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# An algorithm for the Baker-Campbell-Hausdorff formula

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Abstract: A simple algorithm, which exploits the associativity of the BCH formula, and that can be generalized by iteration, extends the remarkable simplification of the Baker-Campbell-Hausdorff (BCH) formula, recently derived by Van-Brunt and Visser. We show that if $[X,Y]=uX+vY+cI$, $[Y,Z]=wY+zZ+dI$, and, consistently with the Jacobi identity, $[X,Z]=mX+nY+pZ+eI$, then

$$\exp(X)\exp(Y)\exp(Z)=\exp(aX+bY+cZ+dI)$$

where $a$, $b$, $c$ and $d$ are solutions of four equations. In particular, the Van-Brunt and Visser formula

$$\exp(X)\exp(Z)=\exp(aX+bZ+c[X,Z]+dI)$$

extends to cases when $[X,Z]$ contains also elements different from $X$ and $Z$. Such a closed form of the BCH formula may have interesting applications both in mathematics and physics. As an application, we provide the closed form of the BCH formula in the case of the exponentiation of the Virasoro algebra, with $\mathrm{SL}_2(\mathbb{C})$ following as a subcase. We also determine three-dimensional subalgebras of the Virasoro algebra satisfying the Van-Brunt and Visser condition. It turns out that the exponential form of $\mathrm{SL}_2(\mathbb{C})$ has a nice representation in terms of its eigenvalues and of the fixed points of the corresponding Möbius transformation. This may have applications in Uniformization theory and Conformal Field Theories.

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## Contents





## 1 The algorithm

Very recently Van-Brunt and Visser [1] (see also [2] for related issues) found a remarkable relation that simplifies, in important cases, the Baker-Campbell-Hausdorff (BCH) formula. Namely, if $X$ and $Y$ are elements of a Lie algebra with commutator

$$[X, Y] = uX + vY + cI\,, \tag{1.1}$$

with $I$ a central element and $u$, $v$, $c$, complex parameters, then [1]

$$\exp(X)\exp(Y) = \exp(X + Y + f(u, v)[X, Y])\,, \tag{1.2}$$

where

$$f(u, v) = f(v, u) = \frac{(u - v)e^{u+v} - (ue^u - ve^v)}{uv(e^u - e^v)}\;. \tag{1.3}$$

In the following we formulate an algorithm which exploits the associativity of the BCH formula. Set $\alpha + \beta = 1$ and consider the identity

$$\exp(X)\exp(Y)\exp(Z) = \exp(X)\exp(\alpha Y)\exp(\beta Y)\exp(Z)\;. \tag{1.4}$$

If

$$[X, Y] = uX + vY + cI\,, \qquad [Y, Z] = wY + zZ + dI\,, \tag{1.5}$$

then, by eq. (1.2),

$$\exp(X)\exp(\alpha Y) = \exp(\tilde{X})\,, \qquad \exp(\beta Y)\exp(Z) = \exp(\tilde{Y})\,, \tag{1.6}$$

where

$$\begin{aligned}\tilde{X} &:= g_\alpha(u, v)X + h_\alpha(u, v)Y + l_\alpha(u, v)cI\,,\\ \tilde{Y} &:= h_\beta(z, w)Y + g_\beta(z, w)Z + l_\beta(z, w)dI\,,\end{aligned} \tag{1.7}$$

with $g_\alpha(u, v) := 1 + \alpha u f(\alpha u, v)$, $h_\alpha(u, v) := \alpha(1 + v f(\alpha u, v))$ and $l_\alpha(u, v) := \alpha f(\alpha u, v)$. Imposing

$$[\tilde{X}, \tilde{Y}] = \tilde{u}\tilde{X} + \tilde{v}\tilde{Y} + \tilde{c}I\,, \tag{1.8}$$

fixes $\alpha$, $\tilde{u}$, $\tilde{v}$ and $\tilde{c}$. This solves the BCH problem, since, by (1.1), (1.2), (1.4) and (1.6)

$$\exp(X)\exp(Y)\exp(Z) = \exp(\tilde{X})\exp(\tilde{Y}) = \exp(\tilde{X} + \tilde{Y} + f(\tilde{u},\tilde{v})[\tilde{X},\tilde{Y}]) \ . \tag{1.9}$$

Note that the commutator between $X$ and $Z$ may contain also $Y$

$$[X, Z] = mX + nY + pZ + eI \ . \tag{1.10}$$

This is consistent with the Jacobi identity

$$[X,[Y,Z]] + [Y,[Z,X]] + [Z,[X,Y]] = 0 \, , \tag{1.11}$$

that constrains $e, m, n$ and $p$ by a linear system. Setting $Y = \lambda_0 Q$ and $\lambda_- := \lambda_0\alpha$, $\lambda_+ := \lambda_0\beta$, eq. (1.4) includes, as a particular case,

$$\exp(X)\exp(Z) = \lim_{\lambda_0\to 0} \exp(X)\exp(\lambda_- Q)\exp(\lambda_+ Q)\exp(Z) \ . \tag{1.12}$$

This explicitly shows that the algorithm solves the BCH problem for $\exp(X)\exp(Z)$ in some of the cases when $[X, Z]$ includes elements of the algebra different from $X$ and $Z$.

The complete classification of the commutator algebras, leading to the closed form (1.9) of the BCH formula, is investigated in [3]. The algorithm has been applied to the case of semisimple complex Lie algebras in [4].

In the next section we implement the above algorithm. In particular, we write down the linear system coming from the Jacobi identity and then find the explicit expression for $\tilde{c}, \tilde{u}$ and $\tilde{v}$. We also find the equation, which is the key result of the algorithm, satisfied by $\alpha$. In section 3, as an application, we consider the exponentiation of the Virasoro algebra and derive the solution of the corresponding BCH problem. This includes, as a particular case, the closed form of the BCH formula for $\mathrm{SL}_2(\mathbb{C})$. We also determine three-dimensional subalgebras of the Virasoro algebra satisfying the Van-Brunt and Visser condition (1.1). In the last section we apply the algorithm to find the exponential form of $\mathrm{SL}_2(\mathbb{C})$ matrices. Furthermore, we reproduce the same results, using an alternative method, and extending them to $\mathrm{GL}_2(\mathbb{C})$ matrices. In this respect, it seems that in the literature, similar expressions for $\gamma \in \mathrm{SL}_2(\mathbb{C})$ are usually given only separately for three distinguished cases, depending if $\gamma_{11}^2 + \gamma_{12}\gamma_{21}$ is negative, vanishing or positive. It turns out that the expression of $X$ in $\gamma = \exp(X)$ has a geometrical representation which is not directly evident in $\gamma$. Namely, it turns out that $X$ can be nicely expressed in terms of its eigenvalues and of the fixed points $z_\pm$, solutions of the equation $z = \gamma z$, where $\gamma z$ is the Möbius transformation

$$\gamma z := \frac{\gamma_{11}z + \gamma_{12}}{\gamma_{21}z + \gamma_{22}} \ . \tag{1.13}$$

Such a geometrical representation of $\gamma$ may be of interest in the framework of Uniformization theory and Conformal Field Theories.



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## 2 Implementation of the algorithm

Let us write down the linear system that follows by the Jacobi identity (1.11)

$$\begin{aligned} uw + mz &= 0\,, \\ vm - wp + n(z-u) &= 0\,, \\ pu + zv &= 0\,, \\ c(w+m) + e(z-u) - d(p+v) &= 0\;. \end{aligned} \tag{2.1}$$

Replacing $\tilde{X}$ and $\tilde{Y}$ on the right hand side of (1.8) by their expressions in terms of $X$, $Y$ and $I$, and comparing the result with the direct computation, by (1.5), (1.7) and (1.10), of $[\tilde{X}, \tilde{Y}]$, yields

$$\begin{aligned} &\tilde{c} = (h_\beta(z,w) - g_\beta(z,w) l_\alpha(u,v) m)c + (h_\alpha(u,v) - g_\alpha(u,v) l_\beta(z,w) p)d + g_\alpha(u,v) g_\beta(z,w) e\,, \\ &\tilde{u} = h_\beta(z,w) u + g_\beta(z,w) m\,, \\ &\tilde{v} = g_\alpha(u,v) p + h_\alpha(u,v) z\,, \\ &\tilde{u} h_\alpha(u,v) + \tilde{v} h_\beta(z,w) = g_\alpha(u,v) h_\beta(z,w) v + g_\alpha(u,v) g_\beta(z,w) n + h_\alpha(u,v) g_\beta(z,w) w\;. \end{aligned} \tag{2.2}$$

The first three equations fix $\tilde{c}$, $\tilde{u}$ and $\tilde{v}$ in terms of $\alpha = 1 - \beta$. Replacing the expressions of $\tilde{u}$ and $\tilde{v}$ in the fourth equation provides the following equation for $\alpha$

$$h_\alpha(u,v)[h_\beta(z,w)(u+z) + g_\beta(z,w)(m-w)] + g_\alpha(u,v)[h_\beta(z,w)(p-v) - g_\beta(z,w) n] = 0\;. \tag{2.3}$$

This is the basic equation of the algorithm and is further investigated, together with the linear system (2.1), in [3]. Note that

$$\begin{aligned} g_\alpha(u,v) &= \frac{v - \alpha u}{v} \frac{e^{\alpha u/2} \sinh(v/2)}{\sinh[(v-\alpha u)/2]}\,, \\ h_\alpha(u,v) &= \frac{v - \alpha u}{u} \frac{e^{v/2} \sinh(\alpha u/2)}{\sinh[(v-\alpha u)/2]}\;. \end{aligned} \tag{2.4}$$

We then obtained a closed form of the BCH formula for cases in which the commutator contains other elements than the ones in the commutator. The above algorithm can be extended to more general cases, e.g. by considering decompositions like the one in (1.4) for

$$\exp(X_1) \cdots \exp(X_n)\;. \tag{2.5}$$

## 3 Exponentiating the Virasoro algebra

In this section we apply the algorithm, leading to closed forms of the BCH formula, in the case of the exponentiation of the Virasoro algebra

$$[\mathcal{L}_j, \mathcal{L}_k] = (k-j)\mathcal{L}_{j+k} + \frac{c}{12}(k^3 - k)\delta_{j+k,0} I\,, \tag{3.1}$$

$j, k \in \mathbb{Z}$. In particular, we find the closed form for $W$ in

$$\exp(X)\exp(Y)\exp(Z) = \exp(W)\,, \tag{3.2}$$

where

$$X := \lambda_{-k}\mathcal{L}_{-k}\,, \qquad Y := \lambda_0\mathcal{L}_0\,, \qquad Z := \lambda_k\mathcal{L}_k\;. \tag{3.3}$$

This is particularly interesting because we do not know alternative ways to get it. The case of $\mathrm{SL}_2(\mathbb{C})$ follows straightforwardly since the $\mathrm{sl}_2(\mathbb{R})$ algebra

$$[L_j, L_k] = (k-j)L_{j+k}\,, \tag{3.4}$$

$j, k = -1, 0, 1$, is a subalgebra of (3.1). Note that setting $E_- = L_1$, $H = -2L_0$ and $E_+ = -L_{-1}$, reproduces the other standard representation of the $\mathrm{sl}_2(\mathbb{R})$ algebra $[H, E_+] = 2E_+$, $[E_+, E_-] = H$ and $[H, E_-] = -2E_-$. Note that,

$$[X, Y] = k\lambda_0 X\,, \qquad [Y, Z] = k\lambda_0 Z\,, \qquad [X, Z] = \lambda_{-k}\lambda_k\left[\frac{2k}{\lambda_0}Y + \frac{c}{12}(k^3 - k)\right], \tag{3.5}$$

where, besides $c = d = v = w = 0$, we have, consistently with the Jacobi identity, $m = p = 0$. The other commutator parameters are

$$\begin{aligned} u &= z = k\lambda_0\,, \\ n &= \lambda_{-k}\lambda_k\frac{2k}{\lambda_0}\,, \\ e &= \lambda_{-k}\lambda_k\frac{c}{12}(k^3 - k)\;. \end{aligned} \tag{3.6}$$

By (2.3) it follows that $\alpha$ satisfies the equation

$$ng_\alpha(u, 0)g_\beta(u, 0) = 2uh_\alpha(u, 0)h_\beta(u, 0)\,, \tag{3.7}$$

so that, using

$$h_\alpha(u, 0) = \alpha\,, \qquad g_\alpha(u, 0) = \frac{\alpha u}{1 - e^{-\alpha u}}\,, \tag{3.8}$$

and recalling that $\lambda_- := \lambda_0\alpha$, $\lambda_+ := \lambda_0\beta$, one gets

$$e^{-k\lambda_\pm} = \frac{1 + e^{-k\lambda_0} - k^2\lambda_{-k}\lambda_k \pm \sqrt{(1 + e^{-k\lambda_0} - k^2\lambda_{-k}\lambda_k)^2 - 4e^{-k\lambda_0}}}{2}\;. \tag{3.9}$$

Next, observe that, by (2.2), $\tilde{u} = k\lambda_+$, $\tilde{v} = k\lambda_-$, and $c_k \equiv \tilde{c} = eg_\alpha(k\lambda_0, 0)g_\beta(k\lambda_0, 0)$, that is

$$c_k = \frac{\lambda_-\lambda_{-k}}{1 - e^{-k\lambda_-}}\frac{\lambda_+\lambda_k}{1 - e^{-k\lambda_+}}\frac{c}{12}(k^5 - k^3)\,, \tag{3.10}$$

that, by (3.9), is equivalent to

$$c_k = \frac{\lambda_{-k}\lambda_k}{\lambda_+ - \lambda_-}\left(\frac{\lambda_+}{1 - e^{-k\lambda_+}} - \frac{\lambda_-}{1 - e^{-k\lambda_-}}\right)\frac{c}{12}(k^4 - k^2)\;. \tag{3.11}$$



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Finally, eq. (1.9) yields

$$\exp(\lambda_{-k}\mathcal{L}_{-k})\exp(\lambda_0\mathcal{L}_0)\exp(\lambda_k\mathcal{L}_k) = \exp\left\{\frac{\lambda_+ - \lambda_-}{e^{-k\lambda_-} - e^{-k\lambda_+}}\left[k\lambda_{-k}\mathcal{L}_{-k} + \left(2 - e^{-k\lambda_+} - e^{-k\lambda_-}\right)\mathcal{L}_0 + k\lambda_k\mathcal{L}_k + c_k I\right]\right\} . \quad (3.12)$$

In the case $\lambda_0 = 0$ we have

$$\exp(\lambda_{-k}\mathcal{L}_{-k})\exp(\lambda_k\mathcal{L}_k) = \exp\left[\frac{\lambda_+}{\sinh(k\lambda_+)}(k\lambda_{-k}\mathcal{L}_{-k} + k^2\lambda_{-k}\lambda_k\mathcal{L}_0 + k\lambda_k\mathcal{L}_k + c_k I)\right] , \quad (3.13)$$

with

$$c_k = \lambda_{-k}\lambda_k\frac{c}{24}(k^4 - k^2) . \quad (3.14)$$

Let us report the relevant case corresponding to $\mathrm{SL}_2(\mathbb{C})$, obtained by setting $k = 1$ in the above formulas. We have

$$\exp(\lambda_{-1}L_{-1})\exp(\lambda_0 L_0)\exp(\lambda_1 L_1) = \exp\left\{\frac{\lambda_+ - \lambda_-}{e^{-\lambda_-} - e^{-\lambda_+}}\left[\lambda_{-1}L_{-1} + \left(2 - e^{-\lambda_+} - e^{-\lambda_-}\right)L_0 + \lambda_1 L_1\right]\right\} , \quad (3.15)$$

and

$$\exp(\lambda_{-1}L_{-1})\exp(\lambda_1 L_1) = \exp\left[\frac{\lambda_+}{\sinh(\lambda_+)}(\lambda_{-1}L_{-1} + \lambda_{-1}\lambda_1 L_0 + \lambda_1 L_1)\right] . \quad (3.16)$$

Interestingly, the algorithm for the BCH formula may be extended to any three-dimensional subalgebras of the Virasoro algebra. In this respect, it can be easily seen that the highest finite dimensional subalgebras of the Virasoro algebra are the four-dimensional ones generated by $\mathcal{L}_{-n}$, $\mathcal{L}_0$, $\mathcal{L}_n$ and the central element $I$, for all $n \in \mathbb{Z}\backslash\{0\}$. Above we solved the BCH problem for all such subalgebras. Since all the remanent non-trivial Virasoro subalgebras are either three-dimensional, each one containing $I$, or two-dimensional, it follows that all of them satisfy the condition (1.1) and therefore the corresponding BCH problem of finding $Z$ such that $\exp(X)\exp(Y) = \exp(Z)$ is easily solved by (1.2). In this respect, note that it can be easily seen that the two dimensional subalgebras are generated by $\mathcal{L}_n$ and $\mathcal{L}_0$, for all $n \in \mathbb{Z}\backslash\{0\}$. A two-parameter family of three-dimensional subalgebras of the Virasoro algebra, is the one where each subalgebra is generated by

$$X_n(\delta,\epsilon) := \delta\mathcal{L}_0 + \epsilon\mathcal{L}_n , \qquad X_{-n}(\epsilon,\delta) , \qquad I , \quad (3.17)$$

whose commutator is

$$[X_n(\delta,\epsilon), X_{-n}(\epsilon,\delta)] = -n\epsilon X_n(\delta,\epsilon) - n\delta X_{-n}(\epsilon,\delta) + \delta\epsilon\frac{c}{12}(n^3 - n)I . \quad (3.18)$$

Other three-dimensional subalgebras of the Virasoro algebra are the one-parameter family of subalgebras, each one generated by

$$X_n(\alpha) := \mathcal{L}_{2n} + \alpha\mathcal{L}_n + \frac{2}{9}\alpha^2\mathcal{L}_0 , \qquad Y_{-n}(\alpha) := \mathcal{L}_{-n} + \frac{3}{\alpha}\mathcal{L}_0 , \qquad I , \quad (3.19)$$

whose commutator is

$$[X_n(\alpha), Y_{-n}(\alpha)] = -6\frac{n}{\alpha}X_n(\alpha) - \frac{2}{9}n\alpha^2 Y_{-n}(\alpha) + \alpha\frac{c}{12}(n^3 - n)I . \quad (3.20)$$



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## 4 Geometrical constructions

Note that, in the case of $\mathrm{SL}_2(\mathbb{R})$, therefore for real $\lambda_k$'s, depending on the values of the $\lambda_k$'s, the factor $\frac{\lambda_+-\lambda_-}{e^{-\lambda_-}-e^{-\lambda_+}}$ in (3.15) may take complex values. This is in agreement with the non-surjectivity of the exponential map for $\mathrm{sl}_2(\mathbb{R})$ into $\mathrm{SL}_2(\mathbb{R})$. In particular, exponentiating $\mathrm{sl}_2(\mathbb{R})$ cannot give $\mathrm{SL}_2(\mathbb{R})$ matrices whose trace is less than $-2$. The case with trace $-2$ and non-diagonalizable matrices is critical, both for $\mathrm{SL}_2(\mathbb{R})$ and $\mathrm{SL}_2(\mathbb{C})$.

Let us express eq. (3.15) in terms of the associated $\mathrm{SL}_2(\mathbb{C})$ matrix

$$\gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} . \tag{4.1}$$

Replace then the $L_k$'s in the left hand side of (3.15) by their matrix representation

$$L_{-1} = \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix} , \qquad L_0 = \begin{pmatrix} -\frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix} , \qquad L_1 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} , \tag{4.2}$$

using

$$\exp(L_{-1}) = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} , \quad \exp(L_0) = \begin{pmatrix} e^{-\frac{1}{2}} & 0 \\ 0 & e^{\frac{1}{2}} \end{pmatrix} , \quad \exp(L_1) = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} . \tag{4.3}$$

Comparing the result with $\gamma$ yields

$$A = e^{\lambda_0/2}(e^{-\lambda_0} - \lambda_{-1}\lambda_1) , \qquad B = -\lambda_{-1}e^{\lambda_0/2} , \qquad C = \lambda_1 e^{\lambda_0/2} , \qquad D = e^{\lambda_0/2} , \tag{4.4}$$

so that

$$e^{-\lambda_\pm} = \frac{t \pm \sqrt{t^2-1}}{D} , \tag{4.5}$$

where $t := \frac{1}{2}\,\mathrm{tr}\,\gamma$. Since the eigenvalues of $\gamma$, solutions of the characteristic polynomial $\nu^2 - 2t\nu + 1 = 0$, are $\nu_\pm = t \pm \sqrt{t^2-1}$, we have

$$e^{-\lambda_\pm} = \frac{\nu_\pm}{D} , \tag{4.6}$$

so that

$$\exp(\lambda_{-1}L_{-1}) \exp(\lambda_0 L_0) \exp(\lambda_1 L_1) = \exp\left\{ \frac{\ln(\nu_+/\nu_-)}{\nu_+ - \nu_-} [CL_1 + (D-A)L_0 - BL_{-1}] \right\} . \tag{4.7}$$

Note that when $\gamma$ is parabolic, that is for $|t| = 1$, and therefore $\nu_+ = \nu_-$, we have

$$\frac{\ln(\nu_+/\nu_-)}{\nu_+ - \nu_-} = t . \tag{4.8}$$

Also, note that when $\gamma$ is elliptic, that is for $|t| < 1$, one has $\nu_+ = \bar{\nu}_- = \rho e^{i\theta}$, so that

$$\frac{\ln(\nu_+/\nu_-)}{\nu_+ - \nu_-} = \frac{\theta}{\rho \sin\theta} . \tag{4.9}$$



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Eq. (4.7) implies

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \exp\left[\frac{\ln(\nu_+/\nu_-)}{\nu_+ - \nu_-}\begin{pmatrix} (A-D)/2 & B \\ C & (D-A)/2 \end{pmatrix}\right], \tag{4.10}$$

equivalently, using $\nu_+\nu_- = 1$,

$$\gamma = \exp\left[\frac{\ln(t+\sqrt{t^2-1})}{\sqrt{t^2-1}}(\gamma - tI_2)\right], \tag{4.11}$$

which is indefinitely iterable by replacing $\gamma$ on the right hand side, by its exponential form, that is by the expression on the right hand side itself.

The relation (4.11) can be derived in an alternative way. First note that $\gamma = \exp(X)$ does not uniquely fix $X$. For example, $\exp(X) = \exp(X)\exp(2\pi ikI_2) = \exp(X + 2\pi ikI_2)$, $k \in \mathbb{Z}$. However, $X$ can be consistently fixed to be traceless, so that, being $X^2$ proportional to $I_2$, one has $\gamma = \exp(X) = aX + tI_2$. Therefore $\gamma = \exp[a^{-1}(\gamma - tI_2)]$ for some $a$. For distinct eigenvalues the diagonalization of both sides (by the same matrix), reproduces (4.11), since it fixes

$$a = \frac{\nu_+ - \nu_-}{2\ln\nu_+}. \tag{4.12}$$

A particular case of (4.11) is when $D = A^{-1}$

$$\gamma = \exp\left[\ln A\begin{pmatrix} 1 & \frac{2B}{A-A^{-1}} \\ \frac{2C}{A-A^{-1}} & -1 \end{pmatrix}\right], \tag{4.13}$$

where, since $AD - BC = 1$, either $B = 0$ or $C = 0$. It follows that, when $D = A^{-1}$, with $A^2 = 1$, $\gamma$ admits exponentiation only if $A = 1$. For example, $\begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix}$ cannot be expressed as the exponential neither of $\mathrm{sl}_2(\mathbb{R})$ nor of $\mathrm{sl}_2(\mathbb{C})$. Of course, this is not a problem in the case such a matrix is seen as an element of $\mathrm{PSL}_2(\mathbb{R}) = \mathrm{SL}_2(\mathbb{R})/\{\pm\mathbb{I}\}$.

Let us derive a more geometrical representation of (4.11), useful, e.g., in the framework of Uniformization Theory and in Conformal Field Theories. Consider the Möbius transformation

$$\gamma z := \frac{Az+B}{Cz+D}, \tag{4.14}$$

and the solutions of the fixed point equation $\gamma z = z$

$$z_\pm = \frac{(A-D)/2 \pm \sqrt{t^2-1}}{C}. \tag{4.15}$$

If $B = 0$, then

$$\begin{pmatrix} A & 0 \\ C & D \end{pmatrix} = \exp\left[\frac{\ln(\nu_+/\nu_-)}{z_+ - z_-}\begin{pmatrix} z_+ + z_- & 0 \\ 1 & -z_+ - z_- \end{pmatrix}\right], \tag{4.16}$$

otherwise, in the case of Möbius transformations, one can fix $B = 1$ to get

$$\begin{pmatrix} A & 1 \\ C & D \end{pmatrix} = \exp\left[\frac{\ln(\nu_+/\nu_-)}{z_+ - z_-}\begin{pmatrix} z_+ + z_- & \frac{z_+ - z_-}{\nu_+ - \nu_-} \\ 1 & -z_+ - z_- \end{pmatrix}\right]. \tag{4.17}$$

We conclude by observing that our findings trivially extend to $\mathrm{GL}_2(\mathbb{C})$. In particular, multiplying both sides of (4.11) by $\sqrt{|\gamma|}I_2$, $|\gamma| := \det\gamma$, one gets for $\gamma \in \mathrm{GL}_2(\mathbb{C})$

$$\gamma = \exp\left[\frac{\ln(t + \sqrt{t^2 - |\gamma|})}{\sqrt{t^2 - |\gamma|}}(\gamma - tI_2) + \frac{1}{2}\ln(|\gamma|)I_2\right] . \tag{4.18}$$

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