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# INTEGRAL REPRESENTATIONS FOR BRACKET-GENERATING MULTI-FLOWS

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Abstract. If $f_1, f_2$ are smooth vector fields on an open subset of an Euclidean space and $[f_1, f_2]$ is their Lie bracket, the asymptotic formula

$$\Psi_{[f_1,f_2]}(t_1,t_2)(x) - x = t_1t_2[f_1,f_2](x) + o(t_1t_2), \tag{1}$$

where we have set $\Psi_{[f_1,f_2]}(t_1,t_2)(x) \stackrel{\text{def}}{=} \exp(-t_2f_2) \circ \exp(-t_1f_1) \circ \exp(t_2f_2) \circ \exp(t_1f_1)(x)$, is valid for all $t_1, t_2$ small enough. In fact, the integral, exact formula

$$\Psi_{[f_1,f_2]}(t_1,t_2)(x) - x = \int_0^{t_1}\int_0^{t_2} [f_1,f_2]^{(s_2,s_1)}(\Psi(t_1,s_2)(x))ds_1\,ds_2, \tag{2}$$

where $[f_1,f_2]^{(s_2,s_1)}(y) \stackrel{\text{def}}{=} D\Big(\exp(s_1f_1)\circ\exp(s_2f_2)\Big)^{-1}(y)\cdot[f_1,f_2](\exp(s_1f_1)\circ \exp(s_2f_2)(y))$, has also been proven. Of course (2) can be regarded as an improvement of (1). In this paper we show that an integral representation like (2) holds true for any iterated Lie bracket made of elements of a family $f_1,\dots,f_m$ of vector fields. In perspective, these integral representations might lie at the basis for extensions of asymptotic formulas involving non-smooth vector fields.

## 1. Introduction and preliminaries.

### 1.1. A notational premise.

Let us begin with a few notational conventions which are consistent with the so-called Agrachev-Gamkrelidze formalism (see [1, 2, 9]). First, in the formulas involving flows and vector fields, we shall write the argument of a function on the left. For instance, if $M$ is a differentiable manifold, $x \in M$ and $f$ is a vector field on $M$, we shall use $xf$ to denote the evaluation of $f$ at $x$. Similarly, for the (assumed unique) value at $t$ of the Cauchy problem $\dot{x} = f(x) \quad x(0) = \bar{x}$ we shall write $\bar{x}e^{tf}$ (so in particular, the differential equation itself will be written $\frac{d}{dt}(\bar{x}e^{tf}) = \bar{x}e^{tf}f$). Secondly, if $t \in \mathbb{R}$, and $f, g$ are $C^1$ vector fields, the notation $\bar{x}fe^{tg}$ stands for the tangent vector at $\bar{x}e^{tg}$ obtained by i) evaluating $f$ at $\bar{x}$ (so obtaining the vector $\bar{x}f$) and then ii) by mapping $\bar{x}f$ though the differential (at $\bar{x}$) of the map $x \to xe^{tg}$. Finally, the vector fields $f, g$ can be regarded as first order operators, so the notation $fg$ reasonably stands for the second order operator which, in the conventional notation, would map any $C^2$ function $\phi$ to $D(D\phi\cdot g)\cdot f$[1]. In particular, the Lie bracket $[f,g]$, which is a first order operator resulting as a difference between two second order operators, in this notation has the following

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[1] In terms of Lie derivatives, this operator maps $\phi$ into $L_fL_g\phi$

expression: $[f,g]\stackrel{\text{def}}{=}fg-gf$. These conventions turn out to be particularly convenient for the subject we are going to deal with. However, sometimes more conventional notation will be utilized as well and the context will be sufficient to avoid any confusion.

1.2. **The main question.** Let $n$ be a positive integer and let $M \subseteq \mathbb{R}^n$ be an open subset. If $f_1, f_2$ are $C^1$ vector fields and $x \in M$, the Lie bracket $[f_1, f_2]$ verifies the well-known asymptotic formula

$$x\Psi_{[f_1,f_2]}(t_1,t_2) = x + t_1t_2 \cdot (x[f_1,f_2]) + o(t_1t_2), \tag{3}$$

where we have set $x\Psi_{[f_1,f_2]}(t_1,t_2) \stackrel{\text{def}}{=} xe^{t_1f_1}e^{t_2f_2}(e^{t_1f_1})^{-1}(e^{t_2f_2})^{-1}$ ( $= xe^{t_1f_1}e^{t_2f_2}$ $e^{-t_1f_1}e^{-t_2f_2}$). ((3) is the same as (1), just rewritten in the above-introduced formalism). Similarly, for a bracket of degree 3 one has

$$x\Psi_{[[f_1,f_2],f_3]}(t_1,t_2,t_3) = x + t_1t_2t_3 \cdot (x[[f_1,f_2],f_3]) + o(t_1t_2t_3). \tag{4}$$

where $x\Psi_{[[f_1,f_2],f_3]} := x\Psi_{[f_1,f_2]}(t_1,t_2)e^{t_3f_3}\left(\Psi_{[f_1,f_2]}(t_1,t_2)\right)^{-1}(e^{t_3f_3})^{-1}$. [2] Asymptotic estimates like (3)-(4) can be utilized, through a suitable application of open mapping arguments, to deduce various controllability results.

In this paper we aim at replacing asymptotic estimates for multiflows like the above ones with *integral, exact* formulas. For a bracket of degree two such a formula has been provided in [8]. More precisely, if $f_1, f_2$ are vector fields of class $C^1$ then for every $t_1, t_2$ sufficiently small the equality

$$x\Psi_{[f_1,f_2]}(t_1,t_2) = x + \int_0^{t_1}\int_0^{t_2} x\Psi_{[f_1,f_2]}(t_1,s_2)[f_1,f_2]^{(t_2,s_1)}\,ds_1\,ds_2 \tag{5}$$

holds true, where we have set

$$[f_1,f_2]^{(t_2,s_1)} \stackrel{\text{def}}{=} e^{t_2f_2}e^{s_1f_1}[f_1,f_2]e^{-s_1f_1}e^{-t_2f_2} \tag{6}$$

Formula (5) says that the composition of flows $x\Psi_{[f_1,f_2]}(t_1,t_2)$ can be calculated as the integral, over the multi-time rectangle $[0,t_1]\times[0,t_2]$. of $x\Psi(t_1,s_2)[f_1,f_2]^{(s_2,s_1)}$, namely the function that maps each $(s_1,s_2)\in[0,t_1]\times[0,t_2]$ to the estimation at $x\Psi_{[f_1,f_2]}(t_1,s_2)$ of the *integrating* bracket $[f_1,f_2]^{(s_2,s_1)}$. Incidentally, let us observe that as a trivial byproduct of (5) one gets the commutativity theorem (stating that the flows of $f_1$ and $f_2$ locally commute if and only if $[f_1,f_2]\equiv 0$).

We shall construct integrating brackets corresponding to every iterated bracket so that formulas analogous to (5) hold true. Though we will set our problem on an open subset of $\mathbb{R}^n$, we will perform such construction in a chart invariant way, so that the resulting formulas are meaningful on a differentiable manifold as well.

Rather than stating here the main theorem (see Theorem 3.1 below), which would require a certain number of technicalities, we limit ourselves to illustrating the situation in the case of a degree 3 bracket $[[f_1,f_2],f_3]$. Let us assume that $f_1$ and $f_2$ are of class $C^2$ and $f_3$ is of class $C^1$, and let us define the integrating bracket

[2]Notice that the left-hand side can be written as the product of 10 (=4+1+4+1) flows:

$$x\Psi_{[[f_1,f_2],f_3]} = xe^{t_1f_1}e^{t_2f_2}e^{-t_1f_1}e^{-t_2f_2}e^{t_3f_3}e^{-t_2f_2}e^{t_1f_1}e^{t_2f_2}e^{t_1f_1}$$

$[[f_1,f_2],f_3]^{(t_1,t_3,s_1,s_2)}$ by setting, for every $t_1,t_3,s_1,s_2$ sufficiently small,

$$[[f_1,f_2],f_3]^{(t_1,t_3,s_1,s_2)} \stackrel{\text{def}}{=} (e^{t_3f_3}e^{t_1f_1}e^{s_2f_2}e^{-t_1f_1}e^{-s_2f_2}) \Big[[f_1\,,\,f_2]^{(s_1,s_2)}\,,\,f_3\Big](e^{t_3f_3}e^{t_1f_1}e^{s_2f_2}e^{-t_1f_1}e^{-s_2f_2})^{-1}\,, \quad (7)$$

where $[f_1\,,\,f_2]^{(s_1,s_2)}$ is defined as in (6) with $t_2=s_1,\ s_1=s_2$. Then Theorem 3.1 says that there exists $\delta>0$ such that for all $t_1,t_2,t_3\in[-\delta,\delta]$

$$x\Psi_{[[f_1,f_2],f_3]}(t_1,t_3,t_3) = x+\int_0^{t_1}\int_0^{t_2}\int_0^{t_3} x\Psi_{[[f_1,f_2],f_3]}(t_1,t_2,s_3)[[f_1,f_2],f_3]^{(t_1,s_3,s_1,s_2)}\,ds_1\,ds_2\,ds_3 \quad (8)$$

Let us point out two main facts:

**i)** on one hand, formula (8) is similar to (5)

**ii)** on the other hand, there is a crucial difference in the definition of integrating bracket passing from the degree 2 to a degree $>2$; indeed while the integrating bracket$[f_1,f_2]^{(t_2,s_1)}$ is defined as an integral (over $[0,t_1]\times[0,t_2]$) of a suitable adjoint of the classical bracket at the points $x\Psi_{[f_1,f_2]}(t_1,s_2)$, the integrand in $[[f_1,f_2],f_3]^{(t_1,t_3,s_1,s_2)}$ contains the bracket of $[f_1,f_2]^{(t_2,s_1)}$ –instead of $[f_1,f_2]$– and $f_3$. In fact, the definition of higher degree integrating bracket is given by induction and involves various adjoint of classical brackets (see Definition 2.3).

1.3. **A motivation.** Integral representations may be regarded as improvements of asymptotic formulas. In fact, our interest for this issue was raised by the aim of laying down a basic setting on which one can reasonably investigate families of vector fields that are less regular than what is required by the classical definition of (iterated) Lie bracket. A typical case where such an investigation might prove interesting is provided by the Chow-Rashevski's Theorem, which, for $C^\infty$ vector fields $f_1,\dots,f_k$, guarantees small-time local controllability at $x\in M$ for driftless control systems $\dot y=\sum_{i=1}^k u_if_i(y)\ |u_i|\le 1$ as soon as a condition like $Lie\{f_1,\dots,f_k\}(x)=T_xM$ is verified [3]. Akin results are valid for vector fields $f_i$ of class $C^{r_i}$, $r_i$ being the maximal order of differentiation needed to define all the (classical) brackets that make the Lie algebra rank condition to hold true. (see Subsection 4).

So, a natural question might be the following: what about a Chow-Rashevski's Theorem in the case when, say, the vector fields $f_1,\dots,f_k$ are such that each $f_i$, $i=1,\dots,m$, is just of class $C^{r_i-1}$ with locally Lipschitz $r_i-1$-th order derivatives? Some different answers have been proposed e.g. in [7, 6], [8], [10]. In particular, in [8] a set-valued notion of bracket has been introduced for locally Lipschitz vector fields. However, a mere recursive definition of bracket of degree greater than two would not work (see e.g. [9]*Section 7, where it is shown that such an iterated bracket would be *too small* for an asymptotic formula to hold true). We think that the study of integral representations in the smooth case may represent a first step towards a useful definition of iterated bracket in the non smooth case (see Subsection 5.2).

[3] $Lie\{f_1,\dots,f_k\}$ is the Lie algebra generated by the family $\{f_1,\dots,f_k\}$

1.4. **Outline of the paper.** The paper is organized as follows: in the remaining part of the present section we recall the concept of formal iterated bracket of letters $X_1, X_2, \ldots$. In Section 2 we introduce the notion of *integrating bracket* Section 3 is devoted to the main result of the paper, namely Theorem 3.1, which provides exact representations for bracket-generating multi-flows through integrals involving integrating brackets. In Section 4 we discuss the question of regularity in connection with the validity of integral formulas. As a byproduct of the main result we state a Chow-Rashewski theorem with low regularity assumptions. In Section 5 we provide a simple example remarking the crucial difference between integrating brackets of degree 2 and those of higher degree. Finally we discuss some motivations of the present article coming from the aim of extending asymptotic formulas (possibly, via flows's regularization) to a nonsmooth setting.

1.5. **Formal brackets.** Given a fixed sequence $\mathbf{X} = (X_1, X_2, \ldots)$ of distinct objects called *variables*, or *indeterminates*, let $W(\mathbf{X})$ be the set of all words in the alphabet consisting of the $X_i$, the left bracket, the right bracket, and the comma. The *bracket* of two members $W_1, W_2$ of $W(\mathbf{X})$ is the word $[W_1, W_2]$ obtained by writing first a left bracket, then $W_1$, then a comma, then $W_2$, and then a right bracket. We call *iterated brackets* of $\mathbf{X}$ the elements of the smallest subset $S$ of $W(\mathbf{X})$ that contains the single-letter words $X_j$ and is such that whenever $W_1$ and $W_2$ belong to $S$ it follows that $[W_1, W_2] \in S$. The *degree* $deg(W)$ of a word $W \in W(\mathbf{X})$ is the length of the *letter sequence* of $W$, namely of the sequence $Seq(B)$ obtained from $W$ by deleting all the brackets and commas. Clearly, if $W_1, W_2 \in W(\mathbf{X})$ then $deg([W_1, W_2]) = deg(W_1) + deg(W_2)$.

An iterated bracket $B \in ITB(\mathbf{X})$ is *canonical* if $Seq(B) = X_1 X_2 \cdots X_{deg(B)}$.

Given a canonical bracket $B \in ITB(\mathbf{X})$ of $deg(B) = m$, and any finite sequence $\sigma = (\sigma_1, \ldots, \sigma_n)$ of objects (possibly with repetitions) such that $n \geq m$, we use $B(\sigma)$, or $B(\sigma_1, \ldots, \sigma_m)$, to denote the expression obtained from $B$ by substituting $\sigma_j$ for $X_j$ for $j = 1, \ldots, m$. (For example, (a) if $B = [[X_1, X_2], [X_3, X_4]]$ then $B(f_1, f_2, g, h) = [[f_1, f_2, [g, h]], B(f_1, f_2, f_1, f_2) = [[f_1, f_2, [f_1, f_2]$, (b) if $B$ is any canonical bracket of degree $m$, then $B(X_1, X_2, \ldots, X_m) = B$, (c) if $B = [X_1, X_2]$ and $\mathbf{f} = (f_1, f_2, f_3)$ then $B(\mathbf{f}) = [f_1, f_2]$.)

Given any canonical bracket $B$ of degree $m$ and any nonnegative integer $\mu$, the *$\mu$-shift* of $B$ is the iterated bracket

$$B^{(\mu)} = B(X_{1+\mu}, X_{2+\mu}, \ldots, X_{m+\mu}) .$$

(For example, if $B = [[[X_1, X_2], [X_3, [X_4, X_5]]], X_6]$, then the 4-shift of $B$ is the bracket $B^{(4)}$ given by $B^{(4)} = [[[X_5, X_6], [X_7, [X_8, X_9]]], X_{10}]$ .)

A *semicanonical bracket* is an iterated bracket $B$ which coincides with a $\mu$-shift of a canonical bracket for some nonnegative integer $\mu$.

For every iterated bracket $B$ of degree $m > 1$ there exists a unique pair $(B_1, B_2)$ of brackets such that $B = [B_1, B_2]$. The pair $(B_1, B_2)$ is the *factorization* of $B$, and the brackets $B_1$, $B_2$ are known, respectively, as the *left factor* and the *right factor* of $B$.

If $B$ is semicanonical then both factors of $B$ are semicanonical as well. If $B$ is canonical then the left factor of $B$ is canonical and the right factor of $B$.is semicanonical. Hence, if $B$ is canonical of degree $m > 1$ and $(B_1, \tilde{B}_2)$ is its factorization, there exists a canonical bracket $B_2$ such that $\tilde{B}_2 = B_2^{(deg(B_1))}$, so that $B = [B_1, B_2^{(deg(B_1))}]$. We will call the pair $(B_1, B_2)$ the *canonical factorization* of $B$. (For example, if $B = [[X_1, X_2], [[X_3, X_4], X_5]]$, then the factorization of $B$ is the pair $\Big([X_1, X_2], [[X_3, X_4], X_5]\Big)$, and the canonical factorization is the pair $([X_1, X_2], [[X_1, X_2], X_3])$.

Let $B = B_0^{(\mu)}$ be a semicanonical bracket, where $B_0$ is a canonical bracket of degree $m$. Let $M$ be a differential manifold and let $\mathbf{f} = (f_1, \dots, f_\nu)$ $(\nu \geq m + \mu)$ be a finite sequence of vector fields on $M$. We use $B(\mathbf{f})$ to denote the expression obtained from $B$ by substituting $f_j$ for $X_j + \mu$, for $j = 1, \dots, m$. If the manifold $M$ and the vector fields $f_j$ are sufficiently regular, then we can regard $B(\mathbf{f})$ as an iterated Lie bracket, in the common sense. For instance, if $B = [[X_7, X_8], X_9]$ and $\mathbf{f} = (f, g, h, k))$ is a 4-tuple of vector fields, then

$$B(\mathbf{f}) = [[f, g], h] = [f, g]h - h[f, g] = fgh - gfh - hfg + hgf.$$

Of course the regularity of the vector field $B(\mathbf{f})$ depends on both the regularity of the fields $(f_1, \dots, f_\nu)$ and on the structure of $B$.

## 2. **Integrating brackets.**

### 2.1. **Bracket generating multi-flows.**

To simplify our exposition, when not otherwise specified *we shall assume the vector fields involved in the formulas are defined on a open subset $M \subseteq \mathbb{R}^n$ and are of class $C^\infty$*. However, the regularity question is obviously quite important and will be treated in Section 5. In particular, vector fields will be assumed as regular as required by the structure of the involved formal brackets.

**Definition 2.1.** Let us associate with a formal bracket $B$ of degree $m$ and a $m$-tuple $\mathbf{f} = (f_1, \dots, f_m)$ of vector fields a product $\Psi_B^{\mathbf{f}}(t_1, \dots, t_m)$ of exponentials $e^{t_i f_i}$, $i = 1, \dots, m$. We proceed recursively:

(i) If $B = X_1$ (so that $\mathbf{f}$ consists of a single vector field $f$) we set

$$\Psi_B^{\mathbf{f}}(t) \stackrel{\text{def}}{=} e^{tf},$$

i.e., for each $x \in M$ and each sufficiently small $t$, $x\Psi_B^{\mathbf{f}}(t) = y(t)$ where (in the conventional notation) $y(\cdot)$ is the solution to the Cauchy problem $\dot{y} = f(y)$, $y(0) = x$.

(ii) If $deg(B) = m > 1$ and $B = [B_1, B_2^{(m_1)}]$ is the canonical factorization of $B$, for any $\mathbf{t} = (t_1, \dots, t_m)$, we set

$$\begin{array}{ll} \mathbf{f}_{(1)} \stackrel{\text{def}}{=} (f_1, \dots, f_{m_1}) & \mathbf{f}_{(2)} \stackrel{\text{def}}{=} (f_{m_1+1}, \dots, f_m) \\ \mathbf{t}_{(1)} \stackrel{\text{def}}{=} (t_1, \dots, t_{m_1}) & \mathbf{t}_{(2)} \stackrel{\text{def}}{=} (t_{m_1+1}, \dots, t_m) \end{array} \tag{9}$$

and[4]

$$\Psi_B^{\mathbf{f}}(\mathbf{t}) \stackrel{\text{def}}{=} \boldsymbol{\Psi}_{\mathbf{B_1}}^{\mathbf{f}_{(1)}}(\mathbf{t}_{(1)})\ \boldsymbol{\Psi}_{\mathbf{B_2}}^{\mathbf{f}_{(2)}}(\mathbf{t}_\beta)\ \Big(\boldsymbol{\Psi}_{\mathbf{B_1}}^{\mathbf{f}_{(1)}}(\mathbf{t}_{(2)})\Big)^{-1}\ \Big(\boldsymbol{\Psi}_{\mathbf{B_2}}^{\mathbf{f}_{(2)}}(\mathbf{t}_{(2)})\Big)^{-1}.$$

It is clear that for every precompact subset $K \subset M$ there exist a neighborhood $U$ of $K$ and a $\delta > 0$ such that $x\Psi_B^{\mathbf{f}}(\mathbf{t})$ is defined for every $x \in U$ and $\mathbf{t} \in ]-\delta, \delta[^m$. However, when not otherwise stated, we shall assume that vector fields $f_i$ are *complete*, meaning that their flows $(x, t) \mapsto xe^{tf_i}$ are well-defined for all $x \in M$ and $t \in \mathbb{R}$. Obviously, the general case can be recovered by standard "cut-off function" arguments.

Let us illustrate the above definition of $\Psi^{\mathbf{f}}_B(\mathbf{t})$ by means of simple examples:

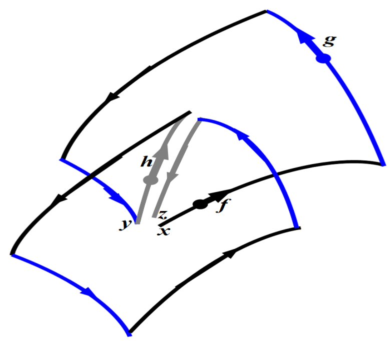


FIGURE 1. $y = x\Psi^{(f,g)}_{[X_1,X_2]}(t_1, t_2)$, $z = x\Psi^{(f,g,h)}_{[[X_1,X_2],X_3]}(t_1, t_2, t_3)$

1. if $B = [X_1, X_2]$ and $\mathbf{f} = (f, g)$, then
$$\Psi^{\mathbf{f}}_B(t_1, t_2) = e^{t_1 f} e^{t_2 g} e^{-t_1 f} e^{-t_2 g}\,;$$
2. if $B = [X_1, [X_2, X_3]]$ and $\mathbf{f} = (f, g, h)$, then
$$\Psi^{\mathbf{f}}_B(t_1, t_2, t_3) = e^{t_1 f} e^{t_2 g} e^{t_3 h} e^{-t_2 g} e^{-t_3 h} e^{-t_1 f} e^{t_3 h} e^{t_2 g} e^{-t_3 h} e^{-t_2 g}\,.$$
3. if $B = [X_1, X_2], [X_3, X_4]]$ and $\mathbf{f} = (f, g, h, k)$, then
$$\Psi^{\mathbf{f}}_B(t_1, t_2, t_3, t_4)$$
$$= e^{t_1 f} e^{t_2 g} e^{-t_1 f} e^{-t_2 g} e^{t_3 h} e^{t_4 k} e^{-t_3 h} e^{-t_4 k} e^{t_2 g} e^{t_1 f} e^{-t_2 g} e^{-t_1 f} e^{t_4 k} e^{t_3 h} e^{-t_4 k} e^{-t_3 h}\,.$$

Observe that the number $N(B)$ of exponential factors of $\Psi^{\mathbf{f}}_B$ is given recursively by $N(B) = 1$ if $deg(B) = 1$ and, for $m > 1$, $N(B) = 2(N(B_1) + N(B_2))$, where $[B_1, B_2]$ is the canonical factorization of $B$.

2.2. **Integrating brackets.** The *integrating bracket* corresponding to $B$ and $\mathbf{f}$ will be defined as a $(2m - 2)$-parameterized (continuous) vector field on $M$
$$x \mapsto xB(\mathbf{f})^{(t_1,\ldots,t_{m_1-1},t_{m_1+1},\ldots,t_m,s_1,\ldots,s_{m-1})},$$
which, in particular, depends continuously on the parameters
$$(t_1, \ldots, t_{m_1-1}, t_{m_1+1}, \ldots, t_m, s_1, \ldots, s_{m-1}) \in \mathbb{R}^{m-1} \times \mathbb{R}^{m-1}\,,$$
and verifies $xB(\mathbf{f})^{(0\ldots,0)} = xB(\mathbf{f})$.

To begin with, let us recall the notion of $Ad$ operator:

**Definition 2.2.** Let $U, V \subseteq M$ be open subsets and let $\Phi : U \to V$ be a $C^r$ diffeomorphism ($r \geq 1$). If $h$ is a vector field on $U$, $Ad_\Phi h$ is the vector field on $U$ defined by
$$x \mapsto xAd_\Phi h \stackrel{\text{def}}{=} x\Phi h \Phi^{-1} \qquad \forall x \in U$$

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[4]In [3], [5, 4], akin maps, usually defined for brackets $B$ of the form
$$[X_1, [X_2[\ldots.[X_{m-1}, X_m], ], \ldots]$$
and $\mathbf{t}$ of the form $(t, \ldots, t)$, are called *quasiexponential*, *almost exponential*, or *approximate exponential maps*.

(In the conventional notation, the vector field $Ad_\Phi h$ would be denoted by $x \mapsto D(\Phi)^{-1}_{|\Phi(x)}(h(\Phi(x)))$

We remind that the $Ad$ operator is bracket preserving, namely

$$Ad_\Phi[h_1, h_2] = [Ad_\Phi h_1, Ad_\Phi h_2],$$

for all vector fields $h_1, h_2$.

Willing to define *integrating brackets* of degree greater than 2, we cannot avoid introducing a few more notation. However, some examples following Definition 2.3 should allow one to get an intuitive idea of the bracket's construction.

If $d$ is any positive integer and $\mathbf{r} = (r_1, \dots, r_d) \in \mathbb{R}^d$ and $\alpha \in \{1, \dots, d\}$, let us use

$$\underset{\alpha}{\mathbf{r}}$$

to denote the $(d-1)$-tuple obtained by $\mathbf{r}$ by deleting the $\alpha$-th element. So, for instance, if $\mathbf{r} = (r_1, r_2, r_3, r_4)$ one has

$$\underset{1}{\mathbf{r}} = (r_2, r_3, r_4) \quad \underset{2}{\mathbf{r}} = (r_1, r_3, r_4), \quad \underset{3}{\mathbf{r}} = (r_1, r_2, r_4) \quad \underset{4}{\mathbf{r}} = (r_1, r_2, r_3)$$

For $\alpha, \beta \in \{1, \dots, 2\}$, $\alpha < \beta$, we also let

$$\underset{\{\alpha,\beta\}}{\mathbf{r}}$$

denote the $(d-2)$-tuple obtained by $\mathbf{r}$ by deleting the $\alpha$-th and $\beta$-th elements, so that, for instance, if $\mathbf{r} = (r_1, r_2, r_3, r_4)$,

$$\underset{\{2,4\}}{\mathbf{r}} = (r_1, r_3).$$

When $d = 1$, we set

$$\underset{1}{\mathbf{r}} \stackrel{\text{def}}{=} \emptyset. \tag{10}$$

Also, if $d = 2$, we set

$$\underset{\{\alpha,\beta\}}{\mathbf{r}} \stackrel{\text{def}}{=} \emptyset. \tag{11}$$

Let $B$ be an iterated bracket and let $B = [B_1, B_2^{(m_1)}]$ be its canonical factorization, with $deg(B) = m$, $deg(B_1) = m_1$, $deg(B_2) = m_2$, $m = m_1 + m_2$. Let $\mathbf{f} = (f_1, \dots, f_m)$ be an $m$-tuple of vector fields . We set, as before,

$$\mathbf{t} = (t_1, \dots, t_m) \qquad \mathbf{t}_{(1)} \stackrel{\text{def}}{=} (t_1, \dots, t_{m_1}) \qquad \mathbf{t}_{(2)} \stackrel{\text{def}}{=} (t_{m_1+1}, \dots, t_m).$$

Moreover, let

$$\mathbf{s} = (s_1, \dots, s_{m-1}) \in \mathbb{R}^{m-1}$$

**Definition 2.3** (**Integrating bracket**)**.** We call *integrating bracket* (*corresponding to the pair* $(B, \mathbf{f})$) the $(2m-2)$-parameterized vector field $B(\mathbf{f})^{\left(\underset{m_1}{\mathbf{t}}, \mathbf{s}\right)}$ defined recursively as follows:

$\boxed{m = 1}$ If $m = 1$ (so that $B = X_1$, $\mathbf{f} = f_1$), we let

$$B(\mathbf{f})^{\left(\underset{1}{\mathbf{t}}, \mathbf{s}\right)} = B(\mathbf{f})^{\emptyset} \stackrel{\text{def}}{=} f_1\,. \tag{12}$$

$\boxed{m>1}$ If $m = m_1 + m_2 \geq 2$, and $B_1 = [B_{11}, B_{12}^{(m_{11})}]$, $B_2 = [B_{21}, B_{22}^{(m_{21})}]$ are the canonical factorizations of $B_1$ and $B_2$, respectively, for some $1 \leq m_{11} < m_1$, $1 \leq m_{21} < m_2$,

$$B(\mathbf{f})^{\left(\underset{m_1}{\mathbf{t}},\mathbf{s}\right)} \stackrel{\text{def}}{=} Ad_{\Psi_{B_2}^{\mathbf{f}_{(2)}}(\mathbf{t}_{(2)})\Psi_{B_1}^{\mathbf{f}_{(1)}}\left(\underset{m_1}{\mathbf{t}_{(1)}},s_{m_1}\right)} \left[ B_1(\mathbf{f}_{(1)})^{\left(\underset{\{m_{11},m_1\}}{\mathbf{t}_{(1)}},s_{m_1},\mathbf{s}_{(1)}\right)}, B_2(\mathbf{f}_{(2)})^{\left(\underset{m_1+m_{21}}{\mathbf{t}_{(2)}},\mathbf{s}_{(2)}\right)} \right]. \tag{13}$$

**Remark 1.** When one of the indexes $m_1, m_2$ is equal to one, formula (13) has to be interpreted as follows:

$\boxed{m_1 = 1, m_2 = 1}$ If $m_1 = m_2 = 1$ (so $m = 2$, $B = [X_1, X_2]$, $\mathbf{f} = (f_1, f_2)$ ),

$$B(\mathbf{f})^{\left(\underset{m_1}{\mathbf{t}},\mathbf{s}\right)} \stackrel{\text{def}}{=} [f_1, f_2]^{(t_2,s_1)} \stackrel{\text{def}}{=} Ad_{e^{t_2 f_2} e^{s_1 f_1}} [f_1\,,\ f_2] \tag{14}$$

$\boxed{m_1 > 1, m_2 = 1}$ If $m_1 > 1$ and $m_2 = 1$ (so $m_1 + 1 = m$, $\mathbf{f}_{(2)} = (f_m)$),

$$B(\mathbf{f})^{\left(\underset{m_1}{\mathbf{t}},\mathbf{s}\right)} \stackrel{\text{def}}{=} Ad_{e^{t_m f_m}\Psi_{B_1}^{\mathbf{f}_{(1)}}(\mathbf{t}_{(1)}^{s_{m_1}})} \left[ B_1(\mathbf{f}_{(1)})^{\left(\underset{\{m_{11},m_1\}}{\mathbf{t}_{(1)}},s_{m_1},\mathbf{s}_{(1)}\right)}, f_m \right] \tag{15}$$

$\boxed{m_1 = 1, m_2 > 1}$ If $m_1 = 1$ and $m_2 > 1$ (so $1 + m_2 = m$, $\mathbf{f}_{(1)} = (f_1)$),

$$B(\mathbf{f})^{\left(\underset{m_1}{\mathbf{t}},\mathbf{s}\right)} \stackrel{\text{def}}{=} Ad_{\Psi_{B_2}^{\mathbf{f}_{(2)}}(\mathbf{t}_{(2)})e^{t_1 f_1}} \left[ f_1\,, B_2(\mathbf{f}_{(2)})^{\left(\underset{m_1+m_{21}}{\mathbf{t}_{(2)}},\mathbf{s}_{(2)}\right)} \right], \tag{16}$$

□

**Examples of integrating brackets:**

$\boxed{m_1 = 1, m_2 = 1}$ If $B = [X_1, X_2]$, $\mathbf{f} = (f_1, f_2)$, then ($m = 2$ and)

$$\begin{aligned} B(\mathbf{f})^{\left(\underset{m_1}{\mathbf{t}},\mathbf{s}\right)} &= [f_1, f_2]^{(t_2,s_1)} \\ &= Ad_{e^{t_2 f_2} e^{s_1 f_1}} [f_1\,,\ f_2] \\ &= e^{t_2 f_2} e^{s_1 f_1} [f_1\,,\ f_2]\, e^{-s_1 f_1} e^{-t_2 f_2} \end{aligned} \tag{17}$$

$\boxed{m_1 = 2, m_2 = 1}$ If $B = [[X_1, X_2], X_3]$, $\mathbf{f} = (f_1, f_2, f_3)$, (so $\mathbf{f}_{(1)} = (f_1, f_2)$ and $\mathbf{f}_{(2)} = (f_3)$), then

$$\begin{aligned} B(\mathbf{f})^{\left(\underset{m_1}{\mathbf{t}},\mathbf{s}\right)} &= [[f_1, f_2], f_3]^{(t_1,t_3,s_1,s_2)} \\ &= Ad_{e^{t_3 f_3}\Psi_{B_1}^{(f_1,f_2)}(t_1,s_2)} \left[ [f_1, f_2]^{(s_2,s_1)},\ f_3 \right] \\ &= Ad_{e^{t_3 f_3} e^{t_1 f_1} e^{s_2 f_2} e^{-t_1 f_1} e^{-s_2 f_2}} \left[ Ad_{e^{s_2 f_2} e^{s_1 f_1}} [f_1\,,\ f_2]\,,\ f_3 \right] \\ &= e^{t_3 f_3} e^{t_1 f_1} e^{s_2 f_2} e^{-t_1 f_1} e^{-s_2 f_2} \left[ e^{s_2 f_2} e^{s_1 f_1} [f_1\,,\ f_2]\, e^{-s_1 f_1} e^{-s_2 f_2},\ f_3 \right] \\ &\quad e^{s_2 f_2} e^{t_1 f_1} e^{-s_2 f_2} e^{-t_1 f_1} e^{-t_3 f_3}. \end{aligned} \tag{18}$$

$\boxed{m_1 = 2, m_2 = 2}$ If $B = [[X_1, X_2], [X_3, X_4]]$, $\mathbf{f} = (f_1, f_2, f_3, f_4)$, (so $\mathbf{f}_{(1)} = (f_1, f_2)$ and $\mathbf{f}_{(2)} = (f_3, f_4)$), then

$$\begin{aligned} B(\mathbf{f})^{\left(\mathbf{t}_{m_1},\mathbf{s}\right)} &= [[f_1, f_2], [f_3, f_4]]^{(t_1,t_3,t_4,s_1,s_2,s_3)} \\ &= Ad_{e^{t_3f_3}e^{t_4f_4}e^{-t_3f_3}e^{-t_4f_4}e^{t_1f_1}e^{s_2f_2}e^{-t_1f_1}e^{-s_2f_2}}\Big[Ad_{e^{s_2f_2}e^{s_1f_1}}[f_1\,,\,f_2]\,, \\ &\qquad\qquad Ad_{e^{t_4f_4}e^{s_3f_3}}[f_3\,,\,f_4]\Big] \\ &= e^{t_3f_3}e^{t_4f_4}e^{-t_3f_3}e^{-t_4f_4}e^{t_1f_1}e^{s_2f_2}e^{-t_1f_1}e^{-s_2f_2} \\ &\quad \Big[e^{s_2f_2}e^{s_1f_1}[f_1\,,\,f_2]\,e^{-s_1f_1}e^{-s_2f_2}\,,\,e^{t_4f_4}e^{s_3f_3}[f_3\,,\,f_4]\,e^{-s_3f_3}e^{-t_4f_4}\Big] \\ &\quad e^{s_2f_2}e^{t_1f_1}e^{-s_2f_2}e^{-t_1f_1}e^{t_4f_4}e^{t_3f_3}e^{-t_4f_4}e^{-t_3f_3}\,. \end{aligned}$$

**Remark 2.** On one hand, we have made small abuses of notation by writing, for instance, $[[f_1, f_2], f_3]^{(t_1,t_3,s_1,s_2)}$ instead of $B(\mathbf{f})^{(t_1,t_3,s_1,s_2)}$, with $B = [[X_1, X_2], X_3]$ and $\mathbf{f} = (f_1, f_2, f_3)$. We shall pursue with such notational simplifications whenever the danger of confusion is ruled out by the context. On the other hand let us point out that the definition of integrating bracket is based on the pair $(B, \mathbf{f})$ rather then on the vector field $B(\mathbf{f})$. It may well happen that an integrating bracket $B^{(\hat{\mathbf{t}},\mathbf{s})}(\mathbf{f})$ of degree $m > 2$ is different from zero while the vector field $B(\mathbf{f})$ (i.e. the corresponding iterated Lie bracket) is identically equal to zero: see Example 5.1.

### 2.3. **Some basic properties of integrating brackets.**

**Lemma 2.4.** *Let $f_1, f_2$ be $C^2$ vector fields on $M$, and let $x \in M$ and $\delta_x > 0$ such that the integrating bracket $x[f_1, f_2]^{t_2,s_1}$ exists for every $(t_2, s_1) \in [-\delta_x, \delta_x]^2$.* [5] *Then*

$$\begin{aligned} &x[f_1, f_2]^{(t_2,s_1)} \\ =&x[f_1, f_2] + \int_0^{t_2} xAd_{e^{\tau f_2}}[f_2, [f_1, f_2]]d\tau + \int_0^{s_1} xAd_{e^{t_2f_2}e^{\sigma f_1}}[f_1, [f_1, f_2]]d\sigma \end{aligned}$$

*In particular,*

$$x[f_1, f_2]^{(t_2,s_1)} = x[f_1, f_2] \qquad \forall x \in M, \quad \forall (t_2, s_1) \in [-\delta_x, \delta_x]^2 \tag{19}$$

*if and only if*

$$x[f_1, [f_1, f_2]] = 0 = x[f_2, [f_1, f_2]] \qquad \forall x \in M \quad {}^{6} \tag{20}$$

*Proof.* The Lemma is just an application to the $C^1$ map $\Psi : [-\delta_x, \delta_x]^2 \to M$ of the following trivial fact:

*If $\Psi(0, 0) = W \in \mathbb{R}^n$, then, for all $(t_2, s_1) \in [-\delta_x, \delta_x]^2$ one has*

$$\Psi(t_2, s_1) = W + \int_0^{t_2} \frac{\partial\Psi(\tau, 0)}{\partial\tau}\, d\tau + \int_0^{s_1} \frac{\partial\Psi(t_2, \sigma)}{\partial\sigma}\, d\sigma$$

In fact, setting $\Psi(t_2, s_1) \overset{\text{def}}{=} x[f_1, f_2]^{(t_2,s_1)}$, one gets

$$\frac{\partial\Psi(t_2, 0)}{\partial\tau} = xAd_{e^{\tau f_2}}[f_2, [f_1, f_2]] \qquad \frac{\partial\Psi(t_2, \sigma)}{\partial\sigma} = xAd_{e^{t_2f_2}e^{\sigma f_1}}[f_1, [f_1, f_2]]$$

□

[5] As remarked above, such a $\delta_x$ does exist, uniformly on precompact subsets of $M$.

[6] Of course this condition is equivalent to the vanishing of *all* brackets of degree $\geq 3$

For simplicity, let us keep the notation

$$\Psi(t_1,s_2)\overset{\text{def}}{=}\Psi^{(f_1,f_2)}_{[X_1,X_2]}(t_1,s_2)$$

As a consequence of Lemma 2.4 one gets:

**Proposition 1.** *Consider vector fields $f_1, f_2$ of class $C^3$. Then, for every $x \in M$ and every vector field $f_3$ of class $C^1$ there exists $\delta_x > 0$ such that $\forall (t_1,t_3,s_1,s_2) \in [-\delta_x,\delta_x]^4$, one has*

$$x[[f_1,f_2],f_3]^{(t_1,t_3,s_1,s_2)} = xAd_{e^{t_3f_3}\Psi(t_1,s_2)}\bigg([[f_1,f_2],f_3] \\ + \int_0^{s_2}\Big[Ad_{e^{\tau f_2}}[f_2,[f_1,f_2]],f_3\Big]d\tau + \int_0^{s_1}\Big[Ad_{e^{s_2f_2}e^{\sigma f_1}}[f_1,[f_1,f_2]],f_3\Big]d\sigma\bigg). \quad (21)$$

*In particular, the following two statements are equivalent:*

- *For every vector field $f_3$ of class $C^1$ in a neighborhood of $x$, there is neighborhood $U$ of $x$ such that*
$$y[[f_1,f_2],f_3]^{(t_1,t_3,s_1,s_2)} = yAd_{e^{t_3f_3}\Psi(t_1,s_2)}\,[[f_1,f_2]\,,\,f_3] \quad (22)$$
*for all $y \in U$ and all 4-tuples $(t_1,t_3,s_1,s_2)$ sufficiently close to the origin.*
- *The identity*
$$y[f_1,[f_1,f_2]] = 0 = y[f_2,[f_1,f_2]] \quad (23)$$
*holds true for every $y$ in a neighborhood of $x$.*

*Proof.* To get (21) it is sufficient to recall the definition

$$[[f_1,f_2],f_3]^{(t_1,t_3,s_1,s_2)} = Ad_{e^{t_3f_3}\Psi(t_1,s_2)}\Big[[f_1,f_2]^{(s_2,s_1)}\,,\;f_3\Big]$$

and to apply Lemma 2.4. Moreover, clearly (23) implies (22) for every $f_3$. To prove the converse claim, observe that by (21) and (22), taking $(t_1,t_3) = (0,0)$, one gets

$$0 = \int_0^{s_2} y\Big[Ad_{e^{\tau f_2}}[f_2,[f_1,f_2]],f_3\Big]d\tau + \int_0^{s_1} y\Big[Ad_{e^{s_2f_2}e^{\sigma f_1}}[f_1,[f_1,f_2]],f_3\Big]d\sigma\Big), \quad (24)$$

for any $y$ in a neighborhood of $x$, for all vector fields $f_3$ of class $C^1$ near $x$, and for all $(s_1,s_2)$ sufficiently close to the origin. By computing the partial derivatives at $(s_1,s_2) = (0,0)$ of the right-hand side, in view of the continuity of integrands one obtains

$$y\Big[[f_2,[f_1,f_2]],f_3\Big] = 0, \qquad y\Big[[f_1,[f_1,f_2]],f_3\Big] = 0$$

for all vector fields $f_3$ of class $C^1$ near $x$ . Then, necessarily, one has

$$y[f_2,[f_1,f_2]] = 0, \qquad y[f_1,[f_1,f_2]] = 0.$$

□

**Remark 3.** The fact that an integrating bracket corresponding to a pair $(B,\mathbf{f})$, with $deg(B) > 2$, *is not*, in general, of the form $Ad_\phi B(\mathbf{f})$ (where $\phi$ depends on $m$ parameters) marks a crucial difference with the case when $B = [X_1,X_2]$, for which, instead, one actually has

$$[X_1,X_2](\mathbf{f})^{(t_2,s_1)} = Ad_\phi[f_1,f_2],$$

with $\phi = e^{t_2f_2}e^{s_1f_1}e^{-t_2f_2}e^{-s_1f_1}$. Incidentally, this fact has strong consequences in the attempt of defining a (set-valued) "Lie bracket" $[[f_1,f_2],f_3]$ when $f_1,f_2$ are of class $C^{1,1}$ and $f_3$ is merely Lipschitz continuous (see the Introduction and Section 5).

**Remark 4.** It is trivial to check that condition (23) remains necessary for (22) to hold even if the latter is verified just for $n$ vector fields that are linearly independent at each $y \in U$. Of course, identity (22) may well be true for a particular $f_3$ even if (23) is not verified, as it is immediately apparent by taking $f_3 \equiv 0$, in which case (22) holds with both sides vanishing. However, unless (23) is verified, it is not true that the vanishing of $[[f_1, f_2], f_3]$ implies the vanishing of $[[f_1, f_2], f_3]^{t_1,t_3,s_1,s_2}$, as shown in Example 5.1 below. As a byproduct of Theorem 3.1 below, this is connected with the (almost obvious) fact that in general the condition $[[f_1, f_2], f_3] \equiv 0$ *does not* imply $\Psi^{\mathbf{f}}_B = Id_M$.

3. **Integral representation.** We are now ready to state the main result. In this section, $m$ will stand for a positive integer, $B$ will represent a formal bracket of degree $m$, $m_1$ will be the degree of the first bracket in the canonical decomposition of $B$, and $\mathbf{f} = (f_1, \ldots, f_m)$ will be a $m$ of vector fields.

**Theorem 3.1** (**Integral representation).** *For every $m$-tuple $\mathbf{t} = (t_1, \ldots, t_m) \in \mathbb{R}^m$ one has*

$$x\Psi^{\mathbf{f}}_B(\mathbf{t}) = x + \int_0^{t_1} \cdots \int_0^{t_m} x\Psi^{\mathbf{f}}_B\left(\underset{m}{\mathbf{t}}, s_m\right) B(\mathbf{f})^{\left(\underset{m_1}{\mathbf{t}}, \mathbf{s}\right)} ds_1 \ldots ds_m \ , \qquad (25)$$

*where, using the notation introduced in previous section, we have set*

$$\underset{m_1}{\mathbf{t}} = (t_1, \ldots, t_{m_1-1}, t_{m_1+1}, \ldots, t_m) \qquad \mathbf{s} = (s_1, \ldots, s_{m-1}).$$

**Example 3.2.** For brackets of degree $2, 3$, see formulas (5) and (7).

If $B = [[X_1, X_2], [X_3, X_4]]$ (so $\mathbf{f}_{(1)} = (f_1, f_2)$ and $\mathbf{f}_{(2)} = (f_3, f_4)$), then (25) reads

$$x\Psi^{\mathbf{f}}_B(\mathbf{t}) = x + \int_0^{t_1}\int_0^{t_2}\int_0^{t_3}\int_0^{t_4} x\Phi\Big[\,[f_1\,,\,f_2]^{(s_1,s_2)}\,,\,[f_3\,,\,f_4]^{(s_3,s_3)}\,\Big]\Gamma\, ds_1\, ds_2\, ds_3\, ds_4\,.$$

where

$$x\Psi^{\mathbf{f}}_B(\mathbf{t}) = xe^{t_1f_1}e^{t_2f_2}e^{-t_1f_1}e^{-t_2f_2}e^{t_3f_3}e^{t_4f_4}e^{-t_3f_3}e^{-t_4f_4} \\ e^{t_2f_2}e^{t_1f_1}e^{-t_2f_2}e^{-t_1f_1}e^{t_4f_4}e^{t_3f_3}e^{-t_4f_4}e^{-t_3f_3}\,,$$

$$\Phi \overset{\text{def}}{=} e^{t_1f_1}e^{t_2f_2}e^{-t_1f_1}e^{-t_2f_2}e^{t_3f_3}e^{s_4f_4}e^{-t_3f_3}e^{-s_4f_4}e^{t_2f_2}e^{t_1f_1}e^{(s_2-t_2)f_2}e^{-t_1f_1}e^{-s_2f_2}\,,$$

$$\Gamma \overset{\text{def}}{=} e^{s_2f_2}e^{t_1f_1}e^{-s_2f_2}e^{-t_1f_1}e^{t_4f_4}e^{t_3f_3}e^{-t_4f_4}e^{-t_3f_3}\,,$$

and $[f_1\,,\,f_2]^{(s_1,s_2)}$, $[f_3\,,\,f_4]^{(s_3,s_3)}$ are defined as in (6).

The proof of Theorem 3.1 will rely on an analogous result (see Theorem 3.4 below) concerning the ($(t_1, \ldots, t_m)$-dependent) *vector field* $x \mapsto xV^{\mathbf{f}}_B(t_1, \ldots, t_m)$ corresponding to the ($\mathbf{t}$-dependent) *local 1-parameter action* $A$ defined as

$$(x, \tau) \mapsto A(\mathbf{t}, x, \tau) \overset{\text{def}}{=} x\left(\Psi^{\mathbf{f}}_B(\mathbf{t})\right)^{-1}\Psi^{\mathbf{f}}_B\left(\underset{m}{\mathbf{t}}, t_m + \tau\right) \qquad (26)$$

**Definition 3.3.** Foe every value of the parameter $\mathbf{t} = (t_1, \ldots, t_m) \in \mathbb{R}^m$ let us define the vector field[7] $x \mapsto xV^{\mathbf{f}}_B(\mathbf{t})$ by setting, for every $x \in M$,

$$xV^{\mathbf{f}}_B(\mathbf{t}) \overset{\text{def}}{=} \frac{\partial}{\partial \tau} A(\mathbf{t}, x, \tau)_{\tau=0} \qquad (27)$$

---

[7]Notice that, for every $\mathbf{t}$, $x \mapsto xV^{\mathbf{f}}_B(\mathbf{t})$ is in fact a (intrisicly defined) vector field, for $A(\mathbf{t}, \cdot, \cdot)$ is a true local *action*: this means that

$$A(\mathbf{t}, x, 0) = x, \qquad A(\mathbf{t}, x, \tau_1 + \tau_2) = A(\mathbf{t}, A(\mathbf{t}, x, \tau_1), \tau_2),$$

**Theorem 3.4.** *Let $m$ be an integer $\geq 1$. For every $m$-tuple $\mathbf{t} = (t_1, \dots, t_m) \in \mathbb{R}^m$ one has*

$$xV_B^{\mathbf{f}}(\mathbf{t}) = \int_0^{t_1} \cdots \int_0^{t_{m-1}} xB(\mathbf{f})^{\left(\underset{m_1}{\mathbf{t}}, \mathbf{s}\right)} ds_1 \dots ds_{m-1} \;{}^{8}, \tag{28}$$

*where, as before,* $\underset{m_1}{\mathbf{t}} = (t_1, \dots, t_{m_1-1}, t_{m_1+1}, \dots, t_m)$, $\mathbf{s} = (s_1, \dots, s_{m-1})$.

*Proof of Theorem 3.1.* Equality (25) is a straightforward consequence of (28) and the following identity

$$x\Psi_B^{\mathbf{f}}(\mathbf{t}) - x = \int_0^{t_m} x\Psi_B^{\mathbf{f}}\left(\underset{m}{\mathbf{t}}, s_m\right) V_B^{\mathbf{f}}\left(\underset{m}{\mathbf{t}}, s_m\right) ds_m\,,$$

which in turn follows by the very definition of $V_B^{\mathbf{f}}$. □

3.1. **Special cases of Theorem 3.4.** . We postpone the general proof of Theorem 3.4 to the next subsection and *let us treat directly the cases when* $m = 1, 2, 3$. Actually the case when $B = [[X_1, X_2], X_3]$ is a bit technical, but, still, we prefer to perform all calculations for the simple reason that they are paradigmatic of those needed in the general proof.

- $(m = 1)$

  The case when $m = 1$ Theorem 3.4 is trivial, since
  $$xV_X^f(t) = xf. \tag{29}$$

- $(m = 2)$

  In the case when $m = 2$ the proof of Theorem 3.4 is straightforward as well. Indeed
  $$\begin{aligned} xV_{[X_1,X_2]}^{f_1,f_2}(t_1,t_2) &= \frac{\partial}{\partial\tau}\left(xe^{t_2}e^{t_1f_1}e^{-t_2f_2}e^{-t_1f_1}e^{t_1f_1}e^{\tau f_2}e^{-t_1f_1}e^{-\tau f_2}\right)_{\tau=t_2} \\ &= xe^{t_2f_2}\left(e^{t_1f_1}f_2e^{-t_1f_1} - f_2\right)e^{-t_2f_2} \\ &= xe^{t_2f_2}\left(\int_0^{t_1} e^{\rho f_1}[f_1,f_2]e^{-\rho f_1}d\rho\right)e^{-t_2f_2} \\ &= \int_0^{t_1} x[f_1\,f_2]^{(t_2,s_1)}ds_1\,. \end{aligned} \tag{30}$$

- (m=3)

  The case when $B = [[X_1, X_2], X_3]$: we have to show that
  $$xV_{[[X_1,X_2],X_3]}^{(f_1,f_2,f_3)}(t_1,t_2,t_3) = \int_0^{t_1}\int_0^{t_2} x[[f_1\,f_2],f_3]^{(t_1,t_3,s_1,s_2)}ds_1ds_2. \tag{31}$$

  To prove (31), let us shorten notation by setting $\Psi(t_1,t_2) = e^{t_1f_1}e^{t_2f_2}e^{-t_1f_1}e^{-t_2f_2}$. One has
  $$\begin{aligned} &xV_{[[X_1,X_2],X_3]}^{(f_1,f_2,f_3)}(t_1,t_2,t_3) \\ &= \frac{\partial}{\partial\tau}\left(xe^{t_3f_3}\Psi(t_1,t_2)e^{-t_3f_3}\Psi(t_1,t_2)^{-1}\Psi(t_1,t_2)e^{\tau f_3}\Psi(t_1,t_2)^{-1}e^{-\tau f_3}\right)_{\tau=t_3} \\ &= \frac{\partial}{\partial\tau}\left(xe^{t_3f_3}\Psi(t_1,t_2)e^{(\tau-t_3)f_3}\Psi(t_1,t_2)^{-1}e^{-\tau f_3}\right)_{\tau=t_3} \end{aligned}$$

---

for all $\tau_1, \tau_2$ sufficiently small.

[8] If $m = 1$, so that $B = X_1$, $\mathbf{f} = f_1$, the formula above should be understood as $xV_B^{\mathbf{f}}(t) = xB^t(\mathbf{f}) = f_1$.

$$
\begin{aligned}
&= xe^{t_3f_3}\Psi(t_1,t_2)f_3\Psi(t_1,t_2)^{-1}e^{-t_3f_3} - xe^{t_3f_3}\Psi(t_1,t_2)\Psi(t_1,t_2)^{-1}f_3e^{-t_3f_3}\\
&= xe^{t_3f_3}\Psi(t_1,t_2)f_3\Psi(t_1,t_2)^{-1}e^{-t_3f_3} - xe^{t_3f_3}f_3e^{-t_3f_3}\\
&= xe^{t_3f_3}\Psi(t_1,t_2)f_3\Psi(t_1,t_2)^{-1}e^{-t_3f_3} - xe^{t_3f_3}\Psi(t_1,0)f_3\Psi(t_1,0)^{-1}e^{-t_3f_3}\\
&= \int_0^{t_2}\frac{\partial}{\partial\sigma}\left(xe^{t_3f_3}\Psi(t_1,\sigma)f_3\Psi(t_1,\sigma)^{-1}e^{-t_3f_3}\right)d\sigma\\
&= \int_0^{t_2} xe^{t_3f_3}\frac{\partial}{\partial\sigma}\left(\Psi(t_1,\sigma)f_3\Psi(t_1,\sigma)^{-1}\right)e^{-t_3f_3}d\sigma
\end{aligned}
\tag{32}
$$

To compute the last integral let us begin by observing that, in view of Definition 3.3, one has

$$
\frac{\partial}{\partial\sigma}\Psi(t_1,\sigma) = \Psi(t_1,\sigma)V^{(f_1,f_2)}_{[X_1,X_2]}(t_1,\sigma) \tag{33}
$$

Furthermore, let us compute the derivative $\frac{\partial}{\partial\sigma}\left(\Psi(t_1,\sigma)^{-1}\right)$, by differentiating the relation

$$
\Psi(t_1,\sigma)\Psi(t_1,\sigma)^{-1} = Id_M
$$

with respect to $\sigma$. We obtain

$$
\begin{aligned}
0 = \frac{\partial}{\partial\sigma}\left(\Psi(t_1,\sigma)\Psi(t_1,\sigma)^{-1}\right) &= \frac{\partial}{\partial\sigma}\left(\Psi(t_1,\sigma)\right)\Psi(t_1,\sigma)^{-1} + \Psi(t_1,\sigma)\frac{\partial}{\partial\sigma}\left(\Psi(t_1,\sigma)^{-1}\right)\\
&= \Psi(t_1,\sigma)V^{(f_1,f_2)}_{[X_1,X_2]}(t_1,\sigma)\Psi(t_1,\sigma)^{-1} + \Psi(t_1,\sigma)\frac{\partial}{\partial\sigma}\left(\Psi(t_1,\sigma)^{-1}\right),
\end{aligned}
$$

from which we get

$$
\frac{\partial}{\partial\sigma}\left(\Psi(t_1,\sigma)^{-1}\right) = -V^{(f_1,f_2)}_{[X_1,X_2]}(t_1,\sigma)\Psi(t_1,\sigma)^{-1}. \tag{34}
$$

Using (33), (34), we can continue the row of equalities in (32), so obtaining

$$
\begin{aligned}
&xV^{(f_1,f_2,f_3)}_{[[X_1,X_2],X_3]}(t_1,t_2,t_3)\\
&\quad= \int_0^{t_2}\Big(xe^{t_3f_3}\frac{\partial}{\partial\sigma}\left(\Psi(t_1,\sigma)\right)f_3\Psi(t_1,\sigma)^{-1}e^{-t_3f_3}\\
&\qquad\qquad + xe^{t_3f_3}\Psi(t_1,\sigma)f_3\frac{\partial}{\partial\sigma}\left(\Psi(t_1,\sigma)^{-1}\right)e^{-t_3f_3}\Big)d\sigma\\
&\quad= \int_0^{t_2}\Big(xe^{t_3f_3}\Psi(t_1,\sigma)V^{(f_1,f_2)}_{[X_1,X_2]}(t_1,\sigma)f_3\Psi(t_1,\sigma)^{-1}e^{-t_3f_3}\\
&\qquad\qquad - xe^{t_3f_3}\Psi(t_1,\sigma)f_3V^{(f_1,f_2)}_{[X_1,X_2]}(t_1,\sigma)\Psi(t_1,\sigma)^{-1}e^{-t_3f_3}\Big)d\sigma\\
&\quad= \int_0^{t_2} xe^{t_3f_3}\Psi(t_1,\sigma)\left[V^{(f_1,f_2)}_{[X_1,X_2]}(t_1,\sigma), f_3\right]\Psi(t_1,\sigma)^{-1}e^{-t_3f_3}d\sigma.
\end{aligned}
$$

Then, using (30), we get

$$
\begin{aligned}
&xV^{(f_1,f_2,f_3)}_{[[X_1,X_2],X_3]}(t_1,t_2,t_3)\\
&\quad= \int_0^{t_2} xe^{t_3f_3}\Psi(t_1,\sigma)\left[\int_0^{t_1}[f_1\, f_2]^{(\sigma,s_1)}ds_1, f_3\right]\Psi(t_1,\sigma)^{-1}e^{-t_3f_3}d\sigma\\
&\quad= \int_0^{t_1}\int_0^{t_2} xe^{t_3f_3}\Psi(t_1,s_2)\left[[f_1\, f_2]^{(s_2,s_1)}, f_3\right]\Psi(t_1,s_2)^{-1}e^{-t_3f_3}ds_1ds_2
\end{aligned}
$$

having set $\sigma = s_2$. Taking into account (18), this is precisely (31).

3.2. **Proof of Theorem 3.4.** Theorem 3.4 will be proved as a consequence of the following result, which establishes a recursive structure for the vector fields $V_B^{\mathbf{f}}(\mathbf{t})$.

**Proposition 2.** *If $B$ is a canonical bracket of degree $m$ and $B = [B_1, B_2^{(m_1)}]$ is its canonical factorization, then, for all $x \in M$,*

$$xV_B^{\mathbf{f}}(\mathbf{t}) = \int_0^{t_{m_1}} x\mathrm{Ad}_{\Psi_{B_2}^{\mathbf{f}_{(2)}}(\mathbf{t}_{(2)})}\mathrm{Ad}_{\Psi_{B_1}^{\mathbf{f}_{(1)}}\left(\underset{m_1}{\mathbf{t}_{(1)}},\sigma\right)}\left(\left[V_{B_1}^{\mathbf{f}_{(1)}}\left(\underset{m_1}{\mathbf{t}_{(1)}},\sigma\right), V_{B_2}^{\mathbf{f}_{(2)}}\left(\mathbf{t}_{(2)}\right)\right]\right) d\sigma\,. \tag{35}$$

*Proof.* ${}^{\mathbf{f}}_{B}(t_1, \ldots, t_{m-1}, t)^{-1}$. Since for all $t$ one has

$$x\Psi_B^{\mathbf{f}}(t_1, \ldots, t_{m-1}, t)\Psi_B^{\mathbf{f}}(t_1, \ldots, t_{m-1}, t)^{-1} = x\,,$$

one gets

$$\begin{aligned}
0 &= \frac{\partial}{\partial t}\Big(x\Psi_B^{\mathbf{f}}(t_1, \ldots, t_{m-1}, t)\Psi_B^{\mathbf{f}}(t_1, \ldots, t_{m-1}, t)^{-1}\Big) \\
&= \frac{\partial}{\partial t}\Big(x\Psi_B^{\mathbf{f}}(t_1, \ldots, t_{m-1}, t)\Big)\Psi_B^{\mathbf{f}}(t_1, \ldots, t_{m-1}, t)^{-1} \\
&\quad + x\Psi_B^{\mathbf{f}}(t_1, \ldots, t_{m-1}, t)\frac{\partial}{\partial t}\Big(\Psi_B^{\mathbf{f}}(t_1, \ldots, t_{m-1}, t)^{-1}\Big) \\
&= x\Psi_B^{\mathbf{f}}(t_1, \ldots, t_{m-1}, t)V_B^{\mathbf{f}}(t_1, \ldots, t_{m-1}, t)\Psi_B^{\mathbf{f}}(t_1, \ldots, t_{m-1}, t)^{-1} \\
&\quad + x\Psi_B^{\mathbf{f}}(t_1, \ldots, t_{m-1}, t)\frac{\partial}{\partial t}\Big(\Psi_B^{\mathbf{f}}(t_1, \ldots, t_{m-1}, t)^{-1}\Big)\,.
\end{aligned}$$

If we write $y = x\Psi_B^{\mathbf{f}}(t_1, \ldots, t_{m-1}, t)$, then we have shown that

$$yV_B^{\mathbf{f}}(t_1, \ldots, t_{m-1}, t)\Psi_B^{\mathbf{f}}(t_1, \ldots, t_{m-1}, t)^{-1} + \frac{\partial}{\partial t}\Big(y\Psi_B^{\mathbf{f}}(t_1, \ldots, t_{m-1}, t)^{-1}\Big) = 0\,.$$

As $x$ varies over $M$, so does $y$, and we can rewrite the above using the variable $x$ instead of $y$, obtaining

$$xV_B^{\mathbf{f}}(t_1, \ldots, t_{m-1}, t)\Psi_B^{\mathbf{f}}(t_1, \ldots, t_{m-1}, t)^{-1} + \frac{\partial}{\partial t}\Big(x\Psi_B^{\mathbf{f}}(t_1, \ldots, t_{m-1}, t)^{-1}\Big) = 0\,,$$

from which it follows that

$$\begin{aligned}
&\frac{\partial}{\partial t}\Big(x\Psi_B^{\mathbf{f}}(t_1, \ldots, t_{m-1}, t)^{-1}\Big) \\
&\qquad = -xV_B^{\mathbf{f}}(t_1, \ldots, t_{m-1}, t)\Psi_B^{\mathbf{f}}(t_1, \ldots, t_{m-1}, t)^{-1}\,.
\end{aligned} \tag{36}$$

By (27), (36) we obtain

$$\begin{aligned}
&\frac{\partial}{\partial t}x\Psi_B^{\mathbf{f}}\left(\underset{m}{\mathbf{t}}, t\right) \\
&= \frac{\partial}{\partial t}\left(x\Psi_{B_1}^{\mathbf{f}_{(1)}}(\mathbf{t}_{(1)})\Psi_{B_2}^{\mathbf{f}_{(2)}}\left(\underset{m}{\mathbf{t}_{(2)}}, t\right)\Psi_{B_1}^{\mathbf{f}_{(1)}}(\mathbf{t}_{(1)})^{-1}\Psi_{B_2}^{\mathbf{f}_{(2)}}\left(\underset{m}{\mathbf{t}_{(2)}}, t\right)^{-1}\right) \\
&= x\Psi_{B_1}^{\mathbf{f}_{(1)}}(\mathbf{t}_{(1)})\Psi_{B_2}^{\mathbf{f}_{(2)}}\left(\underset{m}{\mathbf{t}_{(2)}}, t\right)V_{B_2}^{\mathbf{f}_{(2)}}\left(\underset{m}{\mathbf{t}_{(2)}}, t\right)\Psi_{B_1}^{\mathbf{f}_{(1)}}(\mathbf{t}_{(1)})^{-1}\Psi_{B_2}^{\mathbf{f}_{(2)}}\left(\underset{m}{\mathbf{t}_{(2)}}, t\right)^{-1} \\
&\qquad - x\Psi_{B_1}^{\mathbf{f}_{(1)}}(\mathbf{t}_{(1)})\Psi_{B_2}^{\mathbf{f}_{(2)}}\left(\underset{m}{\mathbf{t}_{(2)}}, t\right)\Psi_{B_1}^{\mathbf{f}_{(1)}}(\mathbf{t}_{(1)})^{-1}V_{B_2}^{\mathbf{f}_{(2)}}\left(\underset{m}{\mathbf{t}_{(2)}}, t\right)\Psi_{B_2}^{\mathbf{f}_{(2)}}\left(\underset{m}{\mathbf{t}_{(2)}}, t\right)^{-1} \\
&= x\Psi_{B_1}^{\mathbf{f}_{(1)}}(\mathbf{t}_{(1)})\Psi_{B_2}^{\mathbf{f}_{(2)}}\left(\underset{m}{\mathbf{t}_{(2)}}, t\right)\Psi_{B_1}^{\mathbf{f}_{(1)}}(\mathbf{t}_{(1)})^{-1}\Psi_{B_2}^{\mathbf{f}_{(2)}}\left(\underset{m}{\mathbf{t}_{(2)}}, t\right)^{-1}
\end{aligned}$$

$$
\begin{aligned}
&\left(\Psi^{\mathbf{f}_{(2)}}_{B_2}\left(\underset{m}{\mathbf{t}_{(2)}},t\right)\Psi^{\mathbf{f}_{(1)}}_{B_1}(\mathbf{t}_{(1)})V^{\mathbf{f}_{(2)}}_{B_2}\left(\underset{m}{\mathbf{t}_{(2)}},t\right)\Psi^{\mathbf{f}_{(1)}}_{B_1}(\mathbf{t}_{(1)})^{-1}\Psi^{\mathbf{f}_{(2)}}_{B_2}\left(\underset{m}{\mathbf{t}_{(2)}},t\right)^{-1}\right)\\
&\qquad - x\Psi^{\mathbf{f}_{(1)}}_{B_1}(\mathbf{t}_{(1)})\Psi^{\mathbf{f}_{(2)}}_{B_2}\left(\underset{m}{\mathbf{t}_{(2)}},t\right)\Psi^{\mathbf{f}_{(1)}}_{B_1}(\mathbf{t}_{(1)})^{-1}\Psi^{\mathbf{f}_{(2)}}_{B_2}\left(\underset{m}{\mathbf{t}_{(2)}},t\right)^{-1}\\
&\qquad\qquad\left(\Psi^{\mathbf{f}_{(2)}}_{B_2}\left(\underset{m}{\mathbf{t}_{(2)}},t\right)V^{\mathbf{f}_{(2)}}_{B_2}\left(\underset{m}{\mathbf{t}_{(2)}},t\right)\Psi^{\mathbf{f}_{(2)}}_{B_2}\left(\underset{m}{\mathbf{t}_{(2)}},t\right)^{-1}\right)\\
&= x\Psi^{\mathbf{f}}_{B}\left(\underset{m}{\mathbf{t}},t\right)\left(\Psi^{\mathbf{f}_{(2)}}_{B_2}\left(\underset{m}{\mathbf{t}_{(2)}},t\right)\Psi^{\mathbf{f}_{(1)}}_{B_1}(\mathbf{t}_{(1)})V^{\mathbf{f}_{(2)}}_{B_2}\left(\underset{m}{\mathbf{t}_{(2)}},t\right)\Psi^{\mathbf{f}_{(1)}}_{B_1}(\mathbf{t}_{(1)})^{-1}\right.\\
&\qquad\qquad\qquad\left.\Psi^{\mathbf{f}_{(2)}}_{B_2}\left(\underset{m}{\mathbf{t}_{(2)}},t\right)^{-1}\right)\\
&\qquad - x\Psi^{\mathbf{f}}_{B}\left(\underset{m}{\mathbf{t}},t\right)\left(\Psi^{\mathbf{f}_{(2)}}_{B_2}\left(\underset{m}{\mathbf{t}_{(2)}},t\right)V^{\mathbf{f}_{(2)}}_{B_2}\left(\underset{m}{\mathbf{t}_{(2)}},t\right)\Psi^{\mathbf{f}_{(2)}}_{B_2}\left(\underset{m}{\mathbf{t}_{(2)}},t\right)^{-1}\right),
\end{aligned}
$$

from which it follows that

$$
\begin{aligned}
V^{\mathbf{f}}_{B}\left(\underset{m}{\mathbf{t}},t\right) &= \Psi^{\mathbf{f}_{(2)}}_{B_2}\left(\underset{m}{\mathbf{t}_{(2)}},t\right)\Psi^{\mathbf{f}_{(1)}}_{B_1}(\mathbf{t}_{(1)})V^{\mathbf{f}_{(2)}}_{B_2}\left(\underset{m}{\mathbf{t}_{(2)}},t\right)\Psi^{\mathbf{f}_{(1)}}_{B_1}(\mathbf{t}_{(1)})^{-1}\Psi^{\mathbf{f}_{(2)}}_{B_2}\left(\underset{m}{\mathbf{t}_{(2)}},t\right)^{-1}\\
&\qquad - \Psi^{\mathbf{f}_{(2)}}_{B_2}\left(\underset{m}{\mathbf{t}_{(2)}},t\right)V^{\mathbf{f}_{(2)}}_{B_2}\left(\underset{m}{\mathbf{t}_{(2)}},t\right)\Psi^{\mathbf{f}_{(2)}}_{B_2}\left(\underset{m}{\mathbf{t}_{(2)}},t\right)^{-1},
\end{aligned}
$$

so that

$$
V^{\mathbf{f}}_{B}\left(\underset{m}{\mathbf{t}},t\right) = \mathrm{Ad}_{\Psi^{\mathbf{f}_{(2)}}_{B_2}\left(\underset{m}{\mathbf{t}_{(2)}},t\right)}\left(\mathrm{Ad}_{\Psi^{\mathbf{f}_{(1)}}_{B_1}(\mathbf{t}_{(1)})}\left(V^{\mathbf{f}_{(2)}}_{B_2}\left(\underset{m}{\mathbf{t}_{(2)}},t\right)\right) - V^{\mathbf{f}_{(2)}}_{B_2}\left(\underset{m}{\mathbf{t}_{(2)}},t\right)\right), \tag{37}
$$

For any $y \in M$,one has

$$
\begin{aligned}
&\frac{\partial}{\partial\sigma}\left(y\Psi^{\mathbf{f}_{(1)}}_{B_1}\left(\underset{m_1}{\mathbf{t}_{(1)}},\sigma\right)V^{\mathbf{f}_{(2)}}_{B_2}\left(\underset{m}{\mathbf{t}_{(2)}},t\right)\Psi^{\mathbf{f}_{(1)}}_{B_1}\left(\underset{m_1}{\mathbf{t}_{(1)}},\sigma\right)^{-1}\right)\\
&\quad = y\Psi^{\mathbf{f}_{(1)}}_{B_1}\left(\underset{m_1}{\mathbf{t}_{(1)}},\sigma\right)V^{\mathbf{f}_{(1)}}_{B_1}\left(\underset{m_1}{\mathbf{t}_{(1)}},\sigma\right)V^{\mathbf{f}_{(2)}}_{B_2}\left(\underset{m}{\mathbf{t}_{(2)}},t\right)\Psi^{\mathbf{f}_{(1)}}_{B_1}\left(\underset{m_1}{\mathbf{t}_{(1)}},\sigma\right)^{-1}\\
&\qquad\qquad - y\Psi^{\mathbf{f}_{(1)}}_{B_1}\left(\underset{m_1}{\mathbf{t}_{(1)}},\sigma\right)V^{\mathbf{f}_{(2)}}_{B_2}\left(\underset{m}{\mathbf{t}_{(2)}},t\right)V^{\mathbf{f}_{(1)}}_{B_1}\left(\underset{m_1}{\mathbf{t}_{(1)}},\sigma\right)\Psi^{\mathbf{f}_{(1)}}_{B_1}\left(\underset{m_1}{\mathbf{t}_{(1)}},\sigma\right)^{-1}\\
&\quad = y\Psi^{\mathbf{f}_{(1)}}_{B_1}\left(\underset{m_1}{\mathbf{t}_{(1)}},\sigma\right)\left[V^{\mathbf{f}_{(1)}}_{B_1}\left(\underset{m_1}{\mathbf{t}_{(1)}},\sigma\right),V^{\mathbf{f}_{(2)}}_{B_2}\left(\underset{m}{\mathbf{t}_{(2)}},t\right)\right]\Psi^{\mathbf{f}_{(1)}}_{B_1}\left(\underset{m_1}{\mathbf{t}_{(1)}},\sigma\right)^{-1},
\end{aligned}
$$

If we take $y = x\Psi^{\mathbf{f}_{(2)}}_{B_2}\left(\underset{m}{\mathbf{t}_{(2)}},t\right)$, $x \in M$, and use the fact that

$$
\Psi^{\mathbf{f}_{(1)}}_{B_1}\left(\underset{m_1}{\mathbf{t}_{(1)}},\sigma\right)V^{\mathbf{f}_{(2)}}_{B_2}\left(\underset{m}{\mathbf{t}_{(2)}},t\right)\Psi^{\mathbf{f}_{(1)}}_{B_1}\left(\underset{m_1}{\mathbf{t}_{(1)}},\sigma\right)^{-1} = V^{\mathbf{f}_{(2)}}_{B_2}\left(\underset{m}{\mathbf{t}_{(2)}},t\right) \qquad \text{when} \quad \sigma = 0,
$$

by (37) we get (35), if $t = t_m$. This concludes the proof. □

*Proof of Theorem 3.4.* Let us proceed by induction on $m = deg(B)$. For $m = 1$, the thesis simply follows from (29). For $m > 1$ let us assume the inductive hypothesis for each of the two subbrackets $B_1$, $B_2$ appearing in the canonical factorization $[B_1, B_2^{(m_1)}]$ of $B$. So,

$$xV_{B_1}^{\mathbf{f}_{(1)}}\left(\underset{m_1}{\mathbf{t}_{(1)}}, s_{m_1}\right) = \int_0^{t_1}\cdots\int_0^{t_{m_1-1}} xB_1(\mathbf{f}_{(1)})^{\left(\underset{\{m_{11},m_1\}}{\mathbf{t}_{(1)}}, s_{m_1}, \mathbf{s}_{(1)}\right)} ds_1\dots ds_{m_1-1}\ , \tag{38}$$

$$xV_{B_2}^{\mathbf{f}_{(2)}}\left(\mathbf{t}_{(2)}\right) = \int_0^{t_{m_1+1}}\cdots\int_0^{t_{m-1}} xB_2(\mathbf{f}_{(2)})^{\left(\underset{m_{21}}{\mathbf{t}_{(1)}}, \mathbf{s}_{(2)}\right)} ds_{m_1+1}\dots ds_{m-1}\,. \tag{39}$$

If $m_1 = 1$ (resp. if $m_1 = m-1$, i.e., $m_2 = m - m_1 = 1$ ), we mean that formula (38) (resp. (39)) reads $xV_{B_1}^{\mathbf{f}_{(1)}}(\mathbf{t}_{(1)}^{s_{m_1}}) = xV_{X_1}^{f_1}(s_1) = xf_1$ (resp. $xV_{B_2}^{\mathbf{f}_{(2)}}(\mathbf{t}_{(2)}) = xV_{X_1}^{f_m}(t_m) = xf_m$).

By applying (35) we obtain

$$xV_B^{\mathbf{f}}(\mathbf{t}) = \int_0^{t_1}\cdots\int_0^{t_{m-1}} x\mathrm{Ad}_{\Psi_{B_2}^{\mathbf{f}_{(2)}}(\mathbf{t}_{(2)})}\mathrm{Ad}_{\Psi_{B_1}^{\mathbf{f}_{(1)}}\left(\underset{m_1}{\mathbf{t}_{(1)}}, s_{m_1}\right)} \left[B_1(\mathbf{f}_{(1)})^{\left(\underset{\{m_{11},m_1\}}{\mathbf{t}_{(1)}}, s_{m_1}, \mathbf{s}_{(1)}\right)}, B_2(\mathbf{f}_{(2)})^{\left(\underset{m_{21}}{\mathbf{t}_{(1)}}, \mathbf{s}_{(2)}\right)}\right] ds_1\dots ds_{m-1}\,.$$

Since, by Definition 2.3,

$$x\mathrm{Ad}_{\Psi_{B_2}^{\mathbf{f}_{(2)}}(\mathbf{t}_{(2)})}\mathrm{Ad}_{\Psi_{B_1}^{\mathbf{f}_{(1)}}\left(\underset{m_1}{\mathbf{t}_{(1)}}, s_{m_1}\right)} \left[B_1(\mathbf{f}_{(1)})^{\left(\underset{\{m_{11},m_1\}}{\mathbf{t}_{(1)}}, s_{m_1}, \mathbf{s}_{(1)}\right)}, B_2(\mathbf{f}_{(2)})^{\left(\underset{m_{21}}{\mathbf{t}_{(1)}}, \mathbf{s}_{(2)}\right)}\right] = xB(\mathbf{f})^{\left(\underset{m_1}{\mathbf{t}}, \mathbf{s}\right)},$$

the proof is concluded. $\square$

## 4. "$C^B$" regularity.

The main results of this paper remain valid also when vector fields $f_i$ fail to be $C^\infty$, provided suitable $C^r$ hypotheses are assumed. To state them, given an $m$-tuple of vector fields $\mathbf{f} = (f_1, \dots, f_m)$ on $M$ and a canonical bracket $B$ of degree $m$, we shall define the notion of $\mathbf{f}$ *of class* $C^B$. Roughly speaking, it means that all components $f_i$, for $i = 1, \dots, m$, possess the minimal order of differentiation for which $B(\mathbf{f})$ (can be computed everywhere and) is continuous.

As a byproduct of the integral representation provided in Theorem 3.1 we get versions of the asymptotic formulas (and of Chow-Rashevski's controllability theorem) under quite low regularity hypotheses.

### 4.1. Number of differentiations.

To give a precise meaning to the expression *"* $\mathbf{f}$ *possess the minimal order of differentiation for which* $B(\mathbf{f})$ *(can be computed everywhere and) is continuous"* we need some formalism concerning the way any bracket $B$ can be regarded as constructed in a recursive way by iterated bracketings. In this recursive construction, each subbracket $S$ undergoes a certain number of bracketings until $B$ is obtained. When we plug in vector fields $f_j$ for the indeterminates $X_j$, each bracketing involves a differentiation. So we will refer to this "number of bracketings" as "the number of differentiations of $S$ in $B$," and use the expression $\Delta(S;B)$ to denote it. Naturally, this will only make sense for brackets $B$ and subbrackets $S$ such that $S$ only occurs once as a substring of $B$. For more general brackets, one must define a "subbracket of $B$" to be not just a string that occurs as a substring of $B$ and is a bracket, but as an *occurrence* of such a string, so that, for example, the two occurrences of $X_1$ in $B = [X_1,[X_1,X_2]]$ count as different subbrackets. Notice that the number of differentiations of $X_1$ in $B$ is 1 for the first one and 2 for the second one. In order to avoid this extra complication, we will confine ourselves to semicanonical brackets, for which this problem does not arise, because a subbracket $S$ of a semicanonical bracket $B$ can only occur once as a substring of $B$.

The precise definition of $\Delta(S;B)$, $B$ being canonical, and $S \in \mathit{Subb}(B)$, is by a backwards recursion on $S$:

- $\Delta(B;B) \stackrel{\text{def}}{=} 0$;
- $\Delta(S_1;B) \stackrel{\text{def}}{=} \Delta(S_2;B) \stackrel{\text{def}}{=} 1 + \Delta([S_1,S_2];B)$.

It is then easy to prove by induction that

$$\Delta(S;B) = \mathit{nrbr}(S;B) - \mathit{nlbr}(S;B)\,,$$

where $\mathit{nrbr}(S;B)$ is the number of right brackets that occur in $B$ to the right of $S$, and $\mathit{nlbr}(S;B)$ is the number of left brackets that occur in $B$ to the right of $S$.

For example, if

$$B = [X_3,[[[[X_4,X_5],X_6],X_7],[X_8,[X_9,X_{10}]]]]\,,$$

then $\Delta([X_4,X_5];B) = 4$.

It is easy to see that, if $(B_1,B_2)$ is the canonical factorization of $B$, and $\mathit{deg}(B_1) = m_1$, $\mathit{deg}(B_2) = m_2$, then

$$\Delta(X_j;B) = \begin{cases} \Delta(X_j;B_1)+1 & \text{if} \quad j \in \{1,\ldots,m_1\} \\ \Delta(X_{j-m_1};B_2)+1 & \text{if} \quad j \in \{m_1+1,\ldots,m_1+m_2\}\,. \end{cases}$$

The following trivial but important identity then holds:

If $X_j$ is a subbracket of $S$ and $S$ is a subbracket $B$,

$$\Delta_j(B) = \Delta_j(S) + \Delta(S;B) \quad \text{if} \quad X_j \quad \text{and } S \quad \text{are subbrackets of } B.$$

**Definition 4.1** (Class $C^{B+k}$)**.** Let $m,\mu,\nu,k$ be nonnegative integers such that $\nu \geq m+\mu$, $m \geq 1$. Given a semicanonical bracket $B$ of degree $m$ such that $B = B_0^{(\mu)}$, $B_0$ being canonical, and a $\nu$-tuple $\mathbf{f} = (f_1,\ldots,f_\nu)$ of vector fields, we say that $\mathbf{f}$ *is of class* $C^{B+k}$ if $f_j$ is of class $C^{\Delta_j(B)+k}$ for each $j \in \{1+\mu,\ldots,m+\mu\}$. We also write $\mathbf{f} \in C^{B+k}$ to indicate that $\mathbf{f}$ is of class $C^{B+k}$. Finally, we simplify the notation by just writing $C^B$ instead of $C^{B+0}$.

**Remark 5.** The above definition can be adapted in a obvious way to the case when $M$ is just a manifold of class $C^\ell$ for $\ell \geq 1+k+\max\Big\{\Delta_j(B) : j \in \{1+\mu,\ldots,m+\mu\}\Big\}$.

For example, suppose that

$$B = [[X_1,X_2],[[X_3,X_4],X_5]] \quad \text{and} \quad \mathbf{f} = (f_1,f_2,f_3,f_4,f_5)\,.$$

Then $\mathbf{f} \in C^B$ if and only if $f_1, f_2, f_5 \in C^2$ and $f_3, f_4 \in C^3$.

It is then easy to verify the following result:

**Proposition 3.** *Assume that we are given data $B$, $m$, $B_0$, $\mu$, $k$, $\nu$, and an $\nu$-tuple $\mathbf{f} = (f_1, \dots, f_\nu)$ as in Definition 4.1. Let $(B_1, B_2)$ be the factorization of $B$. Then* $\mathbf{f} \in C^{B+k}$ *if and only if* $\mathbf{f} \in C^{B_1+k+1}$ *and* $\mathbf{f} \in C^{B_2+k+1}$.

It then follows, by an easy induction on the subbrackets $S$ of $B$, that one can define $S(\mathbf{f})$ for every subbracket $S$ of $B$ as a true vector field, by simply letting

$$S(\mathbf{f}) = [S_1(\mathbf{f}), S_2(\mathbf{f})] \qquad \text{if} \quad S = [S_1, S_2]\,.$$

The resulting vector field $S(\mathbf{f})$ is of class $C^{\Delta(S;B)+k}$ as soon as $\mathbf{f}$ is of class $C^{B+k}$.

In particular:

- If $\mathbf{f} \in C^{B+k}$, then $B(\mathbf{f})$ is a vector field on $M$ of class $C^k$.

In particular,

- if $\mathbf{f} \in C^B$ then $B(\mathbf{f})$ is a continuous vector field.

4.2. **Representation with low regularity, asymptotic formulas, and Chow-Rashevski's theorem.** One can easily verify that $\left(x, \underset{m_1}{\mathbf{t}}, \mathbf{s}\right) \mapsto B(\mathbf{f})^{\left(\underset{m_1}{\mathbf{t}}, \mathbf{s}\right)}$ is well-defined and continuous for any canonical bracket $B$ with canonical factorization $B = [B_1, B_2^{(m_1)}]$, where $1 \leq m_1 < deg(B)$, and any $\mathbf{f} \in C^B$. Moreover, with obvious reinterpretation of the notation one easily obtains the following low regularity version of Theorem 3.1:

**Theorem 4.2** (Integral representation with $C^B$ regularity)**.** *Let $B$ be a canonical iterated bracket of $deg(B) = m \geq 1$, and let $\mathbf{f}$ be an $m$-tuple of vector fields on $M$ of class $C^B$. Then, for every $m-tuple$ $\mathbf{t} = (t_1, \dots, t_m) \in \mathbb{R}^m$ one has*

$$x\Psi_B^{\mathbf{f}}(\mathbf{t}) = x + \int_0^{t_1} \cdots \int_0^{t_m} x\Psi_B^{\mathbf{f}}\left(\underset{m}{\mathbf{t}}, s_m\right) B(\mathbf{f})^{\left(\underset{m_1}{\mathbf{t}}, \mathbf{s}\right)} ds_1 \dots ds_m \ ,$$

As an almost obvious byproduct we get the following asymptotic formulas under low regularity assumptions.

**Theorem 4.3** (Asymptotic formulas)**.** *Let $B$ be a canonical iterated bracket of $deg(B) = m \geq 1$, and $\mathbf{f}$ an $m$-tuple of vector fields on $M$. Assume that $\mathbf{f} \in C^B$. Then we have*

$$x\Psi_B^{\mathbf{f}}(t_1, \dots, t_m) = x + t_1 \cdots t_m B(\mathbf{f})(x) + o(t_1 \cdots t_m) \tag{40}$$

*as* $|(t_1, \dots, t_m)| \to 0$.

In turn, as a consequence of the asymptotic formulas above (and via a standard application of the open mapping theorem, see e.g. [8]), one gets a low-regularity version of Chow-Rashevski's controllability theorem:

**Theorem 4.4** (Chow-Rashevski)**.** *Let $\{f_1, \dots f_r\}$ be a family of ($C^1$) vector fields on $M$. Let us consider the driftless control system*

$$\dot{x} = \sum_{i=1}^{r} u_i f_i(x) \tag{41}$$

*with control constraints $|u_i| \leq 1$ for $i = 1, \dots, r$. Let $x_* \in M$, and let $B_1, \dots, B_\ell$ and $\mathbf{f}_1, \dots, \mathbf{f}_\ell$ be canonical iterated brackets, and finite collections of the vector fields $f_i$, $i = 1, \dots r$, respectively, such that:*

- *i) for every* $j = 1, \dots, \ell$, $\mathbf{f}_j \in C^{B_j}$,
- *ii)*

$$span\Big\{B_1(\mathbf{f}_1)(x_*), \dots, B_\ell(\mathbf{f}_\ell)(x_*)\Big\} = T_{x_*}M \tag{42}$$

*Then the control system* (41) *is locally controllable from* $x_*$ *in small time. More precisely, if* $d$ *is a Riemannian distance defined on an open set* $A$ *containing the point* $x_*$*, and if* $k$ *is the maximum of the degrees of the iterated Lie brackets* $B_j$*, then there exist a neighborhood* $U \subset A$ *of* $x_*$ *and a positive constant* $C$ *such that for every* $x \in U$ *one has*

$$T(x) \le C(d(x, x_*))^{\frac{1}{k}}, \tag{43}$$

*where* $T(x)$ *denotes the minimum time to reach* $x$ *over the set of admissible controls, provided that this set contains the piecewise constant controls* $t \to (u_1(t), \dots, u_m(t))$ *such that at each time* $t$ *only one of the numbers* $u_i(t)$, $i = 1, \dots, m$*, is nonzero.*

**Remark 6.** In view of some arguments utilized in [8], the $C^1$-regularity assumption for the vector fields $f_i$ in Theorem 4.4 may be further weakened: in fact, the only needed regularity hypotheses are those stated at point $i$). The latter, in turn, allow for some of the vector fields $f_i$ to be just continuous, so that the corresponding flows are set-valued maps. We refer to Subsection 5.2 for other considerations on the regularity question.

## 5. **Concluding remarks.**

### 5.1. **On the "adjoint" structure of integrating brackets.**

The crucial difference between integrating brackets of degree 2 and brackets of degree greater than 2 consists in the fact that while the former are *adjoint* to the corresponding Lie brackets, namely $[f_1, f_2]^{(t_2,s_1)} \stackrel{\text{def}}{=} Ad_{e^{t_2 f_2} e^{s_1 f_1}} [f_1\,,\, f_2]$, the latter in general include intermediate adjoining operations. Indeed this is already true when the degree is equal to three, in that two adjoinings are needed. For instance:

$$[[f_1, f_2], f_3]^{(t_1,t_3,s_1,s_2)} = Ad_{e^{t_3 f_3} e^{t_1 f_1} e^{s_2 f_2} e^{-t_1 f_1} e^{-s_2 f_2}} \Big[ Ad_{e^{s_2 f_2} e^{s_1 f_1}} [f_1\,,\, f_2]\,,\, f_3 \Big]$$

In fact, this would not obstruct the possibility that

$$[[f_1, f_2], f_3]^{(t_1,t_3,s_1,s_2)} = Ad_{\phi(t_1,t_3,s_1,s_2)} \Big[ [f_1\,,\, f_2]\,,\, f_3 \Big] \tag{44}$$

for some $(t_1, t_3, s_1, s_2)$-dependent diffeomorphism $\phi$. However, let us point out that if (44) were standing, then, in view of the integral representation provided by Theorem 3.1, the vanishing of the iterated Lie bracket $[[f_1, f_2], f_3]$ would imply that $\Psi^{(f_1,f_2,f_3)}_{[[X_1,X_2],X_3]} = Id_M$. Notice incidentally that by Proposition 1, (44) holds true for any $f_3$, as soon as $0 = [[f_1, f_2], f_2] = [[f_1, f_2], f_1] = 0$.

Yet, in general (44) does not hold, so that in general one has $\Psi^{(f_1,f_2,f_3)}_{[[X_1,X_2],X_3]} \neq Id_M$. This is in fact what we get from the following simple example:

**Example 5.1.** In $M = \mathbb{R}^2$ let us consider the linear vector fields $f_i(x, y) = A_i \begin{pmatrix} x \\ y \end{pmatrix}$, $i = 1, 2, 3$, where

$$A_1 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad A_3 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

Let us show that $[[f_1,f_2],f_3] \equiv 0$, and nevertheless $\Psi^{(f_1,f_2,f_3)}_{[[X_1,X_2],X_3]} \neq Id_M$. Clearly $[f_1,f_2]$, $[f_1,f_2]^{(t_2,s_1)}$, $[[f_1,f_2],f_3]$, $[[f_1,f_2],f_3]^{(t_1,t_3,s_1,s_2)}$ are also linear (parameterized) vector fields: the corresponding matrices are defined, respectively, as [9]

$$\begin{aligned}
[A_1,A_2] &\stackrel{\text{def}}{=} A_2A_1 - A_1A_2,\\
[A_1,A_2]^{(t_2,s_1)} &\stackrel{\text{def}}{=} e^{-t_2A_2}e^{-s_1A_1}[A_1,A_2]e^{s_1A_1}e^{t_2A_2},\\
[[A_1,A_2],A_3] &\stackrel{\text{def}}{=} A_3A_2A_1 - A_3A_1A_2 - A_2A_1A_3 + A_1A_2A_3\\
[[A_1,A_2],A_3]^{(t_1,t_3,s_1,s_2)} &\stackrel{\text{def}}{=} e^{-t_3A_3}e^{-t_1A_1}e^{-s_2A_2}e^{t_1A_1}e^{s_2A_2}\left[[A_1,A_2]^{(s_2,s_1)},\, A_3\right]\\
&\qquad e^{-s_2A_2}e^{-t_1A_1}e^{s_2A_2}e^{t_1A_1}e^{t_3A_3}.
\end{aligned}$$

One finds

$$[A_1,A_2] = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad [[A_1,A_2],A_3] = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix},$$

$$[A_1,A_2]^{(t_2,s_1)} = \begin{pmatrix} 2s_1t_2+1 & (2s_1t_2+1)t_2+t_2 \\ -2s_1 & -2s_1t_2-1 \end{pmatrix}$$

$$[[A_1,A_2],A_3]^{(t_1,t_3,s_1,s_2)}$$
$$= \begin{pmatrix} 2s_2t_1(s_2t_1+1)(-2s_1s_2+t_1s_2-1) & 2e^{-t_3}s_2{}^2(s_1s_2-t_1s_2+1) \\ -2e^{t_3}t_1\left(s_2{}^2t_1{}^3 - s_1\left(2s_2{}^2t_1{}^2+2s_2t_1+1\right)\right) & -2s_2\left(-s_2t_1{}^2+s_1s_2t_1+s_1\right) \end{pmatrix}.$$

By definition (or by formula (8)), one gets

$$\begin{aligned}
\begin{pmatrix} x \\ y \end{pmatrix} \Psi^{(f_1,f_2,f_3)}_{[[X_1,X_2],X_3]}(t_1,t_2,t_3) &= \begin{pmatrix} x \\ y \end{pmatrix}\\
&+ \begin{pmatrix} t_1{}^3t_2{}^3e^{-t_3}\left(1-e^{t_3}\right)x - t_1{}^2t_2e^{-t_3}\left(e^{t_3}-1\right)\left(t_1{}^2t_2{}^2+t_1t_2+1\right)y \\ t_1t_2{}^2\left(e^{t_3}-1\right)(t_1t_2-1)x + t_1{}^3t_2{}^3\left(e^{t_3}-1\right)y \end{pmatrix}
\end{aligned}$$

for all $(x,y) \in \mathbb{R}^2$ and $t_1,t_2,t_3 \in \mathbb{R}$.

Hence, although $[[f_1,f_2],f_3]$ vanishes (identically), $\Psi^{(f_1,f_2,f_3)}_{[[X_1,X_2],X_3]}(t_1,t_2,t_3) \neq Id_M$, so that (44) cannot hold.

5.2. **Nonsmooth vector fields and set-valued Lie brackets.** Let us conclude with a theme already mentioned in the Introduction. In [8] the following notion of *set valued* Lie bracket $[f,g]_{set}$ has been proposed for locally Lipschitz continuous vector fields $f,g$: for every $x \in M$, one lets

$$[f,g]_{set}(x) \stackrel{\text{def}}{=} co\left\{ \lim_{j\to\infty}[f,g](x_j) \mid (x_j)_{j\in\mathbb{N}} \subset DIFF(f)\cap DIFF(g),\ \lim_{j\to\infty} x_j = x \right\}, \tag{45}$$

[9] More generally, if $B$ is a canonical bracket of $deg(B) = m \geq 1$ with canonical factorization $B = [B_1, B_2^{(m_1)}]$ for $1 \leq m_1 < m$, and $\mathbf{f} = (f_1,\dots,f_m)$ an $m$-tuple of linear vector fields on some linear space $M$, then $B(\mathbf{f})$, $B(\mathbf{f})^{\left(\mathbf{t}_{m_1},\mathbf{s}\right)}$ are also linear vector fields on $M$, for all $\mathbf{t} \in \mathbb{R}^m$, $\mathbf{s} \in \mathbb{R}^{m-1}$. Denoting by $\mathbf{A} = (A_1,\dots,A_m)$ the $m$-tuple of matrices associated in order to the components of $\mathbf{f}$ with respect to some fixed basis of $M$, then one can define in an obvious way matrices $B(\mathbf{A})$, and $B(\mathbf{A})^{\left(\mathbf{t}_{m_1},\mathbf{s}\right)}$ in such a way that they be the associated matrices of $B(\mathbf{f})$ and $B(\mathbf{f})^{\left(\mathbf{t}_{m_1},\mathbf{s}\right)}$ with respect to that same basis of $M$.

where *co* means convex enveloping, and $DIFF(f) \subset M$ and $DIFF(g) \subset M$ denote the subsets where $f$ and $g$, respectively, are differentiable. By Rademacher's theorem, these sets have full measure, so, in particular $DIFF(f) \cap DIFF(g)$ is dense in $M$. The set-valued map $x \mapsto [f,g]_{set}(x)$ turns out to be upper semicontinuous with compact, convex nonempty values. In [8] this bracket has been utilized to provide a nonsmooth generalization of Chow-Rashevski's theorem. Successively it has been also used to prove Frobenius-like and commutativity results for nonsmooth vector fields. Therefore, a natural issue might be a generalization of this notion to formal brackets $B$ of degree $m \geq 3$ and vector fields that fail to be of class $C^B$. As mentioned in the introduction, a mere iteration of (45), produces a (set-valued) bracket that is *too small* for various purposes, notably for asymptotic formulas. For instance (see the example in [9]*Section 7) one can find a point $x \in M$ and vector fields $f, g$ with locally Lipschitz derivatives such that, setting $h \stackrel{\text{def}}{=} [f,g]$, *the map* $t \mapsto x\Psi^{(f,g,h)}_{[[X_1,X_2],X_3]}(t,t,t) - x$ *is not* $o(t^3)$ [10], while

$$x[[f,g],h]_{set} = x[h,h]_{set} = 0.$$

Our guess is that a suitable notion of iterated (set-valued) bracket should contain more tangent vectors than those prescribed by definition (45) (namely the limits of sequences $([[f,g],h](x_j))_{j\in\mathbb{N}}$ for $(x_j)_{j\in\mathbb{N}} \subset DIFF(h)$) with $x_j \to x$ as $j \to \infty$). More specifically, we think that the nested structure of a formal bracket $B$, and in particular, the recursive definition of integrating brackets, suggests a new notion of set-valued bracket giving rise to asymptotic formulas that might prove useful for obtaining a higher order, non-smooth, Chow-Rashevski type result.

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[10] If $f, g$ where of class $C^2$ and hence h of class $C^1$, this map *would be* $o(t^3)$, namely

$$x\Psi^{(f,g,h)}_{[[X_1,X_2],X_3]}(t,t,t) - x = x[[f,g],h] \cdot t^3 + o(t^3) = o(t^3).$$

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