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INVARIABLE GENERATION OF PROSOLUBLE GROUPS

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ABSTRACT

A group G is *invariably generated* by a subset S of G if $G = \langle s^{g(s)} \mid s \in S \rangle$ for each choice of $g(s) \in G$, $s \in S$. Answering two questions posed by Kantor, Lubotzky and Shalev in [8], we prove that the free prosoluble group of rank $d \geq 2$ cannot be invariably generated by a finite set of elements, while the free solvable profinite group of rank d and derived length l is invariably generated by precisely $l(d-1)+1$ elements.

1. Introduction

Following [2] we say that a subset S of a group G invariably generates G if $G = \langle s^{g(s)} \mid s \in S \rangle$ for each choice of $g(s) \in G$, $s \in S$. We also say that a group G is invariably generated (IG for short) if G is invariably generated by some subset S of G; when S can be chosen to be finite, we say that G is FIG. A group G is IG if and only if it cannot be covered by a union of conjugates of a proper subgroup, which amount to saying that in every transitive permutation representation of G on a set with more than one element there is a fixed-point-free element. Using this characterization, Wiegold [13] proved that the free group on two (or more) letters is not IG. Kantor, Lubotzky and Shalev studied invariable generation in finite and infinite groups. For example in [7] they proved that every finite group G is invariably generated by at most $\log_2 |G|$ elements. In [8] they studied invariable generation of infinite groups, with emphasis on linear groups, proving that a finitely generated linear group is FIG if and only if it is virtually soluble.

Let G be a profinite group. Then generation and invariable generation in G are interpreted topologically. Just as every finite group is IG, every profinite group G is also IG. Indeed every proper subgroup of a profinite group G is contained in a maximal open subgroup M, and, since M has finite index, G cannot coincide with the union $\cup_{g\in G}M^g$. On the other hand, finitely generated profinite groups are not necessarily FIG. In fact by [7, Proposition 2.5], there exist 2-generated finite groups H with $d_I(H)$ (the minimal number of invariable generators) arbitrarily large. This implies that the free profinite of rank $d \geq 2$ is not FIG. In [8] the following questions are asked: Are finitely generated prosoluble groups FIG? Are finitely generated soluble profinite groups FIG?

We prove that the first question has in general a negative answer:

THEOREM 1: *The free prosoluble group of rank $d \geq 2$ is not FIG.*

We will deduce Theorem 1 from the following result (see Theorem 8). Let G be a finite 2-generated soluble group and let p be the smallest prime divisor of $|G|$. Then either $d_I(G) \geq p$ or there exists a prime $q > p$ such that $d_I(G) < d_I(C_q \wr G)$, where $C_q \wr G$ is the wreath product with respect to the regular permutation representation of G.

In contrast, the second question has a positive answer. More precisely we can adapt the arguments used in the proof of Theorem 1 to show:

THEOREM 2: *Let F be the free soluble profinite group of rank d and derived length l. Then $d_I(F) = l(d-1)+1$.*

Denote by $d(G)$ the smallest cardinality of a generating set of a finitely generate profinite group G. Clearly if G is pronilpotent, then $d(G) = d_I(G)$. More precisely, by [7, Proposition 2.4] a finitely generated profinite group G is pronilpotent if and only if every generating set of G invariably generates G. But what can we say about the difference $d_I(G) - d(G)$ when G is a prosupersoluble group? In this case $G/\operatorname{Frat}(G)$ is metabelian, so Theorem 2 implies that $d_I(G) - d(G) \leq d(G) - 1$. Although supersolubility is a quite strong property and in particular a metabelian group is not in general supersoluble, the previous estimate is sharp.

Theorem 3: *Let F be the free prosupersoluble group of rank d. Then $d_I(F) = 2d-1$.*

2. Preliminaries

A profinite group is a topological group that is isomorphic to an inverse limit of finite groups. The textbooks [11] and [14] provide a good introduction to the theory of profinite groups. In the context of profinite groups, generation and invariable generation are interpreted topologically. By a standard argument (see e.g. [14, Proposition 4.2.1]) it can be proved that a profinite group G is invariably generated by d elements if and only if G/N is invariably generated by d elements for every open normal subgroup N of G. Therefore in the following we will mainly work on finite groups.

If G is a finite soluble group, the minimal number of generators for G can be computed in term of the structure of G-modules of the chief factors of G with the following formula due to Gaschütz [4].

Proposition 4: *Let G be a finite soluble group. For every irreducible G-module V define $r_G(V) = \dim_{\mathrm{End}_G(V)} V$, set $\theta_G(V) = 0$ if V is a trivial G-module, and $\theta_G(V) = 1$ otherwise, and let $\delta_G(V)$ be the number of chief factors G-isomorphic to V and complemented in an arbitrary chief series of G. Then*

$$d(G) = \max_V \left(\theta_G(V) + \left\lceil \frac{\delta_G(V)}{r_G(V)} \right\rceil \right)$$

where V ranges over the set of non G-isomorphic complemented chief factors of G and $\lceil x \rceil$ denotes the smallest integer greater or equal to x.

There is no similar formula for the minimal size of the invariable generating sets. The best result in this direction is a criterion we gave in [1] to decide whether an invariable generating set of a group G can be lifted to an extension over an abelian normal subgroup. To formulate this result, we need to recall some notation from [1].

Let G be a finite group acting irreducibly on an elementary abelian finite p-group V. For a positive integer u we consider the semidirect product $V^u \rtimes G$: unless otherwise stated, we assume that the action of G is diagonal on V^u, that is, G acts in the same way on each of the u direct factors. In [1, Proposition 8] we proved the following.

Proposition 5: *Suppose G acts faithfully and irreducibly on V and $\mathrm{H}^1(G,V)=0$. Assume that $g_1,\dots,g_d$ invariably generate G. There exist some elements $w_1,\dots,w_d\in V^u$ such that $g_1w_1,g_2w_2,\dots,g_dw_d$ invariably generate $V^u\rtimes G$ if and only if*

$$u\le\sum_{i=1}^{d}\dim_{\mathrm{End}_G(V)}C_V(g_i).$$

The assumption $\mathrm{H}^1(G,V)=0$ in the case of soluble groups is assured by the following unpublished result by Gaschütz (see [12, Lemma 1]).

Lemma 6: *Let $G\neq 1$ be a finite soluble group and let V be an irreducible G-module. Then $\mathrm{H}^1(G,V)=0$.*

In the following we will use this straightforward consequence of Proposition 5.

Corollary 7: *Let $G\neq 1$ be a finite soluble group and let V be an irreducible G-module. Assume that $x_1,\dots,x_d$ invariably generate $V^u\rtimes G$, where $x_i=v_ig_i$ with $v_i\in V^u$ and $g_i\in G$. Then $g_1,\dots,g_d$ invariably generate G and*

$$u\le\sum_{i=1}^{d}\dim_{\mathrm{End}_{G/C_G(V)}(V)}C_V(g_i).$$

Proof. Clearly, $g_1,\dots,g_d$ invariably generate G. Denote by $\overline{g}_i$ the image of g_i in the quotient group $G/C_G(V)$. By Lemma 6 and Proposition 5 we have

$$u\le\sum_{i=1}^{d}\dim_{\mathrm{End}_{G/C_G(V)}(V)}C_V(\overline{g}_i).$$

Since $\dim C_V(\overline{g}_i)=\dim C_V(g_i)$, the result follows. ∎

3. Proof of Theorem 1

If G is a finite group, $\pi(G)$ is the set of primes dividing the order of G.

Theorem 8: *Let G be a 2-generated finite soluble group. Either $d_I(G)\ge\min\pi(G)$ or there exists a finite soluble group H having G as an epimorphic image and such that*

- *$d(H)=2$;*
- *$d_I(H)>d_I(G)$;*

- $\min \pi(H) = \min \pi(G)$.

Proof. By Dirichlet's theorem on primes in arithmetic progressions, there exists a prime q such that the exponent of G divides $q-1$. Let $\mathbb{F}$ be the field of order q. By a result of Brauer (see e.g. [3, B 5.21]) $\mathbb{F}$ is a splitting field for G so

$$V := \mathbb{F}G = V_1^{n_1} \oplus \cdots \oplus V_r^{n_r}$$

where the V_j are absolutely irreducible $\mathbb{F}G$-modules no two of which are G-isomorphic, and $n_j = \dim_{\mathbb{F}} V_j$. Consider the semidirect product $H = V \rtimes G$; note that H is isomorphic to $C_q \wr G$ with respect to the regular permutation representation of G. By [9, Corollary 2.4], as C_q and G have coprime orders, $d(C_q \wr G) = \max(d(G), d(C_q) + 1) = 2$.

Clearly $d_I(G) \leq d_I(H)$. Assume $d_I(G) = d_I(H) = d$. By Corollary 7 applied to each homomorphic image $V_j^{n_j} \rtimes G$, it follows that there exists an invariable generating set $g_1, \ldots, g_d$ of G such that, for any j

$$n_j \leq \sum_{i=1}^{d} \dim_{\mathbb{F}} C_{V_j}(g_i).$$

Multiplying by n_j we get

$$n_j^2 \leq \sum_{i=1}^{d} n_j \dim_{\mathbb{F}} C_{V_j}(g_i).$$

It follows that:

$$|G| = \sum_{j=1,\ldots,r} n_j^2 \leq \sum_{\substack{i=1,\ldots,d \\ j=1,\ldots,r}} n_j \dim_{\mathbb{F}} C_{V_j}(g_i) = \sum_{i=1,\ldots,d} \dim_{\mathbb{F}} C_{\mathbb{F}G}(g_i).$$

On the other hand, by Lemma 9 below,

$$\dim_{\mathbb{F}} C_{\mathbb{F}G}(g_i) = \frac{|G|}{|g_i|}$$

and therefore

$$1 \leq \sum_{i=1}^{d} \frac{1}{|g_i|}.$$

Since $d = d_I(G)$ we have $g_i \neq 1$ for every i, hence $|g_i| \geq p = \min \pi(G)$. Therefore

$$1 \leq \sum_{i=1}^{d} \frac{1}{|g_i|} \leq \frac{d}{p}$$

which implies that $p \leq d$, as required. ∎

Lemma 9: *If $g \in G$, then* $\dim_{\mathbb{F}} C_{\mathbb{F}G}(g) = |G : \langle g \rangle|$.

Proof. Let $t_1, \dots, t_r$ be a left transversal of $\langle g \rangle$ in G. Assume that $x \in C_{\mathbb{F}G}(g)$. As every element of G can be uniquely written in the form $t_i g^j$, we can write $x = \sum_{i,j} a_{t_i g^j} t_i g^j$, where $a_{t_i g^j} \in \mathbb{F}$, and, since $xg = x$, we have in particular

$$a_{t_i g^j} = a_{t_i g^{j+1}}$$

for every i and j. Hence $x = \sum_i b_i t_i (1 + g + \cdots + g^{|g|-1})$, for some $b_i \in \mathbb{F}$. Conversely, every $\mathbb{F}$-linear combination of the elements $t_i(1+g+\cdots+g^{|g|-1})$ is centralized by g. In other words the elements $t_i(1+g+\cdots+g^{|g|-1})$, $1 \le i \le r$, are a basis for $C_{\mathbb{F}G}(g)$. ∎

Corollary 10: *For every $d \in \mathbb{N}$, there exists a finite 2-generated soluble group G with $d_I(G) \ge d$.*

Proof. Let p be a prime number with $d \le p$ and consider the set Ω_p of the finite 2-generated soluble groups whose order is divisible by no prime smaller than p. Assume by contradiction, that $d_I(G) < d$ for every $G \in \Omega_p$ and let G^* be a group in Ω_p such that $d_I(G^*) = \max_{G \in \Omega_p} d_I(G)$. Since $d_I(G^*) \le d$ and $d \le p$, by the Theorem 8 there exists H in Ω_p with $d_I(G^*) < d_I(H)$, and this contradicts the maximality of $d_I(G^*)$. ∎

Proof of Theorem 1. Let F be the d-generated free prosoluble group, with $d \ge 2$. Assume that F is FIG. In particular $d_I(H) \le d_I(F)$ for every 2-generated finite soluble group H, but this contradicts Corollary 10. ∎

4. Proof of Theorem 2

We need, as a preliminary result, a formula for the minimal number of generators of a G-module.

Lemma 11: *Let G be a finite group. Assume that A is a direct product*

$$A = A_1^{n_1} \times \cdots \times A_r^{n_r}$$

where, for each i, A_i is a finite elementary abelian p_i-group for a prime number p_i, A_i is an irreducible $\mathbb{F}_{p_i}G$-module and A_i is not G-isomorphic to A_j for $i \neq j$. Then the minimal number of elements needed to generate A as G-module is

$$d_G(A) = \max_{i \in \{1,\dots,r\}} \left(\left\lceil \frac{n_i}{r_G(A_i)} \right\rceil \right),$$

where $\lceil x \rceil$ denotes the smallest integer greater or equal to x.

Proof. If J_i is the Jacobson radical of $\mathbb{F}_{p_i}G$, then $\mathbb{F}_{p_i}G/J_i$ is semisimple and Artinian, hence we can apply the Wedderburn-Artin theorem (see e.g. [6, Lemma 1.11, Theorems 1.14 and 3.3]) and we conclude that A_i occurs precisely $\dim_{\mathrm{End}_G(A_i)}(A_i) = r_G(A_i)$ times in $\mathbb{F}_{p_i}G/J_i$. Then, by [5, Lemma 7.12], A can be generated, as G-module, by

$$d_G(A) = \max_{i\in\{1,\ldots,r\}} \left(\left\lceil \frac{n_i}{r_G(A_i)} \right\rceil \right)$$

elements. ∎

PROPOSITION 12: *Let G be a finite soluble d-generated group of derived length l. Then $d_I(G) \le l(d-1)+1$.*

Proof. The proof is by induction on l. If $l = 1$, then G is abelian and $d_I(G) = d(G) \le d = 1(d-1)+1$.

Assume $l > 1$ and let A be the last non-trivial term of the derived series of G. Then $dl(G/A) = l-1$. Since $d_I(G) = d_I(G/\operatorname{Frat}(G))$, without loss of generality we can assume $\operatorname{Frat}(G) = 1$. Then A is a direct product of complemented minimal normal subgroups of G and we can write

$$A = A_1^{n_1} \times \cdots \times A_r^{n_r}$$

where each A_i is an elementary abelian p_i-group, for a prime number p_i, A_i is an irreducible $\mathbb{F}_{p_i}G$-module and A_i is not G-isomorphic to A_j for $i \ne j$. Therefore by Lemma 11

$$d_G(A) = \max_{i\in\{1,\ldots,r\}} \left(\left\lceil \frac{n_i}{r_G(A_i)} \right\rceil \right). \tag{4.1}$$

On the other hand, by Proposition 4,

$$d \ge d(G) = \max_V \left(\theta_G(V) + \left\lceil \frac{\delta_G(V)}{r_G(V)} \right\rceil \right) \tag{4.2}$$

where V ranges over the set of non G-isomorphic complemented chief factors of G. Note that $\theta_G(A_i) = 1$ for every i. Indeed, if we assume that A_i is a trivial G-module, then, as $\operatorname{Frat}(G) = 1$, we have $G = A_i \times H$ for a complement H of A_i in G. Hence $G' = H'$ and G' does not contain A_i, contradicting the fact that A_i is a subgroup of the last term of the derived series of G.

Since $n_i \le \delta_G(A_i)$, by equations 4.1 and 4.2 we deduce that

$$d \ge \max_{i\in\{1,\dots,r\}} \left(1 + \left\lceil \frac{n_i}{r_G(A_i)} \right\rceil\right) = 1 + d_G(A)$$

hence $d_G(A) \le d-1$. Let $a_1, \dots, a_{d-1}$ be a set of generators for A as G-module and let $g_1, \dots, g_t$ be invariable generators for G modulo A with $t = d_I(G/A)$. Then it is straightforward to check that the the elements

$$g_1, \dots, g_t, a_1, \dots, a_{d-1}$$

invariably generate G, hence

$$d_I(G) \le t + (d-1) = d_I(G/A) + (d-1).$$

Since $dl(G/A) = l-1$, by inductive hypothesis we have that

$$d_I(G/A) \le (l-1)(d-1) + 1,$$

and we conclude that

$$d_I(G) \le (l-1)(d-1) + 1 + (d-1) = l(d-1) + 1,$$

as required. ∎

Denote by $dl(G)$ the derived length of a soluble group G. It follows from the previous proposition, that if G is a finitely generated solvable profinite group, then $d_I(G) \le dl(G)(d(G)-1)+1$. In order to complete the proof of Theorem 2 it suffices to prove the following result:

Theorem 13: *Let d be a positive integer and let p be a prime number. For every positive integer $l < \frac{p-1}{d-1} + 1$ there exists a finite soluble group G_l such that*

- $p = \min \pi(G_l)$,
- $dl(G_l) = l$,
- $d(G_l) = d$,
- $d_I(G_l) = l(d-1)+1$.

Proof. We prove the theorem by induction on l. If $l = 1$, then we can take $G_1 = C_p^d$. So suppose that a group G_l, with the desired properties, has been constructed for $l < \frac{p-1}{d-1}$. As in the proof of Theorem 8, if we take a prime q

such that the exponent of G_l divides $q-1$ and we consider the field $\mathbb{F}$ be the field of order q, then

$$V := \mathbb{F}G_l = V_1^{n_1} \oplus \cdots \oplus V_r^{n_r}$$

where the V_j are absolutely irreducible $\mathbb{F}G$-modules no two of which are G-isomorphic, and $n_j = \dim_{\mathbb{F}} V_j$. Consider the semidirect product $G_{l+1} = V^{d-1} \rtimes G_l$. It can be easily seen that $dl(G_{l+1}) = dl(G_l) + 1 = l + 1$ and that G_{l+1} is isomorphic to the wreath product $C_q^{d-1} \wr G_l$ with respect to the regular permutation representation of G_l. In particular, by [9, Corollary 2.4], as C_q^{d-1} and G_l have coprime orders,

$$d(G_{l+1}) = d(C_q^{d-1} \wr G_l) = \max(d(G_l), d(C_q^{d-1}) + 1)) = d.$$

Now let $t = d_I(G_{l+1})$ and suppose that $w_1g_1, \ldots, w_tg_t$, with $w_i \in V^{d-1}$ and $g_i \in G_l$, invariably generate G_{l+1}. By Corollary 7, for any $j \in \{1, \ldots, t\}$

$$(d-1)n_j \leq \sum_{i=1}^{t} \dim_{\mathbb{F}} C_{V_j}(g_i).$$

As in the proof of Theorem 8, this implies

$$d - 1 \leq \sum_{i=1}^{t} \frac{\dim_{\mathbb{F}} C_{\mathbb{F}G_l}(g_i)}{|G_l|}. \tag{4.3}$$

Notice that $g_1, \ldots, g_t$ must invariably generate G_l so $t \geq d_I(G_l) = l(d-1) + 1$ and in particular we may assume $g_i \neq 1$ for every $i \leq l(d-1) + 1$. Therefore, by Lemma 9,

$$\frac{\dim_{\mathbb{F}} C_{\mathbb{F}G_l}(g_i)}{|G_l|} \leq \frac{1}{p} \quad \text{if } i \leq l(d-1) + 1.$$

Since the trivial bound $\dim_{\mathbb{F}} C_{\mathbb{F}G_l}(g_i)/|G_l| \leq 1$ holds for all $i = l(d-1)+2, \ldots, t$, it follows from (4.3) that

$$d - 1 \leq \frac{l(d-1)+1}{p} + t - l(d-1) - 1$$

i.e.

$$t \geq \left\lceil (l+1)(d-1) + 1 - \frac{l(d-1)+1}{p} \right\rceil.$$

Since we are assuming $l < \frac{p-1}{d-1}$, we have $\frac{l(d-1)+1}{p} < 1$ and consequently $d_I(G_{l+1}) = t \geq (l+1)(d-1) + 1$. On the other hand, since $dl(G_l) = l + 1$, by Proposition 12 we have $d_I(G_{l+1}) \leq (l+1)(d-1) + 1$ and therefore the equality $d_I(G_{l+1}) = (l+1)(d-1) + 1$ has been proved. ∎

5. Proof of Theorem 3

PROPOSITION 14: *For every $d \in \mathbb{N}$ there exists a finite supersoluble group G such that $d(G) = d$ and $d_I(G) \geq 2d - 1$.*

Proof. Let $K = C_2^d$. There are $\alpha := 2^d - 1$ different epimorphisms $\sigma_1, \ldots, \sigma_\alpha$ from K to C_2 ($\sigma_i : K \to C_2$ is uniquely determined by $M_i = \ker \sigma_i$, a $(d-1)$-dimensional subspace of K). To any i, there corresponds a K-module V_i defined as follows: $V_i \cong C_3$ and $v_i^k = v_i$ if $k \in M_i$, $v_i^k = v_i^2$ otherwise. Let $W_i = V_i^{d-1}$ and consider $G = \left(\prod_{1 \leq i \leq \alpha} W_i\right) \rtimes K$. The group G is supersoluble and, by Proposition 4, it is easy to see that $d(G) = d$. Now assume that $g_1, \ldots, g_r$ invariably generate G. We write $g_i = (w_{i1}, \ldots, w_{i\alpha})k_i$ with $k_i \in K$ and $w_{ij} \in W_j$. In particular $k_1, \ldots, k_r$ generate K and, up to reordering the elements $g_1, \ldots, g_r$, we can assume that the first d-elements $k_1, \ldots, k_d$ are a basis for K. Let $M = \langle k_1^{-1}k_2, \ldots, k_{d-1}^{-1}k_d \rangle$. It can be easily checked that M is a maximal subgroup of K, so $M = M_j$ for some $j \in \{1, \ldots, \alpha\}$. Moreover $k_i \notin M_j$ for every $i \in \{1, \ldots, d\}$, in particular $C_{V_j}(k_i) = 0$ for every $i \in \{1, \ldots, d\}$. On the other hand $w_{1j}k_1, \ldots, w_{rj}k_r$ invariably generate G, so, by Corollary 7,

$$d - 1 \leq \sum_{1 \leq i \leq r} \dim_{\mathbb{F}_3} C_{V_j}(k_i) = \sum_{d+1 \leq i \leq r} \dim_{\mathbb{F}_3} C_{V_j}(k_i) \leq r - d.$$

Hence $r \geq 2d - 1$. ∎

Proof of Theorem 3. Let F be the free prosupersoluble group of rank $d \geq 2$. By Proposition 14, there exists a finite supersoluble d-generated group G such that $d_I(G) \geq 2d - 1$. Hence $d_I(F) \geq 2d - 1$.

To prove the converse, since $d_I(F) = d_I(F/\operatorname{Frat}(F))$, it suffices to consider $G = F/\operatorname{Frat} F$. By [10, Proposition 3.3], G' is abelian hence $dl(G) \leq 2$ and it follows from Proposition 12 that $d_I(G) \leq 2d-1$. Therefore $d_I(F) = 2d-1$. ∎

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