A terminating evaluation-driven variant of G3i

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Abstract. We present **Gbu**, a terminating variant of the sequent calculus **G3i** for intuitionistic propositional logic. **Gbu** modifies **G3i** by annotating the sequents so to distinguish rule applications into two phases: an unblocked phase where any rule can be backward applied, and a blocked phase where only right rules can be used. Derivations of **Gbu** have a trivial translation into **G3i**. Rules for right implication exploit an *evaluation* relation, defined on sequents; this is the key tool to avoid the generation of branches of infinite length in proof-search. To prove the completeness of **Gbu**, we introduce a refutation calculus **Rbu** for unprovability dual to **Gbu**. We provide a proof-search procedure that, given a sequent as input, returns either a **Rbu**-derivation or a **Gbu**-derivation of it.

1 Introduction

It is well-known that **G3i** [10], the sequent calculus for intuitionistic propositional logic with weakening and contraction "absorbed" in the rules, is not suited for proof-search. Indeed, the naïve proof-search strategy, consisting in applying the rules of the calculus bottom-up until possible, is not terminating. This is because the rule for left implication retains the main formula $A \to B$ in the left-hand side premise, hence such a formula might be selected for application more and more times. A possible solution to this problem is to support the proof-search procedure with a *loop-checking* mechanism [5–7]: whenever the "same" sequent occurs twice along a branch of the proof under construction, the search is cut. An efficient implementation of loop-checking exploits *histories* [6, 7]. In the construction of a branch, the formulas decomposed by right rules are stored in the history; loops are avoided by preventing the application of some right rules to formulas in the history.

In this paper we propose a different and original approach: we show that terminating proof-search for **G3i** can be accomplished only exploiting the information contained in the sequent to be proved by means of a suitable *evaluation relation*. Our proof-search strategy alternates two phases: an unblocked phase (u-phase), where all the rules of **G3i** can be backward applied, and a blocked phase (b-phase), where only right-rules can be used. To improve the presentation, we embed the strategy inside the calculus by annotating sequents with the label u (*unblocked*) or b (*blocked*); we call **Gbu** the resulting calculus (see Fig. 1). A **Gbu**-derivation can be straightforwardly mapped to a **G3i**-derivation by erasing the labels and, possibly, by padding the left contexts; from this, the soundness of **Gbu** immediately follows. Unblocked sequents, characterizing an u-phase, behave as the ordinary sequents of **G3i**: any rule of **Gbu** can be (backward) applied to them. Instead, b-sequents resemble focused-right sequents (see, e.g., [2]): they only allow backward right-rule applications (thus, the left context is "blocked"). Proof-search starts from an u-sequent (u-phase); the transition to a b-phase is determined by the application of one of the rules for left implication or right disjunction. For instance, let $[A \to B, \Gamma \stackrel{\mathrm{u}}{\Rightarrow} H]$ be the u-sequent to be proved and suppose we apply the rule $\to L$ with main formula $A \to B$. The next goals are the b-sequent $[A \to B, \Gamma \stackrel{\mathrm{b}}{\Rightarrow} A]$ and the u-sequent $[B, \Gamma \stackrel{\mathrm{u}}{\Rightarrow} H]$, corresponding to the two premises of $\to L$. While the latter goal continues the u-phase, the former one starts a new b-phase, which focuses on A. Similarly, if we apply the rule $\vee R_k$ (with $k \in \{0, 1\}$) to $[\Gamma \stackrel{\mathrm{u}}{\Rightarrow} H_0 \vee H_1]$, the phase changes to b and the next goal is $[\Gamma \stackrel{\mathrm{b}}{\Rightarrow} H_k]$, the only premise of $\vee R_k$.

Rules for right implication have two possible outcomes determined by the evaluation relation. Indeed, let $[\Gamma \stackrel{l}{\Rightarrow} A \to B]$ be the current goal ($l \in \{\mathrm{u}, \mathrm{b}\}$) and let $A \to B$ be the selected main formula: if A is evaluated in Γ, then we continue the search with $[\Gamma \stackrel{l}{\Rightarrow} B]$ and the phase does not change (see rule $\to R_1$); note that the formula A is dropped out. If A is not evaluated in Γ the next goal is $[A, \Gamma \stackrel{\mathrm{u}}{\Rightarrow} B]$. Moreover, if $l = \mathrm{b}$, we switch from a b-phase to an u-phase and this is the only case where a b-sequent is "unblocked". The crucial point is that, due to the side conditions on the application of rules $\to R_1$ and $\to R_2$ (which rely on the evaluation relation), every branch of a **Gbu**-tree has finite length (Section 3); this implies that our proof-search strategy always terminates. We point out that we do not bound ourselves to a specific evaluation relation, but we admit any evaluation relation satisfying properties $(\mathcal{E}1)$–$(\mathcal{E}6)$ defined in Section 2.

The proof of completeness ($[\Gamma \Rightarrow H]$ provable in **G3i** implies $[\Gamma \stackrel{\mathrm{u}}{\Rightarrow} H]$ provable in **Gbu**) involves non-trivial aspects. Following [3, 9], we introduce a refutation calculus **Rbu** for asserting intuitionistic unprovability (Section 4). From an **Rbu**-derivation of an u-sequent $\sigma^{\mathrm{u}} = [\Gamma \stackrel{\mathrm{u}}{\Rightarrow} H]$ we can extract a Kripke countermodel for σ^{u}, namely a Kripke model such that, at its root, all formulas in Γ are forced and H is not forced; from this, it follows that σ^{u} is not intuitionistically valid. In Section 5 we introduce the function `F` which implements the proof-search strategy outlined above; if the search for a **Gbu**-derivation of σ^{u} fails, an **Rbu**-derivation of σ^{u} is built. To sum up, $\mathtt{F}(\sigma^{\mathrm{u}})$ returns either a **Gbu**-derivation or an **Rbu**-derivation of σ^{u}; in the former case we get a **G3i**-derivation of the sequent $\sigma = [\Gamma \Rightarrow H]$, in the latter case we can build a countermodel for σ.

2 Preliminaries and evaluations

We consider the propositional language $\mathcal{L}$ based on a denumerable set of propositional variables $\mathcal{V}$, the connectives $\wedge$, $\vee$, $\to$ and the logical constant $\bot$. We denote with $\mathcal{V}(A)$ the set of propositional variables occurring in A, with $|A|$ *the size of* A, that is the number of symbols occurring in A, and with $\mathrm{Sf}(A)$ the set of subformulas of A (including A itself).

A *(finite) Kripke model* for $\mathcal{L}$ is a structure $\mathcal{K} = \langle P, \leq, \rho, V \rangle$, where $\langle P, \leq, \rho \rangle$ is a finite partially ordered set with minimum ρ and $V : P \to 2^{\mathcal{V}}$ is a function such that $\alpha \leq \beta$ implies $V(\alpha) \subseteq V(\beta)$. The *forcing relation* $\Vdash \subseteq P \times \mathcal{L}$ is defined as follows:

- $\mathcal{K}, \alpha \nVdash \bot$ and, for every $p \in \mathcal{V}$, $\mathcal{K}, \alpha \Vdash p$ iff $p \in V(\alpha)$;
- $\mathcal{K}, \alpha \Vdash A \wedge B$ iff $\mathcal{K}, \alpha \Vdash A$ and $\mathcal{K}, \alpha \Vdash B$;
- $\mathcal{K}, \alpha \Vdash A \vee B$ iff $\mathcal{K}, \alpha \Vdash A$ or $\mathcal{K}, \alpha \Vdash B$;
- $\mathcal{K}, \alpha \Vdash A \to B$ iff, for every $\beta \in P$ such that $\alpha \leq \beta$, $\mathcal{K}, \beta \nVdash A$ or $\mathcal{K}, \beta \Vdash B$.

Given a set Γ of formulas, $\mathcal{K}, \alpha \Vdash \Gamma$ iff $\mathcal{K}, \alpha \Vdash A$ for every $A \in \Gamma$. *Monotonicity property* holds for arbitrary formulas, i.e.: $\mathcal{K}, \alpha \Vdash A$ and $\alpha \leq \beta$ imply $\mathcal{K}, \beta \Vdash A$. A formula A is *valid* in $\mathcal{K}$ iff $\mathcal{K}, \rho \Vdash A$. Intuitionistic propositional logic coincides with the set of the formulas valid in all (finite) Kripke models [1].

As motivated in the Introduction, we use (labelled) sequents of the form $\sigma = [\Gamma \xRightarrow{l} H]$ where $l \in \{\mathrm{b}, \mathrm{u}\}$, Γ is a finite set of formulas and H is a formula. We adopt the usual notational conventions; e.g., $[A, \Gamma \xRightarrow{l} H]$ stands for $[\{A\} \cup \Gamma \xRightarrow{l} H]$. The *size* of σ is $|\sigma| = \sum_{A \in \Gamma} |A| + |H|$; the set of subformulas of σ is $\mathrm{Sf}(\sigma) = \bigcup_{A \in \Gamma \cup \{H\}} \mathrm{Sf}(A)$.

The semantics of formulas extends to sequents as follows. Given a Kripke model $\mathcal{K}$ and a world α of $\mathcal{K}$, α *refutes* $\sigma = [\Gamma \xRightarrow{l} H]$ *in* $\mathcal{K}$, written $\mathcal{K}, \alpha \triangleright \sigma$, iff $\mathcal{K}, \alpha \Vdash \Gamma$ and $\mathcal{K}, \alpha \nVdash H$; σ is *refutable* if there exists a Kripke model $\mathcal{K}$ with root ρ such that $\mathcal{K}, \rho \triangleright \sigma$; in this case $\mathcal{K}$ is a *countermodel* for σ. It is easy to check that σ is refutable iff the formula $\wedge\Gamma \to H$ is not intuitionistically valid iff, by soundness and completeness of **G3i** [10], $[\Gamma \Rightarrow H]$ is not provable in **G3i**.

Evaluations An *evaluation relation* $\vdash_{\mathcal{E}}$ is a relation between a set Γ of formulas and a formula A satisfying the following properties:

($\mathcal{E}1$) $\Gamma \vdash_{\mathcal{E}} A$ iff $\Gamma \cap \mathrm{Sf}(A) \vdash_{\mathcal{E}} A$.
($\mathcal{E}2$) $A, \Gamma \vdash_{\mathcal{E}} A$.
($\mathcal{E}3$) $\Gamma \vdash_{\mathcal{E}} A$ and $\Gamma \vdash_{\mathcal{E}} B$ implies $\Gamma \vdash_{\mathcal{E}} A \wedge B$.
($\mathcal{E}4$) $\Gamma \vdash_{\mathcal{E}} A_k$, with $k \in \{0, 1\}$, implies $\Gamma \vdash_{\mathcal{E}} A_0 \vee A_1$.
($\mathcal{E}5$) $\Gamma \vdash_{\mathcal{E}} B$ implies $\Gamma \vdash_{\mathcal{E}} A \to B$.
($\mathcal{E}6$) Let $\mathcal{K} = \langle P, \leq, \rho, V \rangle$ and $\alpha \in P$; if $\mathcal{K}, \alpha \Vdash \Gamma$ and $\Gamma \vdash_{\mathcal{E}} A$, then $\mathcal{K}, \alpha \Vdash A$.

Conditions ($\mathcal{E}1$)–($\mathcal{E}5$) concern syntactical properties; note that, by ($\mathcal{E}1$), the evaluation of A w.r.t. Γ only depends on the subformulas in Γ which are subformulas of A. Intuitively, the role of an evaluation relation is to check if the "information contained" in A is semantically implied by Γ (see ($\mathcal{E}6$)). In the sequel, we also write $[\Gamma \xRightarrow{l} H] \vdash_{\mathcal{E}} A$ to mean $\Gamma \vdash_{\mathcal{E}} A$.

In the examples we use the evaluation relation $\vdash_{\tilde{\mathcal{E}}}$ defined below. Let $\mathcal{L}_\top$ be the language extending $\mathcal{L}$ with the constant $\top$ ($\mathcal{K}, \alpha \Vdash \top$, for every $\mathcal{K}$ and every α in $\mathcal{K}$). To define $\vdash_{\tilde{\mathcal{E}}}$, we introduce the function $\mathcal{R}$ which simplifies a formula $A \in \mathcal{L}_\top$ w.r.t. a set Γ of formulas of $\mathcal{L}$ (see [4]):

$$\mathcal{R}(A,\Gamma) = \begin{cases} \top & A \in \Gamma \\ A & \text{if } A \notin \Gamma \text{ and } A \in \mathcal{V} \cup \{\bot,\top\} \\ \mathcal{B}\left(\mathcal{R}(A_0,\Gamma) \cdot \mathcal{R}(A_1,\Gamma)\right) & \text{if } A \notin \Gamma \text{ and } A = A_0 {\cdot} A_1 \text{ with } \cdot \in \{\wedge,\vee,\to\} \end{cases}$$

$\mathcal{B}(A)$ performs the *boolean simplification* of A [4, 8], consisting in applying the following reductions inside A:

$$\begin{array}{llllll} K \wedge \top \rightsquigarrow K & K \wedge \bot \rightsquigarrow \bot & K \vee \top \rightsquigarrow \top & K \vee \bot \rightsquigarrow K & K \to \top \rightsquigarrow \top & K \to K \rightsquigarrow \top \\ \top \wedge K \rightsquigarrow K & \bot \wedge K \rightsquigarrow \bot & \top \vee K \rightsquigarrow \top & \bot \vee K \rightsquigarrow K & \top \to K \rightsquigarrow K & \bot \to K \rightsquigarrow \top \end{array}$$

We set $\Gamma \vdash_{\tilde{\mathcal{E}}} A$ iff $\mathcal{R}(A,\Gamma) = \top$.

Theorem 1. *$\vdash_{\tilde{\mathcal{E}}}$ is an evaluation relation.*

Proof. We have to prove that $\vdash_{\tilde{\mathcal{E}}}$ satisfies properties $(\mathcal{E}1)$–$(\mathcal{E}6)$ of Section 2.

– $(\mathcal{E}1)$ It is easy to prove, by induction on the structure of A, that $\mathcal{R}(A,\Gamma) = \mathcal{R}(A,\Gamma \cap \mathrm{Sf}(A))$, thus $\Gamma \vdash_{\tilde{\mathcal{E}}} A$ iff $\Gamma \cap \mathrm{Sf}(A) \vdash_{\tilde{\mathcal{E}}} A$.
– $(\mathcal{E}2)$ It immediately follows by the definition of $\vdash_{\tilde{\mathcal{E}}}$ and $\mathcal{R}$.
– $(\mathcal{E}3)$ Let $\Gamma \vdash_{\tilde{\mathcal{E}}} A$ and $\Gamma \vdash_{\tilde{\mathcal{E}}} B$. By definition of $\vdash_{\tilde{\mathcal{E}}}$, $\mathcal{R}(A,\Gamma) = \mathcal{R}(B,\Gamma) = \top$. To prove $\Gamma \vdash_{\tilde{\mathcal{E}}} A \wedge B$, we must show that $\mathcal{R}(A \wedge B,\Gamma) = \top$. If $A \wedge B \in \Gamma$, this immediately follows. Otherwise: $\mathcal{R}(A \wedge B,\Gamma) = \mathcal{B}(\mathcal{R}(A,\Gamma) \wedge \mathcal{R}(B,\Gamma)) = \mathcal{B}(\top \wedge \top) = \top$. The proof of properties $(\mathcal{E}4)$ and $(\mathcal{E}5)$ is similar.
– $(\mathcal{E}6)$ Let $\mathcal{K} = \langle P, \leq, \rho, V\rangle$ and $\alpha \in P$ such that $\mathcal{K},\alpha \Vdash \Gamma$. It is easy to prove, by induction on A, that $\mathcal{K},\alpha \Vdash A \leftrightarrow \mathcal{R}(A,\Gamma)$. Now, if $\Gamma \vdash_{\tilde{\mathcal{E}}} A$ then $\mathcal{R}(A,\Gamma) = \top$; hence by the above property $\mathcal{K},\alpha \Vdash A \leftrightarrow \top$ and this implies $\mathcal{K},\alpha \Vdash A$. □

3 The sequent calculus Gbu

We present the **G3**-style [10] calculus **Gbu** for intuitionistic propositional logic. The calculus consists of the *axiom rules* (rules with zero premises) $\bot L$ and Id, and the left and right introduction rules in Fig. 1. The *main formula* of a rule is the one put in evidence in the conclusion of the rule. In the conclusion of a rule, when we write C,Γ we assume that $C \notin \Gamma$; e.g., in the rule $\wedge L$ it is assumed that $A \wedge B \notin \Gamma$, hence the formula $A \wedge B$ is not retained in the premise. The choice between $\to R_1$ and $\to R_2$ depends on the relation $\vdash_{\mathcal{E}}$. In the application of $\to L$ to $\sigma = [A \to B, \Gamma \overset{\mathrm{u}}{\Rightarrow} H]$, contraction of $A \to B$ is explicitly introduced in the leftmost premise σ_A; as a consequence we might have $|\sigma_A| \geq |\sigma|$. In all the other cases, passing from the conclusion to a premise of a rule, the size of the sequents strictly decreases. The rule $\to R_2$ is the only rule that, when applied backward, can turn a b-sequent into an u-sequent.

A **Gbu**-*tree* π is a tree of sequents such that: if σ is a node of π with $\sigma_1,\ldots,\sigma_n$ as children, then there exists a rule of **Gbu** having premises $\sigma_1,\ldots,\sigma_n$ and conclusion σ. The *root rule of* π is the one having as conclusion the root sequent

$$\dfrac{}{[\bot, \Gamma \stackrel{l}{\Rightarrow} H]}\ \bot L \qquad \dfrac{}{[H, \Gamma \stackrel{l}{\Rightarrow} H]}\ \mathrm{Id}$$

$$\dfrac{[A, B, \Gamma \stackrel{\mathrm{u}}{\Rightarrow} H]}{[A \wedge B, \Gamma \stackrel{\mathrm{u}}{\Rightarrow} H]}\ \wedge L \qquad \dfrac{[\Gamma \stackrel{l}{\Rightarrow} A] \quad [\Gamma \stackrel{l}{\Rightarrow} B]}{[\Gamma \stackrel{l}{\Rightarrow} A \wedge B]}\ \wedge R$$

$$\dfrac{[A, \Gamma \stackrel{\mathrm{u}}{\Rightarrow} H] \quad [B, \Gamma \stackrel{\mathrm{u}}{\Rightarrow} H]}{[A \vee B, \Gamma \stackrel{\mathrm{u}}{\Rightarrow} H]}\ \vee L \qquad \dfrac{[\Gamma \stackrel{\mathrm{b}}{\Rightarrow} H_k]}{[\Gamma \stackrel{l}{\Rightarrow} H_0 \vee H_1]}\ \vee R_k \quad k \in \{0, 1\}$$

$$\dfrac{[A \to B, \Gamma \stackrel{\mathrm{b}}{\Rightarrow} A]\, [B, \Gamma \stackrel{\mathrm{u}}{\Rightarrow} H]}{[A \to B, \Gamma \stackrel{\mathrm{u}}{\Rightarrow} H]}\ \to L \qquad \dfrac{[\Gamma \stackrel{l}{\Rightarrow} B]}{[\Gamma \stackrel{l}{\Rightarrow} A \to B]}\ \to R_1 \qquad \dfrac{[A, \Gamma \stackrel{\mathrm{u}}{\Rightarrow} B]}{[\Gamma \stackrel{l}{\Rightarrow} A \to B]}\ \to R_2$$

$$\text{if } \Gamma \vdash_{\mathcal{E}} A \qquad\qquad \text{if } \Gamma \nvdash_{\mathcal{E}} A$$

Fig. 1. The calculus **Gbu**.

of π. A **Gbu**-*derivation* of σ is a **Gbu**-tree π with root σ and having conclusions of an axiom rule as leaves. A sequent σ is provable in **Gbu** iff there exists a **Gbu**-derivation of σ; H is provable in **Gbu** iff $[\stackrel{\mathrm{u}}{\Rightarrow} H]$ is provable in **Gbu**. Note that **Gbu** has the *subformula property*: given a **Gbu**-tree π with root σ, for every sequent σ' occurring in π it holds that $\mathrm{Sf}(\sigma') \subseteq \mathrm{Sf}(\sigma)$.

A **Gbu**-derivation π can be translated into a **G3i**-derivation $\tilde{\pi}$ applying the following steps: erase the labels from the sequents in π; when rule $\to R_1$ is applied, add the formula A to the left context; rename all occurrences of $\to R_1$ and $\to R_2$ to $\to R$. From this translation and the soundness of **G3i** [10] we get the soundness of **Gbu**. Semantically, this means that, if σ is provable in **Gbu**, then σ is not refutable.

Here we provide an example of a **Gbu**-derivation, then we prove that **Gbu** is terminating. The completeness of **Gbu** (Theorem 4) is proved in Section 5 as a consequence of the correctness of the proof-search procedure.

Example 1. Let $W = ((((p \to q) \to p) \to p) \to q) \to q$ be an instance of the *Weak Pierce Law* [1]. In Fig. 2 we give a **Gbu**-derivation[1] π_1 of $\sigma_1 = [\stackrel{\mathrm{u}}{\Rightarrow} W]$, using the evaluation $\vdash_{\tilde{\mathcal{E}}}$ of Section 2. Sequents are indexed by integers; by π_i we denote the subderivation of π_1 with root σ_i. When ambiguities can arise, we underline the main formula of a rule application. Building the derivation bottom-up, the only choice points are in the (backward) application of rule $\to L$ to σ_4 and σ_7, since we can select both A and B as main formula. If at sequent σ_6 we choose B instead of A, we get the **Gbu**-tree with root σ_6 sketched on the right. We have $\sigma_{7'} \vdash_{\tilde{\mathcal{E}}} p$ (indeed, p occurs on the left in $\sigma_{7'}$), hence the rule $\to R_1$ must be applied to $\sigma_{7'}$, which

$$\dfrac{\dfrac{[p, B, A \stackrel{\mathrm{b}}{\Rightarrow} q]_{8'}}{[p, B, A \stackrel{\mathrm{b}}{\Rightarrow} p \to q]_{7'}}\ \to R_1 \qquad \dfrac{\vdots}{[p, A \stackrel{\mathrm{u}}{\Rightarrow} q]_{9'}}}{[p, \underbrace{\underline{(p \to q) \to p}}_{B}, A \stackrel{\mathrm{u}}{\Rightarrow} q]_6}\ \to L$$

[1] The derivations and their LaTeX rendering are generated with g3ibu, an implementation of **Gbu** and **Rbu** available at `http://www.dista.uninsubria.it/~ferram/`.

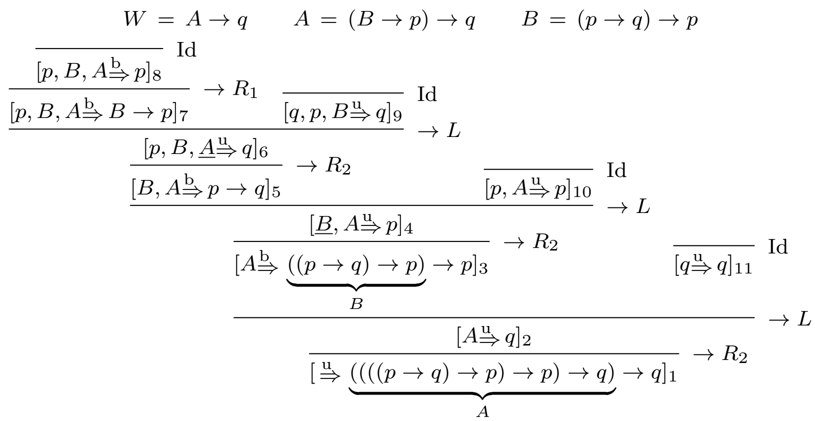


Fig. 2. **Gbu**-derivation of Weak Pierce Law

yields the b-sequent $\sigma_{8'}$. Since $\sigma_{8'}$ is blocked, we cannot decompose again left implications; thus the proof-search fails without entering an infinite loop. $\Diamond$

Termination of Gbu We show that every **Gbu**-tree has finite depth. A **Gbu**-*branch* is a sequence of sequents $\mathcal{B} = (\sigma_1, \sigma_2, \dots)$ such that, for every $i \geq 1$, there exists a rule $\mathcal{R}$ of **Gbu** having σ_i as conclusion and σ_{i+1} among its premises. The *length* of $\mathcal{B}$ is the number of sequents in it. Let $\gamma = (\sigma_i, \sigma_{i+1})$ be a pair of successive sequents in $\mathcal{B}$ with labels l_i and l_{i+1} respectively; γ is a bu-pair if $l_i = \mathrm{b}$ and $l_{i+1} = \mathrm{u}$; γ is an ub-pair if $l_i = \mathrm{u}$ and $l_{i+1} = \mathrm{b}$. By $\mathrm{BU}(\mathcal{B})$ and $\mathrm{UB}(\mathcal{B})$ we denote the number of bu-pairs and ub-pairs occurring in $\mathcal{B}$ respectively. Note that the only rule generating bu-pairs is $\to R_2$. Moreover, $|\sigma_{i+1}| \geq |\sigma_i|$ can happen only if (σ_i, σ_{i+1}) is an ub-pair generated by $\to L$: σ_{i+1} is the leftmost premise of an application of $\to L$ with conclusion σ_i. As a consequence, every subbranch of $\mathcal{B}$ not containing ub-pairs is finite. Hence, if we show that $\mathrm{UB}(\mathcal{B})$ is finite, we get that $\mathcal{B}$ has finite length.

We prove a kind of persistence of $\vdash_{\mathcal{E}}$, namely: if A occurs in the left-hand side of a sequent σ occurring in $\mathcal{B}$, then $\sigma' \vdash_{\mathcal{E}} A$ for every σ' following σ in $\mathcal{B}$.

Lemma 1. *Let $\mathcal{B} = (\sigma_1, \sigma_2, \dots)$ be a* **Gbu**-*branch where, for every $i \geq 1$, $\sigma_i = [\Gamma_i \stackrel{l_i}{\Rightarrow} H_i]$. Let $n \geq 1$ and $A \in \bigcup_{1 \leq i \leq n} \Gamma_i$. Then, $\Gamma_n \vdash_{\mathcal{E}} A$.*

Proof. By induction on $|A|$. If $A \in \Gamma_n$, by $(\mathcal{E}2)$ we immediately get $\Gamma_n \vdash_{\mathcal{E}} A$. If $A \notin \Gamma_n$, there exists $i : 1 \leq i < n$ such that $A \in \Gamma_i$ and $A \notin \Gamma_{i+1}$. This implies $A = B \cdot C$ with $\cdot \in \{\wedge, \vee, \to\}$. Let $\cdot = \wedge$; then σ_{i+1} is obtained from σ_i by an application of $\wedge L$ with main formula $B \wedge C$, hence $B \in \Gamma_{i+1}$ and $C \in \Gamma_{i+1}$. By induction hypothesis, $\Gamma_n \vdash_{\mathcal{E}} B$ and $\Gamma_n \vdash_{\mathcal{E}} C$; by $(\mathcal{E}3)$, $\Gamma_n \vdash_{\mathcal{E}} B \wedge C$. The cases $\cdot \in \{\vee, \to\}$ are similar and require properties $(\mathcal{E}4)$ and $(\mathcal{E}5)$. $\square$

Now, we provide a bound on $\mathrm{BU}(\mathcal{B})$.

$$\frac{}{[\Gamma^{\to}, \Gamma^{\mathrm{At}} \stackrel{l}{\Rightarrow} H]}\ \mathrm{Irr} \qquad \text{if } [\Gamma^{\to}, \Gamma^{\mathrm{At}} \stackrel{l}{\Rightarrow} H] \text{ is irreducible} \quad \begin{cases} H = \bot \text{ or } H \in \mathcal{V} \setminus \Gamma^{\mathrm{At}} \\ l = \mathrm{b} \text{ or } \Gamma^{\to} = \emptyset \end{cases}$$

$$\frac{[A, B, \Gamma \stackrel{\mathrm{u}}{\Rightarrow} H]}{[A \wedge B, \Gamma \stackrel{\mathrm{u}}{\Rightarrow} H]}\ \wedge L \qquad \frac{[\Gamma \stackrel{l}{\Rightarrow} H_k]}{[\Gamma \stackrel{l}{\Rightarrow} H_0 \wedge H_1]}\ \wedge R_k \quad k \in \{0,1\}$$

$$\frac{[A_k, \Gamma \stackrel{\mathrm{u}}{\Rightarrow} H]}{[A_0 \vee A_1, \Gamma \stackrel{\mathrm{u}}{\Rightarrow} H]}\ \vee L_k \quad k \in \{0,1\} \qquad \frac{[\Gamma \stackrel{\mathrm{b}}{\Rightarrow} H_0] \quad [\Gamma \stackrel{\mathrm{b}}{\Rightarrow} H_1]}{[\Gamma \stackrel{\mathrm{b}}{\Rightarrow} H_0 \vee H_1]}\ \vee R$$

$$\frac{[B, \Gamma \stackrel{\mathrm{u}}{\Rightarrow} H]}{[A \to B, \Gamma \stackrel{\mathrm{u}}{\Rightarrow} H]}\ \to L \qquad \frac{[\Gamma \stackrel{l}{\Rightarrow} B]}{[\Gamma \stackrel{l}{\Rightarrow} A \to B]}\ \to R_1 \ \text{ if } \Gamma \vdash_{\mathcal{E}} A \qquad \frac{[A, \Gamma \stackrel{\mathrm{u}}{\Rightarrow} B]}{[\Gamma \stackrel{l}{\Rightarrow} A \to B]}\ \to R_2 \ \text{ if } \Gamma \nvdash_{\mathcal{E}} A$$

$$\frac{\{\, [\Gamma^{\to}, \Gamma^{\mathrm{At}} \stackrel{\mathrm{b}}{\Rightarrow} A] \,\}_{A \to B \in \Gamma^{\to}}}{[\Gamma^{\to}, \Gamma^{\mathrm{At}} \stackrel{\mathrm{u}}{\Rightarrow} H]}\ \mathrm{S}_u^{\mathrm{At}} \qquad \text{where } \Gamma^{\to} \neq \emptyset \text{ and } (H = \bot \text{ or } H \in \mathcal{V} \setminus \Gamma^{\mathrm{At}})$$

$$\frac{\{\, [\Gamma^{\to}, \Gamma^{\mathrm{At}} \stackrel{\mathrm{b}}{\Rightarrow} A] \,\}_{A \to B \in \Gamma^{\to}} \quad [\Gamma^{\to}, \Gamma^{\mathrm{At}} \stackrel{\mathrm{b}}{\Rightarrow} H_0] \quad [\Gamma^{\to}, \Gamma^{\mathrm{At}} \stackrel{\mathrm{b}}{\Rightarrow} H_1]}{[\Gamma^{\to}, \Gamma^{\mathrm{At}} \stackrel{\mathrm{u}}{\Rightarrow} H_0 \vee H_1]}\ \mathrm{S}_u^{\vee}$$

Fig. 3. The refutation calculus **Rbu**.

Lemma 2. *Let* $\mathcal{B} = (\sigma_1, \sigma_2, \dots)$ *be a* **Gbu**-*branch. Then,* $\mathrm{BU}(\mathcal{B}) \leq |\sigma_1|$.

Proof. Let $(\sigma_i^{\mathrm{b}}, \sigma_{i+1}^{\mathrm{u}})$ be a bu-pair in $\mathcal{B}$. Since bu-pairs are generated by applications of $\to R_2$, we have: $\sigma_i^{\mathrm{b}} = [\Gamma \stackrel{\mathrm{b}}{\Rightarrow} A \to B]$, $\sigma_{i+1}^{\mathrm{u}} = [A, \Gamma \stackrel{\mathrm{u}}{\Rightarrow} B]$ and $\Gamma \nvdash_{\mathcal{E}} A$. By Lemma 1, for every $j \geq i+1$ it holds that $\Gamma_j \vdash_{\mathcal{E}} A$. Thus, any bu-pair following $(\sigma_i^{\mathrm{b}}, \sigma_{i+1}^{\mathrm{u}})$ must treat an implication $C \to D$ with $C \neq A$. Since **Gbu** has the subformula property, the main formulas of $\to R_2$ applications belong to $\mathrm{Sf}(\sigma_1)$. Thus, $\mathrm{BU}(\mathcal{B})$ is bounded by the number $\#\mathrm{Sf}(\sigma_1)$ of subformulas of σ_1. Since $\#\mathrm{Sf}(\sigma_1) \leq |\sigma_1|$, we get $\mathrm{BU}(\mathcal{B}) \leq |\sigma_1|$. □

Since between two ub-pairs of $\mathcal{B}$ a bu-pair must occur, $\mathrm{UB}(\mathcal{B}) \leq \mathrm{BU}(\mathcal{B}) + 1$; by Lemma 2, $\mathrm{UB}(\mathcal{B})$ is finite. We can conclude:

Proposition 1. *Every* **Gbu**-*branch has finite length.* □

As a consequence, every **Gbu**-tree has finite depth and **Gbu** is terminating.

4 The refutation calculus Rbu

In this section, following the ideas of [3, 9], we introduce the refutation calculus **Rbu** for deriving intuitionistic unprovability. Intuitively, an **Rbu**-derivation π of a sequent σ^{u} is a sort of "constructive proof" of refutability of σ^{u} in the sense that from π we can extract a countermodel $\mathrm{Mod}(\pi)$ for σ^{u}.

We denote with Γ^{At} a finite set of propositional variables and with $\Gamma^{\to}$ a finite set of implicative formulas. A sequent σ is *irreducible* iff $\sigma = [\Gamma^{\to}, \Gamma^{\mathrm{At}} \stackrel{l}{\Rightarrow} H]$

with $H \in \{\bot\} \cup (\mathcal{V} \setminus \Gamma^{\mathrm{At}})$ and ($l = \mathrm{b}$ or $\Gamma^{\rightarrow} = \emptyset$). The rules of **Rbu** are given in Fig. 3. As in **Gbu**, writing C, Γ in the conclusion of a rule, we assume that $C \notin \Gamma$. The notions of **Rbu**-tree, **Rbu**-derivation and **Rbu**-branch are defined analogously to those for **Gbu**.

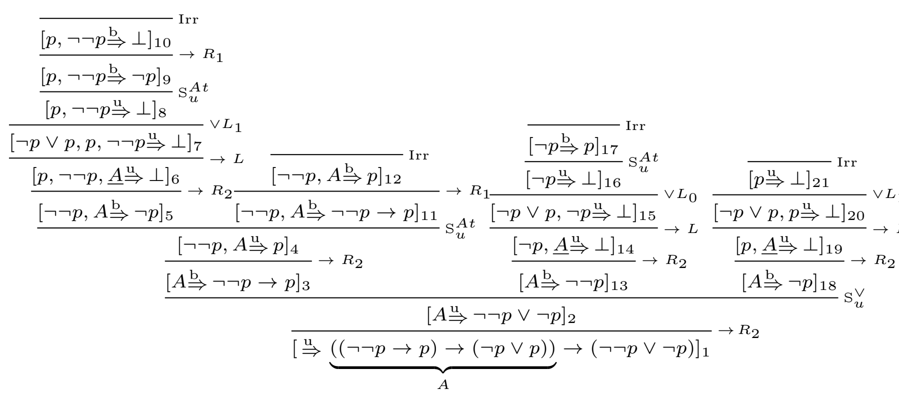
The rule $\mathrm{S}_u^{\mathrm{At}}$ has a premise $[\Gamma^{\rightarrow}, \Gamma^{\mathrm{At}} \overset{\mathrm{b}}{\Rightarrow} A]$ for every A such that $A \rightarrow B \in \Gamma^{\rightarrow}$; since $\Gamma^{\rightarrow} \neq \emptyset$, there exists at least one premise. The rule $\mathrm{S}_u^{\vee}$ is similar and has at least two premises. All the premises of $\mathrm{S}_u^{\mathrm{At}}$ and $\mathrm{S}_u^{\vee}$ are b-sequents.

It is easy to check that an **Rbu**-branch is also a **Gbu**-branch[2]. Accordingly, Proposition 1 implies that the calculus **Rbu** is terminating. In the following we prove that **Rbu** is sound in the following sense:

Theorem 2 (Soundness of Rbu). *If an* u-*sequent* σ^{u} *is provable in* **Rbu**, *then* σ^{u} *is refutable.* □

Example 2. Let $S = ((\neg\neg p \rightarrow p) \rightarrow (\neg p \vee p)) \rightarrow (\neg\neg p \vee \neg p)$ be an instance of the *Scott principle* [1], where $\neg Z = Z \rightarrow \bot$. We show the **Rbu**-derivation π_1 of $[\overset{\mathrm{u}}{\Rightarrow} S]$.

$S = A \rightarrow (\neg\neg p \vee \neg p) \qquad A = (\neg\neg p \rightarrow p) \rightarrow (\neg p \vee p)$



◊

Soundness of Rbu Let π be an **Rbu**-derivation with root $\sigma^{\mathrm{b}} = [\Gamma^{\rightarrow}, \Gamma^{\mathrm{At}} \overset{\mathrm{b}}{\Rightarrow} H]$. By $\Pi(\pi, \sigma^{\mathrm{b}})$ we denote the maximal subtree of π having root σ^{b} and only containing b-sequents (that is, any subtree of π with root σ^{b} extending $\Pi(\pi, \sigma^{\mathrm{b}})$ contains at least one u-sequent). Since only the rules $\wedge R_k$, $\vee R$ and $\rightarrow R_1$ can be applied in $\Pi(\pi, \sigma^{\mathrm{b}})$, every leaf σ' of $\Pi(\pi, \sigma^{\mathrm{b}})$ has the form $[\Gamma^{\rightarrow}, \Gamma^{\mathrm{At}} \overset{\mathrm{b}}{\Rightarrow} H']$, where $H' \in \mathrm{Sf}(H)$; moreover, σ' is either an irreducible sequent (hence a leaf of π) or the conclusion of an application of $\rightarrow R_2$ (the only rule of **Rbu** which, read bottom-up, "unblocks" a b-sequent). Thus, π can be displayed as in Fig. 4. The sequents $\sigma_1^{\mathrm{u}}, \ldots, \sigma_n^{\mathrm{u}}$ ($n \geq 0$) are called the u-*successors* of σ^{b} in π, while the sequents $\tau_1^{\mathrm{b}}, \ldots, \tau_m^{\mathrm{b}}$ ($m \geq 0$) are the *i-successors* (irreducible successors) of σ^{b} in π. Let $\mathrm{d}(\pi)$ be the depth of π; if $\mathrm{d}(\pi) = 0$, then σ^{b} coincides with τ_1^{b}, hence σ^{b} has no u-successors and has itself as only i-successor.

[2] The converse in general does not hold since the rule $\vee R$ of **Rbu** requires a b-sequent as conclusion.

$$\cdots \quad \dfrac{\begin{array}{c}\vdots\ \pi_i\\ \sigma_i^{\mathrm{u}} = [\Gamma^{\to}, \Gamma^{\mathrm{At}}, A_i \overset{\mathrm{u}}{\Rightarrow} B_i]\end{array}}{\sigma_i^{\mathrm{b}} = [\Gamma^{\to}, \Gamma^{\mathrm{At}} \overset{\mathrm{b}}{\Rightarrow} A_i \to B_i]}\to R_2 \quad \cdots \quad \dfrac{}{\tau_j^{\mathrm{b}} = [\Gamma^{\to}, \Gamma^{\mathrm{At}} \overset{\mathrm{b}}{\Rightarrow} H_j]}\,\mathrm{Irr} \quad \cdots$$

$$\vdots\ \Pi(\pi, \sigma^{\mathrm{b}})$$

$$\sigma^{\mathrm{b}} = [\Gamma^{\to}, \Gamma^{\mathrm{At}} \overset{\mathrm{b}}{\Rightarrow} H]$$

where $i \in \{1, \dots, n\}$, $j \in \{1, \dots, m\}$, $n \geq 0$, $m \geq 0$, $n + m \geq 1$ and:

- the **Rbu**-tree $\Pi(\pi, \sigma^{\mathrm{b}})$ only contains b-sequents;
- π_i is an **Rbu**-derivation of σ_i^{u}.

Fig. 4. Structure of an **Rbu**-derivation π of $\sigma^{\mathrm{b}} = [\Gamma^{\to}, \Gamma^{\mathrm{At}} \overset{\mathrm{b}}{\Rightarrow} H]$.

Now, let us consider an **Rbu**-derivation π of an u-sequent σ^{u} having root rule $\mathcal{R} = \mathrm{S}_u^{\mathrm{At}}$ or $\mathcal{R} = \mathrm{S}_u^{\vee}$. Every premise σ' of $\mathcal{R}$ is a b-sequent and the subderivation of π with root σ' has the structure shown in Fig. 4. The set of the u-*successors* of σ^{u} in π is the union of the sets of u-successors in π of the premises of $\mathcal{R}$; the set of the i-*successors* of σ^{u} in π is defined analogously. To display a proof π of this kind we use the concise notation of Fig. 5.

Example 3. Let us consider the **Rbu**-derivation π_1 in Ex. 2. The u-successors and i-successors are defined as follows:

u-sequent	**u-successors**	**i-successors**
σ_2	σ_4 , σ_{14} , σ_{19}	
σ_4	σ_6	σ_{12}
σ_8		σ_{10}
σ_{16}		σ_{17}

$\Diamond$

Now we describe how to extract from an **Rbu**-derivation of an u-sequent σ^{u} a Kripke countermodel $\mathrm{Mod}(\pi)$ for σ^{u}. $\mathrm{Mod}(\pi)$ is defined by induction on $\mathrm{d}(\pi)$. By $\mathcal{K}^1(\rho, \Gamma^{\mathrm{At}})$ we denote the Kripke model $\mathcal{K} = \langle \{\rho\}, \{(\rho, \rho)\}, \rho, V \rangle$ consisting of only one world ρ such that $V(\rho) = \Gamma^{\mathrm{At}}$. Let $\mathcal{R}$ be the root rule of π.

(K1) If $\mathcal{R} = \mathrm{Irr}$, then $\mathrm{d}(\pi) = 0$ and $\sigma^{\mathrm{u}} = [\Gamma^{\mathrm{At}} \overset{\mathrm{u}}{\Rightarrow} H]$ (being σ^{u} irreducible, $\Gamma^{\to} = \emptyset$). We set $\mathrm{Mod}(\pi) = \mathcal{K}^1(\rho, \Gamma^{\mathrm{At}})$, with ρ any element.

(K2) Let $\mathcal{R}$ be different from Irr, $\mathrm{S}_u^{\mathrm{At}}$, $\mathrm{S}_u^{\vee}$ and let π' be the only immediate subderivation of π. Then, $\mathrm{Mod}(\pi) = \mathrm{Mod}(\pi')$.

(K3) Let $\mathcal{R}$ be $\mathrm{S}_u^{\mathrm{At}}$ or $\mathrm{S}_u^{\vee}$ and let π be displayed as in Fig. 5.
If $n = 0$, then $\mathcal{K}$ is the model $\mathcal{K}^1(\rho, \Gamma^{\mathrm{At}})$, with ρ any element.
Let $n > 0$ and, for every $i \in \{1, \dots, n\}$, let $\mathrm{Mod}(\pi_i) = \langle P_i, \leq_i, \rho_i, V_i \rangle$. Without loss of generality, we can assume that the P_i's are pairwise disjoint. Let ρ be an element not in $\bigcup_{i \in \{1,\dots,n\}} P_i$ and let $\mathcal{K} = \langle P, \leq, \rho, V \rangle$ be the model such that:
– $P = \{\rho\} \cup \bigcup_{i \in \{1,\dots,n\}} P_i$;

$$\pi \;=\; \cfrac{\begin{array}{c}\vdots\;\pi_1 \\ \sigma_1^{\mathrm{u}}\end{array} \quad \dots \quad \begin{array}{c}\vdots\;\pi_n \\ \sigma_n^{\mathrm{u}}\end{array} \quad \tau_1^{\mathrm{b}} \dots \quad \tau_m^{\mathrm{b}}}{\sigma^{\mathrm{u}} = [\Gamma^{\to}, \Gamma^{\mathrm{At}} \overset{\mathrm{u}}{\Rightarrow} H]}\ \mathcal{R}$$

- $\mathcal{R} \in \{\mathrm{S}_u^{\mathrm{At}}, \mathrm{S}_u^{\vee}\}$, $n \geq 0$, $m \geq 0$, $n + m \geq 1$.
- $\sigma_1^{\mathrm{u}}, \dots, \sigma_n^{\mathrm{u}}$ are all the u-successors of σ^{u} in π.
- π_i is an **Rbu**-derivation of σ_i^{u} $(1 \leq i \leq n)$.
- $\tau_1^{\mathrm{b}}, \dots, \tau_m^{\mathrm{b}}$ are all the i-successors of σ^{u} in π.

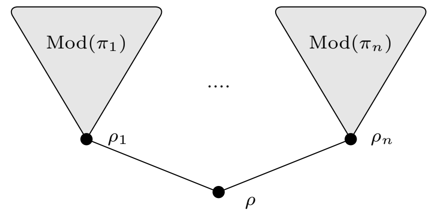
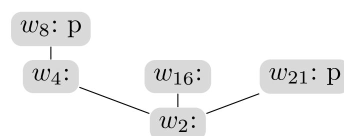


Fig. 5. An **Rbu**-derivation π with root rule $\mathrm{S}_u^{\mathrm{At}}$ or $\mathrm{S}_u^{\vee}$ and the model Mod(π).

- $\leq \,=\, \{\,(\rho, \alpha) \mid \alpha \in P\,\} \,\cup\, \bigcup_{i \in \{1,\dots,n\}} \leq_i$;
- $V(\rho) \,=\, \Gamma^{\mathrm{At}}$ and, for every $i \in \{1, \dots, n\}$ and $\alpha \in P_i$, $V(\alpha) = V_i(\alpha)$.

Then $\mathrm{Mod}(\pi) = \mathcal{K}$. The model $\mathrm{Mod}(\pi)$ is represented in Fig. 5.

Example 4. We show the Kripke model $\mathrm{Mod}(\pi_1)$ extracted from the **Rbu**-derivation π_1 of Ex. 2. The model is displayed as a tree with the convention that $w < w'$ if the world w is drawn below w'. For each w_i, we list the propositional variables in $V(w_i)$. We inductively define the models $\mathrm{Mod}(\pi_i)$ for every i such that $\sigma_i = [\Gamma_i \overset{\mathrm{u}}{\Rightarrow} H_i]$ is an u-sequent. At each step one can check that $\mathrm{Mod}(\pi_i), \rho_i \rhd \sigma_i$, where ρ_i is the root of $\mathrm{Mod}(\pi_i)$. Hence, $\mathrm{Mod}(\pi_1), w_2 \nVdash S$ ($\mathrm{Mod}(\pi_1)$ is a countermodel for S).



- By Point (K3), since σ_8 has no u-successors (see Ex. 3), $\mathrm{Mod}(\pi_8) = \mathcal{K}^1(w_8, \{p\})$. Similarly, $\mathrm{Mod}(\pi_{16}) = \mathcal{K}^1(w_{16}, \emptyset)$.
- Since σ_{21} is irreducible, by Point (K1) $\mathrm{Mod}(\pi_{21}) = \mathcal{K}^1(w_{21}, \{p\})$.
- By Point (K2), $\mathrm{Mod}(\pi_6) = \mathrm{Mod}(\pi_7) = \mathrm{Mod}(\pi_8)$. Similarly, $\mathrm{Mod}(\pi_{14}) = \mathrm{Mod}(\pi_{15}) = \mathrm{Mod}(\pi_{16})$ and $\mathrm{Mod}(\pi_{19}) = \mathrm{Mod}(\pi_{20}) = \mathrm{Mod}(\pi_{21})$.
- By Point (K3), $\mathrm{Mod}(\pi_4)$ is obtained by extending with w_4 the model $\mathrm{Mod}(\pi_6)$ (indeed, σ_6 is the only u-successor of σ_4) and $V(w_4) = \Gamma_4 \cap \mathcal{V} = \emptyset$. Similarly, $\mathrm{Mod}(\pi_2)$ is obtained by gluing on w_2 the models generated by the u-successors σ_4, σ_{14} and σ_{19} of σ_2 and $V(w_2) = \Gamma_2 \cap \mathcal{V} = \emptyset$.
- Finally, $\mathrm{Mod}(\pi_1) = \mathrm{Mod}(\pi_2)$ by Point (K2). $\Diamond$

We prove the soundness of **Rbu**. Given an **Rbu**-tree π with root $[\Gamma^{\to}, \Gamma^{\mathrm{At}} \overset{\mathrm{b}}{\Rightarrow} H]$ and only containing b-sequents, every leaf of π has the form $[\Gamma^{\to}, \Gamma^{\mathrm{At}} \overset{\mathrm{b}}{\Rightarrow} H']$.

Lemma 3. *Let π be an* **Rbu***-tree with root $\sigma^{\mathrm{b}} = [\Gamma^{\to}, \Gamma^{\mathrm{At}} \overset{\mathrm{b}}{\Rightarrow} H]$ and only containing* b*-sequents, let $\sigma_1^{\mathrm{b}} = [\Gamma^{\to}, \Gamma^{\mathrm{At}} \overset{\mathrm{b}}{\Rightarrow} H_1], \dots,\ \sigma_n^{\mathrm{b}} = [\Gamma^{\to}, \Gamma^{\mathrm{At}} \overset{\mathrm{b}}{\Rightarrow} H_n]$ be the leaves of π. Let $\mathcal{K} = \langle P, \leq, \rho, V\rangle$ be a Kripke model and $\alpha \in P$ such that:*

(H1) $\mathcal{K}, \alpha \nVdash H_i$, for every $i \in \{1, \dots, n\}$;
(H2) $\mathcal{K}, \alpha \Vdash Z$, for every $Z \in \Gamma^{\to} \cap \mathrm{Sf}(H)$;

(H3) $V(\alpha) = \Gamma^{\mathrm{At}}$.

Then, $\mathcal{K}, \alpha \nVdash H$.

Proof. By induction on $\mathrm{d}(\pi)$. If $\mathrm{d}(\pi) = 0$, then $\sigma^{\mathrm{b}} = \sigma_1^{\mathrm{b}}$ and the assertion immediately follows by (H1). Let us assume that $\mathrm{d}(\pi) > 0$ and let $\mathcal{R}$ be the root rule of π. Since both the conclusion and the premises of $\mathcal{R}$ are b-sequents, $\mathcal{R}$ is one of the rules $\wedge R_k$, $\vee R$ and $\to R_1$. The proof proceeds by cases on $\mathcal{R}$. The cases $\mathcal{R} \in \{\wedge R_k, \vee R\}$ immediately follow by the induction hypothesis.

If $\mathcal{R}$ is $\to R_1$, then $\sigma^{\mathrm{b}} = [\Gamma^{\to}, \Gamma^{\mathrm{At}} \overset{\mathrm{b}}{\Rightarrow} A \to B]$, the premise of $\mathcal{R}$ is $\sigma' = [\Gamma^{\to}, \Gamma^{\mathrm{At}} \overset{\mathrm{b}}{\Rightarrow} B]$ and, by the side condition, $\Gamma^{\to}, \Gamma^{\mathrm{At}} \vdash_{\mathcal{E}} A$. By induction hypothesis on the subderivation of π having root σ', we get $\mathcal{K}, \alpha \nVdash B$. We show that $\mathcal{K}, \alpha \Vdash A$. Let $\Gamma_A = (\Gamma^{\to} \cap \mathrm{Sf}(A)) \cup \Gamma^{\mathrm{At}}$. Since $\Gamma_A \cap \mathrm{Sf}(A) = (\Gamma^{\to} \cup \Gamma^{\mathrm{At}}) \cap \mathrm{Sf}(A)$ and $\Gamma^{\to}, \Gamma^{\mathrm{At}} \vdash_{\mathcal{E}} A$, by ($\mathcal{E}1$) we get $\Gamma_A \vdash_{\mathcal{E}} A$. By the hypothesis (H2) and (H3) of the lemma, it holds that $\mathcal{K}, \alpha \Vdash \Gamma_A$; by ($\mathcal{E}6$), we deduce $\mathcal{K}, \alpha \Vdash A$. Thus $\mathcal{K}, \alpha \Vdash A$ and $\mathcal{K}, \alpha \nVdash B$, which implies $\mathcal{K}, \alpha \nVdash A \to B$. ◻

Now, we show that the model $\mathrm{Mod}(\pi)$ is a countermodel for σ^{u}.

Theorem 3. *Let* π *be an* **Rbu***-derivation of an* u*-sequent* σ^{u} *and let* ρ *be the root of* $\mathrm{Mod}(\pi)$*. Then* $\mathrm{Mod}(\pi), \rho \rhd \sigma^{\mathrm{u}}$.

Proof. By induction on $\mathrm{d}(\pi)$. If $\mathrm{d}(\pi) = 0$, then $\mathrm{Mod}(\pi)$ is defined as in (K1) and the assertion immediately follows.

Let $\mathrm{d}(\pi) > 0$ and let $\mathcal{R}$ be the root rule of π. If $\mathcal{R} \notin \{\mathrm{S}_u^{\mathrm{At}}, \mathrm{S}_u^{\vee}\}$, the assertion immediately follows by induction hypothesis (the case $\mathcal{R} = \to R_1$ requires ($\mathcal{E}6$)).

Let $\mathcal{R} = \mathrm{S}_u^{\vee}$ (the case $\mathcal{R} = \mathrm{S}_u^{\mathrm{At}}$ is similar). Let $\sigma^{\mathrm{u}} = [\Gamma^{\to}, \Gamma^{\mathrm{At}} \overset{\mathrm{u}}{\Rightarrow} H_0 \vee H_1]$ and let $\mathcal{K} = \langle P, \leq, \rho, V \rangle$ be the model $\mathrm{Mod}(\pi)$. By a secondary induction hypothesis on the structure of formulas, we prove that:

(B1) $\mathcal{K}, \rho \nVdash A$, for every $A \to B \in \Gamma^{\to}$;
(B2) $\mathcal{K}, \rho \Vdash A \to B$, for every $A \to B \in \Gamma^{\to}$;
(B3) $\mathcal{K}, \rho \nVdash H_0$ and $\mathcal{K}, \rho \nVdash H_1$.

To prove Point (B1), let $A \to B \in \Gamma^{\to}$. By definition of $\mathrm{S}_u^{\vee}$, π has an immediate subderivation π_A of $\sigma_A^{\mathrm{b}} = [\Gamma^{\to}, \Gamma^{\mathrm{At}} \overset{\mathrm{b}}{\Rightarrow} A]$ of the form (see Fig. 4):

$$\cdots \quad \dfrac{\begin{array}{c}\vdots\ \pi_i \\ \sigma_i^{\mathrm{u}} = [\Gamma^{\to}, \Gamma^{\mathrm{At}}, A_i \overset{\mathrm{u}}{\Rightarrow} B_i]\end{array}}{\sigma_i^{\mathrm{b}} = [\Gamma^{\to}, \Gamma^{\mathrm{At}} \overset{\mathrm{b}}{\Rightarrow} A_i \to B_i]} \to R_2 \quad \cdots \quad \dfrac{}{\tau_j^{\mathrm{b}} = [\Gamma^{\to}, \Gamma^{\mathrm{At}} \overset{\mathrm{b}}{\Rightarrow} H_j]}\ \mathrm{Irr} \quad \cdots$$
$$\vdots\ \Pi(\pi_A, \sigma_A^{\mathrm{b}})$$
$$\sigma_A^{\mathrm{b}} = [\Gamma^{\to}, \Gamma^{\mathrm{At}} \overset{\mathrm{b}}{\Rightarrow} A]$$

We show that $\Pi(\pi_A, \sigma_A^{\mathrm{b}})$ meets the hypothesis (H1)–(H3) of Lemma 3 w.r.t. the root ρ of $\mathcal{K}$, so that we can apply the lemma to infer $\mathcal{K}, \rho \nVdash A$. We prove (H1). Let us assume $n \geq 1$ and let $i \in \{1, \ldots, n\}$; we must show that $\mathcal{K}, \rho \nVdash A_i \to B_i$.

Since σ_i^{u} is an u-successor of σ^{u}, the root ρ_i of $\mathrm{Mod}(\pi_i)$ is an immediate successor of ρ in $\mathcal{K}$. By the main induction hypothesis $\mathrm{Mod}(\pi_i), \rho_i \triangleright \sigma_i^{\mathrm{u}}$; this implies that $\mathrm{Mod}(\pi_i), \rho_i \Vdash A_i$ and $\mathrm{Mod}(\pi_i), \rho_i \nVdash B_i$. Since $\mathrm{Mod}(\pi_i)$ is a submodel of $\mathcal{K}$, we get $\mathcal{K}, \rho_i \Vdash A_i$ and $\mathcal{K}, \rho_i \nVdash B_i$, which implies $\mathcal{K}, \rho \nVdash A_i \to B_i$. Let $m \geq 1$ and $j \in \{1, \ldots, m\}$. By definition of τ_j^{b}, either $H_j = \bot$ or $H_j \in \mathcal{V} \setminus \Gamma^{\mathrm{At}}$; in both cases $\mathcal{K}, \rho \nVdash H_j$. This proves that hypothesis (H1) of Lemma 3 holds. To prove hypothesis (H2), let $Z \in \Gamma^{\to} \cap \mathrm{Sf}(A)$. Since $|Z| < |A \to B|$, by the secondary induction hypothesis on Point (B2) we get $\mathcal{K}, \rho \Vdash Z$. The hypothesis (H3) follows by the definition of V in $\mathcal{K}$. We can apply Lemma 3 to deduce $\mathcal{K}, \rho \nVdash A$, and this proves Point (B1).

We prove Point (B2). Let π and $\mathrm{Mod}(\pi)$ be as in Fig. 5 (with $H = H_0 \vee H_1$). Let $A \to B \in \Gamma^{\to}$ and let α be a world of $\mathcal{K}$ such that $\mathcal{K}, \alpha \Vdash A$; we show that $\mathcal{K}, \alpha \Vdash B$. By Point (B1), α is different from ρ. Thus, $n \geq 1$ and, for some $i \in \{1, \ldots, n\}$, α belongs to $\mathrm{Mod}(\pi_i)$. Let ρ_i be the root of $\mathrm{Mod}(\pi_i)$. By the main induction hypothesis, $\mathrm{Mod}(\pi_i), \rho_i \triangleright \sigma_i^{\mathrm{u}}$; since $A \to B$ belongs to the left-hand side of σ_i^{u}, we get $\mathrm{Mod}(\pi_i), \rho_i \Vdash A \to B$, which implies $\mathcal{K}, \rho_i \Vdash A \to B$. Since $\rho_i \leq \alpha$ and $\mathcal{K}, \alpha \Vdash A$, we get $\mathcal{K}, \alpha \Vdash B$; thus $\mathcal{K}, \rho \Vdash A \to B$ and Point (B2) holds.

The proof of Point (B3) is similar to the proof of Point (B1), considering the immediate subderivations of π with root sequents $[\Gamma^{\to}, \Gamma^{\mathrm{At}} \overset{\mathrm{b}}{\Rightarrow} H_0]$ and $[\Gamma^{\to}, \Gamma^{\mathrm{At}} \overset{\mathrm{b}}{\Rightarrow} H_1]$. By Points (B2) and (B3) we conclude $\mathcal{K}, \rho \triangleright \sigma^{\mathrm{u}}$. □

By Theorem 3, we get the soundness of **Rbu** stated in Theorem 2.

5 The proof-search procedure

We show that, given an u-sequent σ^{u}, either a **Gbu**-derivation or an **Rbu**-derivation of σ^{u} can be built; from this, the completeness of **Gbu** follows. To this aim, we introduce the function `F` of Fig. 6. A sequent $[\Gamma \overset{l}{\Rightarrow} H]$ is in *normal form* if $l = \mathrm{b}$ implies $\Gamma = \Gamma^{\to}, \Gamma^{\mathrm{At}}$; given a sequent σ in normal form, `F`(σ) returns either a **Gbu**-derivation or an **Rbu**-derivation of σ. To construct a derivation, we use the auxiliary function `B`: given a calculus $\mathcal{C} \in \{\mathbf{Gbu}, \mathbf{Rbu}\}$, a sequent σ, a set $\mathcal{P}$ of $\mathcal{C}$-trees and a rule $\mathcal{R}$ of $\mathcal{C}$, `B`$(\mathcal{C}, \sigma, \mathcal{P}, \mathcal{R})$ is the $\mathcal{C}$-tree having root sequent σ, root rule $\mathcal{R}$, and all the $\mathcal{C}$-trees in $\mathcal{P}$ as immediate subtrees.

Proof-search is performed by applying backward the rules of **Gbu**. For instance, the recursive call `F(`$[A, B, \Gamma' \overset{\mathrm{u}}{\Rightarrow} H]$`)` at line 3 corresponds to the backward application of the rule $\wedge L$ to $\sigma = [A \wedge B, \Gamma' \overset{\mathrm{u}}{\Rightarrow} H]$; according to the outcome, at lines 4–5 a **Gbu**-derivation or an **Rbu**-derivation of σ with root rule $\wedge L$ is built. We remark that the input sequent of `F` must be in normal form; to guarantee that the recursive invocations are sound, the rules $\vee R_k$ and $\to L$, generating b-sequents, can be backward applied to $[\Gamma \overset{\mathrm{u}}{\Rightarrow} H]$ only if Γ has the form $\Gamma^{\to}, \Gamma^{\mathrm{At}}$.

To save space, some instructions are written in a high-level compact form (see, e.g., line 8); the rules used in lines 1 and 32 are defined as follows:

$$\mathcal{R}_{\mathrm{ax}}([\Gamma \overset{l}{\Rightarrow} H]) \;=\; \begin{cases} \bot L & \text{if } \bot \in \Gamma \\ \mathrm{Id} & \text{otherwise} \end{cases} \qquad \mathcal{R}_{\mathrm{s}}([\Gamma \overset{l}{\Rightarrow} H]) \;=\; \begin{cases} \vee R & \text{if } l = \mathrm{b} \\ \mathrm{S}_u^{\mathrm{At}} & \text{if } l = \mathrm{u} \text{ and } H \in \mathcal{V} \\ \mathrm{S}_u^{\vee} & \text{otherwise} \end{cases}$$

Precondition : σ is in normal form ($l = \mathrm{b}$ implies $\Gamma = \Gamma^{\rightarrow}, \Gamma^{\mathrm{At}}$)
1 **if** $\bot \in \Gamma$ *or* $H \in \Gamma$ **then return** B(**Gbu**, σ, $\emptyset$, $\mathcal{R}_{\mathrm{ax}}(\sigma)$) // $\mathcal{R}_{\mathrm{ax}}(\sigma)$ is $\bot L$ or Id
2 **else if** $\sigma = [A \wedge B, \Gamma' \overset{\mathrm{u}}{\Rightarrow} H]$ *where* $\Gamma' = \Gamma \setminus \{A \wedge B\}$ **then**
3 $\quad \pi' \leftarrow$ F($[A, B, \Gamma' \overset{\mathrm{u}}{\Rightarrow} H]$)
4 $\quad$ **if** π' *is a* **Gbu**-*tree* **then return** B(**Gbu**, σ, $\{\pi'\}$, $\wedge L$)
5 $\quad$ **else return** B(**Rbu**, σ, $\{\pi'\}$, $\wedge L$)
6 **else if** $\sigma = [A_0 \vee A_1, \Gamma' \overset{\mathrm{u}}{\Rightarrow} H]$ *where* $\Gamma' = \Gamma \setminus \{A_0 \vee A_1\}$ **then**
7 $\quad \pi_0 \leftarrow$ F($[A_0, \Gamma' \overset{\mathrm{u}}{\Rightarrow} H]$), $\quad \pi_1 \leftarrow$ F($[A_1, \Gamma' \overset{\mathrm{u}}{\Rightarrow} H]$)
8 $\quad$ **if** $\exists k \in \{0,1\}$ *s.t.* π_k *is an* **Rbu**-*tree* **then return** B(**Rbu**, σ, $\{\pi_k\}$, $\vee L_k$)
9 $\quad$ **else return** B(**Gbu**, σ, $\{\pi_0, \pi_1\}$, $\vee L$)
10 **else if** $\sigma = [\Gamma \overset{l}{\Rightarrow} A \to B]$ **then**
11 $\quad$ **if** $\Gamma \vdash_{\mathcal{E}} A$ **then** $\pi' \leftarrow$ F($[\Gamma \overset{l}{\Rightarrow} B]$), $\quad k \leftarrow 1$
12 $\quad$ **else** $\pi' \leftarrow$ F($[A, \Gamma \overset{\mathrm{u}}{\Rightarrow} B]$), $\quad k \leftarrow 2$
13 $\quad$ **if** π' *is a* **Gbu**-*tree* **then return** B(**Gbu**, σ, $\{\pi'\}$, $\to R_k$)
14 $\quad$ **else return** B(**Rbu**, σ, $\{\pi'\}$, $\to R_k$)
15 **else if** $\sigma = [\Gamma \overset{l}{\Rightarrow} H_0 \wedge H_1]$ **then**
16 $\quad \pi_0 \leftarrow$ F($[\Gamma \overset{l}{\Rightarrow} H_0]$), $\quad \pi_1 \leftarrow$ F($[\Gamma \overset{l}{\Rightarrow} H_1]$)
17 $\quad$ **if** $\exists k \in \{0,1\}$ *s.t.* π_k *is an* **Rbu**-*tree* **then return** B(**Rbu**, σ, $\{\pi_k\}$, $\wedge R_k$)
18 $\quad$ **else return** B(**Gbu**, σ, $\{\pi_0, \pi_1\}$, $\wedge R$)
19 // Here $\sigma = [\Gamma^{\rightarrow}, \Gamma^{\mathrm{At}} \overset{l}{\Rightarrow} H]$, where $H = \bot$ or $H \in \mathcal{V} \setminus \Gamma^{\mathrm{At}}$ or $H = H_0 \vee H_1$
20 **else if** *($l = \mathrm{u}$ and $\Gamma^{\rightarrow} \neq \emptyset$) or $H = H_0 \vee H_1$* **then**
21 $\quad$ Refs $\leftarrow \emptyset$ // set of **Rbu**-trees
22 $\quad$ **if** $H = H_0 \vee H_1$ **then**
23 $\quad\quad \pi_0 \leftarrow$ F($[\Gamma \overset{\mathrm{b}}{\Rightarrow} H_0]$), $\quad \pi_1 \leftarrow$ F($[\Gamma \overset{\mathrm{b}}{\Rightarrow} H_1]$)
24 $\quad\quad$ **if** $\exists k \in \{0,1\}$ *s.t.* π_k *is a* **Gbu**-*tree* **then return** B(**Gbu**, σ, $\{\pi_k\}$, $\vee R_k$)
25 $\quad\quad$ **else** Refs $\leftarrow$ Refs $\cup \{\pi_0, \pi_1\}$
26 $\quad$ **if** $l = \mathrm{u}$ **then**
27 $\quad\quad$ **foreach** $A \to B \in \Gamma^{\rightarrow}$ **do**
28 $\quad\quad\quad \pi_A \leftarrow$ F($[\Gamma^{\rightarrow}, \Gamma^{\mathrm{At}} \overset{\mathrm{b}}{\Rightarrow} A]$), $\pi_B \leftarrow$ F($[B, \Gamma^{\rightarrow} \setminus \{A \to B\}, \Gamma^{\mathrm{At}} \overset{\mathrm{u}}{\Rightarrow} H]$)
29 $\quad\quad\quad$ **if** π_B *is an* **Rbu**-*tree* **then return** B(**Rbu**, σ, $\{\pi_B\}$, $\to L$)
30 $\quad\quad\quad$ **else if** π_A *is a* **Gbu**-*tree* **then return** B(**Gbu**, σ, $\{\pi_A, \pi_B\}$, $\to L$)
31 $\quad\quad\quad$ **else** Refs $\leftarrow$ Refs $\cup \{\pi_A\}$
32 $\quad$ **return** B(**Rbu**, σ, Refs, $\mathcal{R}_{\mathrm{s}}(\sigma)$) // $\mathcal{R}_{\mathrm{s}}(\sigma)$ is $\vee R$ or $\mathrm{S}_u^{\mathrm{At}}$ or $\mathrm{S}_u^{\vee}$
33 // Here ($H = \bot$ or $H \in \mathcal{V} \setminus \Gamma^{\mathrm{At}}$) and ($l = \mathrm{b}$ or $\Gamma^{\rightarrow} = \emptyset$)
34 **else return** B(**Rbu**, σ, $\emptyset$, Irr)

Fig. 6. $\mathrm{F}(\sigma = [\Gamma \overset{l}{\Rightarrow} H])$

By $\|\sigma\|$ we denote the maximal length of a **Gbu**-branch starting from σ (by Prop. 1, $\|\sigma\|$ is finite). Note that, whenever a recursive call F(σ') occurs along the computation of F(σ), it holds that $\|\sigma'\| < \|\sigma\|$.

In the next lemma we prove the correctness of F.

Lemma 4. *Let σ be a sequent in normal form. Then,* F(σ) *returns either a* **Gbu**-*derivation or an* **Rbu**-*derivation of σ.*

Proof. By induction on $\|\sigma\|$. If $\|\sigma\| = 1$, F(σ) does not execute any recursive invocation and the computation ends at line 1 or at line 34. In the former case, a **Gbu**-derivation of σ is returned. In the latter case, since σ is in normal form and none of the conditions at lines 1, 2, 6, 10 15, 20 holds, the sequent σ is irreducible and the tree built at line 34 is an **Rbu**-derivation of σ.

Let $\|\sigma\| > 1$. Whenever a recursive call $\mathtt{F}(\sigma')$ occurs, we have that $\|\sigma'\| < \|\sigma\|$ and σ' is in normal form, hence the induction hypothesis applies to $\mathtt{F}(\sigma')$. Using this, one can easily show that the arguments of function $\mathtt{B}$ are correctly instantiated. We only analyse some cases.

Let us assume that one of the return instructions at lines 8–9 is executed. By induction hypothesis, for every $k \in \{0,1\}$, π_k is either a **Gbu**-proof or an **Rbu**-derivation of $\sigma_k = [A_k, \Gamma' \overset{\mathrm{u}}{\Rightarrow} H]$. If, for some k, π_k is an **Rbu**-derivation of σ_k, then the **Rbu**-tree returned at line 8 is an **Rbu**-derivation of σ. Otherwise, both π_0 and π_1 are **Gbu**-derivations, hence the value returned at line 9 is a **Gbu**-derivation of σ.

Let us assume that $\mathtt{F}(\sigma)$ ends at line 32; in this case σ satisfies the conditions at lines 19 and 20. If $l = \mathrm{b}$, then $H = H_0 \vee H_1$. Since the condition at line 24 is false, we have $\mathsf{Refs} = \{\pi_0, \pi_1\}$ and, by induction hypothesis, both π_0 and π_1 are **Rbu**-derivations. Accordingly, the value returned at line 32 is an **Rbu**-derivation of σ with root rule $\mathcal{R}_{\mathrm{s}}(\sigma) = \vee R$. Let $l = \mathrm{u}$ and let us assume that $H = \bot$ or $H \in \mathcal{V} \setminus \Gamma$. In this case $\sigma = [\Gamma^{\rightarrow}, \Gamma^{\mathrm{At}} \overset{\mathrm{u}}{\Rightarrow} H]$ and the set Refs contains an **Rbu**-tree π_A of $\sigma_A = [\Gamma^{\rightarrow}, \Gamma^{\mathrm{At}} \overset{\mathrm{b}}{\Rightarrow} A]$ for every $A \rightarrow B \in \Gamma^{\rightarrow}$. By induction hypothesis, π_A is an **Rbu**-derivation of σ_A, hence line 32 returns an **Rbu**-derivation of σ with root rule $\mathcal{R}_{\mathrm{s}}(\sigma) = \mathrm{S}_u^{\mathrm{At}}$. The subcase ($l = \mathrm{u}$ and $H = H_0 \vee H_1$) is similar. $\square$

Finally, we get the completeness of **Gbu**:

Theorem 4. *An* u*-sequent* σ^{u} *is provable in* **Gbu** *iff* σ^{u} *is not refutable.*

Proof. The $\Rightarrow$-statement follows by the soundness of **Gbu**. Conversely, let σ^{u} be not refutable. Then, there is no **Rbu**-derivation π of σ^{u}; otherwise, by Theorem 3, from π we could extract a countermodel for σ^{u}. Since σ^{u} is in normal form, by Lemma 4 the call $\mathtt{F}(\sigma^{\mathrm{u}})$ returns a **Gbu**-derivation of σ^{u}. $\square$

6 Conclusions and future works

We have presented **Gbu**, a terminating sequent calculus for intuitionistic propositional logic. **Gbu** is a notational variant of **G3i**, where sequents are labelled to mark the right-focused phase. Note that focusing techniques reduce the search space limiting the use of contraction, but they do not guarantee termination of proof-search (see, e.g., the right-focused calculus *LJQ* [2]). To get this, one has to introduce extra machinery. An efficient solution is loop-checking implemented by history mechanisms [6, 7]. Here we propose a different approach, based on an evaluation relation defined on sequents. Histories require space to store the right formulas already used so to direct and possibly stop the proof-search. Instead, we have to compute evaluation relations when right-implication is treated. We remark that, with an appropriate implementation of the involved data structures (see [4]), the evaluation relation $\vdash_{\tilde{\mathcal{E}}}$ defined in Section 2 can be computed in time linear in the size of the arguments. Hence, we get by means of computation what history mechanisms get using memory. Although a strict comparison is hard, to stress the difference between the two approaches we provide an example where **Gbu** outperforms history-based calculi. Let $\sigma = [\Gamma^{\rightarrow} \overset{\mathrm{u}}{\Rightarrow} \bot]$, where $\Gamma^{\rightarrow} = \{p_1 \rightarrow \bot, \ldots, p_n \rightarrow \bot\}$ and the p_i's are distinct propositional variables. The only rule that can be used to derive σ is $\rightarrow L$. For every $p_i \rightarrow \bot$ chosen as main formula, the right-hand premise is provable in **Gbu**, while the left-hand premise $\sigma_i^{\mathrm{b}} = [\Gamma^{\rightarrow} \overset{\mathrm{b}}{\Rightarrow} p_i]$ is not. Thus, we have a backtrack point which forces the application of $\rightarrow L$ in all possible ways. Being σ_i^{b} blocked, the unprovability of σ_i^{b} is immediately certified. With the calculi in [7], the search process is similar, but to assert the unprovability of $[\Gamma^{\rightarrow} \Rightarrow p_i]$ one has to chain up to n applications of $\rightarrow L$ and build an history set containing all the p_i's.

Differently from the history mechanisms, **Gbu** only exploits the information in the left-hand side of a sequent. We are investigating the use of more expressive evaluation relations to better grasp the information conveyed by a sequent and further reduce the search space. Finally, we aim to extend the use of these techniques to other logics having a Kripke semantics.

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