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Eikonal equations on the Sierpinski gasket

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Abstract We study the eikonal equation on the Sierpinski gasket in the spirit of the construction of the Laplacian in Kigami (Analysis on fractals. Cambridge Tracts in Mathematics, 2001): we consider graph eikonal equations on the prefractals and we show that the solutions of these problems converge to a function defined on the fractal set. We characterize this limit function as the unique metric viscosity solution to the eikonal equation on the Sierpinski gasket according to the definition introduced in Giga et al. (Trans Am Math Soc 367:49–66, 2015).

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1 Introduction

In this paper we consider the eikonal equation

$$|Du| = f(x) \quad \text{in } S, \tag{1.1}$$

where S is the Sierpinski gasket. The eikonal equation is the prototype for a general class of first order Hamilton–Jacobi equations with convex Hamiltonian (see [1]) and, moreover, it arises in several applications in connection with geometric optics, wave front propagation, interfaces evolution, granular matter theory, etc.

The analysis of differential operators on irregular sets such as the Sierpinski gasket has been developed since the late 80's, starting with the pioneering works of Goldstein [4], Kusuoka [9] and Lindstrøm [13]. Since there is no natural notion of derivative on general closed sets, the notion of Laplacian on fractals is introduced by approximating the set from within and performing a limiting process. The probabilistic version of this approach was introduced by Kusuoka [9] and Lindstrøm [13] who considered suitable scaled random walks on the prefractal and then passed to the limit as the graph approaches the fractal so to define a Brownian motion on S. The corresponding analytic approach was taken by Kigami (see [10] and references therein) who defined Δu on the class of post-critically finite (pcf in short) fractals as the uniform limit of suitable scaled finite difference schemes on the prefractal. On pcf fractals the two approaches give rise to the same self-adjoint differential operator which is identified as the Laplacian (see [10,18]). The interior approximation preserves some of the classical properties satisfied by the Laplacian in the Euclidean setting, but other properties are typical of the fractal one ([15]).

In this paper we pursue an interior approximation approach on the Sierpinski gasket S for the eikonal equation (1.1). The starting point of our analysis is the work of Manfredi et al. [14], who consider a general class of nonlinear elliptic difference equations on graphs. This class includes the graph Laplacian, which coincides up to a rescaling factor with the operator considered in [10], and the graph eikonal equation. The principle underlying the definition of Laplacian and harmonic functions on the Sierpinski gasket is the minimization of the energy. For the eikonal equation the principle is the optimal control interpretation of the solution at all the different levels (discrete and continuous on the prefractal and continuous on the fractal).

We consider the graph eikonal equation on the prefractal S^n with a Dirichlet boundary condition on the vertices of the simplex generating the Sierpinski gasket; we characterize the unique solution of the problem by a representation formula which, for $f \equiv 1$ and null boundary data, gives the minimal vertex distance from the boundary. Moreover we infer that the sequence of the solutions of the graph eikonal equations on S^n is compact hence, up to a subsequences, converges uniformly to a continuous function u defined on S. By classical stability properties of viscosity solution theory it is only reasonable to expect the function u to be the solution in some appropriate sense of the eikonal equation (1.1) on S.

In [3], the authors give a definition of viscosity solution for the eikonal equation (1.1) in a general metric space $\mathcal{X}$, which is consistent with the usual notion in the Euclidean space. The unique solution of the associated Dirichlet problem is charac-

terized by a representation formula of control type. This definition obviously applies to the Sierpinski gasket endowed with the metric induced by the path distance. We show that the limit of the sequence of the solutions of the graph eikonal equations on the prefractals converges to the solution defined as in [3] on S. To establish this link, we introduce a class of continuous eikonal equation on the prefractal S^n and we obtain an uniform estimate of the distance between the solutions of the discrete and the continuous eikonal equations on S^n. Then, thanks to the stability property of the definition in [3], we are able to pass to the limit and to show that the solution of the continuous eikonal equation (and therefore also of the graph eikonal equation) on S^n converges to the solution of (1.1) on S. Note that this intermediate step can be also seen as a sort of homogenization result for the continuous eikonal equation on the Sierpinski gasket.

The crucial point in [3] is a notion of metric derivative $|\xi'(t)|$ for a given path $\xi = \xi(t)$ in $\mathcal{X}$ although in general $\xi'(t)$ may be not well defined. It follows that this elegant theory is confined to the case of the eikonal equation and it is difficult to extend to a more general class of Hamilton–Jacobi equation on S. Moreover the notion of viscosity supersolution in [3] is not local, even if consistent with the Euclidean viscosity solution. Instead we are able to extend the interior approximation approach to a more general class of Hamiltonians showing that the corresponding sequence of the solutions of the graph Hamilton–Jacobi equations converges uniformly to a function u on S. Lacking a definition of viscosity solution for general convex Hamiltonian on the Sierpinski gasket, our approach can be seen as a constructive way to define the solution to Hamilton–Jacobi equations on S. From this point of view the previous result for the eikonal equation can be also interpreted as a test that the construction gives the correct solution on the limit fractal.

The paper is organized as follows. In Sect. 2 we recall the definition of Sierpinski gasket. Section 3 and 4 are devoted to the study of the discrete and respectively continuous eikonal equation on the prefractal. In Sect. 5 we study the eikonal equation on the Sierpinski gasket and prove the convergence of the problems on the prefractal. Finally, in Sect. 6 we consider some possible extensions of the results here developed.

2 The Sierpinski gasket

Consider a unit regular D-dimensional simplex of vertices $\Gamma = \{a_1, \ldots, a_{D+1}\}$ in $\mathbb{R}^D$ (e.g., for $D = 1, 2, 3$, the simplex is respectively an interval, an equilateral triangle and a regular tetrahedron), and the $D + 1$ mappings $\psi_i : \mathbb{R}^D \to \mathbb{R}^D$ defined by

$$\psi_i(x) := a_i + \frac{1}{2}(x - a_i) \quad i = 1, \ldots, D + 1.$$

Iterating the ψ_i's, we get the set

$$\Gamma^\infty = \cup_{n=0}^\infty \Gamma^n$$

where $\Gamma^0 \equiv \Gamma$ and each Γ^n is given by the union of the images of Γ under the action of the maps $\psi_{i_1} \circ \cdots \circ \psi_{i_n}$ with $i_h \in \{1, \ldots, D+1\}, h = 1, \ldots, n$. Then the Sierpinski
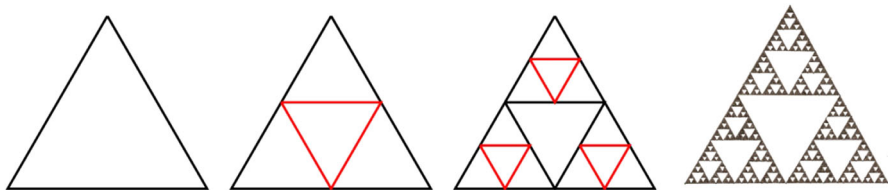
Fig. 1 Respectively: Γ, Γ^1, Γ^2 and Γ^5

gasket S is the closure of Γ^∞ (with respect to the Euclidean topology) (Fig. 1) and it is the unique non empty compact set K which satisfies

$$K = \cup_{i=1}^{D+1} \psi_i(K).$$

For any n, we can identify Γ^n with the graph $(V^n, \sim_n)$, where $V^n = \Gamma^n$ and $\sim_n$ is the following relation on V^n: for $x, y \in V^n$, $x \sim_n y$ if and only if the segment connecting x and y is the image of a side of the starting simplex under the action of some $\psi_{i_1} \circ \cdots \circ \psi_{i_n}$. When there is no possibility of confusion, for simplicity we shall write "$\sim$" instead of "$\sim_n$". Moreover, the graph $(V^n, \sim)$ can be naturally embedded in the network $S^n := V^n \cup E^n$, where E^n is the set of all segments of endpoints x, y with $x, y \in V^n$ and $x \sim_n y$. In other words, S^n is formed by the union of the images of the initial simplex under all maps $\psi_{i_1} \circ \cdots \circ \psi_{i_n}$. Hence $S = V \cup E$ where $V := \cup_{i=1}^\infty V^n$ and $E := \cup_{i=1}^\infty E^n$. The $D + 1$ vertices of the initial simplex assume the role of the boundary of S^n and of the curve S itself:

$$\partial S := \Gamma;$$

on this set a boundary condition will be prescribed.

We note that, for any $x \in S \backslash \Gamma$, there are only two possibilities:

- either $x \in V \backslash \Gamma$, hence, there exists $n \in \mathbb{N}$ and $v \in V^n \backslash \Gamma$ such that $x = v$; in particular, x is a ramification point with $2D$ incident edges in S^m, for any $m \geq n$;
- or $x \in S \backslash (V \cup \Gamma)$, hence for n sufficiently large, there exists $e_n \in E^n$ such that $x \in e_n$ and x is not a ramification point in any S^n (however, it is the limit of ramification points).

We define the geodesic distance on the network S.

Definition 2.1 For any $x, y \in S$, set

$$d(x, y) := \inf \{\ell(\gamma) : \gamma \text{ is a path joining } x \text{ to } y\} \tag{2.1}$$

where $\ell(\gamma)$ is the length of γ.

We refer the reader to [6,17] (see also [11]) for a complete characterization of the geodesic distance on the Sierpinski gasket. See [6] for the next result

Proposition 2.1 *For any* $x, y \in S$, *there holds*

$$|x - y| \leq d(x, y) \leq 1.$$

Moreover d *is continuous and the* inf *in* (2.1) *is a minimum.*

Since now on, we shall require the following hypotheses

Assumptions $D = 2$. The function $f : S \to \mathbb{R}$ is continuous with

$$\lambda := \inf_S f(x) > 0.$$

3 A discrete eikonal equation on the prefractal

In this section, for $n \in \mathbb{N}$ fixed, we tackle the discrete eikonal equation on the graph $(V^n, \sim)$. We denote $V_0^n := V^n \backslash \Gamma$ the set of *interior* vertices while Γ will stand for the set of boundary vertices.

The distance between two adjacent vertices x, y is

$$d_n(x, y) = \frac{1}{2^n} =: h_n$$

while the distance for two arbitrary vertices $x, y \in V^n$ is the *vertex distance*

$$d_n(x, y) := \inf\{d(x_0, x_1) + d(x_1, x_2) + \cdots + d(x_{N-1}, x_N)\} \tag{3.1}$$

where the infimum is taken over all the finite path $\{x_0 = x, x_1, \ldots, x_N = y\}$ with $x_i \sim x_{i+1}, i = 0, \ldots, N - 1$ connecting x to y. In fact, since V^n is a finite set, the "inf" is a "min". Note that d_n is the restriction of the geodesic distance d defined in (2.1) to V^n.

For a function $u : V^n \to \mathbb{R}$, we consider the discrete eikonal equation (see [14])

$$|Du(x)|_n = f(x) \quad \text{in } V_0^n \tag{3.2}$$

where

$$|Du(x)|_n := \max_{y \sim x}\left\{-\frac{1}{h_n}(u(y) - u(x))\right\}.$$

Definition 3.1 A function $u : V^n \to \mathbb{R}$ is said a subsolution (respectively, a supersolution) of (3.2) if

$$|Du(x)|_n \leq f(x) \quad (\text{resp., } |Du(x)|_n \geq f(x)) \quad \forall x \in V_0^n.$$

A function u is said a solution of (3.2) if it is both a sub- and a supersolution of (3.2).

In the next proposition we prove a comparison theorem for (3.2).

Proposition 3.1 *Let u and v be a subsolution and, respectively, a supersolution of (3.2) such that $u \leq v$ on Γ. Then $u \leq v$ in V^n.*

Proof We shall proceed using some ideas of [14]. Assume by contradiction that $\max_{V^n}\{u - v\} = \delta > 0$ and set $W = \arg\max_{V^n}\{u - v\}$, $m = \min\{v(x) : x \in W\}$.

Let $x \in W$ be such that $v(x) = m$; in particular, x belongs to V_0^n. Let $z \in V^n$ be such that $z \sim x$ and $|Dv(x)|_n = -h_n^{-1}(v(z) - v(x))$. By the definition of sub- and supersolution we have

$$-\frac{1}{h_n}(v(z) - v(x)) - f(x) = |Dv(x)|_n - f(x) \geq 0 \geq |Du(x)|_n - f(x)$$

$$\geq -\frac{1}{h_n}(u(z) - u(x)) - f(x);$$

hence $u(z) - v(z) \geq u(x) - v(x) = \delta$ and therefore $z \in W$. Moreover $-\frac{1}{h_n}(v(z) - v(x)) \geq f(x) > 0$, hence $m = v(x) > v(z)$ which contradicts the definition of m. $\square$

Proposition 3.2 *Let $g : \Gamma \to \mathbb{R}$ be such that*

$$g(x) \leq \inf\left\{\sum_{k=0}^{N-1} h_n f(x_k) + g(y)\right\} \quad \forall x, y \in \Gamma, \tag{3.3}$$

where the infimum is taken over all the finite paths $\{x_0 = x, x_1, \ldots, x_N = y\}$ with $x_i \sim x_{i+1}$, $i = 0, \ldots, N-1$. Then, the unique solution of (3.2) with the boundary condition

$$u = g \quad \text{on } \Gamma \tag{3.4}$$

is given by

$$u(x) = \min\left\{\sum_{k=0}^{N-1} h_n f(x_k) + g(y)\right\} \tag{3.5}$$

where the minimum is taken over all $y \in \Gamma$ and over all the finite paths $\{x_0 = x, x_1, \ldots, x_N = y\}$ with $x_i \sim x_{i+1}$, $i = 0, \ldots, N-1$.

Proof We observe that the function u is well defined (i.e., the minimum is achieved) because, for x fixed, the set of admissible paths is finite. Uniqueness is an immediate consequence of Proposition 3.1. In order to show that u is a subsolution, given $x \in V_0^n$, for each $z \sim x$ consider a path $\{x_0 = z, x_1, \ldots, x_N = y_z\}$ such that $u(z) = g(y_z) + \sum_{k=0}^{N} h_n f(x_k)$. Then $\{x, x_0, \ldots, x_N\}$ is a path connecting x to Γ and therefore

$$-(u(z) - u(x)) \leq -\left(g(y_z) + h_n \sum_{k=0}^{N-1} f(x_k) - (g(y_z) + h_n f(x) + h_n \sum_{k=0}^{N-1} f(x_k))\right)$$

$$= h_n f(x).$$

It follows that $|Du(x)|_n \leq f(x)$.

In order to prove that u is a supersolution, given $x \in V_0^n$ consider a path $\{x_0 = x, x_1, \ldots, x_N = y\}$ with $y \in \Gamma$ such that $u(x) = g(y) + \sum_{k=0}^{N-1} h_n f(x_k)$. Then $x_1 \sim x$ and $\{x_1, \ldots, x_N\}$ is a path connecting x_1 to Γ. Therefore

$$-(u(x_1) - u(x)) = -\left(u(x_1) - (g(y) + h_n \sum_{k=1}^{N-1} f(x_k)) \right) + h_n f(x) \geq h_n f(x)$$

which implies $|Du(x)|_n \geq f(x)$.

To show (3.4), assume by contradiction that there exists $x \in \Gamma$ and a path $\{x_0 = x, x_1, \ldots, x_N = y\}$ with $y \in \Gamma$ such that

$$u(x) = g(y) + h_n \sum_{k=0}^{N-1} f(x_k) < g(x).$$

By (3.3), we have

$$g(y) + h_n \sum_{k=0}^{N-1} f(x_k) \geq g(y) + g(x) - g(y) = g(x)$$

and therefore a contradiction. □

It is worth to observe that u verifies a dynamical programming principle.

Lemma 3.1 *The function u defined in* (3.5) *verifies*

$$u(x) = \min \left\{ \sum_{k=0}^{M-1} h_n f(x_k) + u(z) \right\} \tag{3.6}$$

where the minimum is taken over all $z \in V_0^n$ and over all the finite paths $\{x_0 = x, x_1, \ldots, x_M = z\}$ with $x_i \sim x_{i+1}$, $i = 0, \ldots, M - 1$.

Proof We shall argue using some ideas of [1, Prop. III.2.5]. Denote w the right hand side of (3.6). For any path $x_0 = x, \ldots, x_N = y \in \Gamma$, there holds

$$\sum_{k=0}^{N-1} h_n f(x_k) + g(y) = \sum_{k=0}^{M-1} h_n f(x_k) + \sum_{k=M}^{N-1} h_n f(x_k) + g(y) \geq \sum_{k=0}^{M-1} h_n f(x_k) + u(x_M);$$

hence, taking the minimum over x_M and over the admissible paths, we get $u \geq w$.

On the other hand, consider $z \in V_0^n$ and a path as in (3.6) such that $w(x) = \sum_{k=0}^{M-1} h_n f(x_k) + u(z)$; consider an admissible path $z_0 = z, z_1, \ldots, z_N \in \Gamma$ such that $u(z) = \sum_{k=0}^{N-1} h_n f(z_k) + g(z_N)$. We have: $w(x) = \sum_{k=0}^{M+N-1} h_n f(\tilde{x}_k) + g(z_N)$ where $\tilde{x}_k = x_k$ if $k \in \{0, \ldots, M\}$ and $\tilde{x}_k = z_{k-M}$ if $k \in \{M, \ldots, N\}$. Then, $u \leq w$. □

Remark 3.1 Observe that by (3.5), the distance from the boundary

$$d_n(x) := \inf\{d_n(x, y) : y \in \Gamma\} \quad x \in V^n \tag{3.7}$$

is the unique solution of

$$|Du(x)|_n - 1 = 0, \quad x \in V_0^n, \quad u = 0, \quad x \in \Gamma.$$

Proposition 3.3 *Let u be the solution of* (3.2)–(3.4)*. Then*

$$|u(x)| \leq \max_\Gamma |g| + d_n(x) \max_{V^n}\{f\} \quad \forall x \in V^n \tag{3.8}$$

$$\frac{|u(y) - u(x)|}{h_n} \leq \max_{V^n}\{f\} \quad \forall x, y \in V^n, x \sim y \tag{3.9}$$

where $d_n(x)$ is the distance from the boundary introduced in (3.7)*.*

Proof The functions $u^\pm(x) := \pm(\max_\Gamma |g| + d_n(x) \max_{V^n}\{f\})$ are a supersolution and a subsolution of (3.2)–(3.4), then estimate (3.8) follows by Proposition 3.1. The estimate (3.9) is an immediate consequence of the equation (3.2). □

Let $\bar{u}_n$ be a continuous piecewise linear reconstruction of the solution u_n of (3.2)–(3.4) on S^n. The previous proposition yields that the sequence $\{\bar{u}_n\}_n$ is compact. In Sect. 5 we will show that it uniformly converges to a function u and we will characterize u in terms of an eikonal equation defined on the Sierpinski gasket.

4 A continuous eikonal equation on the prefractal

In this section we introduce a continuous eikonal equation on the prefractal S^n. Moreover we estimate the distance between the solutions of the discrete problem (3.2) and of the corresponding continuous eikonal equations on S^n. Let us recall that V_0^n stands for the set of the interior vertices of S^n while Γ is the set of the boundary vertices of S^n. For any n, fix arbitrary orderings I^n and J^n of the vertices $x_i \in V^n$ and, respectively, of the edges $e_j \in E^n$ and denote by $\pi_j : [0, \ell_j] \to \mathbb{R}^2$, $\ell_j > 0$, a parametrization of the edge e_j. For any $x_i \in V^n$, we set $Inc_i := \{j \in J^n : x_i \in e_j\}$. The parametrization π_j of the arc e_j induces an orientation along the edge, which can be expressed by the *signed incidence matrix* $A^n = \{a_{ij}^n\}_{i \in I^n, j \in J^n}$ with

$$a_{ij}^n := \begin{cases} 1 & \text{if } x_i \in e_j \text{ and } \pi_j(0) = x_i, \\ -1 & \text{if } x_i \in e_j \text{ and } \pi_j(\ell_j) = x_i, \\ 0 & \text{otherwise.} \end{cases} \tag{4.1}$$

We denote by $\delta_n : S^n \times S^n \to \mathbb{R}^+$ the *path distance* on S^n, i.e.

$$\delta_n(x, y) := \inf\{\ell(\gamma) : \gamma \subset S^n \text{ is a path joining } x \text{ to } y\} \quad \forall x, y \in S^n. \tag{4.2}$$

Note that δ_n coincides with the vertex distance (3.1) when restricted to $x, y \in V^n$. Moreover $\delta_n \geq d$ for any n where d as in (2.1).

We say that u is continuous in S^n and write $u \in C(S^n)$ if u is continuous with respect to the subspace topology of S^n. In a similar way we define the space of upper semicontinuous functions $USC(S^n)$ and the space of lower semicontinuous functions $LSC(S^n)$, respectively. For any function $u : S^n \to \mathbb{R}$, we denote by u_j the restriction of u to e_j, i.e.

$$u_j := u \circ \pi_j : [0, \ell_j] \to \mathbb{R}.$$

Differentiation is defined with respect to the parametrization of the edge. Hence, if $x \in e_j$ for some $j \in J^n$ (i.e., x is not a ramification point), we set

$$D_j u(x) := \frac{du_j}{dy}(y), \quad \text{with } y = (\pi_j)^{-1}(x)$$

while if $x_i \in V_0^n$ (i.e., x is a vertex), we set

$$D_j u(x_i) := \frac{du_j}{dy}(y) \quad \text{for any } e_j \in Inc_i, \ y = (\pi_j)^{-1}(x_i).$$

For a function $u : S^n \to \mathbb{R}$, we consider the eikonal equation

$$|Du| = f(x) \quad \text{in } S^n. \tag{4.3}$$

In the next definitions we introduce the class of admissible test functions and solution of (4.3) (see [16]).

Definition 4.1 Let $\phi \in C(\Gamma)$.

(i) Let $x \in e_j$, $j \in J^n$. We say that ϕ is test function at x, if ϕ_j is differentiable at $\pi_j^{-1}(x)$.

(ii) Let $x \in V_0^n$, $j, k \in Inc_i$, $j \neq k$. We say that ϕ is (j, k)-test function at x, if ϕ_j and ϕ_k are differentiable at $\pi_j^{-1}(x)$ and $\pi_k^{-1}(x)$, with

$$a_{ij}^n D_j \phi(\pi_j^{-1}(x)) + a_{ik}^n D_k \phi(\pi_k^{-1}(x)) = 0,$$

where (a_{ij}^n) as in (4.1).

Definition 4.2 (i) If $x \in e_j$, $j \in J^n$, then a function $u \in USC(S^n)$ (resp., $v \in LSC(S^n)$) is called a subsolution (resp. supersolution) of (4.3) at x if for any test function ϕ for which $u - \phi$ attains a local maximum at x (resp., a local minimum), we have

$$|D_j \phi(x)| \leq f(x) \quad \big(\text{resp., } |D_j \phi(x)| \geq f(x)\big);$$

(ii) If $x = x_i \in V_0^n$, then

- A function $u \in USC(S^n)$ is called a subsolution at x if for any $j, k \in Inc_i$ and any (j, k)-test function ϕ for which $u - \phi$ attains a local maximum at x relatively to $e_j \cup e_k$, we have $|D_j\phi(x)| \leq f(x)$.
- A function $v \in LSC(S^n)$ is called a supersolution at x if for any $j \in Inc_i$, there exists $k \in Inc_i \setminus \{j\}$ (said feasible for j at x) such that for any (j, k)-test function ϕ for which $u - \phi$ attains a local minimum at x relatively to $e_j \cup e_k$, we have $|D_j\phi(x)| \geq f(x)$.

Remark 4.1 It is worth to observe that the definitions of sub- and supersolution are not symmetric at the vertices. As observed in [16] for the equation $|Du| = 1$, a definition of supersolution similar to the one of subsolution would not characterize the "expected" solution, i.e. the distance from the boundary. Also in [3], these definitions are asymmetric: see Definition 5.1 below. Furthermore, let us recall that several definitions of solutions for Hamilton–Jacobi equations on network have been introduced recently; however for Eq. (4.3) they coincide (see [2]).

Remark 4.2 Consider a vertex $v_i \in V_0^n$ and, wlog, assume $\pi_j(0) = v_i \; \forall j \in Inc_i$. Then, a function $u \in USC(S^n)$ is a viscosity subsolution in v_i if, and only if, for every $j, k \in Inc_i$ with $j \neq k$, the function

$$U_{jk}(t) := \begin{cases} u(\pi_j(t)) & t \in [0, l_j) \\ u(\pi_k(-t)) & t \in [-l_k, 0) \end{cases}$$

is a (standard) viscosity subsolution at $t = 0$.

Similarly, a function $u \in LSC(S^n)$ is a viscosity supersolution in v_i if, and only if, for every $j \in Inc_i$ there exists $k \in Inc_i \setminus \{j\}$ such that the function U_{jk} is a (standard) viscosity supersolution at $t = 0$.

In the next propositions we recall from [16] comparison and existence result for (4.3); for the proof we refer the reader to [16, Prop. 6.1].

Proposition 4.1 *Let u and v be a subsolution and, respectively, a supersolution of* (4.3) *such that $u \leq v$ on Γ. Then $u \leq v$ in S^n.*

Proposition 4.2 *Let $g : \Gamma \to \mathbb{R}$ be such that*

$$g(x) \leq \inf \left\{ \int_0^T f(\xi(t))dt + g(y) \right\} \quad \forall x, y \in \Gamma \tag{4.4}$$

where the infimum is taken over all the piecewise differentiable paths ξ such that $\xi(0) = x$, $\xi(T) = y$ and $|\xi'(t)| \leq 1$. Then the unique solution of (4.3) with the boundary condition

$$u = g \quad on \; \Gamma \tag{4.5}$$

is given by

$$u(x) = \inf \left\{ \int_0^T f(\xi(t))dt + g(y) \right\} \quad for \; all \; x \in S^n, \tag{4.6}$$

where the infimum is taken over all $y \in \Gamma$ and over all the paths ξ such that $\xi(0) = x$, $\xi(T) = y$ and $|\xi'(t)| \leq 1$. Moreover, u is bounded and Lipschitz continuous and

$$|u(x)| \leq \max_{\Gamma} |g| + \delta_n(x, \Gamma) \max_{S^n} \{f\} \quad for\ all\ x \in S^n, \tag{4.7}$$

$$\|Du\|_\infty \leq \max_{S^n} \{f\}. \tag{4.8}$$

Remark 4.3 Observe that by (4.6), the distance from the boundary

$$\delta_n(x) := \inf\{\delta_n(x, y) : y \in \Gamma\} \quad x \in S^n \tag{4.9}$$

is the unique solution of

$$|Du(x)| = 1 \quad x \in S^n, \quad u = 0 \quad x \in \Gamma.$$

Remark 4.4 Note that assumption (3.3) for every $n \in \mathbb{N}$ ensures assumption (4.4).

Proposition 4.3 *The sequence $\{u_n\}_n$ of the solutions of (4.3)–(4.5) is decreasing. Moreover there exists a continuous function $\bar{u}$ on S such that $\{u_n\}_n$ converges uniformly to $\bar{u}$ on each prefractal S^m.*

Proof Let u_m be the solution of (4.3)–(4.5) on the set S^m, with $m > n$. Since $S^n \subset S^m$, u_m is a subsolution of (4.3)-(4.5) on S^n. By Proposition 3.1, it follows that $u_m \leq u_n$. By (4.7)–(4.8), the sequence $\{u_n\}_n$ is equi-bounded, equi-continuous and decreasing. Hence it converges uniformly to a function $\bar{u}$ on each S^m. $\qquad\square$

Proposition 4.4 *Let u_{h_n} and u_n be respectively the solution of (3.2)–(3.4) and of (4.3)–(4.5). Then, there holds*

$$|u_n(x) - u_{h_n}(x)| \leq C \left(\omega_f(h_n^{1/2}) + h_n^{1/2} \right) \quad \forall x \in V^n$$

where ω_f is the modulus of continuity of f and $C \in \mathbb{R}$ depends only on $\max_{\Gamma} |g|$ and $\max_S |f|$. Hence the sequences u_{h_n} and u_n converge to the same limit $\bar{u}$ on S.

In order to prove this result, it is expedient to establish a preliminary lemma whose proof is postponed at the end of this section.

Lemma 4.1 *Let u_{h_n} be the solution of (3.2)–(3.4). Then $w_{h_n} = 1 - e^{-u_{h_n}}$ solves*

$$\begin{cases} |Dw|_n + f_n(x)w = f_n(x) & in\ V_0^n \\ w = 1 - e^{-g} & on\ \Gamma \end{cases} \tag{4.10}$$

where $f_n(x) := h_n^{-1}(e^{h_n f(x)} - 1)$. Moreover

$$\max_{S^n} |f_n - f| \leq h_n \max_S f^2. \tag{4.11}$$

Proof of Proposition 4.4 We observe that the function $w_n := 1 - e^{-u_n}$ solves

$$|Dw| + fw = f \quad \text{in } S^n, \quad w = 1 - e^{-g} \quad \text{on } \Gamma.$$

Define $\Psi : V^n \times S^n \to \mathbb{R}$ by

$$\Psi(x, y) = w_{h_n}(x) - w_n(y) - \frac{\delta_n(x, y)^2}{2\varepsilon}$$

where w_{h_n} is the function introduced in Lemma 4.1 while δ_n is the geodesic distance on S^n defined in (4.2). Since Ψ is continuous and $V^n \times S^n$ is compact, there exists $(x_\varepsilon, y_\varepsilon) \in V^n \times S^n$ such that $\Psi(x_\varepsilon, y_\varepsilon) \geq \Psi(x, y)$ for any $(x, y) \in V^n \times S^n$. By $\Psi(x_\varepsilon, y_\varepsilon) \geq \Psi(x, x)$, we get

$$\frac{\delta_n(x_\varepsilon, y_\varepsilon)^2}{2\varepsilon} \leq 2(\|w_{h_n}\|_\infty + \|w_n\|_\infty)$$

and therefore $\lim_{\varepsilon \to 0^+} \delta_n(x_\varepsilon, y_\varepsilon) = 0$. Hence there exists $x_i \in V^n$ such that

$$\lim_{\varepsilon \to 0^+} x_\varepsilon = \lim_{\varepsilon \to 0^+} y_\varepsilon = x_i.$$

Moreover, since V^n is a finite set, wlog we can assume that, for ε sufficiently small $x_\varepsilon = x_i$. By the Lipschitz continuity of u_n and $\Psi(x_\varepsilon, y_\varepsilon) \geq \Psi(x_\varepsilon, x_\varepsilon)$, it follows

$$\frac{\delta_n(x_\varepsilon, y_\varepsilon)^2}{2\varepsilon} \leq w_n(x_\varepsilon) - w_n(y_\varepsilon) \leq L_{w_n} \delta_n(x_\varepsilon, y_\varepsilon)$$

where L_{w_n} is the Lipschitz constant of w_n (see Proposition 4.2). Whence, we have

$$\delta_n(x_\varepsilon, y_\varepsilon) \leq L_{w_n} \varepsilon. \tag{4.12}$$

Assume first that $x_\varepsilon = x_i \in \Gamma$. Then, since $w_n = w_{h_n}$ on Γ, we get

$$w_{h_n}(x_\varepsilon) - w_n(y_\varepsilon) = w_n(x_\varepsilon) - w_n(y_\varepsilon) \leq L_{w_n} \delta_n(x_\varepsilon, y_\varepsilon). \tag{4.13}$$

Assume $x_\varepsilon = x_i \in V_0^n$ and let $z_\varepsilon \in V^n$ be such that y_ε is contained in the edge e_j, $j \in Inc_i$, of endpoints x_ε and z_ε. Since $z_\varepsilon \sim x_\varepsilon$ and $\Psi(x_\varepsilon, y_\varepsilon) \geq \Psi(z_\varepsilon, y_\varepsilon)$, it follows

$$-\frac{1}{h_n}\left(\frac{\delta_n(z_\varepsilon, y_\varepsilon)^2}{2\varepsilon} - \frac{\delta_n(x_\varepsilon, y_\varepsilon)^2}{2\varepsilon}\right) + f_n(x_\varepsilon)w_{h_n}(x_\varepsilon) \leq f_n(x_\varepsilon). \tag{4.14}$$

By $\delta_n(x_\varepsilon, y_\varepsilon) + \delta_n(y_\varepsilon, z_\varepsilon) = h_n$, we get

$$-\left(\frac{\delta_n(z_\varepsilon, y_\varepsilon)^2}{2\varepsilon} - \frac{\delta_n(x_\varepsilon, y_\varepsilon)^2}{2\varepsilon}\right) = h_n \frac{\delta_n(x_\varepsilon, y_\varepsilon)}{\varepsilon} - \frac{h_n^2}{2\varepsilon}$$

and by (4.14)

$$\frac{\delta_n(x_\varepsilon, y_\varepsilon)}{\varepsilon} - \frac{h_n}{2\varepsilon} + f_n(x_\varepsilon) w_{h_n}(x_\varepsilon) \le f_n(x_\varepsilon). \tag{4.15}$$

Defined $\phi(y) = w_{h_n}(x_\varepsilon) - \delta_n(x_\varepsilon, y)^2/2\varepsilon$, by $-\Psi(x_\varepsilon, y_\varepsilon) \le -\Psi(x_\varepsilon, y)$ it follows that $w_n - \phi$ attains a minimum at y_ε. Observe that only two cases may occur: either $y_\varepsilon = x_\varepsilon$ or $y_\varepsilon \in e_j$. Let us consider the former case; the latter one can be dealt with by easy adaptations and we shall omit it. Since $D_j\phi(x_\varepsilon) = 0 = D_k\phi(x_\varepsilon)$, we obtain that for any $k \in Inc_{x_\varepsilon} \setminus \{j\}$, ϕ is a (j, k)-admissible test function for w_n at x_ε. Hence, in both cases, by definition of viscosity supersolution, we obtain

$$\frac{1}{\varepsilon}\delta_n(x_\varepsilon, y_\varepsilon) + f(y_\varepsilon) w_n(y_\varepsilon) \ge f(y_\varepsilon). \tag{4.16}$$

By (4.15) and (4.16) we get

$$f_n(x_\varepsilon) w_{h_n}(x_\varepsilon) - f(y_\varepsilon) w_n(y_\varepsilon) \le f_n(x_\varepsilon) - f(y_\varepsilon) + \frac{h_n}{2\varepsilon}. \tag{4.17}$$

By either (4.13) or (4.17), using (4.12), (3.8) and (4.11) (and our assumptions as well), we get

$$w_{h_n}(x_\varepsilon) - w_n(y_\varepsilon) \le \frac{C}{\lambda}\left(\omega_f(L_{w_n}\varepsilon) + \frac{h_n}{2\varepsilon}\right) \tag{4.18}$$

where C depends only on $\max_\Gamma |g|$, $\max_S |f|$ (note that L_{w_n} depends only on the latter). Taking $\varepsilon = h_n^{1/2}/L_{w_n}$ in (4.18) we get the estimate

$$w_{h_n}(x_\varepsilon) - w_n(y_\varepsilon) \le C(\omega_f(h_n^{1/2}) + h_n^{1/2}).$$

Finally the inequality $\Psi(x_\varepsilon, y_\varepsilon) \ge \Psi(x, x)$ for $x \in V^n$ yields

$$u_{h_n}(x) - u_n(x) \le u_{h_n}(x_\varepsilon) - u_n(y_\varepsilon) \le C(\omega_f(h_n^{1/2}) + h_n^{1/2}).$$

Let us now pass to prove the reverse inequality: $u_n(x) - u_{h_n}(x) \le C(\omega_f(h_n^{1/2}) + h_n^{1/2})$; the arguments are similar to those of the previous case so we shall just sketch them. We consider the function

$$\Psi(x, y) := w_n(x) - w_{h_n}(y) - \frac{\delta_n(x, y)^2}{2\varepsilon} \quad \text{for } (x, y) \in S^n \times V^n$$

and we denote $(x_\varepsilon, y_\varepsilon)$ its maximum point. Since $\phi(x) := w_{h_n}(y_\varepsilon) + \frac{\delta_n(x, y_\varepsilon)^2}{2\varepsilon}$ is an admissible test function for w_n at x_ε, we obtain

$$\delta_n(x_\varepsilon, y_\varepsilon)/\varepsilon + f(x_\varepsilon) w_n(x_\varepsilon) \le f(x_\varepsilon).$$

On the other hand, by the inequality $\Psi(x_\varepsilon, y_\varepsilon) \geq \Psi(x_\varepsilon, z)$ for any $z \in V^n$ and by (4.10), we get

$$\max_{z \sim y_\varepsilon} \left[\frac{\delta_n(x_\varepsilon, z)^2}{2\varepsilon h_n} - \frac{\delta_n(x_\varepsilon, y_\varepsilon)^2}{2\varepsilon h_n} \right] + f_n(y_\varepsilon) w_{h_n}(y_\varepsilon) \geq f_n(y_\varepsilon). \qquad (4.19)$$

Denote by $z_\varepsilon \in V^n$ the vertex where the maximum in the previous relation is attained; obviously, $\delta_n(y_\varepsilon, z_\varepsilon) = h_n$. Then, plugging $\delta_n(x_\varepsilon, z_\varepsilon) \leq \delta_n(x_\varepsilon, y_\varepsilon) + \delta_n(y_\varepsilon, z_\varepsilon) = \delta_n(x_\varepsilon, y_\varepsilon) + h_n$ in relation (4.19), we get

$$\frac{\delta_n(x_\varepsilon, y_\varepsilon)}{\varepsilon} + \frac{h_n}{2\varepsilon} + f_n(y_\varepsilon) w_{h_n}(y_\varepsilon) \geq f_n(y_\varepsilon)$$

and we can conclude as in the previous case. $\qquad \square$

Remark 4.5 Taking advantage of the structure of the equations, Proposition 4.4 can be also proved following the approach introduced by Ishii et al. in [7,8]. Let us sketch the main issues. For $\varepsilon = h_n^{1/2}$, $\theta = 1 - (\omega_f(L_{u_n} h_n^{1/2}) + h_n^{1/2})\lambda^{-1}$, consider the function

$$\Phi(x, y) := \theta u_{h_n}(x) - u_n(y) - \frac{\delta_n(x, y)^2}{2\varepsilon} \qquad \forall (x, y) \in V^n \times S^n.$$

Let $(x_\varepsilon, y_\varepsilon)$ be a maximum point of Φ; as before, we have $\delta_n(x_\varepsilon, y_\varepsilon) \leq L_{u_n}\varepsilon$. We claim that $x_\varepsilon \in \Gamma$. Indeed, assuming by contradiction that $x_\varepsilon \in V_0^n$ and arguing as in (4.15)–(4.16), we get $\frac{\delta_n(x_\varepsilon, y_\varepsilon)}{\varepsilon} - \frac{h_n}{2\varepsilon} \leq \theta f(x_\varepsilon)$ and $\frac{\delta_n(x_\varepsilon, y_\varepsilon)}{\varepsilon} \geq f(y_\varepsilon)$. Subtracting these inequalities, we infer the desired contradiction:

$$-\frac{h_n}{2\varepsilon} \leq \theta f(x_\varepsilon) - f(y_\varepsilon) \leq \omega_f(L_{u_n}\varepsilon) - (1 - \theta)\lambda \leq -\frac{h_n}{\varepsilon}.$$

In conclusion, the inequality $\Phi(x, x) \leq \Phi(x_\varepsilon, y_\varepsilon)$ gives

$$u_{h_n}(x) - u_n(x) \leq u_{h_n}(x_\varepsilon) - u_n(y_\varepsilon) - \delta_n(x_\varepsilon, y_\varepsilon)^2 (2\varepsilon)^{-1} \\ + (1 - \theta)[u_{h_n}(x) - u_{h_n}(x_\varepsilon)].$$

By our choice of ε and θ, the fact that $x_\varepsilon \in \Gamma$ so $u_{h_n}(x_\varepsilon) = u_n(x_\varepsilon)$, estimates (4.7), (3.8) and of $\delta_n(x_\varepsilon, y_\varepsilon)$, we accomplish the proof.

Proof of Lemma 4.1 For $x \in V_0^n$, consider $z \sim x$ such that

$$|Du_{h_n}(x)|_n = -h_n^{-1} \left(u_{h_n}(z) - u_{h_n}(x) \right) = f(x).$$

Whence, by the definition of w_{h_n}, we deduce

$$\log \left(\frac{1 - w_{h_n}(z)}{1 - w_{h_n}(x)} \right) = h_n f(x)$$

and, by easy calculations, we get

$$- \left(w_{h_n}(z) - w_{h_n}(x) \right) - w_{h_n}(x) \left[1 - e^{h_n f(x)} \right] = e^{h_n f(x)} - 1.$$

Dividing by h_n, we accomplish the proof of (4.10). Estimate (4.11) can be obtained by easy calculations and we omit its proof. □

5 The eikonal equation on the Sierpinski gasket

In this section we show that, in analogy with the construction of the Laplacian on fractal sets, the eikonal equation on the Sierpinski gasket can be obtained as the limit of a sequence of discrete problems defined on the prefractal S^n.

We first recall some notations and definitions in [3]. Let $(\mathcal{X}, d)$ be a metric space. Given a curve $\xi : I \subset \mathbb{R} \to \mathcal{X}$, define the metric derivative of ξ by

$$|\xi'|(t) := \lim_{s \to t} \frac{d(\xi(s), \xi(t))}{|s - t|}.$$

A curve ξ is said *absolutely continuous* if $|\xi'|(t)$ exists for a.e. $t \in I$, $|\xi'| \in L^1_{loc}(I)$ and

$$d(\xi(s), \xi(t)) \leq \int_s^t |\xi'|(r) dr.$$

For $\Omega \subset \mathcal{X}$, denote by $\mathcal{A}(I, \Omega)$ the set of absolutely continuous curves such that $\xi(t) \in \Omega$ and $|\xi'|(t) \leq 1$ a.e. in I and by $\mathcal{A}_x(I, \Omega)$ the set of curves in $\mathcal{A}(I, \Omega)$ such that $0 \in I$ and $\xi(0) = x$. For any $\xi \in \mathcal{A}_x(I, \Omega)$, define the exit and the entrance time in Ω by

$$T_\Omega^+[\xi] := \inf\{t \in [0, +\infty) : \xi(t) \in \partial\Omega\} \in [0, +\infty]$$
$$T_\Omega^-[\xi] := \sup\{t \in (-\infty, 0] : \xi(t) \in \partial\Omega\} \in [-\infty, 0].$$

A function u is said arcwise upper (resp., lower) semicontinuous in $\Omega \subset \mathcal{X}$ if for each $\xi \in \mathcal{A}(I, \Omega)$ the function $u \circ \xi$ is upper (lower) semicontinuous in I. The set of arcwise upper (resp., lower) semicontinuous functions in Ω is denoted by $\mathrm{USC}_a(\Omega)$ (resp., $\mathrm{LSC}_a(\Omega)$). It is worth to recall that these definitions of semicontinuity are weaker than the corresponding standard ones (which are given in terms of the distance; see [3]). For a function $w : \mathbb{R} \to \mathbb{R}$, denote by $D^\pm w(t)$ respectively its super- and subdifferential at the point t (see [1] for the precise definition and main properties).

For $\Omega \subset \mathcal{X}$, consider the eikonal equation

$$|Du| = f(x) \quad \text{in } \Omega; \tag{5.1}$$

for the sake of completeness, we recall the definition of (metric viscosity) subsolutions and supersolutions introduced in [3].

Definition 5.1 (i) A function $u \in USC_a(\Omega)$ is said a viscosity subsolution of (5.1) if for each $x \in \Omega$ and for all $\xi \in A_x(I, \Omega)$, the function $w : I \to \mathbb{R}$, with $w(t) := u(\xi(t))$, satisfies

$$|p| \leq f(x) \quad \forall p \in D^+ w(0).$$

(ii) A function $v \in LSC_a(\Omega)$ is said a viscosity supersolution of (5.1) if for each $x \in \Omega$ and $\varepsilon > 0$, there exists $\xi \in A_x(\mathbb{R}, \Omega)$ and $w \in LSC(\mathbb{R})$ such that

$$\begin{cases} T^\pm := T_\Omega^\pm[\xi] & \text{are both finite,} \\ w(0) = v(x), \quad w(t) \geq v(\xi(t)) - \varepsilon \quad \forall t \in (T^-, T^+), \\ |p| \geq f(\xi(t)) - \varepsilon \quad \forall p \in D^- w(t), \, t \in (T^-, T^+). \end{cases}$$

Remark 5.1 The previous definitions apply to the case where $\mathcal{X}$ is the Sierpinski gasket S introduced in Sect. 2 with $\Omega := S\backslash\Gamma$ (recall that Γ is the set of the vertices of the starting simplex). In this case the metric on $\mathcal{X}$ is the one induced by the geodetic distance d defined in (2.1). Moreover arcwise upper and lower semicontinuous functions coincide with upper and lower semicontinuous functions with respect to the metric induced by d.

The following results are immediate consequences of the theory developed in [3] for a generic metric space $(\mathcal{X}, d)$; for the proofs, we refer the reader to [3, Thm 3.1] and to [3, Thm 3.2].

Proposition 5.1 *Let u and v be a subsolution and, respectively, a supersolution of (5.1) such that $u \leq v$ on Γ. Then $u \leq v$ in S.*

Proposition 5.2 *Let $g : \Gamma \to \mathbb{R}$ be such that*

$$g(x) \leq \inf \left\{ \int_0^T f(\xi(t))dt + g(y) \right\} \quad \text{for all } x, y \in \Gamma \tag{5.2}$$

where the infimum is taken over all the paths $\xi \in A((0, T), S)$ such that $\xi(0) = x$, $\xi(T) = y$. Then the unique solution of (5.1) with the boundary condition

$$u = g \quad \text{on } \Gamma \tag{5.3}$$

is given by

$$u(x) = \inf \left\{ \int_0^T f(\xi(t))dt + g(y) \right\} \quad \text{for all } x \in S \tag{5.4}$$

where the infimum is taken over all $y \in \Gamma$ and over all the paths $\xi \in A((0, T), S)$ such that $\xi(0) = x$, $\xi(T) = y$. Moreover u is continuous, bounded and

$$|u(x)| \leq \max_\Gamma |g| + \max_S |f| d(x, \Gamma) \quad \text{for all } x \in S.$$

The Definition 5.1 also applies to the prefractal S^n. In the next propositions we establish the relation between sub and supersolutions in the sense of Definition 5.1 and of Definition 4.2 for the problem

$$|Du| = f(x) \quad \forall x \in \Omega := S^n \backslash \Gamma. \tag{5.5}$$

These results are expedient in order to show the convergence of the discrete problem.

Proposition 5.3 *A function $u \in USC(S^n)$ is a subsolution in the sense of Definition 5.1 to problem (5.5) if and only if it is a subsolution in the sense of Definition 4.2 to (5.5).*

Taking advantage of Remark 4.2, the proof can be easily done adapting to the case of S^n the argument in [3, Prop. 6.1] which shows the equivalence between metric viscosity subsolutions and standard viscosity subsolution in the Euclidean space. Therefore, we shall omit it.

Proposition 5.4 *For $v \in LSC(S^n)$, the following statement are equivalent:*

(i) *v is a supersolution in the sense of Definition 5.1 to problem (5.5),*
(ii) *v is a supersolution in the sense of Definition 4.2 to problem (5.5),*
(iii) *for each $x \in S^n$, there exists $\xi \in \mathcal{A}_x(\mathbb{R}, S^n)$, with $T^\pm := T_\Omega^\pm[\xi]$ both finite, such that the function $w(t) := v(\xi(t))$ satisfies*

$$|p| \geq f(\xi(t)) \quad \forall p \in D^- w(t), \, t \in (T^-, T^+).$$

Remark 5.2 Point (iii) coincides with the notion of supersolution in Definition 5.1 with $\varepsilon = 0$ and with $w := v \circ \xi$.

Proof (i) implies (ii). Assume by contradiction that v does not satisfy the definition of supersolution in Definition 4.2 at $x \in S^n$. We assume that x coincides with a vertex x_i otherwise we can proceed as in [3, Prop. 6.2]. Then there exists $j \in Inc_i$ such that for any $k \in K := Inc_i \backslash \{j\}$, there exists a (j, k)-admissible test function ϕ_k for u at x which satisfies $|D_j \phi_k(x)| \leq f(x) - 2\delta$ for some $\delta > 0$. By adding the term $d(x, y)^2$ to the test function it is not restrictive to assume that

$$v(x) = \phi_k(x), \quad (v - \phi_k)(y) \geq d(x, y)^2 \quad \forall \, y \in S^n.$$

By the regularity of ϕ_k we can find $r > 0$ such that

$$|D\phi_k(y)| \leq f(y) - \delta \quad \forall y \in B(x, 2r) \cap (e_j \cup e_k).$$

Define $\phi : \bigcup_{k \in Inc_i} e_k \to \mathbb{R}$ by

$$\phi(y) := \begin{cases} \max_{k \in K} \phi_k(y), & \text{if } y \in e_j, \\ \phi_k(y), & \text{if } y \in e_k, \, k \in K. \end{cases}$$

As in [16, Prop. 4.1], we have: ϕ is $C(\cup_{k \in Inc_i} e_k)$, it satisfies $\phi(x) = v(x)$ and (in viscosity and a.e. sense)

$$|D\phi(y)| \le f(y) - \delta \qquad \text{for all } y \in B(x, 2r).$$

For $\varepsilon \in (0, r^2/3)$, consider a couple $(\xi_\varepsilon, w_\varepsilon)$ as in the Definition 5.1 with $T^\pm = T^\pm_{B(x,r)}[\xi_\varepsilon]$. Let $\rho_\eta : \mathbb{R} \to \mathbb{R}$ be a mollifier with support in $[-\eta, \eta]$; set $\phi_{\eta,\varepsilon}(t) := (\phi(\xi_\varepsilon(\cdot)) * \rho_\eta)(t)$. Then, as $\eta \to 0$, $\phi_{\eta,\varepsilon}(t) \to \phi(\xi_\varepsilon(\cdot))$ uniformly in $[T^-, T^+]$. Let $t_{\eta,\varepsilon}$ be a minimum point for $w_\varepsilon - \phi_{\eta,\varepsilon}$ in $[T^-, T^+]$. Then for η sufficiently small in such a way that $\|\phi(\xi_\varepsilon(\cdot)) - \phi_{\eta,\varepsilon}(\cdot)\|_{\infty, [T^-, T^+]} \le \varepsilon$, we have

$$\begin{aligned}
0 = v(x) - \phi(x) = w_\varepsilon(0) - \phi(\xi_\varepsilon(0)) &\ge w_\varepsilon(0) - \phi_{\eta,\varepsilon}(0) - \varepsilon \\
&\ge w_\varepsilon(t_{\eta,\varepsilon}) - \phi_{\eta,\varepsilon}(t_{\eta,\varepsilon}) - \varepsilon \ge v(\xi_\varepsilon(t_{\eta,\varepsilon})) - \phi(\xi_\varepsilon(t_{\eta,\varepsilon})) - 3\varepsilon \\
&\ge d(\xi_\varepsilon(t_{\eta,\varepsilon}), x)^2 - 3\varepsilon.
\end{aligned}$$

By our choice of ε, ξ_ε belongs to $B(x, r)$ for $t \in [0, t_{\eta,\varepsilon}]$; hence, $t_{\eta,\varepsilon}$ belongs to (T^-, T^+) and $\phi_{\eta,\varepsilon}$ is a test function for w_ε at $t_{\eta,\varepsilon}$. It follows that

$$\begin{aligned}
f(\xi_\varepsilon(t_{\eta,\varepsilon})) - \varepsilon &\le |D\phi_{\eta,\varepsilon}(t_{\eta,\varepsilon})| = |D(\phi(\xi_\varepsilon(\cdot) * \rho_\eta)(t_{\eta,\varepsilon})| \\
&\le \int_{(t_{\eta,\varepsilon} - \eta, t_{\eta,\varepsilon} + \eta)} |D\phi(\xi_\varepsilon(s))| |\dot{\xi}_\varepsilon(s)| \rho_\eta(t_{\eta,\varepsilon} - s) ds \\
&\le \int_{(t_{\eta,\varepsilon} - \eta, t_{\eta,\varepsilon} + \eta)} f(\xi_\varepsilon(s)) \rho_\eta(t_{\eta,\varepsilon} - s) ds - \delta \\
&\le f(\xi_\varepsilon(t_{\eta,\varepsilon})) + \omega_f(\eta) - \delta
\end{aligned}$$

which gives a contradiction for ε and η small.

(ii) implies (iii). Let us now prove that if v is a supersolution in the sense of Definition 4.2, then it is a supersolution in the sense of Definition 5.1. To this end, fix a point $x \in S^n$ and introduce a map $\xi : \mathbb{R} \to S^n$ as follows. Assume that $x \in e_j$ with $x = \pi_j(y_x)$, $y_x \in [0, l_j]$. Set $\xi(t) := \pi_j(y_x + t)$ for $t \in [-y_x, l_j - y_x]$; in particular we have $\xi(0) = x$. Let us now proceed to introduce ξ for positive t; the case of negative t is similar and we shall omit it. The point $\xi(l_j - y_x)$ is an endpoint of e_j; hence it belongs either to Γ or to V_0^n. In the former case we obtain T^+. In the latter case, by the definition of supersolution in Definition 4.2 there exists a feasible edge for e_j, say e_k; wlog assume $\pi_k(0) = \xi(l_j - y_x)$. We define $\xi(t) := \pi_k(t - l_j + y_x)$ for $t \in (l_j - y_x, l_k + l_j - y_x)$. We iterate this idea of following an edge and then choosing the corresponding feasible edge when arriving at a vertex of V_0^n. Assume now that the initial point x belongs to V_0^n, say $x = v_i$. In this case we choose arbitrarily an edge e_j, incident to v_i and the corresponding feasible edge in v_i, say e_k. Wlog we assume $x = \pi_j(0) = \pi_k(0)$. We introduce $\xi(t) = \pi_j(t)$ for $t \in [0, l_j]$ and $\xi(t) = \pi_k(-t)$ for $t \in [-l_k, 0)$ and we iterate the same idea as before. We claim that

$$T^\pm := T^\pm_{S^n}[\xi] \text{ are both finite.} \tag{5.6}$$

In order to prove this property, we proceed by contradiction assuming that $\xi(t) \notin \Gamma$ for every $t \in \mathbb{R}$. Since S^n is given by a finite collection of edges of finite length, this may happen only if ξ contains a loop, say $L := e_1 \cup \cdots \cup e_k$. Being lower

semicontinuous, the function v attains its minimum with respect to L at some point $\bar{x}$. Since the constant function $\phi = \min_L v$ is an admissible test function for v at $\bar{x}$ we obtain a contradiction with the definition of supersolution in Definition 4.2; hence our claim (5.6) is completely proved.

Finally, by the Definition 4.2 and Remark 4.2, one can easy check that for $w(t) := v(\xi(t))$ $\forall t \in [T^-, T^+]$, there holds: $|p| \geq f(\xi(t))$ for every $p \in D^- w(t)$, $t \in (T^-, T^+)$.

(iii) implies (i). By Remark 5.2, this statement is a straightforward consequence of the definition of supersolution in Definition 5.1. Hence, the proof is accomplished. $\qquad\square$

A natural question in the previous approach is if we need to consider all the paths on the Sierpinski gasket or just the ones that visit a finite number of vertices. We say that a path $\xi : I \to S$ is *piecewise differentiable* when it is absolutely continuous and there exist $n \in \mathbb{N}$, $t_0 < t_1 < \cdots < t_m$ and $j_1, \ldots, j_m \in J^n$ such that there holds: $I = (t_0, t_m)$ and $\xi(t)$ belongs to e_{j_k} for $t \in (t_{k-1}, t_k)$. In other words, a piecewise differentiable path is an absolutely continuous path in some S^n and it changes edge only a finite number of times.

Definition 5.2 We say that a function u is a *mild* subsolution (resp., supersolution) of (5.1) if it verifies the definition of subsolution (resp., supersolution) in Definition 5.1 by considering only piecewise differentiable paths.

Proposition 5.5 *Assume (5.2). Then, the unique mild solution to (5.1)–(5.3) is given by*

$$v(x) = \inf \left\{ \int_0^T f(\xi(t))dt + g(y) \right\} \quad \text{for all } x \in S$$

where the infimum is taken over all $y \in \Gamma$ and over all the piecewise differentiable paths $\xi \in \mathcal{A}((0, T), S)$ such that $\xi(0) = x$, $\xi(T) = y$. Moreover v coincides with the solution u given by (5.4).

Proof Following exactly the same arguments in the proof of [3, Thm 3.1], one can easily check that Proposition 5.1 still holds for mild super- and subsolutions. Similarly, repeating the arguments of the proof of [3, Thm 3.2], one can also obtain that the function v is a mild supersolution.

Finally, we observe that a subsolution (respectively, a supersolution) in the sense of Definition 5.1 is also a mild subsolution (resp., mild supersolution). Hence we deduce that the function u defined in (5.4) is a mild solution and the comparison principle yields that $u = v$. $\qquad\square$

Remark 5.3 Observe that by (5.4), the distance from the boundary

$$d(x) := \inf\{d(x, y) : y \in \Gamma\} \quad x \in S \tag{5.7}$$

is the unique solution of

$$|Du(x)| = 1 \quad x \in S, \quad u = 0 \quad x \in \Gamma.$$

Theorem 5.1 *Assume* (5.2) *and let* u_n *be the viscosity solution of* (4.3)–(4.5)*. Then, as* $n \to +\infty$*,* u_n *tends to* u *uniformly on each prefractal* S^m *where* u *is the viscosity solution of* (5.1)–(5.3)*.*

Proof First observe that if condition (5.2) is satisfied, then, for any n, condition (4.4) is satisfied as well, hence for any n there exists a solution to (4.3)–(4.5). By Proposition 4.3, we already know that u_n converges to a function $\bar{u}$ on S. We want to prove that $u = \bar{u}$.

In order to show that $\bar{u}$ is a supersolution at $x \in S$, given $\varepsilon > 0$, let $n \in \mathbb{N}$ be such that $x \in S^n$ and $0 \le u_n - \bar{u} \le \varepsilon$. Let $\xi \in \mathcal{A}_x(\mathbb{R}, S^n)$ and $w \in \mathrm{LSC}(\mathbb{R})$ be an ε-admissible couple for u_n as in Definition 5.1 with $\mathcal{X} = S^n$. Set $\bar{w} = w + \bar{u}(x) - u_n(x)$. Then $\bar{u}(x) = \bar{w}(0)$ and

$$\bar{u}(\xi(t)) \le u_n(\xi(t)) + \varepsilon \le w(\xi(t)) + 2\varepsilon \le \bar{w}(\xi(t)) + 3\varepsilon.$$

Moreover $|p| \ge f(\xi(t)) - \varepsilon$ for all $p \in D^- w(\xi(t)) = D^- \bar{w}(\xi(t))$. It follows that $(\xi, \bar{w})$ is an ε-admissible couple for $\bar{u}$ in the sense of Definition 5.1.

To show that $\bar{u}$ is a subsolution, by Proposition 5.5, it suffices to show that it is a mild subsolution. To this end, given $x \in S$, consider $\xi \in \mathcal{A}_x(\mathbb{R}, S)$ and $\bar{n} \in \mathbb{N}$ such that $\xi \in \mathcal{A}_x(\mathbb{R}, S^{\bar{n}})$. Set $w(t) := \bar{u}(\xi(t))$ and let $\phi \in C^1(\mathbb{R})$ be supertangent to w in 0. Since $u_n \to u$ uniformly, then $w_n(t) := u_n(\xi(t))$ converges to w uniformly. Hence there exists $t_n \to 0$ such that ϕ is supertangent to w_n at t_n. Moreover $\xi \in \mathcal{A}_x(\mathbb{R}, S^n)$ for $n > \bar{n}$ and therefore $|D\phi(t_n)| \le f(\xi(t_n))$. For $n \to \infty$, since $\xi(t_n) \to \xi(0)$ it follows that $|D\phi(0)| \le f(x)$. $\qquad\square$

As an immediate consequence of Proposition 4.4 and Theorem 5.1, we get the convergence of discrete problems to the continuous problem on S.

Theorem 5.2 *Assume* (5.2) *and* (3.3) *for any* $n \in \mathbb{N}$*. Then, as* $n \to +\infty$*, the sequence* $\{u_{h_n}\}_n$ *of the solutions of* (3.2)–(3.4) *converges uniformly to the solution* u *of* (5.1)–(5.3)*.*

Remark 5.4 Observe that (5.2) does not imply (3.3) except in some particular case (e.g., g constant on Γ). Nevertheless we can always modify f in such a way that (5.2) implies (3.3). Indeed, under condition (5.2), let us show that (3.3) is satisfied with f replaced by $f_n(x) := f(x) + \omega_f(h_n)$. For any $x, y \in \Gamma$ and any path $\{x_0 = x, x_1, \ldots, x_N = y\} \subset V^n$ with $x_i \sim x_{i+1}$, $i = 0, \ldots, N-1$, define an admissible continuous path $\xi : [0, Nh_n] \to S^n$ such $\xi(ih_n) = x_i$, $i = 0, \ldots, N-1$. Then, by (5.2), there holds

$$g(x) \le \int_0^{Nh} f(\xi(t))dt + g(y) \le \sum_{i=0}^{N-1} \int_{ih}^{(i+1)h} f(\xi(t))dt + g(y)$$

$$\le \sum_{i=0}^{N-1} h_n(f(x_i) + \omega_f(h_n)) + g(y) = \sum_{i=0}^{N-1} h_n f_n(x_i) + g(y).$$

Therefore, f_n satisfies (3.3). Moreover, since $\{f_n\}_n$ is decreasing, by the same arguments as those of previous sections, one can prove that the solution u_{h_n} to (3.2)–(3.4) with f replaced by f_n converges to the solution of (5.1)–(5.3).

Corollary 5.1 *Let δ_n, d_n and d be the distance function from Γ respectively on S^n, V^n and S, defined in (4.9), (3.7) and (5.7). Then, as $n \to +\infty$, both δ_n and d_n converge to d.*

Proof By Remark 3.1 (respectively, Remarks 4.3 and 5.3), the function d_n (resp., δ_n and d) is the solution of the eikonal equation with $f \equiv 1$ in S^n (resp., V^n and S) and $g = 0$ on Γ. Invoking Theorem 5.1 (resp., Theorem 5.2), we infer that δ_n (resp., d_n) converge to d as $n \to +\infty$.   □

6 Extensions

In this section we discuss some possible extensions of the previous results.

More general Hamiltonians For the sake of clarity, the present paper only concerns the eikonal equation; it is our purpose to extend these results to more general Hamiltonians in a future work. For instance, we observe that the results on prefractals can be extended to an Hamiltonian $H(x, p)$ convex and coercive with respect to p, obtaining the uniform convergence of the solutions of (3.2)–(3.4) to a function u defined on the Sierpinski gasket. The definition introduced in [3] for the sole eikonal equation can possibly be extended to a class of Hamiltonians of the type $H(x, |p|)$.

Boundary conditions Following the classical definition of Laplacian on the Sierpinski gasket, we imposed the boundary conditions on Γ. Nevertheless, by easy modifications, it is possible to consider as boundary set any finite subset of V. An interesting case considered in [6,17] is when the boundary reduces to one point of Γ and $g = 0$ at this point. The solutions of the (3.2), (4.3) and (5.1) represent respectively the vertex distance on S^n, the path distance on S^n and the path distance on S from the given boundary vertex. More generally the problem (5.1)–(5.3) can be considered in a connected subdomain Ω of S imposing the boundary condition on any finite subset of vertices contained in Ω.

More general fractals The interior approximation method can be extended to some classes of post-critically finite self similar sets (generally speaking, these sets are obtained by subdividing the initial cell into cells of smaller and smaller size and the cells must intersect at isolated points; see [10,18] for the precise definition). We will consider the problem in a future work.

Resistance metric Given the energy form associated to the Laplacian on the Sierpinski gasket, it is possible to introduce a metric, called resistance metric, which plays an important role in several estimates (see [10,18]). While in the Euclidean setting resistance metric and pathwise distance coincide, this is not true on S. On the other hand the Sierpinski gasket can be embedded in $\mathbb{R}^2$ by certain harmonic maps whose image is now called the harmonic Sierpinski gasket and on this set the two distances

coincide (see [5,12]). We aim to consider a characterization of the resistance metric via the eikonal equation in a future work.

References

1. Bardi, M., Capuzzo Dolcetta, I.: Optimal control and viscosity solutions of Hamilton–Jacobi–Bellman equations. Birkäuser, Boston (1997)
2. Camilli, F., Marchi, C.: A comparison among various notions of viscosity solution for Hamilton–Jacobi equations on networks. J. Math. Anal. Appl. **407**(1), 112–118 (2013)
3. Giga, Y., Hamamuki, N., Nakayasu, A.: Eikonal equations in metric spaces. Trans. Am. Math. Soc. **367**(1), 49–66 (2015)
4. Goldstein, S.: Random walks and diffusions on fractals. In: Kesten, H. (ed.) Percolation theory and ergodic theory of infinite particle systems (Minneapolis, Minn., 1984–1985), IMA Vol. Math. Appl., 8, pp. 121–129. Springer, New York (1987)
5. Hino, M.: Geodesic distances and intrinsic distances on some fractal sets. Publ. Res. Inst. Math. Sci. **50**(2), 181–205 (2014)
6. Hinz, A.M., Schief, A.: The average distance on the Sierpinski gasket. Probab. Theory Relat. Fields **87**(1), 129–138 (1990)
7. Ishii, H.: A simple, direct proof of uniqueness for solutions of the Hamilton-Jacobi equations of eikonal type. Proc. Am. Math. Soc. **100**(2), 247–251 (1987)
8. Ishii, H., Koike, S.: Remarks on elliptic singular perturbation problems. Appl. Math. Optim. **23**(1), 1–15 (1991)
9. Kusuoka, S.: A diffusion process on a fractal. In: Itô, K., Ikeda, N. (eds.) Probabilistic methods in mathematical physics (Katata/Kyoto, 1985), pp 251–274. Academic Press, Boston (1987)
10. Kigami, J.: Analysis on fractals. Cambridge Tracts in Mathematics. Cambridge University Press, Cambridge (2001)
11. Kigami, J.: Hausdorff dimensions of self-similar sets and shortest path metric. J. Math. Soc. Jpn. **47**(3), 381–404 (1995)
12. Koskela, P., Zhou, Y.: Geometry and analysis of Dirichlet forms. Adv. Math. **231**(5), 2755–2801 (2012)
13. Lindstrøm, T.: Brownian motion on nested fractals, Memoirs of the American Mathematical Society, vol 83, No 420 (1990)
14. Manfredi, J., Oberman, A., Sviridov, A.: Nonlinear elliptic PDEs on graphs. Differ. Integral Equ. **28**(1–2), 79–102 (2015)
15. Mosco, U.: Energy functionals on certain fractal structures. J. Convex Anal. **9**(2), 581–600 (2002)
16. Schieborn, D., Camilli, F.: Viscosity solutions of Eikonal equations on topological network. Calc. Var. Partial Differ. Equ. **46**(3), 671–686 (2013)
17. Strichartz, R.: Isoperimetric estimates on Sierpinski gasket type fractals. Trans. Am. Math. Soc. **351**(5), 1705–1752 (1999)
18. Strichartz, R.: Differential equations on fractals, a tutorial. Princeton University Press, Princeton (2006)