A rescaled method for RBF approximation

A rescaled method for RBF approximation

Stefano De Marchi*, Andrea Idda†, Gabriele Santin*,‡
demarchi@math.unipd.it, andrea.idda@sgsbp.it, gabriele.santin@mathematik.uni-stuttgart.de
*University of Padova (Italy), †Banco Popolare (Italy), ‡University of Stuttgart (Germany)

Università degli Studi di Padova

Rescaled Interpolant

Let $\mathcal{X}_N = \{\mathbf{x}_1, \dots, \mathbf{x}_N\} \subset \Omega \subset \mathbb{R}^d$ and consider two functions $f, g : \mathbb{R}^d \to \mathbb{R}$, s.t. $g(\mathbb{R}^d) = 1$.
Then consider the function

$$\hat{\mathcal{P}}_f(\mathcal{X}_N, \mathbf{x}) = \frac{\mathcal{P}_f(\mathcal{X}_N, \mathbf{x})}{\mathcal{P}_g(\mathcal{X}_N, \mathbf{x})}$$

where $\mathcal{P}_\cdot(\mathcal{X}_N, \mathbf{x})$ is a standard, generic RBF interpolant.
$\hat{\mathcal{P}}_f(\mathcal{X}_N, \mathbf{x})$ is still an interpolant to f, since

$$\hat{\mathcal{P}}_f(\mathcal{X}_N, \mathcal{X}_N) = \frac{\mathcal{P}_f(\mathcal{X}_N, \mathcal{X}_N)}{\mathcal{P}_g(\mathcal{X}_N, \mathcal{X}_N)} = \frac{f(\mathcal{X}_N)}{g(\mathcal{X}_N)} = f(\mathcal{X}_N)$$

Function Properties

Heuristically we show that

- $\hat{\mathcal{P}}_f$ is powerful when CSRBF are chosen.
- If ϵ is the shape parameter, there exists $\hat{\epsilon}$ and $(\hat{\epsilon} - \delta_1, \hat{\epsilon} + \delta_2)$ s.t. if ϵ stands in the left side, $\hat{\mathcal{P}}_f$ behaves better than $\mathcal{P}_f$, while on the right side it shows the opposite.
- $\hat{\mathcal{P}}_f$ is, in general, very less sensitive to the setting of ϵ than $\mathcal{P}_f$.
- $\hat{\mathcal{P}}_f$ can be not defined if $\mathcal{X}_N$ doesn't have a sufficiently homogeneous density in Ω.

Native Space $\mathcal{N}_{K_r}$

Theorem 1 (Aronszajn). *Let $K : \Omega \times \Omega \to \mathbb{R}$ be a (strictly) positive definite kernel. Let $s : \Omega \to \mathbb{R}$ a continuous and nonvanishing function on Ω. Then*

$$K_s(\mathbf{x}, \mathbf{y}) = s(\mathbf{x})s(\mathbf{y})K(\mathbf{x}, \mathbf{y})$$

is (strictly) positive definite.

Let $s(\cdot) = \frac{1}{\mathcal{P}_g(\mathcal{X}_N, \cdot)}$, if $\mathcal{X}_N$ is s.t. $\mathcal{P}_g(\Omega) \neq 0$, then

$$K_r(\mathbf{x}, \mathbf{y}) = \frac{1}{\mathcal{P}_g(\mathcal{X}_N, \mathbf{x})} \frac{1}{\mathcal{P}_g(\mathcal{X}_N, \mathbf{y})} K(\mathbf{x}, \mathbf{y})$$

is a Kernel. Considering the usual inner product, we can build the associated Native Space $\mathcal{N}_{K_r}$.

Shepard's Method

Let $\{u_j(\mathbf{x}_i) = \delta_{i,j}\}_j$ be the cardinals for $\mathcal{P}_f$,

$$\hat{\mathcal{P}}_f(\mathcal{X}_N, \mathbf{x}) = \frac{\sum_{j=1}^N f(\mathbf{x}_j)u_j}{\sum_{k=1}^N u_k} = \sum_{j=1}^N f(\mathbf{x}_j) \frac{u_j}{\sum_{k=1}^N u_k}$$

leads to a natural definition of $\frac{u_j}{\sum_{k=1}^N u_k} := \hat{u}_j$.

Theorem 2. *The rescaled interpolation method is a Shepard's method, where the weight functions are defined as $\hat{u}_j = u_j / \left(\sum_{k=1}^N u_k\right)$, $\{u_j\}_j$ being the cardinal basis of span$\{K(\cdot, x), x \in X\}$.*

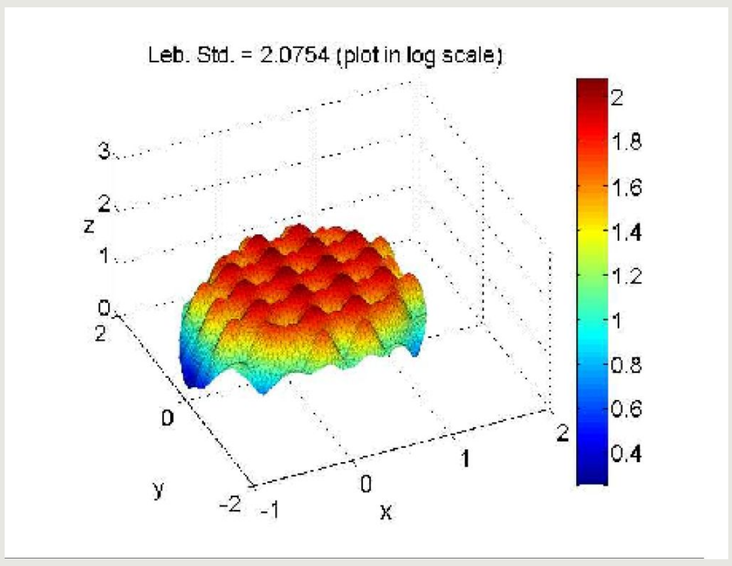
The Lebesgue function and constant are, then,

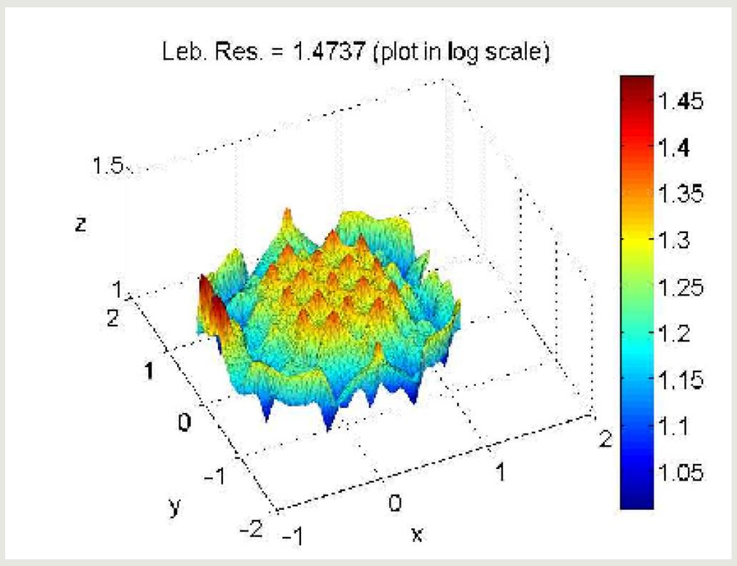
$$\hat{\Lambda}_N(\mathbf{x}) := \sum_{j=1}^N |\hat{u}_j(\mathbf{x})|, \quad \hat{\lambda}_N := \|\hat{\Lambda}_N\|_{\infty,\Omega}$$

that gives an estimate for the stability,

$$\|\hat{P}_f\|_{\infty,\Omega} \leq \hat{\lambda}_N \|f\|_{\infty,X}.$$

Lebesgue Functions Comparison





Standard basis (Left) and rescaled one (Right), $\mathcal{C}^2$ Wendland kernel on the square, $\varepsilon = 3.85$.

Algorithm to get Definiteness of $\hat{\mathcal{P}}_f$ on clustered Sets

Set a tolerance $tol > tol_m$, where tol_m is the workstation precision. In the Rescaling algorithm, after evaluating $\mathcal{P}_g(\mathbf{y})$, $\mathbf{y} \in \Omega$, add the following step :
If $\mathcal{P}_g(\mathbf{y}) < tol$, then set $\mathcal{P}_g(\mathbf{y}) = tol$.
This step will ensure the definitess of $\hat{\mathcal{P}}_f$ on the whole Ω and mantain continuity.

Rescaled PoU Method

Given a *k-stable Partition of Unity* $\{W_j, \Omega_j\}_{j=1}^r$ over Ω, the interpolant given by the Partition of Unity Method is defined as

$$\mathcal{P}_f(\mathcal{X}_N, \mathbf{x}) = \sum_{j=1}^r R_j(\mathbf{x})W_j(\mathbf{x}),$$

$$R_j(\mathbf{x}) = \sum_{l=1}^{N_j} c_l^{(j)} K(\mathbf{x}, \mathbf{x}_l^{(j)}),$$

$$N_j = |\mathcal{X}_N \cap \Omega_j|,$$

with a chosen Radial Kernel K.
By applying the Rescaling Method to every single local interpolant, we obtain what we called Rescaled Partition of Unity Method (RPoU).

$$\mathcal{P}_f(\mathcal{X}_N, \mathbf{x}) = \sum_{j=1}^r \tilde{R}_j(\mathbf{x})W_j(\mathbf{x})$$

$$\tilde{R}_j(\mathbf{x}) = \sum_{l=1}^{N_j} c_l^{(j)} \frac{K(\mathbf{x}, \mathbf{x}_l^{(j)})}{\sum_{i=1}^{N_j} d_i^{(j)} K(\mathbf{x}, \mathbf{x}_i^{(j)})},$$

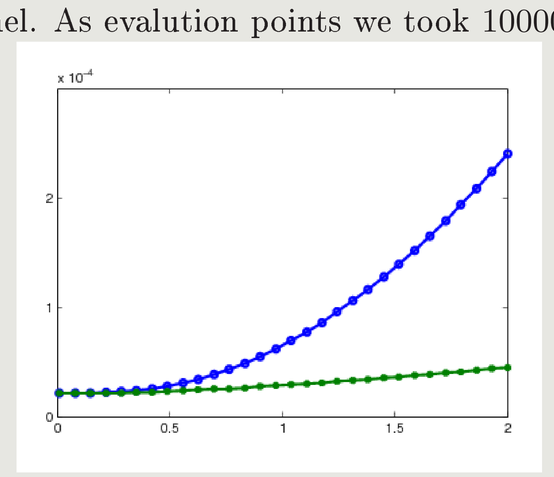
$$N_j = |\mathcal{X}_N \cap \Omega_j|$$

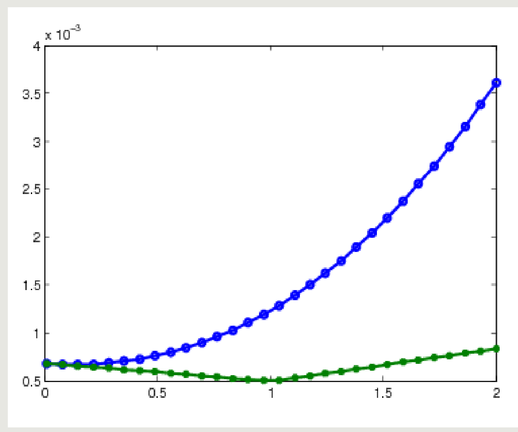
Numerical Evidences

Consider the 2d *Askley's test function*

$$f(x, y) = -20\, e^{-0.2\sqrt{0.5(x^2+y^2)}} - e^{-0.5(\cos(2\pi x)+\cos(2\pi y))} + 20 + e \tag{1}$$

interpolated on 1000 Halton points on the disk centered in $(0.5, 0.5)$ and radius 0.5 with C^2 Wendland kernel. As evalution points we took 10000 uniformly distributed points of the disk.





RMSE (Left) and MAXERR (Right) for the PoU (Blue) and the RPoU (Green) with $\varepsilon \in [0.01, 2]$.

Last but not least, we considered the cost of the process, for which we numerically show that :

- In many cases RPoU can reach the same precision provided by PoU, using a smaller set $\mathcal{X}_N$.
- In most of cases the evaluation time $T_{RPoU} < (1.05) \cdot T_{PoU}$ ($< 5\%$ of increase).
- $\mathcal{P}_f$ and $\mathcal{P}_g$ share the same collocation matrix, so the linear systems to be solved differ by the constant terms. Hence, it would be possible to use specific algorithms to enhance the speed of evaluation.

References

[1] Simone Deparis, Davide Forti, Alfio Quarteroni: *A rescaled localized radial basis function interpolation on non-cartesian and non-conforming grids*, SIAM J. Sci. Comp., 36(6) (2014), A2745–A2762.
[2] Andrea Idda: *A comparison of some RBF interpolation methods: theory and numerics*, Master's degree thesis, Department of Mathematics, University of Padova (2015).
[3] Stefano De Marchi, Andrea Idda, Gabriele Santin: *A rescaled method for RBF approximation*, 2016, draft