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Constitutive Equations and Wave Propagation in Green-Naghdi Type II and III Thermo-Electroelasticity

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Abstract. In this paper we extend the theory of thermo-elasticity devised by Green and Naghdi to the framework of finite thermo-electroelasticity. Both isotropic and transversely isotropic bodies are considered and thermodynamic restrictions on their constitutive relations are obtained by virtue of the reduced energy equality. In the second part a linearized theory for transversely isotropic thermo-piezoelectricity is derived from thermodynamic restrictions by constructing the free energy as a quadratic function of the eleven second-order invariants of the basic fields. The resulting theory provides a natural extension of the (linear) Green and Naghdi theory for type II and type III rigid heat conductors. As a particular case, we derive the linear system which rules the processes depending on the symmetry axis coordinate, only.

Keywords: Heat conduction, Green-Naghdi type II and type III models, Thermal wave propagation, Finite thermo-electro-elasticity, Piezo-thermo-elasticity.

Introduction

We consider a body $\mathcal{B}$ whose particles are identified with their positions $\boldsymbol{X} = (X_1, X_2, X_3) \in \mathcal{E}$ in the region $B = \kappa(\mathcal{B})$ of a fixed reference configuration κ in a three-dimensional Euclidean point space $\mathcal{E}$. Given a scalar- or vector- or tensor-valued field $\varphi(\boldsymbol{X}, t)$, $(\boldsymbol{X}, t) \in \mathcal{E} \times \mathbb{R}^+$, we respectively denote by

$$\dot{\varphi} = \frac{\partial \varphi}{\partial t}\,, \qquad \ddot{\varphi} = \frac{\partial^2 \varphi}{\partial t^2}\,, \qquad \nabla_{\boldsymbol{X}} \varphi = \frac{\partial \varphi}{\partial \boldsymbol{X}}$$

its first and second time derivatives and its spatial gradient. The vector field $\boldsymbol{x}(\boldsymbol{X}, t)$ denotes the current position of $\boldsymbol{X}$ at time t. Accordingly, $\nabla_{\boldsymbol{x}} =$

$\partial/\partial\boldsymbol{x}$ denotes the spatial gradient with respect to the current configuration. Letting $\varphi(\boldsymbol{X},t) = \hat{\varphi}(\boldsymbol{x}(\boldsymbol{X},t),t)$, we have

$$\frac{d\hat{\varphi}}{dt} = \nabla_{\boldsymbol{x}}\hat{\varphi} \cdot \dot{\boldsymbol{x}} + \frac{\partial\hat{\varphi}}{\partial t}$$

Nomenclature for thermo-mechanical magnitudes

ρ mass density in the current configuration
$\boldsymbol{u} = \boldsymbol{x} - \boldsymbol{X}$ displacement field
$\boldsymbol{F} = \nabla_{\boldsymbol{X}}\boldsymbol{x}$ deformation gradient
$\boldsymbol{E} = \frac{1}{2}[\boldsymbol{F}^T\boldsymbol{F} - \boldsymbol{I}]$ strain tensor
$\boldsymbol{\epsilon} = \frac{1}{2}(\nabla_{\boldsymbol{x}}\boldsymbol{u} + \nabla_{\boldsymbol{x}}^T\boldsymbol{u})$ linearized strain tensor
$\boldsymbol{v} = \dot{\boldsymbol{x}}$ velocity field
α thermal displacement [16]
$\boldsymbol{\beta} = \nabla_{\boldsymbol{X}}\alpha$ thermal displacement gradient [16]
$T = \dot{\alpha}$ empirical temperature ('thermal displacement rate' [16])
$\boldsymbol{\gamma} = \nabla_{\boldsymbol{x}}T$ empirical-temperature gradient
θ absolute temperature
$\boldsymbol{g} = \nabla_{\boldsymbol{x}}\theta$ absolute-temperature gradient
k thermal conductivity
$\boldsymbol{f}$ external body force per unit mass
r external rate of supply of heat per unit mass
$s = r/\theta$ external rate of supply of entropy per unit mass
ξ internal rate of supply of entropy per unit mass
$\boldsymbol{\tau}$ Cauchy stress tensor (due to deformation) per unit area
$\boldsymbol{q}$ heat flux vector per unit area
$\boldsymbol{p}$ entropy flux vector per unit area
$\boldsymbol{i}$ extra entropy flux vector per unit area
η density of entropy per unit mass
e internal energy density per unit mass

Nomenclature for electric magnitudes

ϕ electric potential per unit volume
ϵ_0 vacuum electric permittivity
$\boldsymbol{E}^M = -\nabla_{\boldsymbol{x}}\phi$ quasistatic Maxwellian electric field [29, p.589]
$\boldsymbol{P}$ electric polarization vector per unit volume
$\boldsymbol{\pi} = \boldsymbol{P}/\rho$ electric polarization vector per unit mass
$\boldsymbol{D} = \epsilon_0\boldsymbol{E}^M + \boldsymbol{P}$ electric displacement vector
$\boldsymbol{T}^E$ Maxwell stress tensor

The specific *Gibbs free energy* (also known as *free enthalpy* [13, p.101]) density per unit mass is defined as ([29, p.596], [1])

$$\psi = e - \theta\eta - \boldsymbol{E}^M \cdot \boldsymbol{\pi} \tag{1}$$

Classical thermo-electroelasticity. The constitutive relations of piezoelectric ceramics are essentially nonlinear. Indeed, the so-called "piezoelectric strain constants" are not constants but depend on the induced strains. The pioneering paper by Toupin [30] deals mainly with the isothermal case. In view of applications, however, thermal effects have to be taken into account, both in connection with mechanical (thermo-elastic) and electric (thermo-electric) behaviors.

Tiersten [29] derived a natural extension of the coupled field of thermoelasticity to involve electro-coupling. The resulting thermo-electro-elastic equations are cubic with respect to displacement and electric potential gradients and are specialized to isotropic elastic solids by way of the theory of invariants. The involved constitutive equation for the heat flux is the classical Fourier's law. Starting from the isothermal version of this constitutive theory, Yang & Batra [32] developed a second-order theory for the transversely isotropic case which is based on the Taylor expansion of invariant polynomial constitutive relations. Moreover, they obtained isothermal linearized field equations when small deformations and weak electric fields are involved. Due to the importance of heat effects in miniature components of electro- and magneto-mechanical devices (MEMS), more general approach have been recently proposed (see, for instance, [27] and references therein). Unfortunately, the study of wave propagation in classical piezo-thermoelasticity is frustrated by the presence of a first-order-in-time parabolic heat equation induced by Fourier's law.

In order to overcome this difficulty and devise a model with finite wave speeds, different approaches were proposed in the framework of thermoelasticity (see, for instance, [20] and references therein). In connection with thermo-electroelasticity and piezo-thermoelasticity, recent contributions have been proposed by Iesan [19] and Montanaro [25]. In the former paper, heat propagation without energy dissipation is ensured by assuming a linear version of the Green-Nagdhi type II model. In the latter, the finite speed propagation of thermo-electro-elastic waves is ensured by the linearized Cattaneo-Maxwell-Vernotte model. A detailed and updated survey on these subjects can be found in [11, 26].

In the next subsection we compare different models of heat conduction which are compatible with a traveling wave description.

Different models for heat conduction. Fourier's law for the heat conduction states that in a reference configuration the heat flux vector $\mathbf{q}$ is opposite and proportional to the (absolute) temperature gradient,

$$\mathbf{q}(\boldsymbol{X},t) = -k_0 \nabla\theta(\boldsymbol{X},t), \qquad k_0 > 0$$

For simplicity, here and throughout this section we use ∇ instead of $\nabla_{\boldsymbol{X}}$. Assuming that the *absolute temperature* is a given increasing function of the *empirical temperature* T, namely $\theta = \tilde{\theta}(T)$, we obtain

$$\mathbf{q}(\boldsymbol{X},t) = -\tilde{k}(T)\nabla T(\boldsymbol{X},t) \tag{2}$$

where $\tilde{k}(T) = k_0\tilde{\theta}'(T) > 0$ for all T. This constitutive law has served as a reliable model for predicting the temperature in a medium, as well as the rate of heat propagation through a medium, that has been validated by numerous experiments [28]. In a rigid body, Fourier's law may be combined with the internal energy equation

$$\rho\dot{e} + \nabla\cdot\boldsymbol{q} = \rho r$$

to give the conventional heat conduction equation

$$\rho\dot{e} - \tilde{k}(T)\,\Delta T = \rho r$$

where $\Delta = \nabla\cdot\nabla$ and a superposed dot denotes the material time derivative. Assuming $\rho = 1$ and $e = c_0\tilde{\theta}(T) + e_0$, $c_0 > 0$, it provides the *parabolic heat equation*

$$\tilde{c}(T)\,\dot{T} - \tilde{k}(T)\,\Delta T - \tilde{h}(T)\,|\nabla T|^2 = r \tag{3}$$

where $\tilde{h}(T) = k_0\tilde{\theta}''(T)$ and $c(T) = c_0\hat{\theta}'(T) > 0$ for all T. Letting $\tilde{\theta}$ be an affine function, that is

$$\tilde{\theta}(T) = \nu_0 T + \mu_0, \qquad \nu_0, \mu_0 > 0$$

the heat conduction equation takes the usual form of *Fourier's heat equation*

$$c\,\dot{T} - k\,\Delta T = r \tag{4}$$

where $c = c_0\nu_0$ and $k = k_0\nu_0$ are positive constants.

Fourier's heat equation gives a macroscopic description of the microscopic phenomena associated with heat diffusion and is an excellent approximation at length scales much greater than the mean free path and at time scales much greater than the thermal relaxation time, so that the local equilibrium assumption is applicable. Nevertheless, one of the predicted results of Fourier's heat equation, as with all diffusion processes, is that the effect of a source will be instantaneously felt everywhere in a medium, although such a nonzero effect is practically negligible at large distances. Since heat is carried by particles such as electrons, and quanta such as phonons, which are forbidden to propagate at speeds greater than that of light, it is impossible that the response to a sudden heat flux at one location in a medium should be instantaneously felt at all other locations within the medium. This paradox has spurred much academic interest in the last half century towards seeking a model that can predict a finite speed of propagation (see, for instance, [28, 21, 2] and references therein).

In order to extend the macroscopic description of heat transfer to very short time scales at which Fourier's law is no longer appropriate, an additional time derivative of the heat flux term was introduced in the formulation of the rate equation independently by Cattaneo [9] and Vernotte [31]

$$\tau\dot{\mathbf{q}}(\boldsymbol{X},t)+\mathbf{q}(\boldsymbol{X},t)=-\tilde{k}(T)\nabla T(\boldsymbol{X},t),\qquad \tau,k>0 \tag{5}$$

where τ was named *relaxation time*. The Cattaneo-Vernotte model leads to a form of the heat equation known as the *hyperbolic heat equation*,

$$\tau c\ddot{T}+c\dot{T}-k\,\Delta T=r+\tau\dot{r} \tag{6}$$

which is a weakly damped-wave equation and predicts that heat will propagate in non-equilibrium temperature waves with a finite speed [2, 28]. However, there exists no convincing experimental evidence yet to support the validity of (6). For instance, in [3] and [22] it was shown that the (absolute) temperature of a heat wave, reflected from a constant temperature boundary, may take negative values. In addition they proved that the interference of two cold thermal waves may result in negative values of the temperature, so that the maximum principle is violated. More recently, in [8] both Fourier and Cattaneo-Vernotte models are scrutinized within statistical mechanics, classical thermodynamics, and irreversible thermodynamic frameworks.

Green & Naghdi [16, 17] developed a thermo-mechanical theory of deformable continua that relies on an entropy balance law rather than an entropy inequality. A theory of thermoelastic bodies based on the new entropy balance law has been derived. The linearized form of this theory leads to three different models of heat conduction: type I (which is essentially the Fourier's law), type II and type III, respectively. They involve a new variable α, which is called *thermal displacement* and represents a sort of time primitive of the empirical temperature T. In [16, p.180] we read "The temperature T (on the macroscopic scale) is generally regarded as representing (on the molecular scale) some 'mean' velocity magnitude or 'mean' (kinetic energy). With this in mind, we introduce a scalar $\alpha = \alpha(\mathbf{X}, t)$ through an integral of the form"

$$\alpha = \alpha(\mathbf{X}, t) = \int_0^t T(\mathbf{X}, \tau)\, d\tau + \alpha_0(\mathbf{X}), \qquad t > 0 \tag{7}$$

... "In view of the above interpretation associated with T and the physical dimension of the quantity defined by (7), the variable α may justifiably be called thermal displacement magnitude or simply thermal displacement. Alternatively, we may regard the scalar α (on the macroscopic scale) as representing a 'mean' displacement magnitude on the molecular scale and then"

$$T(\boldsymbol{X}, t) = \dot{\alpha}(\boldsymbol{X}, t)$$

The Green-Naghdi linear model of type II (GNII) is given by

$$\mathbf{q}(\boldsymbol{X}, t) = -\lambda \nabla \alpha(\boldsymbol{X}, t), \qquad \lambda > 0$$

In rigid conductors with $\rho = 1$, it leads to a *hyperbolic heat equation*

$$c\,\ddot{\alpha} - \lambda\,\Delta\alpha = r \tag{8}$$

which represents a conservation law in that it does not sustain energy dissipation and permits the transmission of heat as thermal waves at finite speed [17, 18]. Moreover, the heat flux vector is determined by the same potential function that determines the stress [28]. The Green-Naghdi linear model of type III (GNIII) reads

$$\mathbf{q}(\boldsymbol{X}, t) = -\lambda \nabla \alpha(\boldsymbol{X}, t) - k \nabla \dot{\alpha}(\boldsymbol{X}, t)$$

and leads to a *parabolic heat equation* of the second order in time,

$$c\,\ddot{\alpha} + \lambda\,\Delta\alpha + k\,\Delta\dot{\alpha} = r \tag{9}$$

which represents a strongly damped-wave equation and predicts heat propagation with a traveling-wave-like profile [6, 7]. The Green-Naghdi theories have been extensively studied (see [28] and references therein).

A comparative test. In connection with the propagation of thermal disturbances, the difference between Fourier and GNIII models is apparent if a semi-infinite rigid heat conducting rod with no external heat source and a temperature jump ϑ^* at $X = 0$ is considered (see Fig. 1). In particular, the wave-like front is given by $X_0 = v_c t_0$, where $v_c = \sqrt{\lambda/c}$. A similar comparison between the Cattaneo-Vernotte and

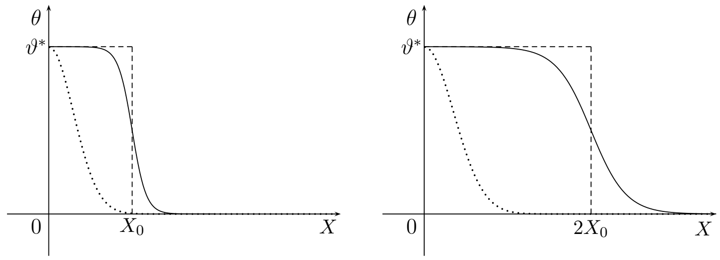

Figure 1: Temperature profiles $\vartheta(X, t_0)$ and $\vartheta(X, 2t_0)$ for Fourier (dotted), Green-Naghdi type II (dashed) and type III (solid), where $\vartheta = T - T_0$, $T_0 = \lim_{X\to\infty} T$. Here, $\lambda = 20k$ and $X_0 = 1$, so that $t_0 = \sqrt{c/\lambda}$.

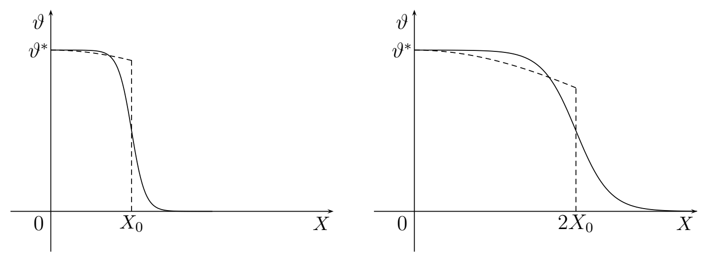

Figure 2: Temperature profiles $\vartheta(X, t_0)$ and $\vartheta(X, 2t_0)$ for Cattaneo-Vernotte (dashed) and Green-Naghdi type III (solid), where $\vartheta = T - T_0$, $T_0 = \lim_{X\to\infty} T$. Here, $\tau = 20$, $\lambda = 20k$ and $X_0 = 1$, so that $t_0 = \sqrt{c/\lambda}$.

type III Green-Naghdi models was performed in [24] (see Fig. 2).

As apparent from Fig. 1 and according to [7], the GNIII linear theory is preferable to Fourier's law of heat conduction if the large-time behavior ($t > \sqrt{c/\lambda}$) is concerned in the propagation of thermal disturbances. In addition, Fig. 2 spotlights that GNIII succeeds in mimicking the behavior of a damped wave. On the other hand, we need to warn the reader that this model predicts an asymptotic behavior of solutions completely different than Fourier's and Cattaneo-Vernotte's laws. Indeed, as λ goes to 0, GNIII solutions tend pointwise to Fourier solutions only in the range $t < c/\lambda$ (see for instance [12]). As a consequence, we can conclude that the GNIII model works well provided that small values of λ ($\lambda < c$) are considered and the time range is restricted to $\sqrt{c/\lambda} < t < c/\lambda$.

Aims and plan of the paper. In the first part of the paper we extend the procedure devised in the theory of thermo-elasticity by Green and Naghdi [14, 16] to the framework of finite thermo-electroelasticity. Then, isotropic and transversely isotropic bodies are separately considered because of the different assumptions on the relation between the heat and entropy fluxes.

Thermodynamic restrictions on the constitutive relations are obtained for isotropic and transversely isotropic bodies, respectively. Unlike theories that provides thermodynamic restrictions by exploiting the Clausius-Duhem inequality, here they are directly derived from the field equations by virtue of the reduced energy equality. The Second Law of Thermodynamics is stated in the form of a dissipation inequality and leads to a sufficient condition that prescribes the constitutive equation of the heat flux vector. As stressed in Remark 0.3, the heat flux constitutive equation (62) generalizes the corresponding relations for Green-Naghdi type II (when $k = 0$) and type III (when $k > 0$) rigid conductors.

The second part of the paper is devoted to deduce a linearized theory for transversely isotropic thermo-piezoelectricity. We first construct the free energy constitutive equation as a quadratic function of the eleven second-order invariants of the basic fields. Then, from thermodynamic restrictions we derive the linearized expressions of the constitutive relations (elastic stress, polarization vector, entropy, etc.). In particular, the linearized heat flux constitutive equation (104) gives rise to a GNII-like model when $k_0 = 0$, but, unlike the original Green and Naghdi framework, here energy dissipation always occurs because of the presence of an extra-flux along the symmetry axis.

The final sections deal with a linearized system which results when processes depend only on the symmetry axis coordinate. In particular, when $k_0 = 0$, all equations of the resulting symmetric system (131) turns out to be linear and hyperbolic, including the heat equation. Accordingly, in this case the study of wave propagation can be performed.

Finite thermo-electroelasticity in isotropic bodies

The body $\mathcal{B}$ is assumed to be an electrically polarizable and finitely deformable heat conducting elastic continuum which interacts with the electric field. According to the Noll's general approach to thermomechanics, we assume the following

Definition 0.1 *The material filling $\mathcal{B}$ is characterized by a given process class $I\!\!P(B)$ of $\mathcal{B}$ as a set of ordered $11-$tuples of functions on $B \times [0, d_p)$,*

$$\mathcal{P} = \Big(\rho(.), \boldsymbol{u}(.), T(.), \phi(.), \psi(.), \eta(.), \boldsymbol{\tau}(.), \boldsymbol{P}(.), \boldsymbol{q}(.), \boldsymbol{b}(.), r(.)\Big) \quad (10)$$

satisfying the balance laws of mass, linear momentum, moment of mo-

mentum, energy, the entropy equality and the field equations of electrostatics listed below in (11).

Above and in the sequel for simplicity the dependence on $\boldsymbol{X}$ (which occurs when the body is not materially homogeneous) is implicit and not written.

Local balance laws in spatial form. Under suitable assumptions of regularity the usual integral forms of the balance laws of mass, linear momentum, moment of momentum, energy, entropy, and the field equations of electrostatics are equivalent to the system

$$\begin{cases} \dot{\rho} + \rho\, \nabla_{\boldsymbol{x}} \cdot \boldsymbol{v} = 0 \\ \rho \dot{\boldsymbol{v}} = \nabla_{\boldsymbol{x}} \cdot \boldsymbol{\tau} + \boldsymbol{P} \cdot \nabla_{\boldsymbol{x}} \boldsymbol{E}^M + \rho \boldsymbol{f} \\ \operatorname{skw} \boldsymbol{\tau} + \operatorname{skw} \boldsymbol{T}^E = \boldsymbol{0} \\ \rho \dot{\eta} = \rho(s + \xi) - \nabla_{\boldsymbol{x}} \cdot \boldsymbol{p} \\ \rho \dot{e} = \boldsymbol{\tau} \cdot \nabla \boldsymbol{v} - \nabla_{\boldsymbol{x}} \cdot \boldsymbol{q} + \boldsymbol{E}^M \cdot \rho \dot{\boldsymbol{\pi}} + \rho r \\ \nabla_{\boldsymbol{x}} \times \boldsymbol{E}^M = \boldsymbol{0} \\ \nabla_{\boldsymbol{x}} \cdot \boldsymbol{D} = 0 \end{cases} \tag{11}$$

where the *internal energy density* e is defined by (1),

$$e = \psi + \theta\eta + \boldsymbol{E}^M \cdot \boldsymbol{\pi}$$

and $\boldsymbol{\pi} = \boldsymbol{P}/\rho$ is the *electric polarization vector per unit mass* and

$$\boldsymbol{T}^E = \boldsymbol{D} \otimes \boldsymbol{E}^M - \frac{1}{2}\epsilon_0 \big(\boldsymbol{E}^M \cdot \boldsymbol{E}^M\big) \boldsymbol{I} \tag{12}$$

is the *Maxwell stress tensor* (cf. [29, Eq. (3.19)], [30]). Note that, of course, eq $(11)_6$ is equivalent to the existence of an electric potential ϕ such that

$$\boldsymbol{E}^M = -\nabla_{\boldsymbol{x}} \phi \tag{13}$$

Following [29], the *total stress tensor* $\boldsymbol{\sigma}$ is defined by

$$\boldsymbol{\sigma} = \boldsymbol{\tau} + \boldsymbol{T}^E \tag{14}$$

Note that the use of $\boldsymbol{\sigma}$ allows to write the field equations of momentum and angular momentum $(11)_{2,3}$ in a form as if the electric fields were missing, i.e.

$$\rho \dot{\boldsymbol{v}} = \nabla_{\boldsymbol{x}} \cdot \boldsymbol{\sigma} + \rho \boldsymbol{f}\,, \qquad \boldsymbol{\sigma} = \boldsymbol{\sigma}^T \tag{15}$$

Finally, eliminating r between equations $(11)_4$, $(11)_5$ and using (1) yields the *reduced energy equation*

$$\rho(\dot{\psi} + \dot{\theta}\eta) + \rho\theta\xi - \boldsymbol{\tau}\cdot\nabla\boldsymbol{v} + \dot{\boldsymbol{E}}^M\cdot\boldsymbol{P} + \nabla_{\boldsymbol{x}}\cdot\boldsymbol{q} - \theta\nabla_{\boldsymbol{x}}\cdot\boldsymbol{p} = 0 \qquad (16)$$

Reduced energy equation and second law of thermodynamics. Constitutive relations arise when the quantities

$$\psi,\ \eta,\ \theta,\ \boldsymbol{p},\ \boldsymbol{q},\ \boldsymbol{\tau},\ \xi,\ \boldsymbol{P} \qquad (17)$$

are assumed to be objective functions of the variables $(T, \boldsymbol{\beta}, \boldsymbol{\gamma}, \boldsymbol{F}, \boldsymbol{E}^M)$. In particular, we assume

$$\begin{cases} \psi = \hat{\psi}(T,\ \boldsymbol{\beta},\ \boldsymbol{\gamma},\ \boldsymbol{F},\ \boldsymbol{E}^M) \\ \eta = \hat{\eta}(T,\ \boldsymbol{\beta},\ \boldsymbol{\gamma},\ \boldsymbol{F},\ \boldsymbol{E}^M) \\ \theta = \hat{\theta}(T,\ \boldsymbol{\beta},\ \boldsymbol{\gamma},\ \boldsymbol{F},\ \boldsymbol{E}^M) \\ \xi = \hat{\xi}(T,\ \boldsymbol{\beta},\ \boldsymbol{\gamma},\ \boldsymbol{F},\ \boldsymbol{E}^M) \\ \boldsymbol{q} = \hat{\boldsymbol{q}}(T,\ \boldsymbol{\beta},\ \boldsymbol{\gamma},\ \boldsymbol{F},\ \boldsymbol{E}^M) \\ \boldsymbol{\tau} = \hat{\boldsymbol{\tau}}(T,\ \boldsymbol{\beta},\ \boldsymbol{\gamma},\ \boldsymbol{F},\ \boldsymbol{E}^M) \\ \boldsymbol{P} = \hat{\boldsymbol{P}}(T,\ \boldsymbol{\beta},\ \boldsymbol{\gamma},\ \boldsymbol{F},\ \boldsymbol{E}^M) \end{cases} \qquad (18)$$

and we take the constitutive relation for the entropy flux in the general form

$$\boldsymbol{p} = \frac{1}{\theta}\boldsymbol{q} + \boldsymbol{i} \qquad (19)$$

where $\boldsymbol{i}$ is usually referred to as *extra entropy flux*,

$$\boldsymbol{i} = \hat{\boldsymbol{i}}(T,\ \boldsymbol{\beta},\ \boldsymbol{\gamma},\ \boldsymbol{F},\ \boldsymbol{E}^M) \qquad (20)$$

Hereafter, a general objective response function will be denoted by

$$\hat{\Omega} \in \left\{ \hat{\psi}\ ,\ \hat{\eta}\ ,\ \hat{\theta}\ ,\ \hat{\xi}\ ,\ \hat{\boldsymbol{p}}\ ,\ \hat{\boldsymbol{\tau}}\ ,\ \hat{\boldsymbol{P}}\ ,\ \hat{\boldsymbol{i}} \right\} \qquad (21)$$

Note that by (19) we have

$$\nabla_{\boldsymbol{x}}\cdot\boldsymbol{q} - \theta\nabla_{\boldsymbol{x}}\cdot\boldsymbol{p} = \boldsymbol{g}\cdot\boldsymbol{p} - \nabla_{\boldsymbol{x}}\cdot(\theta\boldsymbol{i}) \qquad (22)$$

and the reduced energy equality (16) becomes

$$\rho(\dot{\psi} + \dot{\theta}\eta) + \rho\theta\xi - \boldsymbol{\tau}\cdot\nabla\boldsymbol{v} + \dot{\boldsymbol{E}}^M\cdot\boldsymbol{P} + \boldsymbol{g}\cdot\boldsymbol{p} - \nabla_{\boldsymbol{x}}\cdot(\theta\boldsymbol{i}) = 0 \qquad (23)$$

Unlike theories that provides thermodynamic restrictions on the constitutive relations by exploiting the Clausius-Duhem inequality, we here derive such restrictions directly from the field equations by virtue of the reduced energy equality. To this end, we follow here the procedure devised by Green and Naghdi [14], [16] within their theory of thermoelasticity. As they said, "the reduced energy equation ... must be identically satisfied for all processes and will place restrictions on the functional dependence on the constitutive equations " ([16, p.259]).

Accordingly, we apply here the same point of view in the framework of thermo-electroelasticity. Namely, we assume that *the reduced energy equation* (23) *is identically satisfied for all processes and this will place restrictions on the constitutive equations* (18), (19), (20). By paralleling [14, p.257] and [17, p.259], we take the following

General Assumptions. *The process class $\mathbb{P}(B)$ of $\mathcal{B}$ (see Definition 0.1) has sufficiently many processes in the sense that, locally (i.e. at any given point), for each admissible choice of the values for the local state* (T, $\boldsymbol{\beta}$, $\boldsymbol{\gamma}$, $\boldsymbol{F}$, $\boldsymbol{E}^M$) *including, if required, a large enough arbitrariness in the choice of their space and time derivatives, the field equations hold for some process of $\mathbb{P}(B)$. In addition, the heat flux vector $\boldsymbol{q}$ is assumed to be necessarily non zero on physical grounds. Hence, by* (19),

$$\boldsymbol{p} - \boldsymbol{i} \neq \boldsymbol{0} \tag{24}$$

Furthermore, to complete the theory, a statement of the second law of thermodynamics is needed. Following [10, 2], we assume that the second law is expressed in the form of a dissipation inequality.

Dissipation Principle. *The dissipation inequality*

$$\xi \geq 0 \tag{25}$$

must be satisfied for all processes in the constitutive class $\mathbb{P}(B)$.

Although all results of this first part are independent of the dissipation principle, its exploitation is needed to obtain restrictions on the expression of the internal rate of entropy supply ξ and the heat flux vector $\boldsymbol{q}$.

Exploitation of the reduced energy equation. According to Green and Naghdi, in the sequel of this first part concerning isotropic bodies we assume $\hat{\boldsymbol{i}} = \hat{\boldsymbol{0}}$, i.e. $\boldsymbol{p} = \boldsymbol{q}/\theta$, and we deduce the corresponding constitutive restrictions. A brief discussion of the general relation (19) is carried out in Remark 0.2, where the need to assume a non vanishing entropy extra flux, $\hat{\boldsymbol{i}} \neq \hat{\boldsymbol{0}}$, is motivated in connection with transversely isotropic bodies.

Assuming $\boldsymbol{i} \equiv 0$, here we find the restrictions on the response functions implied by the first part of the Dissipation Principle. Thus, the reduced energy equality (23) becomes

$$\rho(\dot{\psi} + \dot{\theta}\eta) + \rho\theta\xi - \boldsymbol{\tau} \cdot \nabla \boldsymbol{v} + \dot{\boldsymbol{E}}^M \cdot \boldsymbol{P} + \boldsymbol{g} \cdot \boldsymbol{p} = 0 \tag{26}$$

Proposition 0.1 *Assume that constitutive equations* (18) *fulfill*

$$\boldsymbol{q} = \theta \boldsymbol{p} \tag{27}$$

$$\frac{\partial \hat{\theta}}{\partial T} > 0 \tag{28}$$

Then the validity of the reduced energy equation (26) *along any smooth enough process* p *implies the following restrictions on the response functions*

$$\psi = \hat{\psi}(T,\, \boldsymbol{\beta},\, \boldsymbol{F},\, \boldsymbol{E}^M)\,, \qquad \theta = \hat{\theta}(T) \tag{29}$$

$$\hat{\boldsymbol{\tau}} = \rho \boldsymbol{F} \frac{\partial \hat{\psi}}{\partial \boldsymbol{F}} \qquad \hat{\boldsymbol{P}} = -\rho \frac{\partial \hat{\psi}}{\partial \boldsymbol{E}^M}\,, \qquad \hat{\eta} = -\frac{\partial \hat{\psi}}{\partial \theta} \tag{30}$$

$$\rho\, \frac{\partial \hat{\psi}}{\partial \boldsymbol{\beta}} \cdot \boldsymbol{F}^T \boldsymbol{\gamma} + \rho\, \hat{\theta}\, \hat{\xi} + \hat{\boldsymbol{p}} \cdot \boldsymbol{g} = 0 \tag{31}$$

Since the assumption (28) implies that $\hat{\theta}$ is an invertible function, in any response function the dependence on T can be replaced by θ.

Remark 0.1 *Note that* (31) *coincides with* (3.12) *in [17, p.259] where, accordingly with the first term to* (3.4) *in [17, p.258], the term* $\boldsymbol{p} \cdot \boldsymbol{\gamma}$ *should be replaced by* $\boldsymbol{p} \cdot \boldsymbol{g}$. *The same remark applies to* (2.79) *in [28, p.56], where* $p_i \dot{\alpha}_{,i}$ *should be replaced by* $p_i g_i$.

Proof. Introducing the constitutive equations (18) into the reduced energy equation (26) we have

$$\begin{aligned}
\rho \left[\left(\frac{\partial \hat{\psi}}{\partial T} + \hat{\eta} \frac{\partial \hat{\theta}}{\partial T} \right) \dot{T} + \left(\frac{\partial \hat{\psi}}{\partial \boldsymbol{\beta}} + \hat{\eta} \frac{\partial \hat{\theta}}{\partial \boldsymbol{\beta}} \right) \cdot \dot{\boldsymbol{\beta}} + \left(\frac{\partial \hat{\psi}}{\partial \boldsymbol{\gamma}} + \hat{\eta} \frac{\partial \hat{\theta}}{\partial \boldsymbol{\gamma}} \right) \cdot \dot{\boldsymbol{\gamma}} \right] \\
+ \left[\rho \left(\frac{\partial \hat{\psi}}{\partial \boldsymbol{E}^M} + \hat{\eta} \frac{\partial \hat{\theta}}{\partial \boldsymbol{E}^M} \right) + \hat{\boldsymbol{P}} \right] \cdot \dot{\boldsymbol{E}}^M \\
+ \left[\rho \left(\frac{\partial \hat{\psi}}{\partial \boldsymbol{F}} + \hat{\eta} \frac{\partial \hat{\theta}}{\partial \boldsymbol{F}} \right) - \boldsymbol{F}^{-1} \hat{\boldsymbol{\tau}} \right] \cdot \dot{\boldsymbol{F}} + \rho \hat{\theta} \hat{\xi} + \hat{\boldsymbol{p}} \cdot \hat{\boldsymbol{g}} = 0
\end{aligned} \tag{32}$$

where the spatial temperature gradient, $\hat{\boldsymbol{g}} = \nabla_{\boldsymbol{x}}\, \hat{\theta}$, by $(18)_3$ reads

$$\hat{\boldsymbol{g}} = \frac{\partial \hat{\theta}}{\partial T} \boldsymbol{\gamma} + \frac{\partial \hat{\theta}}{\partial \boldsymbol{\beta}} \nabla_{\boldsymbol{x}} \boldsymbol{\beta} + \frac{\partial \hat{\theta}}{\partial \boldsymbol{\gamma}} \nabla_{\boldsymbol{x}} \boldsymbol{\gamma} + \frac{\partial \hat{\theta}}{\partial \boldsymbol{F}} \nabla_{\boldsymbol{x}} \boldsymbol{F} + \frac{\partial \hat{\theta}}{\partial \boldsymbol{E}^M} \nabla_{\boldsymbol{x}} \boldsymbol{E}^M \tag{33}$$

Step 1 By the arbitrariness of $\dot{T}$, $\dot{\boldsymbol{\gamma}}$, $\dot{\boldsymbol{F}}$ and $\dot{\boldsymbol{E}}^M$ equation (32) yields

$$\frac{\partial\hat{\psi}}{\partial T}+\hat{\eta}\frac{\partial\hat{\theta}}{\partial T}=0\,,\qquad\qquad \frac{\partial\hat{\psi}}{\partial\boldsymbol{\gamma}}+\hat{\eta}\frac{\partial\hat{\theta}}{\partial\boldsymbol{\gamma}}=\mathbf{0} \tag{34}$$

$$\rho\left(\frac{\partial\hat{\psi}}{\partial\boldsymbol{F}}+\hat{\eta}\frac{\partial\hat{\theta}}{\partial\boldsymbol{F}}\right)-\boldsymbol{F}^{-1}\hat{\boldsymbol{\tau}}=\mathbf{0},\qquad \rho\left(\frac{\partial\hat{\psi}}{\partial\boldsymbol{E}^M}+\hat{\eta}\frac{\partial\hat{\theta}}{\partial\boldsymbol{E}^M}\right)+\hat{\boldsymbol{P}}=\mathbf{0} \tag{35}$$

Thus, using (33), (32) reduces to

$$\begin{aligned}\hat{\boldsymbol{p}}\cdot\left[\frac{\partial\hat{\theta}}{\partial T}\boldsymbol{\gamma}+\frac{\partial\hat{\theta}}{\partial\boldsymbol{\beta}}\nabla_{\boldsymbol{x}}\boldsymbol{\beta}+\frac{\partial\hat{\theta}}{\partial\boldsymbol{\gamma}}\nabla_{\boldsymbol{x}}\boldsymbol{\gamma}+\frac{\partial\hat{\theta}}{\partial\boldsymbol{F}}\nabla_{\boldsymbol{x}}\boldsymbol{F}+\frac{\partial\hat{\theta}}{\partial\boldsymbol{E}^M}\nabla_{\boldsymbol{x}}\boldsymbol{E}^M\right]&\\ +\rho\left(\frac{\partial\hat{\psi}}{\partial\boldsymbol{\beta}}+\hat{\eta}\frac{\partial\hat{\theta}}{\partial\boldsymbol{\beta}}\right)\cdot\dot{\boldsymbol{\beta}}+\rho\hat{\theta}\hat{\xi}=0&\end{aligned} \tag{36}$$

Step 2 Now

$$\nabla_{\boldsymbol{x}}\boldsymbol{\gamma},\quad \nabla_{\boldsymbol{x}}\boldsymbol{F},\quad \nabla_{\boldsymbol{x}}\boldsymbol{E}^M$$

just appear in (36) explicitly as right factors, and thus, according to our General Assumptions, their arbitrariness gives the relations

$$\frac{\partial\hat{\theta}}{\partial\boldsymbol{\gamma}}=\mathbf{0},\quad \frac{\partial\hat{\theta}}{\partial\boldsymbol{F}}=\mathbf{0},\quad \frac{\partial\hat{\theta}}{\partial\boldsymbol{E}^M}=\mathbf{0} \tag{37}$$

i.e.,

$$\theta=\hat{\theta}(T,\,\boldsymbol{\beta})\,,\qquad \hat{\boldsymbol{g}}=\frac{\partial\hat{\theta}}{\partial T}\gamma+\frac{\partial\hat{\theta}}{\partial\boldsymbol{\beta}}\nabla_{\boldsymbol{x}}\boldsymbol{\beta} \tag{38}$$

As a consequence (36) becomes

$$\rho\left(\frac{\partial\hat{\psi}}{\partial\boldsymbol{\beta}}+\hat{\eta}\frac{\partial\hat{\theta}}{\partial\boldsymbol{\beta}}\right)\cdot\dot{\boldsymbol{\beta}}+\rho\hat{\theta}\hat{\xi}+\hat{\boldsymbol{p}}\cdot\left(\frac{\partial\hat{\theta}}{\partial T}\boldsymbol{\gamma}+\frac{\partial\hat{\theta}}{\partial\boldsymbol{\beta}}\nabla_{\boldsymbol{x}}\boldsymbol{\beta}\right)=0 \tag{39}$$

and $(34)_2$, (35) become

$$\frac{\partial\hat{\psi}}{\partial\boldsymbol{\gamma}}=\mathbf{0}\,,\qquad \psi=\hat{\psi}(T,\,\boldsymbol{\beta},\,\boldsymbol{F},\,\boldsymbol{E}^M) \tag{40}$$

$$\hat{\boldsymbol{\tau}}=\rho\boldsymbol{F}\frac{\partial\hat{\psi}}{\partial\boldsymbol{F}}\,,\qquad \hat{\boldsymbol{P}}=-\rho\frac{\partial\hat{\psi}}{\partial\boldsymbol{E}^M} \tag{41}$$

Step 3 Note that $\nabla_{\boldsymbol{x}}\boldsymbol{\beta}=\nabla_{\boldsymbol{x}}\nabla_{\boldsymbol{X}}\alpha$ appears in (39) once as coefficient of $\partial\hat{\theta}/\partial\boldsymbol{\beta}$; hence by its arbitrariness we have

$$\theta=\hat{\theta}(T)\,,\qquad \hat{\boldsymbol{g}}=\frac{\partial\hat{\theta}}{\partial T}\boldsymbol{\gamma} \tag{42}$$

Moreover, the following relation holds,

$$\dot{\boldsymbol{\beta}} = \boldsymbol{F}^T \boldsymbol{\gamma}$$

Accordingly, (39) reduces to (31). Lastly, from (42), using $\partial\hat{\theta}/\partial T > 0$, equation $(34)_1$ becomes

$$\hat{\eta} = -\frac{\partial \hat{\psi}}{\partial \theta} \tag{43}$$

Exploitation with invariant response functions. In order to satisfy the principle of material objectivity the constitutive functions must be scalar invariant under rigid rotations of the deformed and polarized body. The invariance of ψ in a rigid rotation is assured when ψ is an arbitrary function of the referential quantities

$$T, \qquad \boldsymbol{\beta}, \qquad \dot{\boldsymbol{\beta}}, \qquad \boldsymbol{E}, \qquad \boldsymbol{W}$$

where $\boldsymbol{E}$ is the Green-Lagrange strain tensor and

$$\dot{\boldsymbol{\beta}} = \boldsymbol{F}^T \boldsymbol{\gamma} = \boldsymbol{F}^T \nabla_{\boldsymbol{x}} T \tag{44}$$

$$\boldsymbol{E} = \frac{1}{2}(\boldsymbol{F}^T \boldsymbol{F} - \boldsymbol{I})\,, \qquad \boldsymbol{W} = \boldsymbol{F}^T \boldsymbol{E}^M \tag{45}$$

Hence, for any objective response function $\hat{\Omega}$, we define the associated invariant response function

$$\tilde{\Omega} \in \left\{ \tilde{\psi}\,,\ \tilde{\eta}\,,\ \tilde{\theta}\,,\ \tilde{\xi}\,,\ \tilde{\boldsymbol{p}}\,,\ \tilde{\boldsymbol{\tau}}\,,\ \tilde{\boldsymbol{P}} \right\}$$

by putting

$$\tilde{\Omega}(T,\, \boldsymbol{\beta},\, \dot{\boldsymbol{\beta}},\, \boldsymbol{E},\, \boldsymbol{W}) := \hat{\Omega}(T,\, \boldsymbol{\beta},\, \boldsymbol{\gamma},\, \boldsymbol{F},\, \boldsymbol{E}^M) \tag{46}$$

where $(\boldsymbol{\gamma},\, \boldsymbol{F},\, \boldsymbol{E}^M)$ and $(\dot{\boldsymbol{\beta}},\, \boldsymbol{E},\, \boldsymbol{W})$ are related by (44)-(45) and $\dot{\boldsymbol{\beta}}$, $\boldsymbol{E}$, $\boldsymbol{W}$ are independent.

Proposition 0.2 *Assume any constitutive equation in the form*

$$\Omega = \tilde{\Omega}(T,\, \boldsymbol{\beta},\, \dot{\boldsymbol{\beta}},\, \boldsymbol{E},\, \boldsymbol{W}) \tag{47}$$

that is frame-indifferent and invariant under rigid rotations of the deformed and polarized body. Moreover, let

$$\boldsymbol{q} = \theta \boldsymbol{p} \tag{48}$$

$$\frac{\partial \tilde{\theta}}{\partial T} > 0 \tag{49}$$

and let the internal energy response function $\tilde{e}$ be defined as in (1)*. Then the validity of the reduced energy equation* (26) *along any suitably smooth process implies the following conditions for the response functions*

$$\psi = \tilde{\psi}(T,\, \boldsymbol{\beta},\, \boldsymbol{E},\, \boldsymbol{W})\,, \qquad \theta = \tilde{\theta}(T) \tag{50}$$

$$\rho\Big(\frac{\partial\tilde{\psi}}{\partial\boldsymbol{E}}\boldsymbol{F}^T + \frac{\partial\tilde{\psi}}{\partial\boldsymbol{W}} \otimes \boldsymbol{E}^M\Big) - \boldsymbol{F}^{-1}\tilde{\boldsymbol{\tau}} = \boldsymbol{0} \tag{51}$$

$$\rho\boldsymbol{F}\frac{\partial\tilde{\psi}}{\partial\boldsymbol{W}} + \tilde{\boldsymbol{P}} = \boldsymbol{0} \tag{52}$$

$$\tilde{\eta} = -\frac{\partial\tilde{\psi}}{\partial\theta} \tag{53}$$

$$\rho\frac{\partial\tilde{\psi}}{\partial\boldsymbol{\beta}} \cdot \dot{\boldsymbol{\beta}} + \rho\tilde{\theta}\tilde{\xi} + \tilde{\boldsymbol{p}} \cdot \tilde{\boldsymbol{g}} = 0 \tag{54}$$

As in Section we point out that the assumption $\partial\tilde{\theta}/\partial T > 0$ implies that $\tilde{\theta}$ can be inverted and thus in any response function the variable T may be replaced by θ.

Proof. First, we compute the time derivatives appearing in the reduced energy equation (26) by using $\tilde{\Omega}$ instead of $\hat{\Omega}$, according to the identity (46). Thus we find

$$\frac{\partial\hat{\Omega}}{\partial\boldsymbol{F}} \cdot \dot{\boldsymbol{F}} = \Big[\frac{\partial\tilde{\Omega}}{\partial\boldsymbol{E}}\boldsymbol{F}^T + \frac{\partial\tilde{\Omega}}{\partial\boldsymbol{W}} \otimes \boldsymbol{E}^M\Big] \cdot \dot{\boldsymbol{F}}, \quad \frac{\partial\hat{\Omega}}{\partial\boldsymbol{E}^M} \cdot \dot{\boldsymbol{E}}^M = \Big(\boldsymbol{F}\frac{\partial\tilde{\Omega}}{\partial\boldsymbol{W}}\Big) \cdot \dot{\boldsymbol{E}}^M$$

Moreover,

$$\frac{\partial\hat{\Omega}}{\partial\boldsymbol{\gamma}} \cdot \dot{\boldsymbol{\gamma}} = \left(\frac{\partial\tilde{\Omega}}{\partial\dot{\boldsymbol{\beta}}}\frac{\partial\dot{\boldsymbol{\beta}}}{\partial\boldsymbol{\gamma}}\right) \cdot \dot{\boldsymbol{\gamma}} = \boldsymbol{F}\frac{\partial\tilde{\Omega}}{\partial\dot{\boldsymbol{\beta}}} \cdot \dot{\boldsymbol{\gamma}} \tag{55}$$

Consequently, the reduced energy equation (32) becomes

$$\begin{aligned}
&\rho\left[\left(\frac{\partial\tilde{\psi}}{\partial T} + \tilde{\eta}\frac{\partial\tilde{\theta}}{\partial T}\right)\dot{T} + \left(\frac{\partial\tilde{\psi}}{\partial\boldsymbol{\beta}} + \tilde{\eta}\frac{\partial\tilde{\theta}}{\partial\boldsymbol{\beta}}\right) \cdot \dot{\boldsymbol{\beta}} + \boldsymbol{F}\left(\frac{\partial\tilde{\psi}}{\partial\dot{\boldsymbol{\beta}}} + \tilde{\eta}\frac{\partial\tilde{\theta}}{\partial\dot{\boldsymbol{\beta}}}\right) \cdot \dot{\boldsymbol{\gamma}}\right] \\
&+\rho\left[\frac{\partial\tilde{\psi}}{\partial\boldsymbol{E}}\boldsymbol{F}^T + \frac{\partial\tilde{\psi}}{\partial\boldsymbol{W}} \otimes \boldsymbol{E}^M + \tilde{\eta}\Big(\frac{\partial\tilde{\theta}}{\partial\boldsymbol{E}}\boldsymbol{F}^T + \frac{\partial\tilde{\theta}}{\partial\boldsymbol{W}} \otimes \boldsymbol{E}^M\Big) - \frac{\boldsymbol{F}^{-1}}{\rho}\tilde{\boldsymbol{\tau}}\right] \cdot \dot{\boldsymbol{F}} \\
&+ \left[\rho\boldsymbol{F}\left(\frac{\partial\tilde{\psi}}{\partial\boldsymbol{W}} + \tilde{\eta}\frac{\partial\tilde{\theta}}{\partial\boldsymbol{W}}\right) + \tilde{\boldsymbol{P}}\right] \cdot \dot{\boldsymbol{E}}^M + \rho\tilde{\theta}\tilde{\xi} + \tilde{\boldsymbol{p}} \cdot \tilde{\boldsymbol{g}} = 0
\end{aligned} \tag{56}$$

where, by $(18)_3$ and (46), the temperature gradient, $\tilde{\boldsymbol{g}} = \partial\tilde{\theta}/\partial\boldsymbol{x}$, reads

$$\tilde{\boldsymbol{g}} = \frac{\partial\tilde{\theta}}{\partial T}(\boldsymbol{F}^T)^{-1}\dot{\boldsymbol{\beta}} + \frac{\partial\tilde{\theta}}{\partial\boldsymbol{\beta}}\nabla_{\boldsymbol{x}}\,\boldsymbol{\beta} + \frac{\partial\tilde{\theta}}{\partial\dot{\boldsymbol{\beta}}}\nabla_{\boldsymbol{x}}\,\dot{\boldsymbol{\beta}} + \frac{\partial\tilde{\theta}}{\partial\boldsymbol{E}}\nabla_{\boldsymbol{x}}\,\boldsymbol{E} + \frac{\partial\tilde{\theta}}{\partial\boldsymbol{W}}\nabla_{\boldsymbol{x}}\,\boldsymbol{W}$$

The arbitrariness of the time and spatial derivatives of $\boldsymbol{\gamma}$, $\boldsymbol{F}$ and $\boldsymbol{E}^M$, which is required in the proof, is equivalent to the arbitrariness of the corresponding derivatives of $\dot{\boldsymbol{\beta}}$, $\boldsymbol{E}$ and $\boldsymbol{W}$. Then, by virtue of (56) and applying step by step the procedure devised in the proof of Proposition 0.1, we obtain the results from (50) to (54).

Remark 0.2 *In the isotropic case, the assumption $\boldsymbol{q} = \theta \boldsymbol{p}$ is considered as convincing by all authors. On the other hand, some authors contend its validity in the general case, in particular when transversely isotropic bodies are concerned [5, 4, 23].*

Exploitation of the Dissipation Principle. Now we use previous results and the statement of the Dissipation Principle to obtain restrictions on the internal rate of entropy supply ξ. By substituting $\boldsymbol{\gamma}$ from $(42)_2$, (44) into (54) we have

$$\rho\tilde{\xi} = -\frac{1}{\tilde{\theta}}\left(\frac{\partial\tilde{\theta}}{\partial T}\boldsymbol{F}^{-1}\tilde{\boldsymbol{p}} + \rho\frac{\partial\tilde{\psi}}{\partial\boldsymbol{\beta}}\right)\cdot\dot{\boldsymbol{\beta}} \tag{57}$$

so that the requirement (25) of the Dissipation Principle reads

$$\left(\frac{\partial\tilde{\theta}}{\partial T}\boldsymbol{F}^{-1}\tilde{\boldsymbol{p}} + \rho\frac{\partial\tilde{\psi}}{\partial\boldsymbol{\beta}}\right)\cdot\dot{\boldsymbol{\beta}} \leq 0 \tag{58}$$

As well as in the Fourier case, this inequality may be satisfied by letting

$$\frac{\partial\tilde{\theta}}{\partial T}\boldsymbol{F}^{-1}\tilde{\boldsymbol{p}} + \rho\frac{\partial\tilde{\psi}}{\partial\boldsymbol{\beta}} = -k\dot{\boldsymbol{\beta}}, \qquad k = \tilde{k}(T) \geq 0 \tag{59}$$

Since $\tilde{\theta}$ is invertible, we define $\bar{T} = \tilde{\theta}^{-1}$. As a consequence we have

$$\tilde{\boldsymbol{p}} = -\bar{T}'\boldsymbol{F}\left(\rho\frac{\partial\tilde{\psi}}{\partial\boldsymbol{\beta}} + \tilde{k}\dot{\boldsymbol{\beta}}\right) \tag{60}$$

where $\bar{T}' = d\bar{T}/d\theta$. By (57), this choice implies

$$\tilde{\xi} = \frac{\tilde{k}}{\rho\,\theta}\,\dot{\boldsymbol{\beta}}\cdot\dot{\boldsymbol{\beta}} \tag{61}$$

From (44) and (48) it then follows

$$\tilde{\boldsymbol{q}} = -\theta\bar{T}'(\theta)\boldsymbol{F}\left(\rho\frac{\partial\tilde{\psi}}{\partial\boldsymbol{\beta}} + \tilde{k}\,\dot{\boldsymbol{\beta}}\right) = -\theta\bar{T}'(\theta)\boldsymbol{F}\left(\rho\frac{\partial\tilde{\psi}}{\partial\boldsymbol{\beta}} + \tilde{k}\boldsymbol{F}^T\boldsymbol{\gamma}\right) \tag{62}$$

As in the theory of Green and Naghdi, the free energy $\tilde{\psi}$ is strictly related to both $\tilde{\boldsymbol{p}}$ and $\tilde{\boldsymbol{q}}$ by means of the partial derivative $\partial\tilde{\psi}/\partial\boldsymbol{\beta}$.

Remark 0.3 *In a rigid body $\boldsymbol{F} = \boldsymbol{I}$. If we choose ψ quadratic in $\boldsymbol{\beta}$ so that*

$$\rho \frac{\partial \tilde{\psi}}{\partial \boldsymbol{\beta}} = \lambda \, \boldsymbol{\beta}$$

then (62) *becomes*

$$\tilde{\boldsymbol{q}} = -\tilde{\theta} \bar{T}' \big(\lambda \boldsymbol{\beta} + k \dot{\boldsymbol{\beta}} \big) \tag{63}$$

In the special case $\tilde{\theta}(T) = \theta_* \exp T$, $\theta_* > 0$, then $\bar{T}(\theta) = \ln(\theta/\theta_*)$ and (63) *becomes the constitutive equation of a Green-Naghdi type III conductor $\tilde{\boldsymbol{q}} = -\lambda \, \boldsymbol{\beta} - k \, \dot{\boldsymbol{\beta}}$ provided that $k > 0$ (see, for instance, [5, Sect.5]), If $k = 0$, it reduces to the constitutive equation of a Green-Naghdi type II conductor $\tilde{\boldsymbol{q}} = -\lambda \, \boldsymbol{\beta}$ (see, for instance, [5, Sect.4]).*

Elastic stress. The so called *elastic stress* used in [32] is defined as

$$\boldsymbol{T} := \boldsymbol{\tau} + \boldsymbol{P} \otimes \boldsymbol{E}^M \tag{64}$$

where $\boldsymbol{E}^M = (\boldsymbol{F}^T)^{-1} \boldsymbol{W}$. From equalities (51), (52) we have

$$\tilde{\boldsymbol{\tau}} = \rho \boldsymbol{F} \Big(\frac{\partial \tilde{\psi}}{\partial \boldsymbol{E}} \boldsymbol{F}^T + \frac{\partial \tilde{\psi}}{\partial \boldsymbol{W}} \otimes \boldsymbol{E}^M \Big), \qquad \tilde{\boldsymbol{P}} = -\rho \boldsymbol{F} \frac{\partial \tilde{\psi}}{\partial \boldsymbol{W}} \tag{65}$$

thus

$$\boldsymbol{T} = \rho \boldsymbol{F} \Big(\frac{\partial \tilde{\psi}}{\partial \boldsymbol{E}} \boldsymbol{F}^T + \frac{\partial \tilde{\psi}}{\partial \boldsymbol{W}} \otimes \boldsymbol{E}^M \Big) + \boldsymbol{P} \otimes \boldsymbol{E}^M \tag{66}$$

Hence by replacement we find the equality

$$\boldsymbol{T} = \rho \boldsymbol{F} \frac{\partial \tilde{\psi}}{\partial \boldsymbol{E}} \boldsymbol{F}^T \tag{67}$$

that coincides with equation $(1)_3$ in [32].

Now we find the link between elastic stress (64) and total stress (14):

$$\begin{aligned} &\boldsymbol{T} + \epsilon_0 \boldsymbol{E}^M \otimes \boldsymbol{E}^M - \frac{\epsilon_0}{2} (\boldsymbol{E}^M \cdot \boldsymbol{E}^M) \boldsymbol{I} \\ &= \boldsymbol{\tau} + \boldsymbol{P} \otimes \boldsymbol{E}^M + \epsilon_0 \boldsymbol{E}^M \otimes \boldsymbol{E}^M - \frac{\epsilon_0}{2} (\boldsymbol{E}^M \cdot \boldsymbol{E}^M) \boldsymbol{I} \\ &= \boldsymbol{\tau} + (\boldsymbol{D} - \epsilon_0 \boldsymbol{E}^M) \otimes \boldsymbol{E}^M + \epsilon_0 \boldsymbol{E}^M \otimes \boldsymbol{E}^M - \frac{\epsilon_0}{2} (\boldsymbol{E}^M \cdot \boldsymbol{E}^M) \boldsymbol{I} = \boldsymbol{\sigma} \end{aligned}$$

Hence, total stress and elastic stress are related by

$$\boldsymbol{\sigma} = \boldsymbol{T} + \epsilon_0 \boldsymbol{E}^M \otimes \boldsymbol{E}^M - \frac{\epsilon_0}{2} (\boldsymbol{E}^M \cdot \boldsymbol{E}^M) \boldsymbol{I} \tag{68}$$

and the balance of linear momentum $(11)_2$ is equivalent to

$$\rho\dot{\boldsymbol{v}} = \nabla_{\boldsymbol{x}} \cdot \left[\boldsymbol{T} + \epsilon_0 \boldsymbol{E}^M \otimes \boldsymbol{E}^M - \frac{\epsilon_0}{2}(\boldsymbol{E}^M \cdot \boldsymbol{E}^M)\boldsymbol{I}\right] + \rho\boldsymbol{f} \qquad (69)$$

which coincides with equation $(1)_1$ in [32] when $\boldsymbol{f} = \boldsymbol{0}$.

Finite thermo-electroelasticity in transversely isotropic bodies

We start by pointing out that the relation

$$\boldsymbol{p} = \frac{\boldsymbol{q}}{\theta} \qquad (70)$$

is considered as natural and generally valid for all materials by Green and Naghdi. Indeed, they said [15, p. 184]: "Because we use a balance of entropy, and not an entropy inequality, it is noted that the entropy flux vector may be interpreted as the heat flux vector over temperature. It is claimed by a number of writers that such a result cannot be true in general, mainly because they obtain a different expression for entropy flux in second- and higher-order expansions in the kinetic theory of gases. However, this is only because the definition of entropy flux in the kinetic theory is different from that used in our continuum model and there is no reason to accept that it is better or more appropriate".

In spite of these arguments, the general validity of relation (70) has been criticized by some authors [5], [6], [4]. In particular, it has been pointed out (see [5, p.144]) that (70) "holds if both $\boldsymbol{p}$ and $\boldsymbol{q}$ depend *isotropically* on the state variables, whereas it does not hold if that dependence is only *transversely isotropic.* That isotropy guarantees (70), whereas transverse isotropy does not, was first suggested by Liu [23], for elastic bodies studied in a Muller-Liu framework. Recently, one of us has offered a simple argument to prove, in a standard thermodynamic setting, that influx proportionality holds (does not hold) if heat conduction is isotropic (transversely isotropic) [5]". This argument seems more convincing and then we follow the approach devised in [23, p.102] and [4, eq.s (4.11), (4.12)].

Hence, for a transversely isotropic body we assume the relation between entropy flux and heat flux in the form (19)-(20) with $\boldsymbol{i} = \tilde{f}(T, \boldsymbol{X})\boldsymbol{a}$, i.e.

$$\boldsymbol{p} - \frac{\boldsymbol{q}}{\theta} = f\boldsymbol{a}, \qquad f = \tilde{f}(T, \boldsymbol{X}) \qquad (71)$$

where $\boldsymbol{a}$ is a unit vector parallel to the preferred direction of transverse isotropy. According to this different assumption we now perform the exploitation of the reduced energy equation.

Exploitation of the reduced energy equation for a transversely isotropic body. Equality (23) can be rewritten in the form

$$\rho(\dot{\psi}+\dot{\theta}\eta)+\rho\theta\xi-\boldsymbol{\tau}\cdot\nabla\boldsymbol{v}+\dot{\boldsymbol{E}}^M\cdot\boldsymbol{P}+(\boldsymbol{p}-\boldsymbol{i})\cdot\boldsymbol{g}-\theta\nabla_{\boldsymbol{x}}\cdot\boldsymbol{i}=0 \tag{72}$$

where $\boldsymbol{i}=f\boldsymbol{a}$ and $\nabla_{\boldsymbol{x}}\cdot\boldsymbol{a}=\boldsymbol{0}$, by virtue of (71). Then, it follows

$$\rho(\dot{\psi}+\dot{\theta}\eta)+\rho\theta\xi-\boldsymbol{\tau}\cdot\nabla\boldsymbol{v}+\dot{\boldsymbol{E}}^M\cdot\boldsymbol{P}+(\boldsymbol{p}-f\boldsymbol{a})\cdot\boldsymbol{g}-\theta(\nabla_{\boldsymbol{x}}f)\cdot\boldsymbol{a}=0 \tag{73}$$

This expression differs from (26) just for the additional term

$$-\nabla_{\boldsymbol{x}}(\theta f)\cdot\boldsymbol{a}$$

Hence all the steps in the proofs of Proposition 0.1 and Proposition 0.2 remain in force provided that in each occurrence of the reduced energy equation such term be added. In particular, the latter yields the following

Proposition 0.3 *Let the body be transversely isotropic and let* $\boldsymbol{a}$ *be a unit vector of the symmetry axis. Assume constitutive equations* $\tilde{\Omega}$ *in the form* (46)*, with*

$$\frac{\partial\tilde{\theta}}{\partial T}>0 \tag{74}$$

and let the internal energy function be defined by (1)*. Furthermore let*

$$\boldsymbol{p}=\frac{\boldsymbol{q}}{\theta}+f\boldsymbol{a}\,,\qquad f=\tilde{f}(T,\,\boldsymbol{X}) \tag{75}$$

Then the validity of the reduced energy equation (23) *along any suitably smooth process implies the following conditions for the response functions*

$$\psi=\tilde{\psi}(T,\,\boldsymbol{\beta},\,\boldsymbol{E},\,\boldsymbol{W})\,,\qquad \theta=\tilde{\theta}(T) \tag{76}$$

$$\rho\Big(\frac{\partial\tilde{\psi}}{\partial\boldsymbol{E}}\boldsymbol{F}^T+\frac{\partial\tilde{\psi}}{\partial\boldsymbol{W}}\otimes\boldsymbol{E}^M\Big)-\boldsymbol{F}^{-1}\tilde{\boldsymbol{\tau}}=\boldsymbol{0} \tag{77}$$

$$\rho\boldsymbol{F}\frac{\partial\tilde{\psi}}{\partial\boldsymbol{W}}+\tilde{\boldsymbol{P}}=\boldsymbol{0}\,,\quad \tilde{\eta}=-\frac{\partial\tilde{\psi}}{\partial\theta} \tag{78}$$

$$\rho\frac{\partial\tilde{\psi}}{\partial\boldsymbol{\beta}}\cdot\dot{\boldsymbol{\beta}}+\rho\tilde{\theta}\tilde{\xi}+(\tilde{\boldsymbol{p}}-\tilde{f}\boldsymbol{a})\cdot\frac{\partial\tilde{\theta}}{\partial T}\boldsymbol{\gamma}-\tilde{\theta}(\nabla_{\boldsymbol{x}}\tilde{f})\cdot\boldsymbol{a}=0 \tag{79}$$

where

$$\nabla_{\boldsymbol{x}}\tilde{f}=\frac{\partial\tilde{f}}{\partial T}\boldsymbol{\gamma}+\boldsymbol{F}^{-T}\frac{\partial\tilde{f}}{\partial\boldsymbol{X}} \tag{80}$$

Note that, using the elastic stress representation (64), relation (77) turns out to be equivalent to (67). From (1) and (78)

$$\tilde{e} = \tilde{\psi} - \theta \frac{\partial \tilde{\psi}}{\partial \theta} - \boldsymbol{E}^M \cdot \boldsymbol{F} \frac{\partial \tilde{\psi}}{\partial \boldsymbol{W}}$$

so that the *specific heat* is given by

$$c = \frac{\partial \tilde{e}}{\partial \theta} = \theta \frac{\partial \tilde{\eta}}{\partial \theta} + \boldsymbol{W} \cdot \frac{\partial \tilde{\eta}}{\partial \boldsymbol{W}} \tag{81}$$

since $\boldsymbol{E}^M = (\boldsymbol{F}^T)^{-1} \boldsymbol{W}$.

On internal rate of entropy supply and heat flux for transverse isotropy. Here we adapt the arguments previously developed to the case of transverse isotropy. Taking into account (44) and (71), from (79) and (80) we find

$$\rho \tilde{\theta} \tilde{\xi} = -\rho \frac{\partial \tilde{\psi}}{\partial \boldsymbol{\beta}} \cdot \dot{\boldsymbol{\beta}} - \frac{\tilde{\boldsymbol{q}}}{\tilde{\theta}} \cdot \frac{\partial \tilde{\theta}}{\partial T} \boldsymbol{\gamma} + \tilde{\theta} \frac{\partial \tilde{f}}{\partial T} \boldsymbol{\gamma} \cdot \boldsymbol{a} + \tilde{\theta} \boldsymbol{F}^{-T} \frac{\partial \tilde{f}}{\partial \boldsymbol{X}} \cdot \boldsymbol{a}$$

Since $\tilde{\theta}(T)$ is invertible, we introduce $\bar{f}$ such that $\bar{f}(\tilde{\theta}(T), \boldsymbol{X}) = \tilde{f}(T, \boldsymbol{X})$ and we have

$$\frac{\partial \tilde{f}}{\partial T} = \frac{\partial \bar{f}}{\partial \theta} \frac{\partial \tilde{\theta}}{\partial T}, \qquad \frac{\partial \tilde{f}}{\partial \boldsymbol{X}} = \frac{\partial \bar{f}}{\partial \boldsymbol{X}}$$

As a result, it follows

$$\rho \tilde{\xi} = -\frac{1}{\tilde{\theta}} \left[\frac{\partial \tilde{\theta}}{\partial T} \boldsymbol{F}^{-1} \Big(\frac{\tilde{\boldsymbol{q}}}{\tilde{\theta}} - \tilde{\theta} \frac{\partial \bar{f}}{\partial \theta} \boldsymbol{a} \Big) + \rho \frac{\partial \tilde{\psi}}{\partial \boldsymbol{\beta}} \right] \cdot \dot{\boldsymbol{\beta}} + \boldsymbol{F}^{-T} \frac{\partial \bar{f}}{\partial \boldsymbol{X}} \cdot \boldsymbol{a} \tag{82}$$

and the requirement of the second law of thermodynamics (25) becomes

$$\left[\frac{\partial \tilde{\theta}}{\partial T} \boldsymbol{F}^{-1} \Big(\frac{\tilde{\boldsymbol{q}}}{\tilde{\theta}} - \tilde{\theta} \frac{\partial \bar{f}}{\partial \tilde{\theta}} \boldsymbol{a} \Big) + \rho \frac{\partial \tilde{\psi}}{\partial \boldsymbol{\beta}} \right] \cdot \dot{\boldsymbol{\beta}} - \tilde{\theta} \, \boldsymbol{F}^{-T} \frac{\partial \bar{f}}{\partial \boldsymbol{X}} \cdot \boldsymbol{a} \leq 0 \tag{83}$$

This inequality can be satisfied by setting $k = \tilde{k}(T) \geq 0$ and letting

$$\frac{\partial \bar{f}}{\partial \boldsymbol{X}} = \boldsymbol{0}, \qquad \frac{\partial \tilde{\theta}}{\partial T} \boldsymbol{F}^{-1} \Big(\frac{\tilde{\boldsymbol{q}}}{\tilde{\theta}} - \tilde{\theta} \frac{\partial \bar{f}}{\partial \theta} \boldsymbol{a} \Big) + \rho \frac{\partial \tilde{\psi}}{\partial \boldsymbol{\beta}} = -k \dot{\boldsymbol{\beta}} \tag{84}$$

As a consequence, $\tilde{\xi}$ still takes the form (61). These conditions are equivalent to assume $f = \bar{f}(\theta) = \tilde{f}(T)$ and the following constitutive equation for the heat flux,

$$\tilde{\boldsymbol{q}} = -\theta \, \bar{T}' \boldsymbol{F} \left(\rho \frac{\partial \tilde{\psi}}{\partial \boldsymbol{\beta}} + \tilde{k} \dot{\boldsymbol{\beta}} \right) + \theta^2 \bar{f}' \boldsymbol{a} \tag{85}$$

Remark 0.4 *By comparing* (85) *with* (62)*, we infer that the transverse isotropy induces an extra heat-flux along the direction* $\boldsymbol{a}$*. Nevertheless, the resulting expression for* $\tilde{\xi}$ *equals* (61)*. According to Remark 0.3, when either* $k = 0$ *or* $k > 0$*, the resulting constitutive relations for* $\boldsymbol{q}$ *are related to a Green-Naghdi type II or III conductor. In particular, if* $k = 0$ *then* $\xi = 0$*.*

On the other hand, from (71) we find

$$\tilde{\boldsymbol{p}} = \frac{\tilde{\boldsymbol{q}}}{\tilde{\theta}} + \tilde{f}\boldsymbol{a} = -\bar{T}'\boldsymbol{F}\left(\rho\frac{\partial\tilde{\psi}}{\partial\boldsymbol{\beta}} + \tilde{k}\dot{\boldsymbol{\beta}}\right) + \left(\bar{f} + \tilde{\theta}\bar{f}'\right)\boldsymbol{a} \tag{86}$$

and the entropy-flux constitutive relation differs from (60) by an extra-flux contribution along the direction $\boldsymbol{a}$. Since this additional term vanishes provided that $\bar{f}(\theta) = f_0/\theta$, we conclude that if this is the case then anisotropy induces no change in the entropy flux expression with respect to isotropy.

Transversely isotropic thermo-piezoelectricity

In this section we assume an explicit quadratic form of the Gibbs free energy (free entalphy) density and then derive the related linearized theory for transversely isotropic thermo-piezoelectricity.

Field invariants and free energy function. Extending the procedure devised in [32], let us consider the first-and second-order invariants of the triple $(\boldsymbol{E}, \boldsymbol{W}, \boldsymbol{\beta})$,

(1) $\quad I_1 = \boldsymbol{a}\cdot\boldsymbol{E}\cdot\boldsymbol{a}, \quad I_2 = tr\boldsymbol{E}, \quad I_3 = \boldsymbol{a}\cdot\boldsymbol{W}, \quad I_4 = \boldsymbol{a}\cdot\boldsymbol{\beta}$,

(2) $\quad II_1 = \boldsymbol{a}\cdot\boldsymbol{E}^2\cdot\boldsymbol{a}, \quad II_2 = tr\boldsymbol{E}^2, \quad II_3 = \boldsymbol{W}\cdot\boldsymbol{W}, \quad II_4 = \boldsymbol{\beta}\cdot\boldsymbol{\beta},$
$\quad II_5 = \boldsymbol{a}\cdot\boldsymbol{E}\cdot\boldsymbol{W} + \boldsymbol{W}\cdot\boldsymbol{E}\cdot\boldsymbol{a}, \quad II_6 = \boldsymbol{a}\cdot\boldsymbol{E}\cdot\boldsymbol{\beta} + \boldsymbol{\beta}\cdot\boldsymbol{E}\cdot\boldsymbol{a}, \quad II_7 = \boldsymbol{W}\cdot\boldsymbol{\beta}$

where $\boldsymbol{a}$ is the unit vector along the symmetry axis of transverse isotropy. In order to construct a linear theory (see [32]), we assume a reference uniform temperature T_0, which gives rise to the *reference absolute temperature*

$$\theta_0 = \tilde{\theta}(T_0)$$

Then, we choose a free energy function which is quadratic with respect to the invariants of the field variables $\boldsymbol{E}$, $\boldsymbol{W}$, $\boldsymbol{\beta}$, θ, that is

$$\begin{aligned}\Sigma &= \alpha_1 I_1 + \alpha_2 I_2 + \alpha_3 I_3 + \alpha_4 I_4 \\ &+ c_1 I_1^2 + c_2 I_2^2 + c_3 I_1 I_2 + c_4 II_1 + c_5 II_2 \\ &+ \epsilon_1 I_3^2 + \epsilon_2 II_3 + e_1 I_1 I_3 + e_2 I_2 I_3 + e_3 II_5 \\ &+ \nu_1 I_3 I_4 + \nu_2 II_7 + \lambda_1 I_4^2 + \lambda_2 II_4 + \mu_1 I_1 I_4 + \mu_2 I_2 I_4 + \mu_3 II_6 \\ &+ (b_1 I_1 + b_2 I_2 + \kappa_1 I_3 + \kappa_2 I_4)\theta + \frac{1}{2} h \theta^2\end{aligned} \tag{87}$$

where α_1, α_2, α_3, α_4, c_1, c_2, c_3, c_4, c_5, ϵ_1, ϵ_2, e_1, e_2, e_3, ν_1, ν_2, λ_1, λ_2, μ_1, μ_2, μ_3, b_1, b_2, κ_1, κ_2, h are material constants. In particular,

(i) the scalars α_1 and α_2 represent the initial stresses, α_3 the initial polarization, and α_4 the initial entropy flux, i.e. entropy flux for zero temperature gradient $\boldsymbol{g} = \boldsymbol{0} = \boldsymbol{\gamma}$;

(ii) θ is identified with its increment $\theta^{(i)} = \theta - \theta_0$ with respect to the reference absolute temperature.

For the sake of simplicity, hereafter we assume vanishing initial stresses, polarization and entropy flux by putting

$$\alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = 0 \tag{88}$$

Now, we compute the derivatives of the free energy in order to obtain the linearized constitutive equations for elastic stress, polarization vector and entropy. To this end, we first list the non vanishing derivatives of invariants,

$$\frac{\partial I_1}{\partial \boldsymbol{E}} = \frac{\partial(\boldsymbol{a}\cdot\boldsymbol{E}\cdot\boldsymbol{a})}{\partial \boldsymbol{E}} = \boldsymbol{a}\otimes\boldsymbol{a} \qquad \frac{\partial I_2}{\partial \boldsymbol{E}} = \frac{\partial(tr\boldsymbol{E})}{\partial \boldsymbol{E}} = \boldsymbol{I} \qquad \frac{\partial I_3}{\partial \boldsymbol{W}} = \frac{\partial I_4}{\partial \boldsymbol{\beta}} = \boldsymbol{a}$$

$$\frac{\partial I_1^2}{\partial \boldsymbol{E}} = 2I_1\frac{\partial I_1}{\partial \boldsymbol{E}} = 2I_1\boldsymbol{a}\otimes\boldsymbol{a} \qquad \frac{\partial I_2^2}{\partial \boldsymbol{E}} = 2I_2\frac{\partial I_2}{\partial \boldsymbol{E}} = 2(tr\boldsymbol{E})\boldsymbol{I}$$

$$\frac{\partial I_3^2}{\partial \boldsymbol{W}} = 2(\boldsymbol{a}\cdot\boldsymbol{W})\boldsymbol{a} = 2(\boldsymbol{a}\otimes\boldsymbol{a})\boldsymbol{W} \qquad \frac{\partial I_4^2}{\partial \boldsymbol{\beta}} = 2(\boldsymbol{a}\cdot\boldsymbol{\beta})\boldsymbol{a} = 2(\boldsymbol{a}\otimes\boldsymbol{a})\boldsymbol{\beta}$$

$$\frac{\partial II_1}{\partial \boldsymbol{E}} = \boldsymbol{a}\otimes\boldsymbol{E}\boldsymbol{a} + \boldsymbol{E}^T\boldsymbol{a}\otimes\boldsymbol{a} \qquad \frac{\partial II_2}{\partial \boldsymbol{E}} = 2\boldsymbol{E} \qquad \frac{\partial II_3}{\partial \boldsymbol{W}} = 2\boldsymbol{W} \qquad \frac{\partial II_4}{\partial \boldsymbol{\beta}} = 2\boldsymbol{\beta}$$

$$\frac{\partial II_5}{\partial \boldsymbol{E}} = \boldsymbol{a}\otimes\boldsymbol{W} + \boldsymbol{W}\otimes\boldsymbol{a} \qquad \frac{\partial II_5}{\partial \boldsymbol{W}} = \frac{\partial II_6}{\partial \boldsymbol{\beta}} = \boldsymbol{E}^T\boldsymbol{a} + \boldsymbol{E}\boldsymbol{a} = 2\boldsymbol{E}\boldsymbol{a}$$

$$\frac{\partial II_6}{\partial \boldsymbol{E}} = \boldsymbol{a}\otimes\boldsymbol{\beta} + \boldsymbol{\beta}\otimes\boldsymbol{a} \qquad \frac{\partial II_7}{\partial \boldsymbol{\beta}} = \boldsymbol{W} \qquad \frac{\partial II_7}{\partial \boldsymbol{W}} = \boldsymbol{\beta}$$

For the sake of definiteness, hereafter we assume

$$\boldsymbol{a} = \boldsymbol{j}_3 \tag{89}$$

where the set of vectors $(\boldsymbol{j}_1, \boldsymbol{j}_2, \boldsymbol{j}_3)$ represents the standard orthonormal basis related to the coordinate system (O, X_1, X_2, X_3). Accordingly,

$$\boldsymbol{a} \cdot \boldsymbol{E} \cdot \boldsymbol{a} = E_{33}\,, \quad \boldsymbol{a} \cdot \boldsymbol{W} = W_3\,, \quad \boldsymbol{E}\boldsymbol{a} = \sum_{i=1}^{3} E_{i3}\boldsymbol{j}_i \tag{90}$$

As a result, accounting for (88), from (87) we obtain

$$\begin{aligned}\frac{\partial\Sigma}{\partial\boldsymbol{E}} &= 2c_1E_{33}(\boldsymbol{j}_3 \otimes \boldsymbol{j}_3) + 2c_2(tr\boldsymbol{E})\boldsymbol{I} + c_3[(\boldsymbol{j}_3 \otimes \boldsymbol{j}_3)\, tr\boldsymbol{E} \\ &+ E_{33}\,\boldsymbol{I}] + c_4(\boldsymbol{j}_3 \otimes E_{i3}\boldsymbol{j}_i + E_{3i}\boldsymbol{j}_i \otimes \boldsymbol{j}_3) + 2c_5\boldsymbol{E} \\ &+ e_1W_3\boldsymbol{j}_3 \otimes \boldsymbol{j}_3 + e_2W_3\boldsymbol{I} + e_3(\boldsymbol{j}_3 \otimes \boldsymbol{W} + \boldsymbol{W} \otimes \boldsymbol{j}_3) \\ &+ \mu_1\beta_3\boldsymbol{j}_3 \otimes \boldsymbol{j}_3 + \mu_2\beta_3\boldsymbol{I} + \mu_3(\boldsymbol{j}_3 \otimes \boldsymbol{\beta} + \boldsymbol{\beta} \otimes \boldsymbol{j}_3) \\ &+ (b_1\boldsymbol{j}_3 \otimes \boldsymbol{j}_3 + b_2\boldsymbol{I})\theta\end{aligned} \tag{91}$$

$$\begin{aligned}\frac{\partial\Sigma}{\partial\boldsymbol{W}} &= 2(\epsilon_1\boldsymbol{j}_3 \otimes \boldsymbol{j}_3 + \epsilon_2\boldsymbol{I})\boldsymbol{W} + \nu_1\beta_3\boldsymbol{j}_3 + \nu_2\boldsymbol{\beta} \\ &+ (e_1E_{33} + e_2tr\boldsymbol{E} + 2e_3\boldsymbol{E} + \kappa_1\theta)\boldsymbol{j}_3\end{aligned} \tag{92}$$

$$\begin{aligned}\frac{\partial\Sigma}{\partial\boldsymbol{\beta}} &= 2(\lambda_1\boldsymbol{j}_3 \otimes \boldsymbol{j}_3 + \lambda_2\boldsymbol{I})\boldsymbol{\beta} + \nu_1W_3\boldsymbol{j}_3 + \nu_2\boldsymbol{W} \\ &+ (\mu_1E_{33} + \mu_2tr\boldsymbol{E} + 2\mu_3\boldsymbol{E} + \kappa_2\theta)\boldsymbol{j}_3\end{aligned} \tag{93}$$

$$\frac{\partial\Sigma}{\partial\theta} = b_1E_{33} + b_2tr\boldsymbol{E} + h\theta + \kappa_1\,W_3 + \kappa_2\,\beta_3 \tag{94}$$

Linearized theory. By linearization, the spatial and referential descriptions coincide and then we can identify ρ with ρ_0, $\boldsymbol{F}$ with $\boldsymbol{I}$, $\dot{\boldsymbol{\beta}}$ with $\boldsymbol{\gamma}$ and $\boldsymbol{E}^M$ with $\boldsymbol{W}$. Moreover, the Green-Lagrange strain tensor (45) is replaced by

$$\boldsymbol{E} = \frac{1}{2}(\nabla\boldsymbol{u} + \nabla^T\boldsymbol{u}) \tag{95}$$

Let $\rho_0 = \rho_0(\boldsymbol{X})$ be given. We consider the triple $(\boldsymbol{u},\ \alpha,\ \phi)$, where

$$\boldsymbol{u} = \boldsymbol{u}(\boldsymbol{X},\, t), \quad \alpha = \alpha(\boldsymbol{X},\, t), \quad \phi = \phi(\boldsymbol{X},\, t) \tag{96}$$

are the displacement, the thermal displacement and the electric potential, respectively. This triple of fields can be viewed as a thermo-electro-kinetic process superimposed to a *natural state*, that is a reference configuration where the body is stress-free at uniform (absolute) temperature T_0 (θ_0), with uniform electric potential ϕ_0 and such that $\boldsymbol{\beta}_0 = \boldsymbol{0}$. In addition, we assume that in the natural state the body forces and heat supply vanish. Then, by virtue of (44), (74) and $(76)_2$, in the natural state the empirical temperature is constant and the related temperature gradient vanishes,

$$T|_{\text{nat}} = \bar{T}(\theta_0) =: T_0\,, \qquad \dot{\boldsymbol{\beta}}|_{\text{nat}} = \boldsymbol{\gamma}|_{\text{nat}} = \boldsymbol{\gamma}_0 = \mathbf{0} \tag{97}$$

Moreover, under assumption $(84)_1$, the internal rate of entropy supply $\tilde{\xi}$ is given by (61) and then it vanishes in the natural state,

$$\tilde{\xi}|_{\text{nat}} = 0 \tag{98}$$

In order to set up a linearized theory, we let

$$\rho_0 \psi := \Sigma \tag{99}$$

and we point out that the Gibbs free energy Σ is an homogeneous quadratic form with respect to the invariants of the the state variables, because of the vanishing of initial fields (88). Hence, the free energy derivatives are linear and homogeneous. We start by putting

$$\begin{aligned} \phi^{(i)} &:= \phi - \phi_0 \\ \alpha^{(i)} := \alpha - T_0 t, \quad T^{(i)} &:= \dot{\alpha}^{(i)} = T - T_0 \\ \theta^{(i)} := \theta - \theta_0, \quad \zeta := \bar{T}'(\theta), \quad \zeta_0 &:= \bar{T}'(\theta_0) \\ Z := \tilde{\theta}'(T) = 1/\zeta, \quad Z_0 &:= \tilde{\theta}'(T_0) = 1/\zeta_0 \\ f_0 := f(T_0)\,, \quad f' := \tilde{f}'(T)\,, \quad f'_0 &:= \tilde{f}'(T_0) \end{aligned}$$

where $\theta_0 = \tilde{\theta}(T_0)$, $T_0 = \bar{T}(\theta_0)$. Then it follows

$$\bar{T}(\theta) \approx T_0 + \zeta_0 \theta^{(i)}\,, \qquad \tilde{\theta}(T) \approx \theta_0 + Z_0 T^{(i)} \tag{100}$$

$$\dot{\boldsymbol{\beta}} = \boldsymbol{\gamma} = \nabla_{\boldsymbol{X}} T^{(i)}\,, \qquad \boldsymbol{g} = \nabla_{\boldsymbol{X}} \tilde{\theta} \approx Z_0 \nabla_{\boldsymbol{X}} T^{(i)} \tag{101}$$

In the linear approximation, we disregard all infinitesimal terms of order greater than one in the constitutive restrictions (67), (77), (78), (82), so that it follows

$$\tilde{\boldsymbol{T}} = \frac{\partial \tilde{\Sigma}}{\partial \boldsymbol{E}}\,, \quad \tilde{\boldsymbol{P}} = -\frac{\partial \tilde{\Sigma}}{\partial \boldsymbol{W}}\,, \qquad \rho_0 \tilde{\eta} = -\frac{\partial \tilde{\Sigma}}{\partial \theta} \tag{102}$$

$$\rho_0 \theta_0 \tilde{\xi} + \left[\frac{1}{\zeta_0 \theta_0} \tilde{\boldsymbol{q}} - \Big(\theta_0 + \frac{1}{\zeta_0} T^{(i)}\Big) f'_0 \boldsymbol{a} + \frac{\partial \tilde{\Sigma}}{\partial \boldsymbol{\beta}} \right] \cdot \nabla_{\boldsymbol{X}} T^{(i)} = 0 \tag{103}$$

Accordingly, the response functions (85) and (86), respectively for the heat and entropy flux vectors, become

$$\tilde{\boldsymbol{q}} = -\theta_0 \zeta_0 \Big[\frac{\partial \tilde{\Sigma}}{\partial \boldsymbol{\beta}} + k_0 \nabla_{\boldsymbol{X}} T^{(i)} - \Big(\theta_0 + \frac{1}{\zeta_0} T^{(i)}\Big) f'_0 \boldsymbol{a} \Big] \tag{104}$$

$$
\begin{aligned}
\boldsymbol{p} = \frac{1}{\theta_0}\boldsymbol{q} + (f_0 + f_0' T^{(i)})\boldsymbol{a} = & - \zeta_0 \Big[\frac{\partial \tilde{\Sigma}}{\partial \boldsymbol{\beta}} + k_0 \nabla_{\boldsymbol{X}} T^{(i)}\Big] \\
& + (f_0 + \zeta_0\theta_0 f_0' + 2 f_0' T^{(i)})\, \boldsymbol{a}
\end{aligned}
\tag{105}
$$

where $k_0 = \tilde{k}(T_0)$.

Now, taking into account of (91)-(94), we give the explicit linear expression of the elastic stress tensor, the polarization vector and the entropy. The components of the (symmetric) elastic stress are

$$
\begin{aligned}
& T_{11} = 2(c_2 + c_5)E_{11} + 2c_2 E_{22} + (2c_2 + c_3)E_{33} + e_2 W_3 + \mu_2 \beta_3 + b_2 \theta \\
& T_{22} = 2c_2 E_{11} + 2(c_2 + c_5)E_{22} + (2c_2 + c_3)E_{33} + e_2 W_3 + \mu_2 \beta_3 + b_2 \theta \\
& T_{33} = (2c_2 + c_3)E_{11} + (2c_2 + c_3)E_{22} + 2(c_1 + c_2 + c_3 + c_4 + c_5)E_{33} \\
& \quad + (e_1 + e_2 + 2e_3)W_3 + (\mu_1 + \mu_2 + 2\mu_3)\beta_3 + (b_1 + b_2)\theta \\
& T_{23} = T_{32} = (c_4 + 2c_5)E_{23} + e_3 W_2 + \mu_3 \beta_2 \\
& T_{13} = T_{31} = (c_4 + 2c_5)E_{31} + e_3 W_1 + \mu_3 \beta_1 \\
& T_{12} = T_{21} = 2c_5 E_{12}
\end{aligned}
\tag{106}
$$

and generalize [32, eq.s (1), (20)]. The polarization vector can be expressed in components as follows,

$$
\begin{aligned}
P_1 = & - 2\epsilon_2 W_1 - \nu_2 \beta_1 - 2e_3 E_{13} \\
P_2 = & - 2\epsilon_2 W_2 - \nu_2 \beta_2 - 2e_3 E_{23} \\
P_3 = & - 2(\epsilon_1 + \epsilon_2)W_3 - (\nu_1 + \nu_2)\beta_3 \\
& - e_2(E_{11} + E_{22}) - (e_1 + e_2 + 2e_3)E_{33} - \kappa_1 \theta
\end{aligned}
\tag{107}
$$

Finally, the entropy reads

$$
\rho_0 \eta = -[b_2(E_{11} + E_{22}) + (b_1 + b_2)E_{33} + h\theta + \kappa_1 W_3 + \kappa_2 \beta_3] \tag{108}
$$

On axisymmetric processes. Accounting for (89), next we consider waves propagating along the symmetry axis X_3. To this end, we restrict our attention to thermo-electro-kinetic processes having the form

$$
u_i = u_i(X_3,\, t),\ i = 1, 2, 3, \quad \alpha = \alpha(X_3,\, t), \quad \phi = \phi(X_3,\, t) \tag{109}
$$

and, for the sake of simplicity, we neglect the superscript $^{(i)}$. In particular,

$$
\begin{gathered}
\boldsymbol{\beta} = (0,\, 0,\, \beta_3), \qquad \boldsymbol{\gamma} = (0,\, 0,\, \gamma_3), \qquad \boldsymbol{W} = (0,\, 0,\, W_3) \\
\beta_3 = \alpha_{,3}, \quad \gamma_3 = \theta_{,3} \approx Z_0 T_{,3} = Z_0 \dot{\alpha}_{,3}, \quad W_3 = -\phi_{,3}
\end{gathered}
\tag{110}
$$

where $_{,3} = \partial/\partial X_3$. In addition, we assume the vanishing of external sources,

$$\boldsymbol{f} = \mathbf{0}\,, \qquad r = 0$$

For ease of writing, hereafter we adopt the following notations,

$$\begin{aligned} c &= c_1 + c_2 + c_3 + c_4 + c_5 \\ \epsilon &= \epsilon_1 + \epsilon_2\,, \quad \lambda = 2(\lambda_1 + \lambda_2)\,, \quad e = e_1 + e_2 + 2e_3 \\ m &= \mu_1 + \mu_2 + 2\mu_3, \quad \nu = \nu_1 + \nu_2\,, \quad b = b_1 + b_2 \end{aligned} \tag{111}$$

Elastic stress, polarization vector, heat flux, entropy flux, internal rate of entropy supply. In order to write the field equations for processes that just depend on the symmetry axis coordinate, we take advantage of (91)-(94), where all derivatives with respect to X_1 and X_2 vanish. The components of the symmetric elastic stress reduces to

$$\begin{aligned} T_{11} &= (2c_2 + c_3)u_{3,3} - e_2\phi_{,3} + \mu_2\alpha_{,3} + b_2\theta \\ T_{22} &= (2c_2 + c_3)u_{3,3} - e_2\phi_{,3} + \mu_2\alpha_{,3} + b_2\theta \\ T_{33} &= 2cu_{3,3} - e\phi_{,3} + m\alpha_{,3} + b\theta \\ T_{23} &= 0\,, \quad T_{31} = 0\,, \quad T_{12} = 0 \end{aligned} \tag{112}$$

and the polarization vector components become

$$P_1 = P_2 = 0\,, \qquad P_3 = 2\epsilon\phi_{,3} - \nu\alpha_{,3} - eu_{3,3} - \kappa_1\theta \tag{113}$$

As a consequence

$$D_1 = D_2 = 0\,, \qquad D_3 = -(\epsilon_0 - 2\epsilon)\phi_{,3} - \nu\alpha_{,3} - eu_{3,3} - \kappa_1\theta$$

where the material constant

$$\varepsilon = \epsilon_0 - 2\epsilon$$

is positive and represents the *electric permittivity* of the body at rest and at uniform reference temperature.

By virtue of (109)-(110), we note that

$$\boldsymbol{E}\boldsymbol{a} = \boldsymbol{E}\boldsymbol{j}_3 = \frac{1}{2}u_{1,3}\boldsymbol{j}_1 + \frac{1}{2}u_{2,3}\boldsymbol{j}_2 + u_{3,3}\boldsymbol{j}_3 \tag{114}$$

Therefore, using (111), from (93) we obtain

$$\frac{\partial\Sigma}{\partial\beta_3} = -\nu\phi_{,3} + mu_{3,3} + \kappa_2\theta + \lambda\alpha_{,3} \tag{115}$$

Now by substituting the above equalities into (104), we obtain

$$q_1 = q_2 = 0 \tag{116}$$

$$q_3 = \theta_0\zeta_0(\nu\phi_{,3} - mu_{3,3} - \kappa_2\theta - \lambda\alpha_{,3} - k_0T_{,3}) + \theta_0 f_0' T + \theta_0^2\zeta_0 f_0' \tag{117}$$

After substituting the above formulae into (105) we obtain

$$p_1 = p_2 = 0 \tag{118}$$

$$\begin{aligned} p_3 = \frac{q_3}{\theta_0} + f_0 + f_0'T = \zeta_0(\nu\phi_{,3} - mu_{3,3} - \kappa_2\theta - \lambda\alpha_{,3} - k_0T_{,3}) \\ + 2f_0'T + \theta_0\zeta_0 f_0' + f_0 \end{aligned} \tag{119}$$

Linearized field equations for X_3- processes. Note that the total stress tensor

$$\boldsymbol{T} + \epsilon_0 \boldsymbol{W} \otimes \boldsymbol{W} - \frac{\epsilon_0}{2}(\boldsymbol{W}\cdot\boldsymbol{W})\boldsymbol{I} \tag{120}$$

differs from $\boldsymbol{T}$ by quadratic terms in $\boldsymbol{W}$, and therefore in the linear approximation it reduces to the elastic stress. Thus, the linearized Cauchy equation (69) becomes

$$\rho\dot{\boldsymbol{v}} = \nabla_{\boldsymbol{X}}\cdot\boldsymbol{T}\,. \tag{121}$$

Since all components of $\nabla_{\boldsymbol{X}}\cdot\boldsymbol{T}$ vanish except for

$$\frac{\partial}{\partial X_3}T^{33} = 2c\,u_{3,33} - e\phi_{,33} + m\alpha_{,33} + bZ_0\,\dot{\alpha}_{,3}$$

the components of the *linear momentum balance* $(11)_2$ become

$$\rho_0\ddot{u}_1 = 0 \tag{122}$$

$$\rho_0\ddot{u}_2 = 0 \tag{123}$$

$$\rho_0\ddot{u}_3 = 2c\,u_{3,33} - e\phi_{,33} + m\alpha_{,33} + bZ_0\,\dot{\alpha}_{,3} \tag{124}$$

By virtue of (113) and since $\boldsymbol{E}^M = \boldsymbol{W}$, the *electrostatic equation* $(11)_7$ takes the form

$$\varepsilon\phi_{,33} + \nu\alpha_{,33} + e\,u_{3,33} + \kappa_1 Z_0\,\dot{\alpha}_{,3} = 0 \tag{125}$$

Under the assumptions of this section, the entropy (108) becomes

$$\rho_0\eta = -bu_{3,3} - h\theta + \kappa_1\phi_{,3} - \kappa_2\alpha_{,3}$$

and from (81) we obtain the *heat capacity*

$$\rho_0 \tilde{c} = -h\theta + \kappa_1 \phi_{,3} \tag{126}$$

In addition, $\nabla_{\boldsymbol{x}} \cdot \boldsymbol{p} = \nabla_{\boldsymbol{X}} \cdot \boldsymbol{p} = p_{3,3}$ and from (119) it follows

$$p_{3,3} = \zeta_0(\nu\phi_{,33} - m u_{3,33} - \frac{\kappa_2}{\zeta_0} T_{,3} - \lambda\alpha_{,33} - k_0 T_{,33}) + 2 f_0' T_{,3}$$

After substituting these terms into the *entropy balance equation* $(11)_4$ and remembering that $T = \dot{\alpha}$ and $s = r/\theta = 0$, we get

$$\begin{aligned} b\dot{u}_{3,3} + hZ_0\ddot{\alpha} - \kappa_1\dot{\phi}_{,3} + 2(\kappa_2 - f_0')\,\dot{\alpha}_{,3} = \\ + \zeta_0(\nu\phi_{,33} - m u_{3,33} - \lambda\alpha_{,33} - k_0\dot{\alpha}_{,33}) \end{aligned} \tag{127}$$

The system (124)-(127) gives the linearized field equations along the symmetry axis X_3 in the unknowns u_3, ϕ and α, respectively.

A reduced system for X_3 − processes. In general, we can reduce the system (122)-(127) by eliminating the electric variable ϕ. This can be done by integrating the electrostatic equation (125) with respect to X_3, which yields

$$-\varepsilon\phi_{,3} = \nu\alpha_{,3} + e\, u_{3,3} + \kappa_1 Z_0\, \dot{\alpha} + g \tag{128}$$

where g is an arbitrary time function. Provided that $\varepsilon \neq 0$ we replace the resulting expression of $\phi_{,3}$ into (124) and (127), respectively, and we obtain the following *reduced evolution system*

$$\begin{cases} \rho_0\ddot{u}_3 = c_u\, u_{3,33} + m_u\alpha_{,33} + b_u\, \dot{\alpha}_{,3} \\ h_\alpha\ddot{\alpha} = \lambda_\alpha\alpha_{,33} + m_\alpha u_{3,33} - b_\alpha\dot{u}_{3,3} - \kappa_\alpha\dot{\alpha}_{,3} + k_0\dot{\alpha}_{,33} - \frac{\kappa_1}{\varepsilon\zeta_0}\dot{g} \end{cases} \tag{129}$$

where

$$c_u = 2c + \frac{e^2}{\varepsilon}\,, \quad m_u = m_\alpha = m + \frac{e\nu}{\varepsilon}\,, \quad b_u = b_\alpha = \frac{1}{\zeta_0}\Big(b + \frac{e\kappa_1}{\varepsilon}\Big)$$

$$h_\alpha = -\frac{1}{\zeta_0^2}\Big(h + \frac{\kappa_1^2}{\varepsilon}\Big)\,, \quad \lambda_\alpha = \lambda + \frac{\nu^2}{\varepsilon}\,, \qquad \kappa_\alpha = \frac{2}{\zeta_0}\Big(\kappa_2 - f_0' + \frac{\nu\kappa_1}{\varepsilon}\Big)$$

For the sake of simplicity, hereafter we consider the special case

$$\kappa_1 = 0\,, \qquad h < 0$$

The former condition means that the heat capacity (126) is independent of the electric potential, namely

$$\rho_0\tilde{c} = -h\theta$$

and the latter implies that it is a positive quantity, as occurs for the most common materials. In this special case, the reduced system reads

$$\begin{cases} \rho_0 \ddot{u}_3 = c_u\, u_{3,\,33} + m_* \alpha_{,\,33} + b_*\, \dot{\alpha}_{,\,3} \\ h_\alpha \ddot{\alpha} = \lambda_\alpha \alpha_{,\,33} + m_* u_{3,\,33} - b_* \dot{u}_{3,\,3} - \kappa_\alpha \dot{\alpha}_{,\,3} + k_0 \dot{\alpha}_{,\,33} \end{cases} \tag{130}$$

where

$$c_u = 2c + \frac{e^2}{\varepsilon}\,, \quad m_* = m + \frac{e\nu}{\varepsilon}\,, \quad b_* = \frac{b}{\zeta_0}$$

$$h_\alpha = -\frac{h}{\zeta_0^2}\,, \quad \lambda_\alpha = \lambda + \frac{\nu^2}{\varepsilon}\,, \quad \kappa_\alpha = \frac{2}{\zeta_0}\Big(\kappa_2 - f_0'\Big)$$

In order to rewrite the reduced system in a more compact form, we introduce the unknown vector $z = (u_3, \alpha)^T$ and the matrices

$$M_0 = \begin{pmatrix} \rho_0 & 0 \\ 0 & h_\alpha \end{pmatrix}, \qquad M_1 = \begin{pmatrix} c_u & m_* \\ m_* & \lambda_\alpha \end{pmatrix}$$

$$M_2 = \begin{pmatrix} 0 & b_* \\ -b_* & -\kappa_\alpha \end{pmatrix}, \qquad M_3 = \begin{pmatrix} 0 & 0 \\ 0 & k_0 \end{pmatrix}$$

Since $\rho_0, h_\alpha > 0$, we finally obtain

$$\ddot{z} = N_1\, z_{,\,33} + N_2\, \dot{z}_{,\,3} + N_3\, \dot{z}_{,\,33}$$

where $N_i = M_0^{-1} M_i$, $i = 1, 2, 3$.

Alternatively, the reduced system can be rewritten in a symmetric form. Since $\rho_0, h_\alpha > 0$, we are allowed to introduce the unknown vector

$$y = (\sqrt{\rho_0}\, u_3, \sqrt{h_\alpha}\, \alpha)^T$$

and we obtain

$$\ddot{y} = M_1\, y_{,\,33} + M_2\, \dot{y}_{,\,3} + M_3\, \dot{y}_{,\,33} \tag{131}$$

where

$$M_1 = \begin{pmatrix} c_u/\rho_0 & m_*/\sqrt{\rho_0 h_\alpha} \\ m_*/\sqrt{\rho_0 h_\alpha} & \lambda_\alpha/h_\alpha \end{pmatrix}$$

$$M_2 = \begin{pmatrix} 0 & b_*/\sqrt{\rho_0 h_\alpha} \\ -b_*/\sqrt{\rho_0 h_\alpha} & -\kappa_\alpha/h_\alpha \end{pmatrix}, \qquad M_3 = \begin{pmatrix} 0 & 0 \\ 0 & k_0/h_\alpha \end{pmatrix}$$

As apparent, the last term prevents this system to be homogeneous with respect to space-time derivatives.

Wave equations. In the sequel we assume $k_0 = 0$. As a consequence, the reduced system (130) turns out to be linear and homogeneous with respect to space-time derivatives, so that usual techniques to scrutinize wave propagation can be applied. If this is the case, then (131) looks like a *symmetric hyperbolic system* provided that M_1 is positive definite. This condition occurs if

$$c_u > 0\,, \qquad \det M_1 = \frac{1}{\rho_0 h_\alpha}\left[c_u \lambda_\alpha - m_*^2\right] > 0$$

Remembering that $\varepsilon > 0$, $h < 0$ ($h_\alpha > 0$), M_1 is positive definite provided that

$$c > -\frac{e^2}{2\varepsilon}\,, \qquad \lambda > \frac{m_*^2}{c_u} - \frac{\nu^2}{\varepsilon} \tag{132}$$

Alternately, we rewrite (130) as a first-order system. We let $x = X_3$ and

$$\begin{aligned} w(x,t) &= u_{3,3}(X_3,t)\,, \quad & z(x,t) &= \dot{u}_3(X_3,t) \\ \omega(x,t) &= \alpha_{,3}(X_3,t)\,, \quad & \upsilon(x,t) &= \dot{\alpha}(X_3,t) \end{aligned} \tag{133}$$

Then, we have

$$\begin{cases} \quad \dot{w} = z_x \\ \rho_0\,\dot{z} = c_u\,w_x + m_*\,\omega_x + b_*\,\upsilon_x \\ \quad \dot{\omega} = \upsilon_x \\ h_\alpha \dot{\upsilon} = m_* w_x + \lambda_\alpha \omega_x - b_* z_x - \kappa_\alpha \upsilon_x \end{cases} \tag{134}$$

where the subscript $_x$ stands for $\partial/\partial x$. In a more compact form, it can be rewritten as

$$\dot{U} = A\,U_x \tag{135}$$

where

$$U = \begin{pmatrix} w \\ z \\ \omega \\ \upsilon \end{pmatrix}, \quad A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ c_u/\rho_0 & 0 & m_*/\rho_0 & b_*/\rho_0 \\ 0 & 0 & 0 & 1 \\ m_*/h_\alpha & -b_*/h_\alpha & \lambda_\alpha/h_\alpha & -\kappa_\alpha/h_\alpha \end{pmatrix}$$

In particular, since $\det A = \det M_1$, $\det A$ is positive provided that (132) holds. The hyperbolicity of (135) can be established by scrutinizing the eigenvalues of A.

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