \tag{4.25}$$

This yields for the trace part

$$V_\mu dx^\mu = \sqrt{2}(|b|^2 E^+ - E^-)\,, \tag{4.26}$$

while the nonzero components of $\vec{V}_\mu$ read

$$V_\bullet^1 = \frac{b}{\sqrt{2}}\,,\quad V_{\bar\bullet}^1 = \frac{\bar b}{\sqrt{2}}\,,\quad V_\bullet^2 = -\frac{ib}{\sqrt{2}}\,,\quad V_{\bar\bullet}^2 = \frac{i\bar b}{\sqrt{2}}\,,\quad V_+^3 = -\frac{\bar b b}{\sqrt{2}}\,,\quad V_-^3 = -\frac{1}{\sqrt{2}}\,.$$

Using the identities

$$\omega_{X\,i}{}^{j\,*} = -\omega_{X\,j}{}^i\,,\qquad P_{I\,i}{}^{j\,*} = -P_{I\,j}{}^i\,,\qquad S^{ij} = S^{ji}\,,$$

it is straightforward to shew that the linear system (4.7) - (4.22) implies the following constraints on the spin connection:

$$\omega_+^{+-} = \partial_+\ln(\bar b b) = \partial_-(\bar b b)\,,\qquad \omega_-^{+-} = 0\,,\qquad -\bar b b\,\omega^{+\bar\bullet} + \omega_-^{-\bar\bullet} - \omega_\bullet^{+-} = 0\,,$$
$$\bar b b\,\omega_\bullet^{+-} - \partial_\bullet(\bar b b) - \omega_+^{-\bar\bullet} + \bar b b\,\omega_+^{+\bar\bullet} = 0\,,\qquad -\bar b b\,\omega_\bullet^{+\bar\bullet} + \omega_\bullet^{-\bar\bullet} = 0\,,$$
$$-\bar b b(\omega_\bullet^{+\bar\bullet} + \omega_\bullet^{+\bullet}) + \omega_\bullet^{-\bullet} + \omega_\bullet^{-\bar\bullet} = 0\,. \tag{4.27}$$

These are ten real equations, which are easily shown to be equivalent to

$$\partial_A V_B + \partial_B V_A - \omega^C{}_{B|A}V_C - \omega^C{}_{A|B}V_C = 0\,, \tag{4.28}$$

which means that V is Killing. Note that $V^2 = -4\bar b b$, so V is timelike. Moreover, one verifies that the system (4.7) - (4.22) yields the relations

$$dV^x + \epsilon^{xyz}\mathcal{A}^y \wedge V^z = T^x\,, \tag{4.29}$$

with the gauged SU(2) connection

$$\vec{\mathcal{A}}_\mu = 2\partial_\mu q^X \vec{\omega}_X + 2g A_\mu^I \vec{P}_I\,, \tag{4.30}$$

where we switched from SU(2) indices to vector quantities using the conventions of appendix A. The torsion tensor[8] T^x can be written as

$$T^x = -\epsilon^{xyz}B^y \wedge V^z\,, \tag{4.31}$$

with the one-form B^y given by

$$B_+^1 = -2\sqrt{2}g\,\mathrm{Im}(bS^{22})\,,\quad B_-^1 = -2\sqrt{2}g\,\mathrm{Im}\left(\frac{S^{11}}{\bar b}\right)\,,\quad B_\bullet^1 = -\frac{2\sqrt{2}gi}{\bar b}\,\mathrm{Re}(bS^{12})\,,$$
$$B_+^2 = -2\sqrt{2}g\,\mathrm{Re}(bS^{22})\,,\quad B_-^2 = 2\sqrt{2}g\,\mathrm{Re}\left(\frac{S^{11}}{\bar b}\right)\,,\quad B_\bullet^2 = -iB_\bullet^1\,,$$
$$B_+^3 = -2\sqrt{2}g\,\mathrm{Im}(bS^{12})\,,\quad B_-^3 = -\frac{B_+^3}{\bar b b}\,,\qquad\qquad B_\bullet^3 = \frac{\sqrt{2}gi}{\bar b}(bS^{22} - \bar b\bar S^{11})\,. \tag{4.32}$$

---

[8]The reason for choosing this name will be explained in appendix D.

Notice that we are free to include the torsion term in the SU(2) connection, by rewriting (4.29) as

$$dV^x + \epsilon^{xyz}(\mathcal{A}^y + B^y) \wedge V^z = 0\,, \tag{4.33}$$

so that the forms V^x are actually SU(2)-covariantly closed, similar to the ungauged case [15]. If we define

$$\mathcal{A}_\mu^\pm \equiv \mathcal{A}_\mu^1 \pm i\mathcal{A}_\mu^2\,,$$

and similar for B, eqs. (4.9), (4.12), (4.17) and (4.20) can be cast into the form

$$b\,\omega_+^{+\bullet} = -\frac{i}{2}(\mathcal{A}_+^+ + B_+^+)\,, \qquad\qquad \omega_-^{-\bar{\bullet}} = \frac{ib}{2}(\mathcal{A}_-^- + B_-^-)\,,$$

$$\omega_\bullet^{-\bar{\bullet}} = \frac{ib}{2}(\mathcal{A}_\bullet^- + B_\bullet^-)\,, \qquad\qquad b\,\omega_{\bar{\bullet}}^{+\bullet} = -\frac{i}{2}(\mathcal{A}_{\bar{\bullet}}^+ + B_{\bar{\bullet}}^+)\,. \tag{4.34}$$

These equations relate the SU(2) to the spin connection, and tell us how the former is embedded into the latter. Such an embedding is necessary for unbroken supersymmetry; it leads to a (partial) cancellation of the SU(2) and spin connections in the gravitino supersymmetry transformation, and generalizes the mechanism of [41, 42].

Let us now choose coordinates (t, x^1, x^2, x^3) such that $V = \partial_t$. The metric will then be independent of t. Note that $\partial_t = \sqrt{2}\,(|b|^2\partial_- - \partial_+)$. Making use of

$$\omega_{X\,i}{}^i = P_{I\,i}{}^i = 0\,,$$

eqs. (4.7), (4.10), (4.11) and (4.13) give

$$\partial_t \ln b = iA_t\,, \tag{4.35}$$

whose real part implies that $|b|$ is time-independent. In terms of the vierbein E_μ^A the metric is given by

$$ds^2 = 2E^+E^- + 2E^\bullet E^{\bar{\bullet}}\,, \tag{4.36}$$

where

$$E_\mu^+ = \frac{V_\mu - 2V_\mu^3}{2\sqrt{2}|b|^2}\,, \qquad E_\mu^- = -\frac{V_\mu + 2V_\mu^3}{2\sqrt{2}}\,, \qquad E_\mu^\bullet = \frac{V_\mu^1 + iV_\mu^2}{\sqrt{2}b}\,, \qquad E_\mu^{\bar{\bullet}} = \frac{V_\mu^1 - iV_\mu^2}{\sqrt{2}\bar{b}}\,.$$

From $V^2 = -4|b|^2$ and $V = \partial_t$ as a vector we get $V_t = -4|b|^2$, so that $V = -4|b|^2(dt + \sigma)$ as a one-form, with $\sigma_t = 0$. Furthermore, $V^\bullet = 0$ yields $E_t^\bullet = 0$ and thus $V_t^1 = V_t^2 = 0$. Since V and V^3 are orthogonal, $V^\mu V_\mu^3 = 0$, also V_t^3 vanishes, and hence $V_t^x = 0$. The metric (4.36) becomes thus

$$ds^2 = -4|b|^2(dt + \sigma)^2 + |b|^{-2}\delta_{xy}V^xV^y\,. \tag{4.37}$$

In order to proceed one would like to choose the gauge $\mathcal{A}_t^x + B_t^x = 0$, which reduces to the choice made in [15] for $g \to 0$. Then the SU(2)-covariant closure of the V^x (eq. (4.33)) states that the SU(2) connection $\mathcal{A} + B$ and the V^x are time-independent. Eq. (4.33) can then be interpreted as Cartan's first structure equation on the three-dimensional base space. One therefore has to show that the above gauge is always possible. Let us at this point restrict

to the case without hypers and no gauging of special Kähler isometries ($k_I^\alpha = 0$). The inclusion of hypermultiplets will be studied in a forthcoming publication [35]. This leaves two possible solutions for the moment maps [37], namely SU(2) or U(1) Fayet-Iliopoulos (FI) terms. We shall consider here the latter, which satisfy (2.23), where $e^x = \delta_3^x$ without loss of generality.[9] One has then

$$
\begin{aligned}
P_{I\,1}{}^1 = -P_{I\,2}{}^2 = i\,\xi_I \,, &\qquad P_{I\,1}{}^2 = P_{I\,2}{}^1 = 0 \,, \\
S^{12} = S^{21} = i\,\xi_I Z^I e^{\mathcal{K}/2} \,, &\qquad S^{11} = S^{22} = 0 \,,
\end{aligned}
\tag{4.38}
$$

as well as

$$
\mathcal{A}_\mu^1 = \mathcal{A}_\mu^2 = 0 \,, \qquad \mathcal{A}_\mu^3 = 2g A_\mu^I \xi_I \,.
\tag{4.39}
$$

From (4.33) one obtains $dV^3 = 0$, like in minimal gauged supergravity [7, 34]. If we choose the gauge $\mathcal{A}_t^3 + B_t^3 = 0$, the one-forms V^x will be time-independent. Note that the U(1) gauge transformation (3.5) necessary to achieve this does not spoil our choice of representatives: As discussed in section 3.1, the phase factors acquired by the Killing spinors ϵ^i (eq. (3.6)) can be eliminated by a subsequent Spin(3,1) transformation. The above gauge condition implies

$$
A_t^I \xi_I = -4\,\mathrm{Im}(bS^{12}) \,,
\tag{4.40}
$$

and is left invariant by transformations (3.5) with time-independent $\xi_I \alpha^I$. As the SU(2) connection $\mathcal{A} + B$ and the V^x do not depend on t, one can regard (4.33) as Cartan's first structure equation on the three-dimensional base manifold with metric $\delta_{xy} V^x V^y$.

Next we consider the equations coming from the gaugino variation. Using

$$
N_{11}^\alpha = N_{22}^\alpha = 0 \,, \qquad N_{12}^\alpha = N_{21}^\alpha = -2i\,\xi_I e^{\mathcal{K}/2} \mathcal{D}_{\bar\beta} \bar{Z}^I g^{\alpha\bar\beta} \,,
$$

and $\mathcal{D}_\mu z^\alpha = \partial_\mu z^\alpha$, eqs. (4.3) and (4.5) yield

$$
\partial_t z^\alpha = 0 \,,
$$

i. e., the scalar fields are time-independent. Choosing the constants $C_{I,JK} = 0$ and taking into account that the structure constants $f_{IJ}{}^K$ vanish also, eqs. (2.21) and (2.22) imply for the moment map function $P_I^0 = 0$. But then from (2.20) one has for the Kähler U(1)

$$
A_t = -\frac{i}{2}(\partial_\alpha \mathcal{K} \partial_t z^\alpha - \partial_{\bar\alpha} \mathcal{K} \partial_t \bar{z}^{\bar\alpha}) = 0 \,.
$$

Plugging this into (4.35) gives $\partial_t b = 0$, hence b is time-independent as well.

Notice that the system (4.7) - (4.22) allows to express the linear combinations $A^I \xi_I$ and $F^{-I}(\mathrm{Im}\,\mathcal{N})_{IJ} Z^J$ of the gauge potentials and fluxes in terms of the spin connection, the Kähler U(1), the linear combination of scalars $Z^I \xi_I$ and the function b,

$$
\begin{aligned}
ig A_+^I \xi_I = \frac{1}{2}\omega_+^{\bullet\bar\bullet} + \frac{i}{2}A_+ - \frac{1}{2}\partial_+ \ln\frac{b}{\bar{b}} \,, &\qquad ig A_-^I \xi_I = \frac{1}{2}\omega_-^{\bullet\bar\bullet} - \frac{i}{2}A_- \,, \\
ig A_\bullet^I \xi_I = \frac{1}{2}(\omega_\bullet^{+-} + \omega_\bullet^{\bullet\bar\bullet}) - \frac{i}{2}A_\bullet \,,
\end{aligned}
\tag{4.41}
$$

---

[9]$e^x = \delta_3^x$ can always be achieved by a global SU(2) rotation (which is a symmetry of the theory).

$$F^{-I+\bar{\bullet}}(\text{Im}\,\mathcal{N})_{IJ}Z^J e^{\mathcal{K}/2} = \frac{\bar{b}}{\sqrt{2}}\omega_-^{+\bar{\bullet}}\,, \quad F^{-I-\bullet}(\text{Im}\,\mathcal{N})_{IJ}Z^J e^{\mathcal{K}/2} = -\frac{1}{\sqrt{2}b}\omega_+^{-\bullet}\,,$$

$$(F^{-I+-} - F^{-I\bullet\bar{\bullet}})(\text{Im}\,\mathcal{N})_{IJ}Z^J e^{\mathcal{K}/2} = -\sqrt{2}\bar{b}\,\omega_\bullet^{+\bar{\bullet}} - 2ig\,\xi_I e^{\mathcal{K}/2}Z^I\,. \tag{4.42}$$

As the $(n_V+1)\times(n_V+1)$ matrix $(X^I, \mathcal{D}_{\bar{\alpha}}\bar{X}^I)$ is invertible [37], (4.42) together with the gaugino equations (4.3)-(4.6) determine uniquely the fluxes F^{-I}, with the result[10]

$$F^{-I+\bar{\bullet}} = \frac{\sqrt{2}}{b}\bar{X}^I(\partial_\bullet \ln\bar{b} + iA_\bullet) + \frac{\sqrt{2}}{\bar{b}}\mathcal{D}_\alpha X^I \partial_\bullet z^\alpha\,,$$

$$F^{-I-\bullet} = -\sqrt{2}\,\bar{b}\bar{X}^I(\partial_{\bar{\bullet}}\ln\bar{b} + iA_{\bar{\bullet}}) - \sqrt{2}\,b\mathcal{D}_\alpha X^I \partial_{\bar{\bullet}}z^\alpha\,, \tag{4.43}$$

$$F^{-I+-} - F^{-I\bullet\bar{\bullet}} = \frac{2\sqrt{2}}{b}\bar{X}^I(\partial_+\ln\bar{b} + iA_+) + \frac{2\sqrt{2}}{\bar{b}}\mathcal{D}_\alpha X^I \partial_+ z^\alpha + 2ig\,\xi_J(\text{Im}\,\mathcal{N})^{-1|IJ}\,.$$

Moreover, antiselfduality implies that

$$F^{-I+\bullet} = F^{-I-\bar{\bullet}} = F^{-I+-} + F^{-I\bullet\bar{\bullet}} = 0\,.$$

With (4.43), all gaugino equations are satisfied.

Furthermore, the system (4.7) - (4.22) determines almost all components of the spin connection (with the exception of $\omega^{\bullet\bar{\bullet}}$) in terms of A_μ, $Z^I\xi_I$, the function b and its spacetime derivatives,

$$\omega_+^{+-} = \partial_+\ln(\bar{b}b)\,, \qquad \omega_-^{+-} = 0\,, \qquad\qquad \omega_\bullet^{+-} = \partial_\bullet \ln\bar{b} + iA_\bullet\,,$$

$$\omega_+^{+\bullet} = \omega_\bullet^{+\bullet} = 0\,, \qquad\qquad\qquad\qquad \omega_-^{+\bullet} = -\frac{1}{\bar{b}b}(\partial_{\bar{\bullet}}\ln b - iA_{\bar{\bullet}})\,,$$

$$\omega_\bullet^{+\bullet} = \partial_-\ln b - iA_- + \frac{2\sqrt{2}ig}{b}\xi_I e^{\mathcal{K}/2}\bar{Z}^I\,,$$

$$\omega_+^{-\bullet} = -b\,\partial_{\bar{\bullet}}\bar{b} - i\bar{b}bA_{\bar{\bullet}}\,, \qquad\qquad\qquad \omega_-^{-\bullet} = \omega_\bullet^{-\bullet} = 0\,,$$

$$\omega_\bullet^{-\bullet} = \partial_+\ln\bar{b} + iA_+ - 2\sqrt{2}gbi\,\xi_I e^{\mathcal{K}/2}Z^I\,. \tag{4.44}$$

From the gauge condition (4.40) we obtain one more component, namely

$$\omega_t^{\bullet\bar{\bullet}} = \sqrt{2}(|b|^2\omega_-^{\bullet\bar{\bullet}} - \omega_+^{\bullet\bar{\bullet}}) = -\sqrt{2}\,\partial_+\ln\frac{b}{\bar{b}} + 2\sqrt{2}iA_+ - 4ig\xi_I(bX^I + \bar{b}\bar{X}^I)\,. \tag{4.45}$$

The next step is to impose vanishing spacetime torsion,

$$\partial_\mu E_\nu^A - \partial_\nu E_\mu^A + \omega_{\mu B}^A E_\nu^B - \omega_{\nu B}^A E_\mu^B = 0\,.$$

One finds that most of these equations are already identically satisfied, while the remaining ones yield (using the expressions (4.44) for the spin connection)

$$d\sigma + \zeta^x \epsilon^{xyz}V^y \wedge V^z = 0\,, \tag{4.46}$$

where the (real) SU(2) vector ζ^x is defined as

$$\zeta^1 + i\zeta^2 = -\frac{1}{\sqrt{2}\,\bar{b}^2 b}\left(\frac{i}{2}\partial_\bullet \ln\frac{b}{\bar{b}} + A_\bullet\right)\,,$$

$$\zeta^3 = \frac{1}{\sqrt{2}|b|^2}\left[\frac{i}{2}\partial_-\ln\frac{b}{\bar{b}} + A_- - \sqrt{2}g\,\xi_I e^{\mathcal{K}/2}\left(\frac{\bar{Z}^I}{b} + \frac{Z^I}{\bar{b}}\right)\right]\,. \tag{4.47}$$

---

[10]To get this, one has to use (C.15).

We already noted that $dV^3 = 0$, hence there exists a function z such that $V^3 = dz$ locally. Since $V_t^3 = 0$, z must be time-independent. Let us use z as one of the coordinates x^1, x^2, x^3, say $z = x^3$. The remaining spatial coordinates will be denoted by late small latin indices $m, n, \ldots$, i. e., $x^m = x^1, x^2$, while capital late latin indices $M, N, \ldots = 1, 2$ refer to the corresponding tangent space. One can eliminate the components V_z^M by a diffeomorphism

$$x^m = x^m(x'^n, z) \,,$$

with

$$V_m^M \frac{\partial x^m}{\partial z} = -V_z^M \,.$$

As the matrix V_m^M is invertible,[11] one can always solve for $\partial x^m / \partial z$. Notice that the metric (4.37) is invariant under

$$t \to t + \chi(x^m, z) \,, \qquad \sigma \to \sigma - d\chi \,,$$

for an arbitrary function $\chi(x^m, z)$. This second gauge freedom can be used to set $\sigma_z = 0$. Thus, without loss of generality, we can take $\sigma = \sigma_m dx^m$, and the metric (4.37) becomes

$$ds^2 = -4|b|^2(dt + \sigma_m dx^m)^2 + |b|^{-2} \left(dz^2 + \delta_{MN} V^M V^N \right) \,. \tag{4.48}$$

The solution of the Cartan structure equation (4.29) is then given by

$$V_m^1 + iV_m^2 = (\hat{V}_m^1 + i\hat{V}_m^2) \left(\frac{b}{\bar{b}} \right)^{\frac{1}{2}} \exp \Phi \,, \tag{4.49}$$

$$\omega_\bullet^{\bullet\bar{\bullet}} = \frac{|b|}{\sqrt{2}} e^{-\Phi} (\hat{V}_1^m - i\hat{V}_2^m) \left[-i\hat{\omega}_m + \partial_m(\bar{\Phi} - \ln|b|) \right] \,, \tag{4.50}$$

where $\hat{V}_m^M$ denote integration "constants" depending only on x^n but not on z, $\hat{V}_M^m$ is the corresponding inverse zweibein, Φ is a complex function defined by

$$\partial_z \Phi = 2ig\xi_I \left(\frac{\bar{X}^I}{b} - \frac{X^I}{\bar{b}} \right) - \omega_z^{\bullet\bar{\bullet}} \,, \tag{4.51}$$

and $\hat{\omega} \equiv \hat{\omega}^{12}$ is the spin connection following from the zweibein $\hat{V}^M$. At this point it is convenient to use the residual U(1) gauge freedom of a combined local Lorentz and electromagnetic gauge transformation to eliminate $\omega_z^{\bullet\bar{\bullet}}$. This is accomplished by the transformation (3.7), with

$$\psi = \frac{i}{2} \int dz \, \omega_z^{\bullet\bar{\bullet}} \,.$$

Note that ψ is real, as it must be. Moreover, as ψ is time-independent, this does not spoil the gauge choice (4.40). With $\omega_z^{\bullet\bar{\bullet}} = 0$, Φ is real. In what follows, we shall introduce complex coordinates $w = x^1 + ix^2$, $\bar{w} = x^1 - ix^2$, and choose the conformal gauge for the two-metric $\delta_{MN} \hat{V}_m^M \hat{V}_n^N$, i. e.,

$$\delta_{MN} \hat{V}_m^M \hat{V}_n^N = e^{2\gamma} dw d\bar{w} \,, \tag{4.52}$$

---

[11]This follows from $\sqrt{-g} = 2 \det(V_m^M)/|b|^2$.

where $\gamma = \gamma(w, \bar{w})$. From (4.51) it is clear that Φ is defined only up to an arbitrary function of $w, \bar{w}$. This allows to absorb γ into Φ, so one can take $\gamma = 0$ without loss of generality. Then the metric (4.48) simplifies to

$$ds^2 = -4|b|^2(dt + \sigma)^2 + |b|^{-2} \left(dz^2 + e^{2\Phi} dw d\bar{w} \right) , \qquad (4.53)$$

with $\sigma = \sigma_w dw + \sigma_{\bar{w}} d\bar{w}$.

Defining the symplectic vector

$$\mathcal{I} = \text{Im} \left(\mathcal{V}/\bar{b} \right) , \qquad (4.54)$$

where $\mathcal{V}$ is given in (2.1), eq. (4.46) can be cast into the form

$$d\sigma + 2 \star^{(3)} \langle \mathcal{I} , d\mathcal{I} \rangle - \frac{ig}{|b|^2} \xi_I \left(\frac{\bar{X}^I}{b} + \frac{X^I}{\bar{b}} \right) e^{2\Phi} dw \wedge d\bar{w} = 0. \qquad (4.55)$$

Here $\star^{(3)}$ is the Hodge star on the three-dimensional base with dreibein V^x. In the ungauged case $g = 0$, (4.55) reduces correctly to the expression given in [15].

All that remains to be done at this point is to impose the Bianchi identities and the Maxwell equations, which read respectively

$$dF^I = 0, \qquad d\,\text{Re}\,G_I^+ = 0, \qquad (4.56)$$

where $G_I^{\pm} = \mathcal{N}_{IJ} F^{\pm J}$. One finds that the Bianchi identities are equivalent to

$$4\partial\bar{\partial} \left(\frac{X^I}{\bar{b}} - \frac{\bar{X}^I}{b} \right) + \partial_z \left[e^{2\Phi} \partial_z \left(\frac{X^I}{\bar{b}} - \frac{\bar{X}_I}{b} \right) \right] \qquad (4.57)$$

$$-2ig\xi_J \partial_z \left\{ e^{2\Phi} \left[|b|^{-2}(\text{Im}\mathcal{N})^{-1|IJ} + 2 \left(\frac{X^I}{\bar{b}} + \frac{\bar{X}^I}{b} \right) \left(\frac{X^J}{\bar{b}} + \frac{\bar{X}^J}{b} \right) \right] \right\} = 0,$$

while the Maxwell equations yield

$$4\partial\bar{\partial} \left(\frac{F_I}{\bar{b}} - \frac{\bar{F}_I}{b} \right) + \partial_z \left[e^{2\Phi} \partial_z \left(\frac{F_I}{\bar{b}} - \frac{\bar{F}_I}{b} \right) \right]$$

$$-2ig\xi_J \partial_z \left\{ e^{2\Phi} \left[|b|^{-2}\text{Re}\,\mathcal{N}_{IL}(\text{Im}\mathcal{N})^{-1|JL} + 2 \left(\frac{F_I}{\bar{b}} + \frac{\bar{F}_I}{b} \right) \left(\frac{X^J}{\bar{b}} + \frac{\bar{X}^J}{b} \right) \right] \right\}$$

$$-8ig\xi_I e^{2\Phi} \left[\langle \mathcal{I} , \partial_z \mathcal{I} \rangle - \frac{g}{|b|^2} \xi_J \left(\frac{X^J}{\bar{b}} + \frac{\bar{X}^J}{b} \right) \right] = 0. \quad (4.58)$$

Here we defined $\partial = \partial_w$, $\bar{\partial} = \partial_{\bar{w}}$. Note that imposing $dF^I = 0$ is actually not sufficient; we must also ensure that $\xi_I F^I = \xi_I dA^I$, because the linear combination $\xi_I A^I$ is determined by the Killing spinor equations (cf. eq. (4.41)). This gives the additional condition

$$2\partial\bar{\partial}\Phi = ge^{2\Phi} \left[i\xi_I \partial_z \left(\frac{X^I}{\bar{b}} - \frac{\bar{X}^I}{b} \right) + 2g|b|^{-2}\xi_I \xi_J (\text{Im}\mathcal{N})^{-1|IJ} + 4g \left(\frac{\xi_I X^I}{\bar{b}} + \frac{\xi_I \bar{X}^I}{b} \right)^2 \right] , \quad (4.59)$$

which is slightly stronger than the contraction of (4.57) with ξ_I.[12]

_____

[12]Contracting (4.57) with ξ_I and using (4.51), one gets the derivative of (4.59) with respect to z.

Finally, note that the integrability condition for (4.55), namely

$$2\langle \mathcal{I}\,, \Delta^{(3)}\mathcal{I}\rangle = \star^{(3)} d\left[ig|b|^{-2}\xi_I \left(\frac{X^I}{\bar{b}} + \frac{\bar{X}^I}{b} \right) e^{2\Phi} dw \wedge d\bar{w} \right] ,\qquad (4.60)$$

where $\Delta^{(3)}$ denotes the Laplacian on the three-dimensional base manifold, follow from the Bianchi identities and the Maxwell equations. One can show this by using some relations of special Kähler geometry.

In conclusion, the functions b and Φ together with the scalar fields are determined by the equations (4.51), (4.57), (4.58) and (4.59). Then, the shift vector σ is obtained from (4.55) and the metric is given by (4.53). The gauge fields can be read off from (4.43), which can be rewritten as

$$
\begin{aligned}
F^I = {} & 2(dt+\sigma) \wedge d\big[bX^I + \bar{b}\bar{X}^I\big] + |b|^{-2}dz \wedge d\bar{w}\big[\bar{X}^I(\bar{\partial}\bar{b}+iA_{\bar{w}}\bar{b}) + (\mathcal{D}_\alpha X^I)b\bar{\partial}\bar{z}^\alpha \\
& -X^I(\bar{\partial}b - iA_{\bar{w}}b) - (\mathcal{D}_{\bar{\alpha}}\bar{X}^I)\bar{b}\bar{\partial}\bar{z}^{\bar{\alpha}}\big] - |b|^{-2}dz \wedge dw\,\big[\bar{X}^I(\partial\bar{b}+iA_w\bar{b}) \\
& \qquad\qquad + (\mathcal{D}_\alpha X^I)b\partial z^\alpha - X^I(\partial b - iA_w b) - (\mathcal{D}_{\bar{\alpha}}\bar{X}^I)\bar{b}\partial\bar{z}^{\bar{\alpha}}\big] \\
& -\frac{1}{2}|b|^{-2}e^{2\Phi}dw \wedge d\bar{w}\,\big[\bar{X}^I(\partial_z\bar{b}+iA_z\bar{b}) + (\mathcal{D}_\alpha X^I)b\partial_z z^\alpha - X^I(\partial_z b - iA_z b) \\
& \qquad\qquad - (\mathcal{D}_{\bar{\alpha}}\bar{X}^I)\bar{b}\partial_z\bar{z}^{\bar{\alpha}} - 2ig\xi_J(\mathrm{Im}\,\mathcal{N})^{-1|IJ}\big] .
\end{aligned}
\qquad (4.61)
$$

Notice that, in the timelike case, the vanishing of the supersymmetry variations, together with the Bianchi identities and the Maxwell equations, imply all the equations of motion. This is shown in appendix C.

In a forthcoming paper [43] we shall consider various models (specified by a certain prepotential), and give explicit solutions of the above equations that represent supersymmetric AdS black holes with nontrivial scalar fields turned on.

5. Final remarks

In this paper, we applied spinorial geometry techniques to classify all supersymmetric solutions of $\mathcal{N} = 2$ gauged supergravity in four dimensions coupled to abelian vector multiplets. Our results can be used to construct new BPS black holes in AdS$_4$ with nonconstant scalars. Such solutions are, to the best of our knowledge, unknown up to now, and would be important to study the attractor mechanism in AdS [44]. This will be the subject of a future publication [43].

Possible extensions of our work could be to impose the existence of more than one Killing spinor and to determine how this constrains further the geometry of supersymmetric backgrounds, as was done in the minimal case in [34]. It would also be interesting to see if nontrivial preons (i.e., solutions with nearly maximal supersymmetry that are not simply quotients of AdS) exist in matter-coupled gauged supergravity.

In refs. [45, 46], the $\mathcal{N} = 2$, $D = 4$ theory coupled to non-abelian vector multiplets with a gauge group that includes an SU(2) factor was considered, and various supersymmetric solutions, such as embeddings of the 't Hooft-Polyakov monopole and extremal black holes were obtained. These geometries are asymptotically flat, and it would be very interesting

to find similar solutions in the asymptotically AdS case, for instance in $\mathcal{N}=2$ supergravity where the full SU(2) R-symmetry is gauged, which can induce a negative cosmological constant. There are only very few analytically known Einstein-Yang-Mills black holes, and to dispose of more solutions would of course be helpful in probing the validity of the no-hair conjecture. Of particular relevance in this context are black holes with AdS asymptotics, which were recently argued to require an infinite number of parameters for their description [47]. This is one of the reasons that make it desirable to systematically classify all supersymmetric backgrounds of $\mathcal{N}=2$, $D=4$ supergravity with general gauging. Work in this direction is in progress [35].

Acknowledgments

A. Conventions

We use the notations and conventions of [37], which are briefly summarized here. More details can be found in appendix A of [37].

The signature is mostly plus. Late greek letters $\mu, \nu, \ldots$ are curved spacetime indices, while early latin letters $a, b, \ldots = 0, \ldots, 3$ and $A, B, \ldots = +, -, \bullet, \bar{\bullet}$ refer to the corresponding tangent space, cf. also appendix B.

Self-dual and anti-self-dual field strengths are defined by

$$F_{ab}^{\pm I} = \frac{1}{2}(F_{ab}^I \pm \tilde{F}_{ab}^I), \qquad \tilde{F}_{ab}^I \equiv -\frac{i}{2}\epsilon_{abcd}F^{Icd},\tag{A.1}$$

where $\epsilon_{0123} = 1$, $\epsilon^{0123} = -1$. We also introduce

$$\epsilon^{\mu\nu\rho\sigma} = e\, e_a^\mu e_b^\nu e_c^\rho e_d^\sigma \epsilon^{abcd}.\tag{A.2}$$

The p-form associated to an antisymmetric tensor $T_{\mu_1\ldots\mu_p}$ is

$$T = \frac{1}{p!}T_{\mu_1\ldots\mu_p}dx^{\mu_1} \wedge \ldots \wedge dx^{\mu_p},\tag{A.3}$$

and the exterior derivative acts as[13]

$$dT = \frac{1}{p!}T_{\mu_1\ldots\mu_p,\nu}dx^\nu \wedge dx^{\mu_1} \wedge \ldots \wedge dx^{\mu_p}.\tag{A.4}$$

Antisymmetric tensors are often contracted with Γ-matrices as in $\Gamma \cdot F \equiv \Gamma^{ab}F_{ab}$.

---

[13]Our definitions for p-forms, eq. (A.3), and for exterior derivatives, eq. (A.4), are the only points where our conventions differ from those of [37].

$i, j, \ldots = 1, 2$ are SU(2) indices, whose raising and lowering is done by complex conjugation. The Levi-Civita ϵ^{ij} has the property

$$\epsilon_{ij}\epsilon^{jk} = -\delta_i{}^k, \tag{A.5}$$

where in principle ϵ^{ij} is the complex conjugate of ϵ_{ij}, but we can choose $\epsilon = i\sigma_2$, such that

$$\epsilon_{12} = \epsilon^{12} = 1. \tag{A.6}$$

The Pauli matrices $\sigma_{xi}{}^j$ $(x = 1, 2, 3)$ are given by

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{A.7}$$

They allow to switch from SU(2) indices to vector quantities using the convention

$$A_i{}^j \equiv i\vec{A} \cdot \vec{\sigma}_i{}^j. \tag{A.8}$$

At various places in the main text we use σ-matrices with only lower or upper indices, defined by

$$\vec{\sigma}_{ij} \equiv \vec{\sigma}_i{}^k \epsilon_{kj}, \qquad i\vec{\sigma}^{ij} = (i\vec{\sigma}_{ij})^*. \tag{A.9}$$

Notice that both $\vec{\sigma}_{ij}$ and $\vec{\sigma}^{ij}$ are symmetric.

Spinors carrying an index i are chiral, e.g. for the supersymmetry parameter one has

$$\Gamma_5\epsilon^i = \epsilon^i, \qquad \Gamma_5\epsilon_i = -\epsilon_i, \tag{A.10}$$

and the same holds for the gravitino ψ^i_μ. Note however that for some spinors, the upper index denotes negative chirality rather than positive chirality, for instance the gauginos obey

$$\Gamma_5\lambda^{\alpha i} = -\lambda^{\alpha i}, \qquad \Gamma_5\lambda^\alpha_i = \lambda^\alpha_i, \tag{A.11}$$

as is also evident from the supersymmetry transformations. The charge conjugate of a spinor χ is

$$\chi^C = \Gamma_0 C^{-1} \chi^*, \tag{A.12}$$

with the charge conjugation matrix C. Majorana spinors are defined by $\chi = \chi^C$, and chiral spinors obey $\chi_i^C = \chi^i$.

B. Spinors and forms

In this appendix, we summarize the essential information needed to realize the spinors of Spin(3,1) in terms of forms. For more details, we refer to [48]. Let $V = \mathbb{R}^{3,1}$ be a real vector space equipped with the Lorentzian inner product $\langle \cdot, \cdot \rangle$. Introduce an orthonormal basis e_1, e_2, e_3, e_0, where e_0 is along the time direction, and consider the subspace U spanned by the first two basis vectors e_1, e_2. The space of Dirac spinors is $\Delta_c = \Lambda^*(U \otimes \mathbb{C})$, with basis $1, e_1, e_2, e_{12} = e_1 \wedge e_2$. The gamma matrices are represented on Δ_c as

$$\begin{aligned} \Gamma_0 \eta &= -e_2 \wedge \eta + e_2 \lrcorner \eta, & \Gamma_1 \eta &= e_1 \wedge \eta + e_1 \lrcorner \eta, \\ \Gamma_2 \eta &= e_2 \wedge \eta + e_2 \lrcorner \eta, & \Gamma_3 \eta &= ie_1 \wedge \eta - ie_1 \lrcorner \eta, \end{aligned} \tag{B.1}$$

where

$$\eta = \frac{1}{k!}\eta_{j_1...j_k}e_{j_1} \wedge \ldots \wedge e_{j_k}$$

is a k-form and

$$e_i\rfloor\eta = \frac{1}{(k-1)!}\eta_{ij_1...j_{k-1}}e_{j_1} \wedge \ldots \wedge e_{j_{k-1}}\,.$$

One easily checks that this representation of the gamma matrices satisfies the Clifford algebra relations $\{\Gamma_a, \Gamma_b\} = 2\eta_{ab}$. The parity matrix is defined by $\Gamma_5 = i\Gamma_0\Gamma_1\Gamma_2\Gamma_3$, and one finds that the even forms $1, e_{12}$ have positive chirality, $\Gamma_5\eta = \eta$, while the odd forms e_1, e_2 have negative chirality, $\Gamma_5\eta = -\eta$, so that Δ_c decomposes into two complex chiral Weyl representations $\Delta_c^+ = \Lambda^{\text{even}}(U \otimes \mathbb{C})$ and $\Delta_c^- = \Lambda^{\text{odd}}(U \otimes \mathbb{C})$. Note that $\text{Spin}(3,1)$ is isomorphic to $\text{SL}(2, \mathbb{C})$, which acts with the fundamental representation on the positive chirality Weyl spinors.

Let us define the auxiliary inner product

$$\left\langle \sum_{i=1}^{2}\alpha_i e_i, \sum_{j=1}^{2}\beta_j e_j \right\rangle = \sum_{i=1}^{2}\alpha_i^*\beta_i \tag{B.2}$$

on $U \otimes \mathbb{C}$, and then extend it to Δ_c. The $\text{Spin}(3,1)$ invariant Dirac inner product is then given by

$$D(\eta, \theta) = \langle\Gamma_0\eta, \theta\rangle\,. \tag{B.3}$$

The Majorana inner product that we use is[14]

$$A(\eta, \theta) = \langle C\eta^*, \theta\rangle\,, \tag{B.4}$$

with the charge conjugation matrix $C = \Gamma_{12}$. Using the identities

$$\Gamma_a^* = -C\Gamma_0\Gamma_a\Gamma_0 C^{-1}\,, \qquad \Gamma_a^T = -C\Gamma_a C^{-1}\,, \tag{B.5}$$

it is easy to show that (B.4) is $\text{Spin}(3,1)$ invariant as well.

The charge conjugation matrix C acts on the basis elements as

$$C1 = e_{12}\,, \quad Ce_{12} = -1\,, \quad Ce_1 = -e_2\,, \quad Ce_2 = e_1\,. \tag{B.6}$$

In many applications it is convenient to use a basis in which the gamma matrices act like creation and annihilation operators, given by

$$\Gamma_+\eta \equiv \frac{1}{\sqrt{2}}(\Gamma_2 + \Gamma_0)\eta = \sqrt{2}\,e_2\rfloor\eta\,, \qquad \Gamma_-\eta \equiv \frac{1}{\sqrt{2}}(\Gamma_2 - \Gamma_0)\eta = \sqrt{2}\,e_2 \wedge \eta\,,$$

$$\Gamma_\bullet\eta \equiv \frac{1}{\sqrt{2}}(\Gamma_1 - i\Gamma_3)\eta = \sqrt{2}\,e_1 \wedge \eta\,, \qquad \Gamma_{\bar\bullet}\eta \equiv \frac{1}{\sqrt{2}}(\Gamma_1 + i\Gamma_3)\eta = \sqrt{2}\,e_1\rfloor\eta\,. \tag{B.7}$$

The Clifford algebra relations in this basis are $\{\Gamma_A, \Gamma_B\} = 2\eta_{AB}$, where $A, B, \ldots = +, -, \bullet, \bar\bullet$ and the nonvanishing components of the tangent space metric read $\eta_{+-} = \eta_{-+} = $

---

[14]It is known that on even-dimensional manifolds there are two Spin invariant Majorana inner products. The other possibility, based on $C = i\Gamma_{03}$, was used in [25].

	1	e_1	e_2	$e_1 \wedge e_2$
Γ_+	0	0	$\sqrt{2}$	$-\sqrt{2}e_1$
Γ_-	$\sqrt{2}e_2$	$-\sqrt{2}e_1 \wedge e_2$	0	0
$\Gamma_\bullet$	$\sqrt{2}e_1$	0	$\sqrt{2}e_1 \wedge e_2$	0
$\Gamma_{\bar\bullet}$	0	$\sqrt{2}$	0	$\sqrt{2}e_2$
Γ_{+-}	1	e_1	$-e_2$	$-e_1 \wedge e_2$
$\Gamma_{\bar\bullet\bullet}$	1	$-e_1$	e_2	$-e_1 \wedge e_2$
$\Gamma_{+\bullet}$	0	0	$-2e_1$	0
$\Gamma_{+\bar\bullet}$	0	0	0	2
$\Gamma_{-\bullet}$	$-2e_1 \wedge e_2$	0	0	0
$\Gamma_{-\bar\bullet}$	0	$2e_2$	0	0

Table 2: The action of the Gamma matrices and the Lorentz generators Γ_{AB} on the different basis elements.

$\eta_{\bullet\bar\bullet} = \eta_{\bar\bullet\bullet} = 1$. The spinor 1 is a Clifford vacuum, $\Gamma_+ 1 = \Gamma_{\bar\bullet} 1 = 0$, and the representation Δ_c can be constructed by acting on 1 with the creation operators $\Gamma^+ = \Gamma_-, \Gamma^{\bar\bullet} = \Gamma_\bullet$, so that any spinor can be written as

$$\eta = \sum_{k=0}^{2} \frac{1}{k!} \phi_{\bar{a}_1 \dots \bar{a}_k} \Gamma^{\bar{a}_1 \dots \bar{a}_k} 1, \qquad \bar{a} = +, \bar{\bullet}.$$

The action of the Gamma matrices and the Lorentz generators Γ_{AB} is summarized in table 2.

Note that $\Gamma_A = U_A{}^a \Gamma_a$, with

$$(U_A{}^a) = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 1 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 1 & 0 & -i \\ 0 & 1 & 0 & i \end{pmatrix} \in \mathrm{U}(4),$$

so that the new tetrad is given by $E^A = (U^*)^A{}_a E^a$.

C. BPS equations and equations of motion

We will now show that the vanishing of the supersymmetry variations, plus Bianchi identities and Maxwell equations, imply all equations of motion in the timelike case, and all but one in the null case. Without hypermultiplets, the equations of motion are (here we set $8\pi G = 1$)

- Einstein

$$0 = E_{\mu\nu} := \frac{1}{2} R_{\mu\nu} + (\mathrm{Im}\,\mathcal{N})_{IJ} F^{+I}_{\rho\mu} F^{-J\rho}{}_\nu - g_{\alpha\bar\beta} \mathcal{D}_\mu z^\alpha \mathcal{D}_\nu \bar{z}^{\bar\beta} - \frac{1}{2} g_{\mu\nu} V; \quad \text{(C.1)}$$

- Maxwell[15]

$$0 = M_I^\nu := -2\nabla_\mu \left((\text{Im}\,\mathcal{N})_{IJ} F^{-J\mu\nu} \right) + i\partial_\mu \mathcal{N}_{IJ} \tilde{F}^{J\mu\nu} - g g_{\alpha\bar\beta} k_I^\alpha \mathcal{D}^\nu \bar{z}^{\bar\beta}$$
$$-g g_{\alpha\bar\beta} k_I^{\bar\beta} \mathcal{D}^\nu z^\alpha - \frac{g^2}{4e} C_{J,IK} \epsilon^{\nu\mu\rho\sigma} A_\mu^J F_{\rho\sigma}^K ; \tag{C.2}$$

- Scalars

$$0 = G^\alpha := \tilde{\nabla}_\mu \mathcal{D}^\mu z^\alpha - g A^{I\mu} \tilde{\nabla}_\mu k_I^\alpha + \frac{1}{2i} F_{\mu\nu}^{+I} F^{+J\mu\nu} g^{\alpha\bar\gamma} \partial_{\bar{z}^{\bar\gamma}} \mathcal{N}_{IJ}$$
$$-\frac{1}{2i} F_{\mu\nu}^{-I} F^{-J\mu\nu} g^{\alpha\bar\gamma} \partial_{\bar{z}^{\bar\gamma}} \bar{\mathcal{N}}_{IJ} - g^{\alpha\bar\gamma} \partial_{\bar{z}^{\bar\gamma}} V , \tag{C.3}$$

where with $\tilde{\nabla}$ we mean the covariant derivative with respect to the metric connection on both the spacetime and the target manifold of the scalars. Finally

$$V = g^2 e^{\mathcal{K}} \left[k_I^\alpha k_J^{\bar\beta} g_{\alpha\bar\beta} \bar{Z}^I Z^J + 4 \left(g^{\alpha\bar\beta} \mathcal{D}_\alpha Z^I \mathcal{D}_{\bar\beta} \bar{Z}^J - 3\bar{Z}^I Z^J \right) \vec{P}_I \cdot \vec{P}_J \right] \tag{C.4}$$

is the scalar potential.

We set

$$\hat{\mathcal{D}}_\mu \epsilon^i = D_\mu(\omega)\epsilon^i - g\Gamma_\mu S^{ij}\epsilon_j + \frac{1}{4}\Gamma^{ab} F_{ab}^{-I} \epsilon^{ij} \Gamma_\mu (\text{Im}\,\mathcal{N})_{IJ} Z^J e^{\mathcal{K}/2}\epsilon_j . \tag{C.5}$$

where $D_\mu(\omega)$ is defined in (2.19). Then, the gravitini Killing equation is

$$\hat{\mathcal{D}}_\mu \epsilon^i = 0 , \tag{C.6}$$

and its integrability is given by the (holonomy) condition

$$0 = \left[\hat{\mathcal{D}}_\mu , \hat{\mathcal{D}}_\mu \right] \epsilon^i = \hat{\mathcal{D}}_\mu \left(\hat{\mathcal{D}}_\nu \epsilon^i \right) - \hat{\mathcal{D}}_\nu \left(\hat{\mathcal{D}}_\mu \epsilon^i \right) . \tag{C.7}$$

Denoting

$$F_{\mu\nu} := \partial_\mu A_\nu - \partial_\nu A_\mu ,$$
$$\Phi^{ab} := Z^J e^{\mathcal{K}/2} (\text{Im}\,\mathcal{N})_{IJ} F^{-Iab} ,$$
$$\bar{\Phi}^{ab} := \bar{Z}^J e^{\mathcal{K}/2} (\text{Im}\,\mathcal{N})_{IJ} F^{+Iab} , \tag{C.8}$$

and making use of (A.10), we find

$$0 = \frac{1}{4} R_{\mu\nu}{}^{ab} \Gamma_{ab} \epsilon^i + \frac{i}{2} F_{\mu\nu}\epsilon^i + g F_{\mu\nu}^I P_{Ij}{}^i \epsilon^j + g A_\nu^I \partial_\mu P_{Ij}{}^i \epsilon^j - g A_\mu^I \partial_\nu P_{Ij}{}^i \epsilon^j$$
$$-2g^2 \Gamma_{\mu\nu} S^r \bar{S}^s \delta_{rs} \epsilon^i + 2g(\bar{S}^i{}_j \Phi_{\mu\nu} - S^i{}_j \bar{\Phi}_{\mu\nu})\epsilon^j + \left[-\frac{1}{2}\Phi_\nu{}^b \bar{\Phi}_\mu{}^d \Gamma_{bd} \right.$$
$$\left. +\frac{1}{2}\Phi_\mu{}^b \bar{\Phi}_\nu{}^d \Gamma_{bd} - \frac{1}{2}\Phi^{ab}\bar{\Phi}_{\mu\nu}\Gamma_{ab} + \frac{1}{2}\Phi_a{}^b \bar{\Phi}_\mu{}^a \Gamma_{b\nu} - \frac{1}{2}\Phi_a{}^b \bar{\Phi}_\nu{}^a \Gamma_{b\mu} \right]\epsilon^i - g\Gamma_\nu \partial_\mu S^{ij}\epsilon_j$$
$$+g\Gamma_\mu \partial_\nu S^{ij}\epsilon_j + \frac{1}{4}\Gamma^{ab}\epsilon^{ij}(\nabla_\mu \Phi_{ab}\Gamma_\nu - \nabla_\nu \Phi_{ab}\Gamma_\mu)\epsilon_j - igA_\mu\Gamma_\nu S^{ij}\epsilon_j$$
$$+igA_\nu\Gamma_\mu S^{ij}\epsilon_j + \frac{1}{4}A_\mu \Gamma^{ab}\Phi_{ab}\Gamma_\nu \epsilon^{ij}\epsilon_j - \frac{1}{4}A_\nu \Gamma^{ab}\Phi_{ab}\Gamma_\mu \epsilon^{ij}\epsilon_j . \tag{C.9}$$

---

[15]We used the Bianchi identities to put the equations in this form.

Let us now contract this equation with Γ^μ. This leads to

$$
\begin{aligned}
0 = & \frac{1}{2} R_{\nu b} \Gamma^b \epsilon^i + \frac{i}{2} \Gamma^\mu F_{\mu\nu} \epsilon^i + g \Gamma^\mu F^I_{\mu\nu} P_{Ij}{}^i \epsilon^j + g A^I_\nu \Gamma^\mu \partial_\mu P_{Ij}{}^i \epsilon^j \\
& - g A^I_\mu \Gamma^\mu \partial_\nu P_{Ij}{}^i \epsilon^j - 6 g^2 \Gamma_\nu S^r S^s \delta_{rs} \epsilon^i + 2 g \Gamma^\mu (\bar{S}^i{}_j \Phi_{\mu\nu} - S^i{}_j \bar{\Phi}_{\mu\nu}) \epsilon^j \\
& - 2 \bar{\Phi}_a{}^b \Phi^a{}_\nu \Gamma_b \epsilon^i - g \Gamma^\mu{}_\nu \partial_\mu S^{ij} \epsilon_j + 3 g \partial_\nu S^{ij} \epsilon_j - i g A_\mu \Gamma^\mu S^{ij} \epsilon_j \\
& + 3 i g A_\nu S^{ij} \epsilon_j + (\nabla_\mu \Phi^{\mu c} + A_\mu \Phi^{\mu c}) \epsilon^{ij} (\Gamma_{c\nu} + e_{c\nu}) \epsilon_j \,,
\end{aligned}
\tag{C.10}
$$

where we used

$$
F^{I+ab} F^{J-}_{ab} = 0 \,,
\tag{C.11}
$$

$$
F^{I+}_{a[b} F^{J-a}{}_{c]} = 0 \,.
\tag{C.12}
$$

At this point we need to make contact with the equations of motion. To do this, let us first take the gaugini Killing equation (multiplied with Γ^λ)

$$
\begin{aligned}
0 = & - 2 e^{\mathcal{K}/2} g^{\alpha\bar\beta} \mathcal{D}_{\bar\beta} \bar{Z}^I (\operatorname{Im}\mathcal{N})_{IJ} F^{-I\lambda\mu} \Gamma_\mu \epsilon_{ij} \epsilon^j + \Gamma^\lambda \Gamma^\mu \mathcal{D}_\mu z^\alpha \epsilon_i \\
& + g \Gamma^\lambda e^{\mathcal{K}/2} \left[\epsilon_{ij} k^\alpha_I \bar{Z}^I - 2 P_{Iij} \bar{\mathcal{D}}_{\bar\beta} \bar{Z}^I g^{\alpha\bar\beta} \right] \epsilon^j \,,
\end{aligned}
\tag{C.13}
$$

and contract it with $e^{\mathcal{K}/2} \mathcal{D}_\alpha Z^L (\operatorname{Im}\mathcal{N})_{KL} F^{+K}_{\lambda\mu} \epsilon^{il}$. This yields

$$
\begin{aligned}
0 = & - 2 e^{\mathcal{K}} g^{\alpha\bar\beta} \mathcal{D}_{\bar\beta} \bar{Z}^I \mathcal{D}_\alpha Z^L (\operatorname{Im}\mathcal{N})_{IJ} (\operatorname{Im}\mathcal{N})_{KL} F^{-I\lambda\mu} F^{+K}_{\lambda\mu} \Gamma_\mu \epsilon^l \\
& + e^{\mathcal{K}/2} \mathcal{D}_\alpha Z^L (\operatorname{Im}\mathcal{N})_{KL} F^{+K}_{\lambda\mu} \Gamma^\lambda \Gamma^\mu \mathcal{D}_\mu z^\alpha \epsilon^{il} \epsilon_i \\
& + g \Gamma^\lambda e^{\mathcal{K}} \mathcal{D}_\alpha Z^L (\operatorname{Im}\mathcal{N})_{KL} F^{+K}_{\lambda\mu} \epsilon^{il} \left[\epsilon_{ij} k^\alpha_I \bar{Z}^I - 2 P_{Iij} \bar{\mathcal{D}}_{\bar\beta} \bar{Z}^I g^{\alpha\bar\beta} \right] \epsilon^j \,.
\end{aligned}
\tag{C.14}
$$

Now add this to eq. (C.10). Using the relation

$$
g^{\alpha\bar\beta} e^{\mathcal{K}} \mathcal{D}_\alpha Z^I \mathcal{D}_{\bar\beta} \bar{Z}^J = -\frac{1}{2} (\operatorname{Im}\mathcal{N})^{-1|IJ} - e^{\mathcal{K}} \bar{Z}^I Z^J \,,
\tag{C.15}
$$

we see that the first term of (C.14) sums up with the term $-2 \bar{\Phi}_a{}^b \Phi^a{}_\nu \Gamma_b \epsilon^i$ of (C.10) to give

$$
(\operatorname{Im}\mathcal{N})_{IJ} F^{+I}_{\rho\mu} F^{-J\rho}{}_\nu \Gamma^\mu \,.
$$

Some other useful relations are $(X^I = e^{\mathcal{K}/2} Z^I)$

$$
P^0_I = -e^{\mathcal{K}} C_{I,JK} Z^K \bar{Z}^J \,,
\tag{C.16}
$$

$$
X^J k^\alpha_J \mathcal{D}_\alpha X^I + i P^0_J X^J X^I = 0 \,,
\tag{C.17}
$$

and (2.6), from which one also obtains the important identity

$$
X^J \partial_\mu \mathcal{N}_{IJ} = -2i \mathcal{D}_\alpha X^J \operatorname{Im}\mathcal{N}_{IJ} \partial_\mu z^\alpha \,.
\tag{C.18}
$$

After summing up (C.10) and (C.14) and using the above relations we finally find[16]

$$
E_\nu{}^b \Gamma_b \epsilon^i - \frac{1}{2} X^I M^\mu_I \epsilon^{ij} \Gamma_\mu \Gamma_\nu \epsilon_j = 0 \,.
\tag{C.19}
$$

---

[16]This calculation involves a rather mastodontic amount of algebraic manipulations, as well as the use of some further identities of special Kähler geometry that can be found in [37].

Imposing the Maxwell equations one remains with the condition

$$E_\nu{}^b \Gamma_b \epsilon^i = 0 \,.$$ (C.20)

At this point one can proceed in a standard way (see for example [2]). If the Killing spinor is timelike, then (C.20) implies that the Einstein equations are identically satisfied. In the other case, if the Killing spinor is null, thus selecting a null direction "+", then the equation $E_{++} = 0$ must be imposed.

In a similar way we can handle the gaugini equations:

$$0 = \delta\lambda_i^\alpha = -\frac{1}{2}e^{\mathcal{K}/2}g^{\alpha\bar{\gamma}}\mathcal{D}_{\bar{\gamma}}\bar{Z}^I(\mathrm{Im}\,\mathcal{N})_{IJ}F_{\lambda\rho}^{-J}\Gamma^{\lambda\rho}\epsilon_{ij}\epsilon^j + \Gamma^\mu\mathcal{D}_\mu z^\alpha\epsilon_i + gN_{ij}^\alpha\epsilon^j \,.$$ (C.21)

In this case the story is much longer and can be summarized as follows. Let us first apply the operator $\Gamma^\mu D_\mu(\omega)$ (see (2.19)) to (C.21), contracted with $g_{\bar{\beta}\alpha}$. Using (C.5) we get

$$\begin{aligned}
0 = &- \frac{1}{2}\Gamma^\mu\partial_\mu\left[e^{\mathcal{K}/2}\mathcal{D}_{\bar{\beta}}\bar{Z}^I(\mathrm{Im}\,\mathcal{N})_{IJ}F_{\lambda\rho}^{-J}\Gamma^{\lambda\rho}\right]\epsilon_{ij}\epsilon^j \\
&- \frac{1}{8}\Gamma^\mu\omega_\mu^{ab}\Gamma_{ab}e^{\mathcal{K}/2}\mathcal{D}_{\bar{\beta}}\bar{Z}^I(\mathrm{Im}\,\mathcal{N})_{IJ}F_{\lambda\rho}^{-J}\Gamma^{\lambda\rho}\epsilon_{ij}\epsilon^j \\
&+ \frac{1}{8}\Gamma^\mu e^{\mathcal{K}/2}\mathcal{D}_{\bar{\beta}}\bar{Z}^I(\mathrm{Im}\,\mathcal{N})_{IJ}F_{\lambda\rho}^{-J}\Gamma^{\lambda\rho}\omega_\mu^{ab}\Gamma_{ab}\epsilon_{ij}\epsilon^j \\
&- \frac{1}{2}e^{\mathcal{K}}F_{ab}^{-L}F^{-Jab}\mathcal{D}_{\bar{\beta}}\bar{Z}^I(\mathrm{Im}\,\mathcal{N})_{IJ}(\mathrm{Im}\,\mathcal{N})_{LM}Z^M\epsilon_i \\
&+ \nabla_\mu(g_{\bar{\beta}\alpha}\mathcal{D}^\mu z^\alpha)\epsilon_i + 2gg_{\bar{\beta}\alpha}\Gamma^\mu\mathcal{D}_\mu z^\alpha S_{ij}\epsilon^j \\
&- \frac{1}{2}g_{\bar{\beta}\alpha}\Gamma^{ab}F_{ab}^{+I}(\mathrm{Im}\,\mathcal{N})_{IJ}\bar{Z}^J e^{\mathcal{K}/2}\Gamma^\mu\mathcal{D}_\mu z^\alpha\epsilon_{ij}\epsilon^j \\
&+ g\Gamma^\mu\partial_\mu(\mathcal{N}_{\bar{\beta}ij})\epsilon^j + 4g^2\mathcal{N}_{\bar{\beta}ij}S^{jl}\epsilon_l \,.
\end{aligned}$$ (C.22)

At this point there are many possible manipulations which lead to the desired result. However, the most complicated task is to recognize the derivatives of the scalar potential V. To simplify such an effort, it is convenient to express the term $\nabla_\mu(g_{\bar{\beta}\alpha}\mathcal{D}^\mu z^\alpha)$ in terms of G^α by means of (C.3).

A faster way is to work out the first term of (C.22) as follows:

$$-\frac{1}{2}\Gamma^\mu\partial_\mu\left[e^{\mathcal{K}/2}\mathcal{D}_{\bar{\beta}}\bar{Z}^I(\mathrm{Im}\mathcal{N})_{IJ}F_{\lambda\rho}^{-J}\Gamma^{\lambda\rho}\right]\epsilon_{ij}\epsilon^j = -\frac{1}{2}\Gamma^\mu\partial_\mu\left[e^{\mathcal{K}/2}\mathcal{D}_{\bar{\beta}}\bar{Z}^I(\mathrm{Im}\mathcal{N})_{IJ}\right]F_{\lambda\rho}^{-J}\Gamma^{\lambda\rho}\epsilon_{ij}\epsilon^j$$
$$-e^{\mathcal{K}/2}\mathcal{D}_{\bar{\beta}}\bar{Z}^I(\mathrm{Im}\mathcal{N})_{IJ}\nabla_\mu F^{-J\mu\rho}\Gamma_\rho\epsilon_{ij}\epsilon^j \,,$$ (C.23)

where we used the relation

$$\Gamma_{abc} = -i\Gamma_5\epsilon_{abcd}\Gamma^d \,,$$ (C.24)

and the Bianchi identities. Then, we can use (C.2) to rewrite the last term in (C.23) in terms of M_I^μ, so that (C.22) takes the form

$$g_{\bar{\beta}\alpha}G^\alpha\epsilon_i + \frac{1}{2}e^{\mathcal{K}/2}\mathcal{D}_{\bar{\beta}}\bar{Z}^I M_I^\nu\epsilon_{ij}\Gamma_\nu\epsilon^j + \cdots = 0 \,.$$ (C.25)

Next, all the remaining manipulations are very similar to the gravitino case, and have the aim to show that the terms indicated by dots vanish identically, so that we will not report the details here. We only mention that sometimes we found it convenient to use $X^I = e^{\mathcal{K}/2}Z^I$ in place of Z^I to simplify many expressions. Also, the Killing equations for k_I^α (and its conjugate) are often useful in taking account of the Christoffel symbols for the covariant derivative on the scalar target manifold. Both (C.21) and its charge conjugate must be used to eliminate many terms.

As we have anticipated, the final result is that (C.22) reduces to

$$g_{\bar\beta\alpha}G^\alpha\epsilon_i + \frac{1}{2}e^{\mathcal{K}/2}\mathcal{D}_{\bar\beta}\bar Z^I M_I^\nu \epsilon_{ij}\Gamma_\nu\epsilon^j = 0 \; . \tag{C.26}$$

Thus, if the Maxwell equations hold, the scalar fields satisfy their equations of motion as well. Note that the results of this appendix could also be obtained by the Killing spinor identity approach [49, 50].

D. Holonomy of the base manifold

In order to gain a deeper geometrical understanding of the three-dimensional base space with dreibein V^x, some considerations concerning its holonomy are in order. First of all, note that in minimal ungauged $\mathcal{N} = 2$, $D = 4$ supergravity, the base is flat [33] and thus has trivial holonomy. This is still true if one couples the theory to vector multiplets [4]. In five-dimensional minimal ungauged supergravity, the base manifold is hyper-Kähler [2], whereas in the gauged case it is Kähler [6]. Thus, the general pattern in the timelike case is to have a fibration over a base with reduced holonomy. One might therefore ask whether our three-dimensional manifold with metric

$$ds_3^2 = dz^2 + e^{2\Phi}dwd\bar w \, , \tag{D.1}$$

appearing in (4.53), has reduced holonomy. Eqs. (4.32) and (4.38) suggest that the Christoffel connection $\mathcal{A} + B$ (cf. (4.33)) has full holonomy SO(3). In fact, the only nontrivial subgroup of SO(3) is U(1), and integrating the first Cartan structure equation for a Christoffel connection taking values in u(1), one finds the metric (D.1) with $\partial_z\Phi = 0$, which in general will not be the case. From (4.39), however, it is evident that the connection $\mathcal{A}$ (which has nonvanishing torsion, cf. (4.29)), takes values in u(1) $\subseteq$ so(3). The same holds for the corresponding curvature. We can thus interpret the base space as a manifold of reduced holonomy U(1) $\subseteq$ SO(3) with nonzero torsion. Reduced holonomy is equivalent to the existence of parallel tensors, the simplest example being the reduction of GL(D, $\mathbb{R}$) to SO(D) if the metric is covariantly constant, $\nabla g = 0$. In our case, the corresponding parallel tensor is just the vector ∂_z: One easily checks that $\nabla\partial_z = 0$, where ∇ denotes the covariant derivative associated to $\mathcal{A}$.

This is, to the best of our knowledge, the first example of a supersymmetric background that leads to a base space of reduced holonomy for a non-Christoffel connection. Note also that the torsion of this connection is due to the gauging, and not related to the presence of vector multiplets. This is evident from (4.32), which yields vanishing torsion in the ungauged case $g = 0$.

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