Polynomial Meshes: Computation and Approximation

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Polynomial Meshes: Computation and Approximation

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Abstract

We present the software package WAM, written in Matlab, that generates Weakly Admissible Meshes and Discrete Extremal Sets of Fekete and Leja type, for 2d and 3d polynomial least squares and interpolation on compact sets with various geometries. Possible applications range from data fitting to high-order methods for PDEs.

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1 Polynomial meshes

In the field of multivariate polynomial approximation, the notion of polynomial mesh has recently emerged as a significant concept. Originally introduced in the seminal paper [6], it has been studied in several subsequent papers, from both the theoretical and the computational point of view; cf., e.g., [1, 2, 3, 5, 9, 11, 13, 15, 18, 19, 20, 22, 24, 25, 26]. Moreover, approximate Fekete-like points extracted from polynomial meshes have begun to play a role in the framework of high-order methods for PDEs, cf., e.g., [7, 28].

We recall that a polynomial *Weakly Admissible Mesh* (WAM) is a sequence of discrete subsets of a polynomial determining (polynomial vanishing there vanish everywhere) compact set $K \subset \mathbb{R}^d$ (or more generally $K \subset \mathbb{C}^d$) such that the polynomial inequality

$$\|p\|_K \leq C(\mathcal{A}_n)\|p\|_{\mathcal{A}_n} \ , \ \ \forall p \in \mathbb{P}_n^d \tag{1}$$

is satisfied, where both $\mathrm{card}(\mathcal{A}_n) \geq \dim(\mathbb{P}_n^d) = \binom{n+d}{d}$ and $C(\mathcal{A}_n)$ grow at most like a power of n. Here and below, $\mathbb{P}_n^d$ denotes the space of d-variate polynomials of total degree not exceeding n, and $\|f\|_X$ the sup-norm of a function f bounded on the (discrete or continuous) set X. The quantity $C(\mathcal{A}_n)$ is often called "constant" of the WAM. When $C(\mathcal{A}_n)$ is bounded we speak of an Admissible Mesh (AM), which is termed "optimal" if $\mathrm{card}(\mathcal{A}_n) = \mathcal{O}(n^d)$.

Among their properties, it is worth recalling the following ones (cf. [6]), which give also recipes to construct new from known WAMs:

- any *affine transformation* of a WAM is still a WAM, $C(\mathcal{A}_n)$ being invariant;
- any sequence of unisolvent *interpolation sets* whose Lebesgue constant Λ_n grows at most polynomially with n is a WAM, with constant $C(\mathcal{A}_n) = \Lambda_n$;
- a *finite product* of WAMs is a WAM on the corresponding product of compacts, $C(\mathcal{A}_n)$ being the product of the corresponding constants;
- a *finite union* of WAMs is a WAM on the corresponding union of compacts, $C(\mathcal{A}_n)$ being the maximum of the corresponding constants.

A special constructive role is played by some univariate interpolation sets, namely the Chebyshev-Lobatto nodes of an interval $[a,b]$ (via the affine transformation $\tau(s) = \frac{b-a}{2}\,s + \frac{b+a}{2}$),

$$X_n(a,b) = \{\tau(\xi_j)\} \subset [a,b]\ ,\ \ \xi_j = \cos(j\pi/n)\ ,\ 0 \le j \le n\ , \tag{2}$$

the classical Chebyshev nodes

$$Z_n(a,b) = \{\tau(\eta_j)\} \subset (a,b)\ ,\ \ \eta_j = \cos\left(\frac{(2j+1)\pi}{2(n+1)}\right)\ ,\ \ 0 \le j \le n\ , \tag{3}$$

and the Chebyshev-like "subperiodic" angular nodes

$$\Theta_n(\alpha,\beta) = \phi_\omega(Z_{2n}(-1,1)) + \frac{\alpha+\beta}{2} \subset (\alpha,\beta)\ ,\ \ \omega = \frac{\beta-\alpha}{2} \le \pi\ , \tag{4}$$

obtained by the nonlinear tranformation $\phi_\omega(s) = 2\arcsin\left(\sin\left(\frac{\omega}{2}\right)\,s\right)$, $s \in [-1,1]$. These nodal sets satisfy the following fundamental inequalities (the first and second are well-known results of polynomial interpolation theory, the third has been recently proved in the framework of trigonometric interpolation, cf. [8] and the references therein).

Lemma 1 *Let $p \in \mathbb{P}_n^1$ be a univariate algebraic polynomial, and X_n, Z_n the Chebyshev nodal sets (2), (3). Let be $t \in \mathbb{T}_n^1$ be a univariate trigonometric polynomial, and Θ_n the angular nodal set (4). Then the following inequalities hold*

$$\|p\|_{[a,b]} \le c_n\|p\|_{X_n}\ ,\ \ \|p\|_{[a,b]} \le c_n\|p\|_{Z_n}\ ,\ \ \|t\|_{[\alpha,\beta]} \le c_{2n}\|t\|_{\Theta_n}\ ,\ \ c_n = 1 + \frac{2}{\pi}\log(n+1)\ . \tag{5}$$

Below, we recall some recent results on WAMs in $\mathbb{R}^2$ and $\mathbb{R}^3$, privileging the constructive approaches that are at the base of the software package [10] (the case of surface meshes, in particular on algebraic surfaces, is under development). In all the constructions, Z_n can replace X_n (with the possible consequent small increase of the resulting mesh cardinality).

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1.1 Planar meshes

1.1.1 Polygons

Any *convex* quadrangle with vertices $\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3, \boldsymbol{v}_4$, is the image of a *bilinear transformation* of the square, namely $\boldsymbol{x} = \boldsymbol{\sigma}(s_1, s_2) = \frac{1}{4}\ ((1-s_1)(1-s_2)\boldsymbol{v}_1 + (1+s_1)(1-s_2)\boldsymbol{v}_2 + (1+s_1)(1+s_2)\boldsymbol{v}_3 + (1-s_1)(1+s_2)\boldsymbol{v}_4)$, where $(s_1, s_2) \in [-1,1]^2$, with a triangle, e.g. $\boldsymbol{v}_3 = \boldsymbol{v}_4$, as a special degenerate case. Using the fact that $p \circ \boldsymbol{\sigma} \in \mathbb{P}_n^1 \otimes \mathbb{P}_n^1$ for every $p \in \mathbb{P}_n^2$, by Lemma 1 we can easily prove the following

Proposition 1 *The sequence of "oblique" Chebyshev grids $\mathcal{A}_n = \boldsymbol{\sigma}(X_n(-1,1) \times X_n(-1,1))$ is a WAM of the convex quadrangle $Q = \boldsymbol{\sigma}([-1,1]^2)$, with constant $C(\mathcal{A}_n) = c_n^2 = \mathcal{O}(\log^2 n)$ and $\mathrm{card}(\mathcal{A}_n) \leq (n+1)^2$.*

Concerning general polygons, in view of a well-known result of computational geometry and the finite union property of WAMs, we have then immediately the following

Proposition 2 *Let K be a simple polygon (convex or concave) with ℓ sides. Then K has a WAM given by the union of the WAMs of $\ell - 2$ triangles of a minimal triangulation, with constant $C(\mathcal{A}_n) = c_n^2 = \mathcal{O}(\log^2 n)$ and $\mathrm{card}(\mathcal{A}_n) \sim (\ell-2)n^2$.*

The geometric constructions of Proposition 1 and 2, studied in [5, 13], are implemented by the Matlab functions `wamquadrangle` and `wampolygon` of [10] (where minimal polygon triangulation is generated by a Matlab version of the "Ear Clipping" algorithm).

1.1.2 Circular sections

Several circular sections, such as circular *sectors, segments, zones and lenses*, can be described by *linear blending of arcs*, cf. [25]. Let $\boldsymbol{u}(\theta) = \boldsymbol{a}_1 \cos(\theta) + \boldsymbol{b}_1 \sin(\theta) + \boldsymbol{c}_1$, $\boldsymbol{v}(\theta) = \boldsymbol{a}_2 \cos(\theta) + \boldsymbol{b}_2 \sin(\theta) + \boldsymbol{c}_2$, $\theta \in [\alpha, \beta]$, be two trigonometric planar curves of degree one, $\boldsymbol{a}_i = (a_{i1}, a_{i2})$, $\boldsymbol{b}_i = (b_{i1}, b_{i2})$, $\boldsymbol{c}_i = (c_{i1}, c_{i2})$, $i = 1, 2$, being suitable bidimensional vectors (with $\boldsymbol{a}_i, \boldsymbol{b}_i$ not all zero), with the important property that the curves are both parametrized on the *same angular interval* $[\alpha, \beta]$, $0 < \beta - \alpha \leq 2\pi$. Consider the *blending transformation* $\boldsymbol{x} = \boldsymbol{\sigma}(s, \theta) = s\boldsymbol{u}(\theta) + (1-s)\boldsymbol{v}(\theta)$, $(s, \theta) \in [0,1] \times [\alpha, \beta]\}$. Now, for every $p \in \mathbb{P}_n^2$ we have that $p \circ \boldsymbol{\sigma} \in \mathbb{P}_n^1 \otimes \mathbb{T}_n^1$, and we can prove the following

Proposition 3 *The sequence of "blended" grids $\mathcal{A}_n = \boldsymbol{\sigma}(X_n(0,1) \times \Theta_n(\alpha, \beta))$ is a WAM of $K = \boldsymbol{\sigma}([0,1] \times [\alpha, \beta])$, with $C(\mathcal{A}_n) = c_n\, c_{2n} = \mathcal{O}(\log^2 n)$ and $\mathrm{card}(\mathcal{A}_n) \leq (n+1)(2n+1)$.*

Relevant arc related domains that do not fall in the previous class are *circular lunes* (difference of two overlapping disks). A lune, whose boundary is given by two circular arcs, a longer one with semiangle say ω_2, a shorter one with semiangle say ω_1, can be described by

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different *bilinear trigonometric transformations* of rectangles (in angular variables), of the form $\boldsymbol{x} = \boldsymbol{\sigma}(\phi,\theta) = \boldsymbol{a}_1 + \boldsymbol{a}_2\cos(\theta) + \boldsymbol{a}_3\sin(\theta) + \boldsymbol{a}_4\cos(\phi) + \boldsymbol{a}_5\cos(\phi)\cos(\theta) + \boldsymbol{a}_6\cos(\phi)\sin(\theta) + \boldsymbol{a}_7\sin(\phi)\sin(\theta)$, $(\theta,\phi) \in \mathcal{R} = [-\omega_1,\omega_1] \times [-\omega_2,\omega_2]$, where the $\boldsymbol{a}_i = (a_{i,1}, a_{i,2})$ are suitable 2-dimensional vectors depending on ω_1 and ω_2, i.e., each component of $\boldsymbol{\sigma}$ is in the trigonometric space $\mathbb{T}_1^1 \otimes \mathbb{T}_1^1$; cf. [25] and the references therein. Hence for every $p \in \mathbb{P}_n^2$ we have that $p \circ \boldsymbol{\sigma} \in \mathbb{T}_n^1 \otimes \mathbb{T}_n^1$, and by Lemma 1 one can prove the following

Proposition 4 *The sequence of curvilinear grids $\mathcal{A}_n = \boldsymbol{\sigma}(\Theta_n(-\omega_1,\omega_1) \times \Theta_n(-\omega_2,\omega_2))$ is a WAM of the lune $K = \boldsymbol{\sigma}(\mathcal{R})$, with $C(\mathcal{A}_n) = c_{2n}^2 = \mathcal{O}(\log^2 n)$ and $\mathrm{card}(\mathcal{A}_n) \leq (2n+1)^2$.*

WAMs on arc blending domains and on circular lunes can be computed by the Matlab functions `wamblend` and `wamlune` of [10].

1.1.3 Convex and starlike C^2-domains

In [20], a constructive approach has been studied for the generation of optimal Admissible Meshes on smooth starlike planar domains, based on the fulfillment of a tangential-like Markov polynomial inequality.

Proposition 5 *Let $K \subset \mathbb{R}^2$ be a planar compact starlike domain. Assume that K satisfies a* Uniform Interior Ball Condition *(UIBC), i.e., every point of ∂K belongs to the boundary of a disk with radius $\rho > 0$, contained in K (geometrically, there is a fixed disk that can roll along the boundary remaining inside K). Then, for every fixed $\alpha \in (0, 1/\sqrt{2})$, K possesses an optimal admissible mesh $\{\mathcal{A}_n\}$ such that $C(\mathcal{A}_n) \equiv \frac{\sqrt{2}}{1-\alpha\sqrt{2}}$, $\mathrm{card}(\mathcal{A}_n) \sim n^2\,\frac{\mathrm{length}(\partial K)}{\alpha\rho}$.*

When K is convex with C^2 boundary, ρ can be easily computed, being the *minimal radius of curvature* of the boundary, by the well-known "*Rolling Ball Theorem*". The construction of optimal polynomial meshes on convex compact sets, defined by a level set of a smooth convex bivariate function, is implemented by the Matlab function `convomesh` [10].

1.2 Solid meshes

1.2.1 Cones, pyramids, cylinders and solids of rotation

The following geometric constructions have been studied in [11], and implemented by the Matlab functions `wamcone` and `wamrot` of [10]. In both the propositions below, $\Omega \subset \mathbb{R}^3$ is a planar compact set where a 2-dimensional WAM, say $\mathcal{A}_n$, is known.

Proposition 6 *Let $\boldsymbol{v}^*$ be a point in $\mathbb{R}^3$ not belonging to the plane of Ω. Then the generalized cone $\mathcal{C}$ with base Ω and vertex $\boldsymbol{v}^*$, has a WAM, say $\mathcal{B}_n$, with $C(\mathcal{B}_n) = c_n\, C(\mathcal{A}_n)$ and $\mathrm{card}(\mathcal{B}_n) = 1 + n\,\mathrm{card}(\mathcal{A}_n)$, given by the union of the $n+1$ Chebyshev-Lobatto points of the*

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segments joining $\boldsymbol{v}^*$ with each of the points of $\mathcal{A}_n$. If we consider the truncated cone obtained by cutting the cone with a plane parallel to the base, then the WAM $\mathcal{B}_n$ is given by the union of the Chebyshev-Lobatto points of the cut segments, and $\mathrm{card}(\mathcal{B}_n) = (n+1)\mathrm{card}(\mathcal{A}_n)$.

Proposition 7 *Let α be a line in $\mathbb{R}^3$ lying on the plane of Ω and not intersecting Ω (or intersecting Ω only at the boundary), and let $\theta^* \in (0, 2\pi]$ a given angle. Then the solid of rotation $\mathcal{R}$ obtained by rotating Ω around the axis α by an angle θ^* has a WAM, say $\mathcal{B}_n$, with $C(\mathcal{B}_n) = c_{2n}\, C(\mathcal{A}_n)$ and* $\mathrm{card}(\mathcal{B}_n) = (2n+1)\mathrm{card}(\mathcal{A}_n)$*, given by the union of the $2n+1$ copies of $\mathcal{A}_n$ corresponding to rotating $\mathcal{A}_n$ by the angles $\Theta_n(0, \theta^*)$.*

The proofs resort to Lemma 1, together with the relevant affine transformations (in particular, to the classical Thales intercept theorem for the generalized cone case). Observe that generalized *pyramids* (and thus tetrahedra) are special cases of cones, Ω being a polygon. By tetrahedralization and finite union, any *simple polyhedron* has a WAM with constant $\mathcal{O}(\log^3 n)$ and cardinality $\mathcal{O}(n^3)$. Moreover, *generalized cylinders* (up to a rotation compact sets of the form $\Omega \times [a, b]$) by the product property have a WAM $\mathcal{B}_n = \mathcal{A}_n \times X_n(a, b)$ with the same constant and cardinality of that of a truncated cone. A different WAM for a standard cylinder with circular base can be generated by rotation of the Padua interpolation points of a rectangle [9].

1.2.2 Lissajous meshes on the 3-cube

By the product property, $(X_n(-1, 1))^3$ is a natural grid-type WAM for the reference cube $[-1, 1]^3$, with constant $c_n^3 = \mathcal{O}(\log^3 n)$. Recently, a new approach for trivariate polynomial approximation has been explored, based on curves instead of grids. Indeed, it has been proved that

$$\int_{[-1,1]^3} p(\boldsymbol{x}) \frac{d\boldsymbol{x}}{\sqrt{(1-x_1^2)(1-x_2^2)(1-x_3^2)}} = \pi^2 \int_0^\pi p(\boldsymbol{\ell}_n(\theta))\, d\theta \ , \ \ \forall p \in \mathbb{P}_{2n}^3 \ ,$$

where $\boldsymbol{\ell}_n(\theta) = (\cos(\alpha_n\theta), \cos(\beta_n\theta), \cos(\gamma_n\theta))$, $\theta \in [0, \pi]$, is the Lissajous curve with integer frequency parameters $(\alpha_n, \beta_n, \gamma_n) = \left(\frac{3}{4}n^2 + \frac{1}{2}n, \frac{3}{4}n^2 + n, \frac{3}{4}n^2 + \frac{3}{2}n + 1\right)$ for even n, and $(\alpha_n, \beta_n, \gamma_n) = \left(\frac{3}{4}n^2 + \frac{1}{4}, \frac{3}{4}n^2 + \frac{3}{2}n - \frac{1}{4}, \frac{3}{4}n^2 + \frac{3}{2}n + \frac{3}{4}\right)$ for odd n. This entails, via *algebraic cubature* and *hyperinterpolation* (discretized expansion in series of orthogonal polynomials) with respect to the product Chebyshev measure, that

Proposition 8 *The rank-1 Chebyshev cubature lattice $\mathcal{A}_n = \{\boldsymbol{\ell}_n(s\pi/\nu)\}$, $s = 0, \ldots, \nu = n\gamma_n + 1$, is a WAM for the cube, with $C(\mathcal{A}_n) = \mathcal{O}(\log^3 n)$ and* $\mathrm{card}(\mathcal{A}_n) \sim \frac{3}{4}\, n^3$.

The interest for trivariate function approximation by sampling along Lissajous curves arises, for example, in the emerging field of MPI (Magnetic Particle Imaging, cf. [12] and the references therein). The construction of 3d Lissajous WAMs is studied in [4], and is implemented by the Matlab function `wamlissa` of [10].

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2 Interpolation and fitting

We recall here some basic results and algorithms concerning polynomial fitting on WAMs, and polynomial interpolation at discrete extremal sets extracted from WAMs. Let us term $\mathcal{L}_{\mathcal{A}_n}$ the projection operator $C(K) \to \mathbb{P}_n^d$ defined by polynomial least squares on a WAM, and $\mathcal{L}_{\mathcal{F}_n}$ the projection operator defined by interpolation on *Fekete points* of degree n, say $\mathcal{F}_n$, *extracted from a WAM* (these are points that maximize the absolute value of the Vandermonde determinant). Concerning their operator norms with respect to $\|\cdot\|_K$,

$$\|\mathcal{L}_{\mathcal{A}_n}\| \lesssim C(\mathcal{A}_n)\sqrt{\mathrm{card}(\mathcal{A}_n)}\ ,\ \ \|\mathcal{L}_{\mathcal{F}_n}\| \le NC(\mathcal{A}_n)\ ,\ \ N = N_n = \dim(\mathbb{P}_n^d) = \binom{n+d}{d}\ , \quad (6)$$

for polynomial least squares and for polynomial interpolation, respectively, which show that WAMs with slowly increasing constants $C(\mathcal{A}_n)$ and cardinalities are relevant structures for multivariate polynomial approximation, cf. [6]. In practice, however, (6) turn out to be large overestimates of the actual operator norm growth (see Figure 1). A standard calculation for projection operators provides the estimate $\|f - \mathcal{L}f\|_K \le (1 + \|\mathcal{L}\|)\inf_{p\in\mathbb{P}_n^d}\|f-p\|_K$, $\forall f \in C(K)$, from which together with (6) we get convergence, whenever K is a *Jackson compact* and f a sufficiently regular function (cf. [21]).

2.1 Discrete Orthogonal Polynomials and Least Squares

The approximation algorithms start from Vandermonde-like matrices in suitable total-degree polynomial bases. The choice of the standard monomial basis is unappropriate already at small degrees, due to its severe ill-conditioning. A general and more suitable choice is the product Chebyshev basis of the smallest Cartesian rectangle containing the domain (say $\times_{s=1}^d[a_s,b_s] \supseteq K$, $d = 2,3$), namely $\mathcal{T}_{(k_1,\dots,k_d)}(\boldsymbol{x}) = \prod_{s=1}^d T_{k_s}\left(\frac{2x_s-b_s-a_s}{b_s-a_s}\right)$, $0 \le \sum_{s=1}^d k_s \le n$, where $T_k(\cdot) = \cos(k\arccos(\cdot))$ are the classical Chebyshev polynomials of the first kind. By a suitable ordering (in [10] we adopted the *graded lexicographical ordering*), we obtain a polynomial basis that we call $\boldsymbol{p}(\boldsymbol{x}) = (p_1(\boldsymbol{x}),\dots,p_N(\boldsymbol{x}))$, and we can compute the corresponding Vandermonde-like matrix on a WAM of K

$$V(\mathcal{A}_n,\boldsymbol{p}) = (p_j(\boldsymbol{\xi}_i))\ ,\ \ 1 \le i \le M\ ,\ 1 \le j \le N\ , \quad (7)$$

where $\mathcal{A}_n = \{\boldsymbol{\xi}_1,\dots,\boldsymbol{\xi}_M\}$. Notice that $M \ge N$ and $V(\mathcal{A}_n,\boldsymbol{p})$ is full-rank by (1), cf. [6]. The core of the fitting and interpolation procedures is a two-step discrete orthogonalization of the polynomial basis by the QR algorithm, namely $V(\mathcal{A}_n,\boldsymbol{p}) = Q_1R_1$, $Q_1 = QR_2$, where

$$Q = V(\mathcal{A}_n,\boldsymbol{\varphi}) = V(\mathcal{A}_n,\boldsymbol{p})R_1^{-1}R_2^{-1} \quad (8)$$

is the (numerically) orthogonal Vandermonde-like matrix corresponding to the discrete orthonormal polynomial basis $\boldsymbol{\varphi} = (\varphi_1,\dots,\varphi_N) = (p_1,\dots,p_N)R_1^{-1}R_2^{-1}$; cf. [5, 24]. The

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reason for iterating the QR factorization is to cope with the strong ill-conditioning, which is typical of Vandermonde-like matrices and increases with the degree. Two orthogonalization iterations generally suffice, unless the original matrix $V(\mathcal{A}_n, \boldsymbol{p})$ is so severely ill-conditioned (rule of thumb: condition number greater than the reciprocal of machine precision) that the algorithm may fail. In practice, the change of polynomial basis is conveniently implemented by the Matlab matrix right division operator [16], as $\boldsymbol{\varphi} = (\boldsymbol{p}/R_1)/R_2$, in view of the ill-conditioning inherited by the triangular matrices R_1 and R_2. We can compute least squares polynomial projection of $f \in C(K)$ on an array of target points $X = \{\boldsymbol{x}_1, \dots, \boldsymbol{x}_S\} \subset K$, and estimate the norm of the least squares projection operator, that we call its "Lebesgue constant" by analogy with interpolation, on an array of control points Y, as

$$\mathcal{L}_{\mathcal{A}_n} f(X) = V(X, \boldsymbol{\varphi}) Q^t \boldsymbol{f} \ , \ \ \boldsymbol{f} = (f(\boldsymbol{x}_1), \dots, f(\boldsymbol{x}_S))^t \ ; \ \ \|\mathcal{L}_{\mathcal{A}_n}\| \approx \|Q(V(Y, \boldsymbol{\varphi}))^t\|_1 \ , \tag{9}$$

cf. [5]. In the package [10], computation of Discrete Orthogonal Polynomials $\boldsymbol{\varphi}$ and evaluation of $V(X, \boldsymbol{\varphi})$ are implemented by the Matlab functions `wamdop` and `wamdopeval`, least squares fitting by `wamfit`, and estimation of the Lebesgue constant by `wamleb`.

2.1.1 Mesh compression

In many situations, WAMs can be very large sets and thus the fitting procedure becomes computationally heavy. This happens, for example, already in 2d with many-sided polygons (cf. Proposition 2), or with Admissible Meshes of smooth convex domains (cf. Proposition 5). Moreover, when the sampling process is difficult or costly, it could be convenient to reduce in any case the WAM cardinality, resorting to the following "compression" result.

Proposition 9 *Let $\mathcal{A}_n$ be a WAM of a compact set $K \subset \mathbb{R}^d$ with $\mathrm{card}(\mathcal{A}_n) > N_{2n} = \dim(\mathbb{P}_{2n})$. Then there exist a WAM $\mathcal{A}_n^* \subset \mathcal{A}_n$ with $\mathrm{card}(\mathcal{A}_n^*) \le N_{2n}$ and $C(\mathcal{A}_n^*) = C(\mathcal{A}_n)\sqrt{\mathrm{card}(\mathcal{A}_n)}$.*

The proof of Proposition 9 rests on a generalized version of the well-known *Tchakaloff's theorem* (cf. [23]), on the existence of low cardinality algebraic cubature formulas for compactly supported measures, in particular for *discrete measures* (we consider here the discrete measure with unit mass at each WAM point). Indeed, by such a theorem there exist a subset $\mathcal{A}_n^* = \{\boldsymbol{\xi}_{i_1}, \dots, \boldsymbol{\xi}_{i_\mathcal{V}}\} \subset \mathcal{A}_n$, $\mathcal{V} \le N_{2n}$, and positive weights $\boldsymbol{w} = \{w_{i_1}, \dots, w_{i_\mathcal{V}}\}$, such that $\sum_{i=1}^M q(\boldsymbol{\xi}_i) = \sum_{k=1}^{\mathcal{V}} w_{i_k} q(\boldsymbol{\xi}_{i_k})$ for every $q \in \mathbb{P}_{2n}$. Then for every $p \in \mathbb{P}_n$, with $q = p^2$ we get

$$\frac{\|p\|_K}{C(\mathcal{A}_n)} \le \|p\|_{\mathcal{A}_n} \le \|p\|_{\ell^2(\mathcal{A}_n)} = \|p\|_{\ell^2_{\boldsymbol{w}}(\mathcal{A}_n^*)} \le \sqrt{\|\boldsymbol{w}\|_1}\, \|p\|_{\mathcal{A}_n^*} = \sqrt{\mathrm{card}(\mathcal{A}_n)}\, \|p\|_{\mathcal{A}_n^*} \ .$$

The computation of the nodes $\{\boldsymbol{\xi}_{i_k}\}$ and weights $\{w_{i_k}\}$ can be formulated as the problem of finding a *sparse* (nonnegative) solution to the *underdetermined* Vandermonde-like linear system (consider column vectors)

$$V^t \boldsymbol{z} = \boldsymbol{m} \ , \ \ V = V(\mathcal{A}_n; \boldsymbol{p}) \ , \ \ \boldsymbol{m} = V^t \boldsymbol{e} \ , \ \boldsymbol{e} = (1, \dots, 1)^t \ , \tag{10}$$

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cf. (7), where $\boldsymbol{m} = (m_1, \ldots, m_N)^t$ is the vector of discrete moments of the polynomial basis.

Sparsity can be achieved by reformulating (10) as *NonNegative Least Squares* (NNLS) problem $\min_{\boldsymbol{u} \geq \boldsymbol{0}} \|V^t \boldsymbol{u} - \boldsymbol{m}\|_2$, and solving it by the Matlab function `lsqnonneg` which uses a variant of the *active set method* by Lawson and Hanson (cf. [16]). Alternatively, since `lsqnonneg` can be very slow on large problems, we can solve (10) by *QR with column pivoting*, implemented by the Matlab `mldivide` (or backslash) operator. The latter approach does not guarantee positivity, but it turns out in practice that in the considered degree ranges the negative weights are few and small, hence the stability parameter $\rho_n = \sum_{k=1}^{\mathcal{V}} |w_{i_k}| / |\sum_{k=1}^{\mathcal{V}} w_{i_k}|$ is not far from 1, and $C(\mathcal{A}_n^*) = C(\mathcal{A}_n)\sqrt{\rho_n \, \mathrm{card}(\mathcal{A}_n)}$ has a size comparable to that appearing in Proposition 9. Both methods are implemented in the Matlab function `wamcomprex` of [10], where one preliminary orthogonalization step is made to pull down the conditioning of V (see Figure 1 for an example of compression).

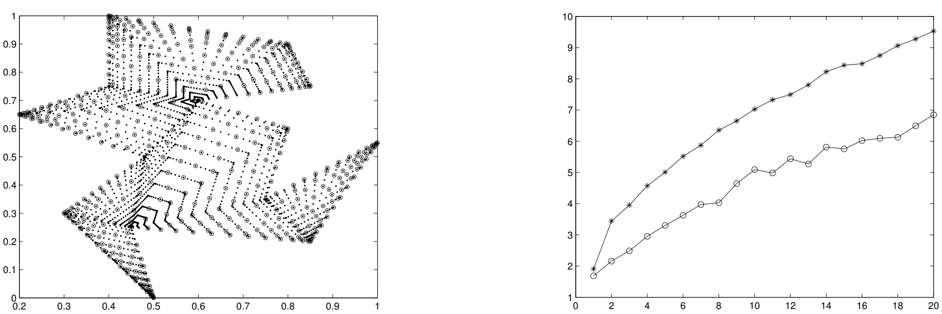

Figure 1: Left: WAM for degree $n = 15$ ($\cdot$, about 2800 points) by triangulation on a polygon with 14 sides, and its compression into $496 = \dim(\mathbb{P}_{2n}^2)$ points ($\circ$); Right: Lebesgue constants of the LS operator on the original ($*$) and the compressed WAM ($\circ$), $n = 1, \ldots, 20$.

2.2 Discrete Extremal Sets

The search for good sets for multivariate polynomial interpolation has received renewed attention in recent years; cf., e.g., [3, 14, 17, 27]. We consider here the approximate versions of Fekete points of K (points that maximize the absolute value of the Vandermonde determinant) studied in [2, 3, 24]. Indeed, the continuum Fekete points are explicitly known only in two univariate instances (interval and complex circle), and are very difficult to compute. By (6), it makes sense to start from a WAM, that is from the corresponding orthogonal Vandermonde-like matrix $Q = V(\mathcal{A}_n, \boldsymbol{\varphi})$ in (8) (which is preferable for conditioning issues). The problem of selecting a $N \times N$ square submatrix with maximal determinant from a given $M \times N$ rectangular matrix is known to be NP-hard, but can be solved in an approximate way by two simple *greedy* algorithms, that are fully described and analyzed in [3]. These

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algorithms produce two interpolation nodal sets, called *Discrete Extremal Sets*.

The first algorithm, that computes the so-called *Approximate Fekete Points* (AFP), tries to maximize iteratively submatrix volumes until a maximal volume $N \times N$ submatrix of Q is obtained, and can be based on *QR factorization with column pivoting*, applied to Q^t (that in Matlab is implemented by the `mldivide` or backslash operator, cf. [16]). The notion of volume generated by a set of vectors generalizes the geometric concept related to parallelograms and parallelepipeds (the volume and determinant notions coincide on a square matrix). The second algorithm, that computes the so-called *Discrete Leja Points* (DLP), tries to maximize iteratively submatrix determinants, and is based simply on *Gaussian elimination with row pivoting* applied to Q. Denoting by A the $M \times 2$ matrix of the WAM nodal coordinates, the corresponding computational steps, in a Matlab-like style, are

$$\boldsymbol{w} = Q \backslash \boldsymbol{v};\ \boldsymbol{i} = \texttt{find}(\boldsymbol{w} \neq \boldsymbol{0});\ \mathcal{F}_n^{AFP} = A(\boldsymbol{i}, :); \tag{11}$$

for AFP, where $\boldsymbol{v}$ is any nonzero N-dimensional vector, and

$$[L, U, \boldsymbol{\pi}] = \texttt{LU}(Q, \text{“}\texttt{vector}\text{”});\ \boldsymbol{i} = \boldsymbol{\pi}(1:N);\ \mathcal{F}_n^{DLP} = A(\boldsymbol{i}, :); \tag{12}$$

for DLP. In (12), we refer to the Matlab version of the LU factorization that produces a row permutation vector $\boldsymbol{\pi}$. In both algorithms, we eventually select an index subset $\boldsymbol{i} = (i_1, \dots, i_N)$, that extracts an approximate discrete extremal set $\mathcal{F}_n$ of the region K from the WAM $\mathcal{A}_n$. Algorithms (11) and (12) are implementd by the Matlab function `dexsets` of [10]. Once one of the discrete extremal sets has been computed, we can simply apply (8)-(9) with the $N \times N$ matrix $V(\mathcal{F}_n, \boldsymbol{p})$ substituting $V(\mathcal{A}_n, \boldsymbol{p})$, in order to compute the interpolation polynomial $\mathcal{L}_{\mathcal{F}_n}$ and to estimate the Lebesgue constant $\|\mathcal{L}_{\mathcal{F}_n}\|$, by the Matlab functions `wamfit` and `wamleb` in [10]. The two families of discrete extremal sets share the same asymptotic behavior which, by a recent deep result in pluripotential theory, is exactly that of the discrete uniform probability measures associated with the continuum Fekete points; cf. [1, 2, 3] and the references therein.

Proposition 10 *Let $\mathcal{F}_n = \{\boldsymbol{\xi}_{i_1}, \dots, \boldsymbol{\xi}_{i_N}\}$ the AFP or DLP extracted from a WAM of a compact set $K \subset \mathbb{R}^d$ (or $K \subset \mathbb{C}^d$). Then $\lim_{n\to\infty} \frac{1}{N}\sum_{k=1}^{N} f(\boldsymbol{\xi}_{i_k}) = \int_K f(\boldsymbol{x})\, d\mu_K$ for every $f \in C(K)$, where μ_K is the equilibrium measure of K.*

Moreover, in all our numerical experiments both AFP and DLP have shown good computational features, with a Lebesgue constant growing more slowly than $NC(\mathcal{A}_n)$, the theoretical upper bound in (6) for Fekete points extracted from a WAM (usually with a better behavior of AFP with respect to DLP); cf., e.g., [2, 3, 4, 9, 11, 25]. See Figures 2 and 3 for some examples of Discrete Extremal Sets on different 2d and 3d geometries.

It is also worth recalling that (differently from AFP) DLP are a *sequence*, i.e., the first $N_s = \dim(\mathbb{P}_s^d)$ points of a set of DLP for degree $n > s$ are unisolvent for interpolation in

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$\mathbb{P}_s^d$. Exploiting this feature, in [10] we provide a file of precomputed DLP for trivariate polynomial interpolation up to degree $n = 30$, on the Lissajous curve of the cube described in Subsection 1.2.2 (by affine transformation these points can be used in any parallelepiped).

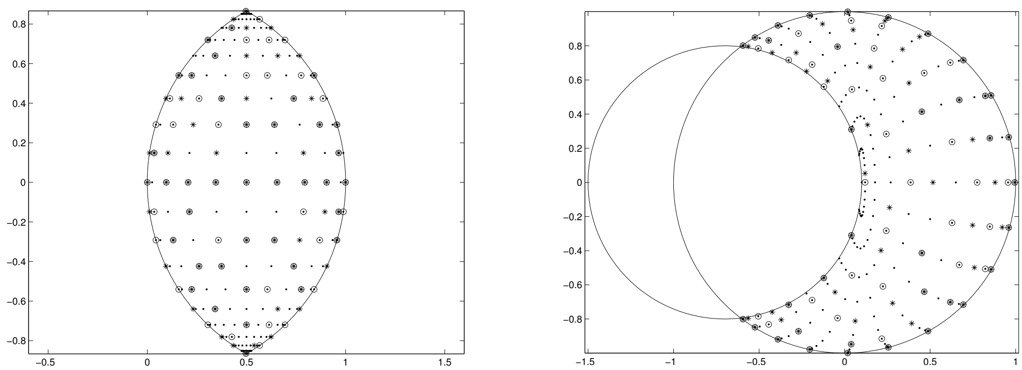

Figure 2: 66 Approximate Fekete Points ($\circ$) and Discrete Leja Points ($*$) extracted from a WAM ($\cdot$) for degree $n = 10$ on a symmetric lens (left) and on a lune (right).

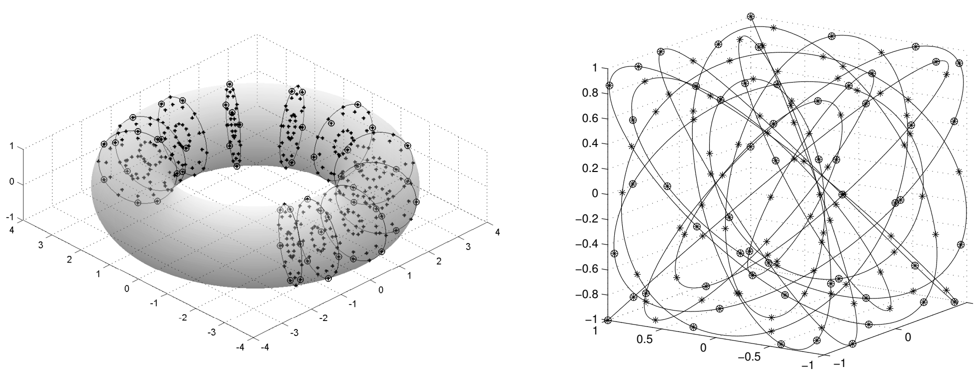

Figure 3: 56 Approximate Fekete Points ($\circ$) for degree $n = 5$ extracted from a WAM ($*$) on a solid torus section and on the Lissajous curve of the cube.

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