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ASYMPTOTIC PROBLEMS IN OPTIMAL CONTROL WITH A VANISHING LAGRANGIAN AND UNBOUNDED DATA

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Abstract. In this paper we give a representation formula for the limit of the finite horizon problem as the horizon becomes infinite, with a nonnegative Lagrangian and unbounded data. It is related to the limit of the discounted infinite horizon problem, as the discount factor goes to zero. We give sufficient conditions to characterize the limit function as unique nonnegative solution of the associated HJB equation. We also briefly discuss the ergodic problem.

1. **Introduction.** The main goal of this paper is to discuss, in the case of a vanishing Lagrangian $l \geq 0$ and truly unbounded data and controls, the limit as t tends to $+\infty$ of *the finite horizon value function*

$$\mathcal{V}(t,x) \doteq \inf_{\alpha(\cdot)} \int_0^t l(y(\tau),\alpha(\tau))\,d\tau,$$

and the limit as δ tends to 0^+ of *the discounted infinite horizon value function*

$$\mathcal{V}_\delta(x) \doteq \inf_{\alpha(\cdot)} \int_0^{+\infty} e^{-\delta\, t} l(y(\tau),\alpha(\tau))\,d\tau,$$

where f, l are given functions, $\alpha(\tau) \in A \subset \mathbb{R}^m$ is the control and the trajectory is given by $\dot{y}(\tau) = f(y(\tau),\alpha(\tau))$, $y(0) = x$.

These limits have been extensively studied in the literature. On the one hand, the approximability of the *infinite horizon value function*

$$\mathcal{V}(x) \doteq \inf_{\alpha(\cdot)} \int_0^{+\infty} l(y(\tau),\alpha(\tau))\,d\tau,$$

by the finite horizon value functions is usually required in most applications (see [13]) and it also represents the key point of several comparison results by viscosity solution methods. On the other hand, much work has recently been devoted to the study of the two ergodic limits $\lim_{t\to+\infty} \mathcal{V}(t,x)/t$ and $\lim_{\delta\to0^+} \delta\, \mathcal{V}_\delta(x)$. We refer to [7] for a presentation of the basic results in the deterministic case, and to [3] for the stochastic case. The same questions have been addressed in L^∞ control problems (see [1] and the references therein).

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The main novelty of this paper is the generality of the hypotheses under which the results are obtained, suitable to a wide range of applications in the framework of optimal control theory. We can consider coercive and non coercive nonnegative Lagrangians, with arbitrary growth in the state variable and without restrictions on the set

$$\mathcal{Z} \doteq \{x : \ l(x,a) = 0 \text{ for some } a\}.$$

For instance, the dynamics can be control-affine, $f(x,a) = f_0(x) + \langle G(x), a\rangle$, where f_0, G are locally Lipschitz functions with linear growth in x. In particular we cover (nonlinear generalizations of) LQR problems with $l(x,a) = x^TQx + a^TRa$, where Q and R are symmetric matrices, R is positive definite and Q is positive semidefinite. We can also allow for control-affine Lagrangians, $l(x,a) = l_0(x) + l_1(x)|a|$ with $l_0 \geq 0$, $l_1 > 0$ continuous and with arbitrary growth in x, used in some economics models, mostly in singular stochastic control (see [15] and the references therein). In general, we do not assume that the set $\{(f(x,a), l(x,a)) : a \in A\}$ is convex. Our results generalize known results for A bounded and for coercive problems. To our knowledge, they are completely new for impulsive control problems, which are included in the non coercive case by setting $\alpha(\cdot) \equiv \dot{U}(\cdot)$, where $U(\cdot)$ is a control with bounded variation (see Remark 1 below).

We show that the function $\Sigma(x) \doteq \lim_{t\to+\infty} \mathcal{V}(t,x)$ is l.s.c. and we characterize it as the minimal nonnegative supersolution to the limit HJB equation at every x where Σ is finite. The representation formula, when A is compact, is given, as expected, by the value function of the so-called *relaxed* infinite horizon problem. Adding some mild assumptions on the data, it is also equal to the l.s.c. envelope of the infinite horizon value function, $\mathcal{V}_*(x)$.

When A is unbounded, the relaxed problem is not defined. In this case, we can still give a representation formula for Σ by introducing an *extended* infinite horizon problem, which has a compact control set. Denoting by V the value function of the extended problem, we prove that Σ coincides with the relaxed version of V and also with its l.s.c. envelope, V_*, under the same assumptions as for A compact. In particular, in classical impulsive control problems, the extended setting is equivalent to replacing the controls with measures. In Theorem 3.1 we give sufficient conditions to have V equal to $\mathcal{V}$.

We obtain the same characterizations for $\lim_{\delta\to 0^+} \mathcal{V}_\delta(x)$, assuming $\mathcal{V}_\delta$ bounded.

In general, Σ is not u.s.c. and the limit HJB equation does not have an unique solution. We give explicit sufficient conditions under which Σ turns out to be continuous and the unique nonnegative solution to the HJB equation.

We now spend a few words on the ergodic problem. Starting from the papers [3] and [2], a large amount of literature has been devoted to the subject, initially in the case of bounded domains or periodic data and under some global controllability assumptions. The first results have been developed and generalized in several directions (see e.g. [9], [17], [25], and the references therein). Here we focus our attention mainly on the case where the set $\mathcal{Z} \neq \emptyset$ and the infinite horizon value function is finite, a case in which the ergodic limits turn out to be zero. We limit ourselves to showing how it is possible, under periodicity of the data and a complete controllability condition, to obtain the results of [2] in our framework.

We give some final bibliographical remarks. When the control set is unbounded, our approach is based on a compactification method introduced for impulsive controls in [11] (see also [19]); for a more complete survey we refer to [10] and the references therein. In particular, the finite horizon problem with both coercive and weakly coercive Lagrangians was treated in [26], while exit-time problems with a nonnegative Lagrangian were investigated in [22]. Moreover some optimality principles were extended in [20] to the HJB equations involved in several optimal control problems of this kind. This approach has also been applied to some stochastic control problems (see e.g. [24] and the references therein).

In Section 2 we state the problem precisely. In Section 3 we introduce the extended setting for A unbounded and give sufficient conditions in order to have the extended infinite horizon value function coinciding with $\mathcal{V}(x)$; then we define the relaxed and the relaxed–extended problems. Section 4 is devoted to characterizing the limit as t tends to $+\infty$ of the finite horizon value functions, whereas Section 6 treats the limit as δ tends to 0^+ of the discounted value functions. In Section 5 we state an uniqueness result for the solution of the limit HJB equation. The ergodic problem is investigated in Section 7. The discounted and the ergodic problems have been considered in the last two sections, since they are studied under assumptions not required for the previous results.

Notation. For any function $u : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$, we will denote the set $\{x \in \mathbb{R}^n : u(x) < +\infty\}$ by $Dom(u)$. $\mathbb{R}_+ \doteq [0, +\infty[$. A function $\omega : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ is called a *modulus* if: $\omega(\cdot, R)$ is increasing in a neighborhood of 0, continuous at 0, and $\omega(0, R) = 0$ for every $R > 0$; $\omega(r, \cdot)$ is increasing for every r. Let $D \subset \mathbb{R}^N$ for some $N \in \mathbb{N}$. $\forall r > 0$ we will denote by D_r the closed set $\overline{B(D, r)}$, while $D_r^c = \mathbb{R}^N \setminus D_r$. Moreover, χ_D will denote the characteristic function of D, namely for any $x \in \mathbb{R}^N$ we set $\chi_D(x) = 1$ if $x \in D$ and $\chi_D(x) = 0$ if $x \notin D$.

2. **Assumptions and preliminaries.**

We consider a nonlinear control system having the form

$$\dot{y}(\tau) = f(y(\tau), \alpha(\tau)), \qquad y(0) = x \tag{1}$$

and an undiscounted payoff

$$\mathcal{J}(t, x, \alpha) = \int_0^t l(y(\tau), \alpha(\tau))\, d\tau, \tag{2}$$

where $\alpha(\tau) \in A \subset \mathbb{R}^m$, and l is nonnegative. For any $x \in \mathbb{R}^n$, we define the infinite horizon value function

$$\mathcal{V}(x) \doteq \inf_{\alpha(\cdot) \in \mathcal{A}} \mathcal{J}(+\infty, x, \alpha), \tag{3}$$

where the admissible controls set $\mathcal{A}$ is given by (7) below.

The following hypotheses (H0), (H1) will be assumed throughout the whole paper.

(H0) *The control set $A \subset \ R^m$ is either compact or a convex, closed, nontrivial cone containing the origin.*
The functions $f : \mathbb{R}^n \times A \to \mathbb{R}^n$, $l : \mathbb{R}^n \times A \to \mathbb{R}$ are continuous; there exist $p, q \in \mathbb{N}$, $q \geq p \geq 1$, $M > 0$, and for any $R > 0$ there are L_R, $M_R > 0$ and a modulus $\omega(\cdot, R)$, such that $\forall x, x_1, x_2 \in \mathbb{R}^n$, $\forall a \in A$,

$$\begin{aligned} |f(x_1, a) - f(x_2, a)| &\leq L_R(1 + |a|^p)|x_1 - x_2|, \\ |l(x_1, a) - l(x_2, a)| &\leq (1 + |a|^q)\, \omega(|x_1 - x_2|, R) \\ 0 \leq l(x, a) &\leq M_R(1 + |a|^q) \quad \textit{if } |x_1|, |x_2|, |x| \leq R, \\ |f(x, a)| &\leq M(1 + |a|^p)(1 + |x|). \end{aligned} \tag{4}$$

If A is compact, the above assumptions reduce to the continuity of l and to the usual hypotheses of sublinear growth and local Lipschitz continuity in x, uniformly w.r.t. a, for f. With a small abuse of notation, in this case we will denote again by L_R the quantity $\max\{L_R(1+|a|^p):\ a\in A\}$ and similarly for the other constants appearing in (H0).

When A is unbounded, we will always assume at least weak coercivity together with a regularity hypothesis in the control variable at infinity:

(H1) *Let f, l, p and q be the same as in* (H0). *There exist some constants $C_1 \geq 0$, $C_2 > 0$ such that*

$$l(x,a) \geq C_2|a|^q - C_1 \qquad \forall (x,a)\in \mathbb{R}^n \times A. \tag{5}$$

There exist two continuous functions f^∞ and l^∞, called the recession functions of f and l *respectively, verifying*

$$f^\infty(x,a) \doteq \lim_{\rho\to 0^+} \rho^q f(x,\rho^{-1}a), \quad l^\infty(x,a) \doteq \lim_{\rho\to 0^+} \rho^q l(x,\rho^{-1}a) \tag{6}$$

uniformly *on compact sets of* $R^n \times A$ (see Example 1)*.*

Remark 1. Condition (5), for $q > p$, is known as *coercivity,* and it is used to yield suitable compactness properties for the set of the admissible controls. It is satisfied, for instance, in the LQR problems anticipated in the Introduction. If $q = p$, instead, (5) is sometimes called *weak coercivity.* In this case, since minimizing sequences of trajectories may converge to a discontinuous function, the natural framework of all our optimization problems is that of generalized controls, which we introduce in Section 3 in terms of some extended problems. Such a general setting provides, under suitable assumptions on f, the correct definition of solution even for *impulsive* control problems, in which $\alpha(\cdot) \equiv \dot{U}(\cdot)$ and $U(\cdot)$ is the control. For instance, if $q = p = 1$, such an extension corresponds to the embedding of the minimization problem over absolutely continuous $U(\cdot)$ into that of bounded variation controls.

Example 1. Functions f and l which are polynomials in the control variable a, admit the recession function introduced in (6). If, for instance, $p = 2$ and there are some continuous functions f_i, F_{ij} such that

$$f(x,a) = f_0(x) + \sum_{i=1}^m f_i(x)a_i + \sum_{i,\,j=1}^m F_{ij}(x)a_i\, a_j \qquad \forall (x,a)\in\mathbb{R}^n\times A,$$

we have $f^\infty(x,a) = \sum_{i,\,j=1}^m F_{ij}(x)a_i\,a_j$ if $q = 2$; $f^\infty(x,a) \equiv 0$ if $q > 2$.

Notice that if $q > p$, then one always has $f^\infty \equiv 0$. For $q = p$ instead, the recession function allows to describe the jumps of the state variable, as explained in Proposition 1 and Remark 3.

Let $\mathcal{B}$ denote the set of the Borel–measurable functions. The controls $\alpha(\cdot)$ are assumed to belong to the set

$$\mathcal{A} \doteq \mathcal{B} \cap L^q_{loc}(\mathbb{R}_+, A), \tag{7}$$

coinciding with $\mathcal{B}$ when A is compact. For any $x \in \mathbb{R}^n$ and for any control $\alpha(\cdot)\in\mathcal{A}$, (1) admits just one solution, defined on the whole interval $\mathbb{R}_+$. We use $y_x(\cdot,\alpha)$ to denote such a solution. When A is unbounded the control set $\mathcal{A}$ is the largest set where both payoff and trajectory are surely defined for all $t \geq 0$. In fact, in view of the coercivity condition (5) (weak, if $q = p$), such a choice is not a restriction, since for any measurable control $\alpha(\cdot)$,

$$J(t, x, \alpha) \geq C_2 \int_0^t |\alpha(\tau)|^q \, d\tau - C_1 t \qquad \forall t > 0$$

so that for controls $\alpha(\cdot) \notin \mathcal{A}$ we will never obtain a finite cost. In particular, if $C_1 = 0$ we can consider merely controls in $L^q(\mathbb{R}_+, A)$.

Let us write two estimates, useful in the sequel, that can be obtained using Gronwall's Lemma. For every $x, z \in \mathbb{R}^n$, $\forall \alpha(\cdot) \in \mathcal{A}$, and $\forall t \geq 0$ one has

$$|y_x(t, \alpha)| \leq \left(|x| + Mt + M \int_0^t |\alpha(t')|^p \, dt' \right) \mathrm{e}^{M(t + \int_0^t |\alpha(t')|^p \, dt')} \tag{8}$$

and, if $\exists R > 0$ such that $|y_x(t', \alpha)|, |y_z(t', \alpha)| \leq R \;\; \forall t' \in [0, t]$, then

$$|y_x(t, \alpha) - y_z(t, \alpha)| \leq |x - z| \mathrm{e}^{L_R(t + \int_0^t |\alpha(t')|^p \, dt')}. \tag{9}$$

For some results we will use the following hypothesis (H2).

(H2) *There is some nonempty closed set $\mathcal{T} \subset \mathbb{R}^n$ with compact boundary such that $\mathcal{V}(x) = 0$ for any $x \in \mathcal{T}$ and*

$$\lim_{x \to \bar{x}} \mathcal{V}(x) = 0 \qquad \forall \bar{x} \in \partial \mathcal{T}. \tag{10}$$

Remark 2. The existence of a point $\bar{x}$ where $\mathcal{V}(\bar{x}) = 0$ is guaranteed, for instance, when $\mathcal{V}$ is lower semicontinuous and there exist some x and $\alpha(\cdot) \in \mathcal{A}$ such that $J(+\infty, x, \alpha) < +\infty$, and $\sup_{t \geq 0} |y_x(t, \alpha)| \leq \bar{R}$ for some $\bar{R} > 0$. Indeed, let $(t_n)_n$ be an increasing sequence of times such that $\lim_n t_n = +\infty$ and let $x_n \doteq y_x(t_n, \alpha)$. The boundedness of $y_x(\cdot, \alpha)$ implies that there exists a subsequence $(x_{n_k})_k$ of $(x_n)_n$ such that $\lim_k x_{n_k} = \bar{x}$ for some $\bar{x}$. Moreover,

$$\mathcal{V}(x_{n_k}) \leq \int_{t_{n_k}}^{+\infty} l(y_x(t, \alpha), \alpha(t)) \, dt \leq J(+\infty, x, \alpha) < +\infty$$

yields that $\mathcal{V}_*(\bar{x}) \leq \lim_k \mathcal{V}(x_{n_k}) = 0$. Since $\mathcal{V}$ is lower semicontinuous, we get $\mathcal{V}(\bar{x}) = 0$.

The hypothesis $\mathcal{V} \equiv 0$ in some compact set $\mathcal{T}$ is satisfied, e.g., if $\mathcal{T} \times \{0\}$ is a viability set for the vector field (f, l). Sufficient conditions for viability in both the cases where $F(x) \doteq \{(f(x, a), l(x, a)) : a \in A\}$ is convex or non convex, can be found in [4].[1]

For the case of A compact, we recall the definition of a local MRF U, given in [21].

Definition 2.1. [MR] Given an open set $\Omega \subset \mathbb{R}^n$, $\Omega \supset \mathcal{T}$ we say that $U : \Omega \setminus \mathring{\mathcal{T}} \to \mathbb{R}_+$ is a local Minimum Restraint Function, in short, a local MRF for l, if U is continuous on $\Omega \setminus \mathring{\mathcal{T}}$, locally semiconcave, positive definite, proper[2] on $\Omega \setminus \mathcal{T}$,

[1] Any closed subset $K \subset \mathbb{R}^n \times \mathbb{R}$ is called a *viability set* for (f, l) if for any $(x_0, \lambda_0) \in K$ there is a solution (y, λ) of the differential inclusion

$$(\dot{y}(t), \dot{\lambda}(t)) \in F(y(t)) \qquad t \geq 0$$

such that $(y(0), \lambda(0)) = (x_0, \lambda_0)$ and $(y(t), \lambda(t)) \in K \;\; \forall t > 0$ (see [4]).

[2] U is said *positive definite on* $\Omega \setminus \mathcal{T}$ if $U(x) > 0 \;\; \forall x \in \Omega \setminus \mathcal{T}$ and $U(x) = 0 \;\; \forall x \in \partial \mathcal{T}$. U is called *proper on* $\Omega \setminus \mathcal{T}$ if $U^{-1}(K)$ is compact for every compact set $K \subset \mathbb{R}_+$.

$\exists U_0 \in]0,+\infty]$ such that

$$\lim_{x\to x_0,\ x\in\Omega} U(x) = U_0 \quad \forall x_0 \in \partial\Omega; \quad U(x) < U_0 \quad \forall x \in \Omega \setminus \mathring{\mathcal{T}},$$

and, moreover, $\exists k > 0$ such that, for every $x \in \Omega \setminus \mathcal{T}$,

$$\min_{a\in A} \{\langle p, f(x,a)\rangle + k\, l(x,a)\} < 0 \qquad \forall p \in D^*U(x), \tag{11}$$

where $D^*U(x)$ is the set of limiting gradients of U at x.

Let us observe that any MRF is a Control Lyapunov function for the system w.r.t. $\mathcal{T}$, which yields local asymptotic controllability to $\mathcal{T}$. For the notions borrowed from nonsmooth analysis, we refer to [12].

For the case A bounded, by [21], the existence of a local MRF U is a sufficient condition for (10). If A is unbounded, by Remark 2.5 of [22], an assumption implying (10) is the following:

There exists a local MRF U for l such that $\forall x \in \Omega \setminus \mathcal{T}$:

$$\min_{a\in A\cap \overline{B(0,R(U(x)))}} \{\langle p, f(x,a)\rangle + k\, l(x,a)\} < 0, \qquad \forall p \in D^*U(x), \tag{12}$$

where $R :]0,\sigma] \to]0,+\infty[$ is a decreasing continuous function (in particular, we may have $\lim_{\delta\to 0^+} R(\delta) = +\infty$).

3. **Generalized and relaxed control problems.** Following the so called graph-completion approach proposed in [11], as developed in [26], when A is unbounded we represent generalized controls and trajectories as reparametrizations (through a time-change, possibly discontinuous in case $q = p$) of controls and trajectories of the extended minimization problems, involving bounded-valued controls, described below. Then we investigate the well-posedness of the generalized setting, that is, when the infima over ordinary and generalized controls are the same. We do this for both the finite and the infinite horizon problem. We remark that dealing with a compact set of controls as the generalized control set is, has two main advantages. On the one hand, it allows to introduce the relaxed problem for which an optimal control exists. On the other hand, the relative Hamiltonian, different from the original, is continuous and satisfies some crucial growth and regularity properties. The exploitation of both these aspects yields many results, even in the coercive case $q > p$.

3.1. **Generalized problems and well posedness.** Throughout this subsection we assume A unbounded. Let us define on $\mathbb{R}^n \times (\mathbb{R}_+ \times A)$ the *extended* dynamics and Lagrangian $\overline{f}$, $\bar{l}$ as follows:

$$\begin{aligned} \overline{f}(x,w_0,w) &\doteq \begin{cases} w_0^q\, f(x, w_0^{-1}w) & \text{if } w_0 \neq 0 \\ f^\infty(x,w) & \text{if } w_0 = 0 \end{cases}, \\ \bar{l}(x,w_0,w) &\doteq \begin{cases} w_0^q\, l(x, w_0^{-1}w) & \text{if } w_0 \neq 0 \\ l^\infty(x,w) & \text{if } w_0 = 0. \end{cases}, \end{aligned} \tag{13}$$

where f^∞, l^∞ are defined in (H1). $\overline{f}$, $\bar{l}$ are continuous, q-positively homogeneous in the control variable (w_0, w) and inherit properties analogous to those of f and l, respectively (see e.g. [20]).

Example 2. Let us consider the same polynomial function f introduced in Example 1, where $p = 2$. Then, $\forall(x, w_0, w) \in \mathbb{R}^n \times (\mathbb{R}_+ \times A)$, for $q = 2$ we have

$$\overline{f}(x, w_0, w) = \begin{cases} f_0(x)w_0^2 + \sum_{i=1}^m f_i(x)w_i w_0 + \sum_{i,\,j=1}^m F_{ij}(x)w_i\, w_j & \text{if } \ w_0 > 0, \\ \sum_{i,\,j=1}^m F_{ij}(x)w_i\, w_j & \text{if } \ w_0 = 0. \end{cases}$$

If instead $q > 2$,

$$\overline{f}(x, w_0, w) = f_0(x)w_0^q + \sum_{i=1}^m f_i(x)w_i w_0^{q-1} + \sum_{i,\,j=1}^m F_{ij}(x)w_i\, w_j w_0^{q-2} \qquad \text{if } \ w_0 > 0,$$

and $\overline{f}(x, w_0, w) = 0$ if $w_0 = 0$.

Let $S(A) \doteq (\mathbb{R}_+ \times A) \cap \{(w_0, w) : \ w_0^q + |w|^q = 1\}$. Define the set of *extended controls* as

$$\Gamma \doteq \{(w_0, w) : \quad (w_0, w) \in \mathcal{B}(\mathbb{R}_+, S(A))\}\,, \tag{14}$$

and $\forall(w_0, w) \in \Gamma$ denote by $\xi(\cdot) \equiv \xi_x(\cdot, w_0, w)$ the *extended trajectory* solving the *extended control system*

$$\xi'(s) = \overline{f}(\xi(s), w_0(s), w(s)) \qquad \xi(0) = x. \tag{15}$$

For any $S > 0$, the *extended payoff* is given by

$$J(S, x, w_0, w) = \int_0^S \bar{l}(\xi(s), w_0(s), w(s))\, ds. \tag{16}$$

Let us recall in Proposition 1 below, a result stated in [22]: the solutions to (15) are simply time-reparametrizations of trajectories of (1) if the controls belong to

$$\Gamma^+ \doteq \Gamma \cap \{(w_0, w) : \ w_0 > 0 \text{ a.e.}\}\,. \tag{17}$$

Proposition 1. [MS] *For any $\alpha(\cdot) \in \mathcal{A}$ let us define $s(t) \doteq \int_0^t (1 + |\alpha(\tau)|^q)\, d\tau$ for all $t \geq 0$ and denote by $t : \mathbb{R}_+ \to [0, +\infty[$ its inverse function. Then (w_0, w) defined by $w(\cdot) \doteq \frac{\alpha(t(\cdot))}{(1+|\alpha(t(\cdot)|^q)^{1/q}}$, $\ w_0(\cdot) \doteq (1 - |w(\cdot)|^q)^{1/q}$, belongs to Γ^+ and $y_x(t(\cdot), \alpha)$ is the solution of (15) associated to (w_0, w).*

Vice-versa, for any $(w_0, w) \in \Gamma^+$ such that

$$\int_0^{+\infty} w_0^q(s)\, ds = +\infty, \tag{18}$$

defining $t(s) \doteq \int_0^s w_0^q(\sigma)\, d\sigma$, and $s : [0, +\infty[\to \mathbb{R}_+$ as the (continuous) inverse function of $t(s)$, the control $\alpha(\cdot) \doteq \frac{w(s(\cdot))}{w_0(s(\cdot))}$ belongs to $\mathcal{A}$ and and $\xi_x(s(\cdot), w_0, w)$ is the solution of (1) corresponding to $\alpha(\cdot)$.

Remark 3. Considering extended controls where $w_0(s) = 0$ for s in some intervals, is a way to introduce a notion of *generalized control,* where the (discontinuous) generalized solution to (1) corresponding to (w_0, w), say y_x^{gen} is defined as $y_x^{gen}(\cdot) \doteq \xi_x(s(\cdot), w_0, w)$, where $s(\cdot)$ is, e.g., the right inverse of $t(s) \doteq \int_0^s w_0^q(\sigma)\, d\sigma$ for $s \geq 0$. It is clear that, for $q > p$, one has $f^\infty \equiv 0$ and $y_x^{gen}(\cdot) \equiv y_x(\cdot)$ (for more details, see [26]).

For any $t \geq 0$, $x \in \mathbb{R}^n$, we define the *extended finite horizon value function*

$$V(t, x) \doteq \inf_{\{(w_0,w)\in\Gamma:\ \exists S>0 \text{ s.t. } \int_0^S w_0^q(s)\, ds = t\}} J(S, x, w_0, w)$$

and the *extended infinite horizon value function*

$$V(x) \doteq \inf_{(w_0,w)\in\Gamma} J(+\infty, x, w_0, w) \quad (\leq +\infty).$$

Remark 4. In Proposition 1, we establish a correspondence between $\alpha(\cdot) \in \mathcal{A}$ and $(w_0, w) \in \Gamma^+$, assuming (18). This is not a restriction, however, since (18) is satisfied by all $(w_0, w) \in \Gamma$ such that $J(+\infty, x, w_0, w) < +\infty$, owing to hypothesis (5) which, in the extended problem, reads as

$$\bar{l}(x, w_0, w) \geq C_2|w|^q - C_1 w_0^q \qquad \forall (x, w_0, w) \in \mathbb{R}^n \times S(A). \tag{19}$$

In fact, if we had $\int_0^{+\infty} w_0^q(s)\, ds = T < +\infty$, (19) together with the constraint $w_0^q + |w|^q = 1$ would yield a cost

$$J(+\infty, x, w_0, w) \geq C_2 \int_0^{+\infty} |w(s)|^q\, ds - C_1 T = +\infty,$$

which is a contradiction.

For this reason in the definition of $V(x)$ we can disregard the constraint (18), which should be naturally assumed, as in the definition of $V(t, x)$. This is a key point: due to hypothesis (5), the extended infinite horizon problem reduces to an *unconstrained* problem with a *compact* control set.

In view of Proposition 1 and Remark 4, in the extended setting we can recover $\mathcal{V}(x)$ and $\mathcal{V}(t, x)$ by restricting the minimization to Γ^+ in the definition of $V(x)$ and $V(t, x)$, respectively. In general, $\mathcal{V}(x)$ is neither l.s.c. nor u.s.c.. Moreover, as shown in the following example, if $q = p$ it may happen that there is a gap between the infimum over generalized and ordinary controls, that is $V(x) < \mathcal{V}(x)$ for some x.

Example 3. Let us consider the bi-dimensional control system

$$\begin{cases} \dot{y}_1(t) = \alpha(t) \\ \dot{y}_2(t) = |y_1(t)| + |y_2(t)| \end{cases}$$

with $y(0) = (y_1(0), y_2(0)) = x \in \mathbb{R}^2$ and $\alpha(\cdot) \in L^1_{loc}(\mathbb{R}_+, \mathbb{R})$, and define the cost function

$$J(t, x, \alpha)) = \int_0^t (|y(\tau)|^2 + |\alpha(\tau)|)\, d\tau.$$

Since any trajectory issuing from $(1, 0)$ has a second component strictly increasing, we get $\mathcal{V}(1, 0) = +\infty$.

Let us now consider the associated extended system, given by

$$\begin{cases} \dot{\xi}_1(s) = w(s) \\ \dot{\xi}_2(s) = (|\xi_1(s)| + |\xi_2(s)|) w_0(s), \end{cases}$$

$\xi(0) = (\xi_1(0), \xi_2(0)) = x$, and the extended cost

$$J(S, x, w_0, w) = \textstyle\int_0^S (|\xi(s)|^2 w_0(s) + |w(s)|)\, ds.$$

Implementing the control $w \doteq -1\chi_{[0,1]}$ the trajectory issuing from $(1, 0)$, in time $S = 1$ reaches the origin, which is an equilibrium point for the extended system, and the corresponding extended cost is

$$J(+\infty, (1, 0), w_0, w) = \textstyle\int_0^{+\infty} |\xi(s)|^2 w_0(s) + |w(s)|\, ds = \int_0^1 |w(s)|\, ds = 1.$$

This yields $V(1, 0) \leq 1$, obviously smaller than $\mathcal{V}(1, 0) = +\infty$.

When $q = p$, we can prove that $\mathcal{V}(x) \equiv V(x)$ using (H2) and the following condition.

(H3) *Assume that there is some closed set $\mathcal{T} \subset \mathbb{R}^n$ with compact boundary such that for any x with $V(x) < +\infty$, there is some $\varepsilon > 0$ for which*

$$\liminf_{s\to+\infty} \mathbf{d}(\xi_x(s, w_0, w)) = 0 \quad \text{for any } \varepsilon\text{-optimal control } (w_0, w) \in \Gamma,^{3} \tag{20}$$

where $\mathbf{d}(\cdot)$ denotes the distance function from $\mathcal{T}$.

When $\mathcal{V} \equiv 0$ in $\mathcal{T}$, both (SC1) and (SC2) below imply (20).

(SC1) *There exists a function $U : \mathbb{R}^n \setminus \mathring{\mathcal{T}} \to \mathbb{R}_+$, C^1 in $\mathbb{R}^n \setminus \mathring{\mathcal{T}}$, positive definite, proper on $\mathcal{T}^c$, such that $\forall x \in \mathcal{T}^c$,*

$$\max_{(w_0,w)\in S(A)} \left\{\langle \nabla U(x), \overline{f}(x, w_0, w)\rangle\right\} \le -m(\mathbf{d}(x)) \tag{21}$$

for some continuous, increasing function $m :]0, +\infty[\to]0, +\infty[$.

(SC2) *There is some continuous, increasing function $c_1 :]0, +\infty[\to]0, +\infty[$ such that*

$$l(x, a) \ge c_1(\mathbf{d}(x)) \qquad \forall (x, a) \in \mathcal{T}^c \times A. \tag{22}$$

(SC1) means that (15) is UGAS (uniformly globally asymptotically stable) w.r.t. $\partial\mathcal{T}$, so that *all* extended trajectories approach $\mathcal{T}$, at least asymptotically, for any $x \in \mathcal{T}^c$ (see e.g. [5]). We point out that (SC1) allows the Lagrangian to be zero outside $\mathcal{T}$.

(SC2) instead, involving just the Lagrangian, implies that l is strictly positive outside $\mathcal{T}$. For $\mathcal{T} \equiv \{0\}$, it is satisfied in LQR problems, where $l(x, a) = x^T Qx + a^T Ra$ and the matrices Q and R are symmetric and positive definite. (SC2) easily implies that $J(+\infty, x, w_0, w) = +\infty$ for any control (w_0, w) not satisfying the lim inf-condition in (20), in view of Remark 4.

We have the following well posedness results.

Theorem 3.1. *For any $t \ge 0$ and $x \in \mathbb{R}^n$, one has*
(i) $\mathcal{V}(t, x) = V(t, x)$ and it is continuous;
(ii) if either $q > p$ or (H2) *and* (H3) *hold for the same $\mathcal{T}$, then $\mathcal{V}(x) = V(x)$.*

Proof. Theorem 3.3 in [26] yields (i) while Proposition 3.4 in [20] implies (ii) for $q > p$. It remains to prove thesis (ii) in case $q = p$. Being $V \le \mathcal{V}$, for any $x \in \mathcal{T}$ the equality $\mathcal{V}(x) = V(x) = 0$ follows trivially from (H2). Let $x \in \mathcal{T}^c$ and $V(x) < +\infty$ (if $V(x) = +\infty$, $\mathcal{V}(x) = +\infty$ too). Assume by contradiction that there is some $\eta > 0$ such that

$$V(x) < \mathcal{V}(x) - 3\eta.$$

By hypothesis (10), $\mathcal{V}$ is continuous on the compact set $\partial\mathcal{T}$, therefore

$$\mathcal{V}(\bar{x}) \le \eta \qquad \forall \bar{x} \in \mathcal{T}^c \text{ such that } \mathbf{d}(\bar{x}) < 3\delta, \tag{23}$$

for some $\delta > 0$. Owing to (H3), there is some $(\tilde{w}_0, \tilde{w}) \in \Gamma$ such that

$$\int_0^{+\infty} \bar{l}(\xi_x(s, \tilde{w}_0, \tilde{w}), \tilde{w}_0(s), \tilde{w}(s))\, ds \le V(x) + \eta$$

and

$$\liminf_{s\to+\infty} \mathbf{d}(\xi_x(s, \tilde{w}_0, \tilde{w})) = 0.$$

[3] Both (H3) and (SC1) below will also be used in the sequel for other results and for A compact. In such a case some obvious changes have to be made ($(w_0, w) \in S(A)$, V, and $\bar{f}$ have to be replaced by $a \in A$, $\mathcal{V}$ and f respectively).

Hence, for some $S > 0$, we have $\mathbf{d}(\xi_x(S, \tilde{w}_0, \tilde{w})) < \delta$ and, using the Gronwall's Lemma, by standard calculations we get that the control $(\tilde{w}_0^n, \tilde{w}^n) \in \Gamma^+$ where $\tilde{w}^n \doteq \frac{n}{n+1}\tilde{w}$, for n large enough satisfies both $\mathbf{d}(\xi_x(S, \tilde{w}_0^n, \tilde{w}^n)) < 2\delta$ and

$$\int_0^S \bar{l}(\xi_x(s, \tilde{w}_0^n, \tilde{w}^n), \tilde{w}_0^n(s), \tilde{w}^n(s))\, ds \le \int_0^S \bar{l}(\xi_x(s, \tilde{w}_0, \tilde{w}), \tilde{w}_0(s), \tilde{w}(s))\, ds + \eta.$$

Thanks to Proposition 1, setting $T \doteq \int_0^S (\tilde{w}_0^n)^q(s)\, ds$, $\exists\, \tilde{\alpha}(\cdot) \in \mathcal{A}$ corresponding to $(\tilde{w}_0^n, \tilde{w}^n)$ such that $\mathbf{d}(y_x(T, \tilde{\alpha})) < 2\delta$ and

$$\int_0^T l(y_x(t, \tilde{\alpha}), \tilde{\alpha}(t))\, dt = \int_0^S \bar{l}(\xi_x(s, \tilde{w}_0^n, \tilde{w}^n), \tilde{w}_0^n(s), \tilde{w}^n(s))\, ds.$$

By (23) it follows that, if $\tilde{x} \doteq y_x(T, \tilde{\alpha})$, there exists a control $\hat{\alpha}(\cdot) \in \mathcal{A}$ such that

$$\int_0^{+\infty} l(y_{\tilde{x}}(t, \hat{\alpha}), \hat{\alpha}(t))\, dt < \eta.$$

Thus the control $\alpha(t) \doteq \tilde{\alpha}(t)\chi_{[0,T[}(t) + \hat{\alpha}(t-T)\chi_{[T,+\infty[}(t)$ belongs to $\mathcal{A}$ and satisfies

$$\int_0^{+\infty} l(y_x(t, \alpha), \alpha(t))\, dt < V(x) + 3\eta < \mathcal{V}(x).$$

At this point the first inequality implies that $\mathcal{V}(x) < +\infty$, which together with the last inequality yields the required contradiction. Statement (ii) for $q = p$ is therefore proved. □

$\mathcal{V}(x)$ is in general neither u.s.c. nor l.s.c., even if A is compact. Sufficient conditions for the upper semicontinuity are given in the following proposition.

Proposition 2. *Assume that* (H2) *and* (H3) *hold for the same $\mathcal{T}$. Then $Dom(\mathcal{V})$ is an open set and $\mathcal{V}$ is locally bounded and u.s.c. in it.*

Proof. If A is unbounded condition (20) is assumed on the extended trajectories. However, (H2) implies that also in this case (and even if $q = p$), for any x with $\mathcal{V}(x) < +\infty$, there is some $\varepsilon > 0$ such that

$$\liminf_{t \to +\infty} \mathbf{d}(y_x(t, \alpha)) = 0 \quad \text{for any } \varepsilon\text{-optimal control } \alpha(\cdot) \in \mathcal{A}. \tag{24}$$

Indeed, if (24) were not satisfied for some x and $\alpha(\cdot)$, Proposition 1 and the equality $\mathcal{V}(x) = V(x)$ proved in Theorem 3.1, would imply a contradiction: (20) would not hold for the extended control (w_0, w) corresponding to such an $\alpha(\cdot)$. From now on, the proof is the same for a compact or non compact set A.

Fix $\eta > 0$ and let $\delta > 0$ be as in (23). Let $x_0 \in Dom(\mathcal{V}) \setminus \mathcal{T}$ and let $\alpha(\cdot) \in \mathcal{A}$ satisfy

$$\int_0^{+\infty} l(y_{x_0}(t), \alpha(t))\, dt \le \mathcal{V}(x_0) + \eta, \tag{25}$$

where $y_{x_0}(\cdot) \doteq y_{x_0}(\cdot, \alpha)$. In view of (24) $\exists \bar{T}$ such that $d(y_{x_0}(\bar{T})) \le \delta\}$. For any $x \in \mathbb{R}^n$, let $y_x(\cdot) \doteq y_x(\cdot, \alpha)$. Estimates (8), (9) imply that one can choose $\delta' > 0$ small enough to have, for all $x \in B(x_0, \delta')$,

$$|y_x(t)|,\ |y_{x_0}(t)| \le \bar{C}, \quad |y_x(t) - y_{x_0}(t)| < \delta'' \qquad \forall t \in [0, \bar{T}] \tag{26}$$

for some $\bar{C} > 0$ and for any $\delta'' > 0$. Now by the Dynamic Programming Principle, in short DPP, choosing $\delta'' \le \delta$, we get

$$\mathcal{V}(x) \le \int_0^{\bar{T}} l(y_x(t), \alpha(t))\, dt + \mathcal{V}(y_x(\bar{T})) \le \int_0^{\bar{T}} M_{\bar{C}}(1 + |\alpha(t)|^q)\, dt + \eta \le C' \tag{27}$$

for some $C' > 0$, where the second inequality holds since $\mathbf{d}(y_x(\bar{T})) < 2\delta$. Therefore $Dom(\mathcal{V})$ is an open set and a simple compactness argument yields that $\mathcal{V}$ is bounded on any compact subset of $Dom(\mathcal{V})$.

The fact that $\mathcal{V}$ is u.s.c. in x_0 can now be easily deduced. Adding and subtracting $\int_0^{\bar{T}} l(y_{x_0}(t), \alpha(t))\, dt$ to the r.h.s. of (27), $\forall x \in B(x_0, \delta')$ one obtains

$$\begin{aligned}\mathcal{V}(x) \le \int_0^{\bar{T}} L_{\bar{C}}(1 + |\alpha(t)|^q)|y_x(t) - y_{x_0}(t)|)\, dt + \int_0^{\bar{T}} l(y_{x_0}(t), \alpha(t))\, dt + \eta \\ \le L_{\bar{C}}(\bar{T} + K)\delta'' + \mathcal{V}(x_0) + 2\eta,\end{aligned}$$

where $K \doteq \int_0^{\bar{T}} |\alpha(t)|^q\, dt$. Taking δ' small enough so that $L_{\bar{C}}(\bar{T} + K)\delta'' \le \eta$ one has $\mathcal{V}(x) \le \mathcal{V}(x_0) + 3\eta$, and with this the upper semicontinuity of $\mathcal{V}$ is proved. □

Let us observe that the continuity on $\partial\mathcal{T}$ prescribed in (H2) plus (H3) does not yield the lower semicontinuity of $\mathcal{V}(x)$. The continuity of $\mathcal{V}$ in its whole domain will be discussed in Remark 6.

3.2. **Relaxed problems.** In this section we introduce the relaxed finite and infinite horizon problems, for the original problems when A is compact, and for the extended problems otherwise. In order to simplify the notation, the corresponding relaxed value functions, $\mathcal{V}^r$ (if A is compact) and V^r (in which A is replaced by $S(A)$ and the extended data are considered), will be always denoted by V^r.

A COMPACT. As usual we define the relaxed controls

$$\mu(\cdot) \in \mathcal{A}^r \doteq L^\infty(\mathbb{R}_+, \mathcal{P}(A)),$$

where $A^r \doteq \mathcal{P}(A)$ is the set of Radon probability measures on the compact set A endowed with the weak*-topology, and we consider $\psi \in \{f, l\}$ extended to $\mathbb{R}^n \times A^r$ by setting

$$\psi^r(x, \mu) \doteq \int_A \psi(x, a)\, d\mu \qquad \forall \mu \in A^r.$$

For any $x \in \mathbb{R}^n$ and $\mu \in \mathcal{A}^r$, $y^r_x(\tau, \mu)$ denotes the relaxed trajectory, solution of

$$\dot{y}^r = f^r(y^r, \mu) \qquad \text{for } \tau > 0, \quad y^r(0) = x. \tag{28}$$

Finally, we introduce

$$V^r(t, x) \doteq \inf_{\mu \in \mathcal{A}^r} \mathcal{J}^r(t, x, \mu) \qquad \forall (t, x) \in]0, +\infty[\times\mathbb{R}^n$$

and

$$V^r(x) \doteq \inf_{\mu \in \mathcal{A}^r} \mathcal{J}^r(+\infty, x, \mu) \qquad \forall x \in \mathbb{R}^n,$$

where

$$\mathcal{J}^r(t, x, \mu) \doteq \int_0^t l^r(y^r_x(\tau, \mu), \mu(\tau))\, d\tau \quad \text{for any } t \in]0, +\infty].$$

Since for A compact,

$$\forall x \in \mathbb{R}^n: \qquad \overline{co}(f(x, A) \times l(x, A)) = f^r(x, A^r) \times l^r(x, A^r), \tag{29}$$

standard arguments yield that the relaxed finite and infinite horizon problems coincide with the original ones under the following convexity hypothesis.

(CV) *Let A be compact. For each $x \in \mathbb{R}^n$, the following set is convex:*

$$\mathcal{L}(x) \doteq \left\{(\lambda, \gamma) \in \mathbb{R}^{n+1}: \ \exists a \in A \ s.\ t.\ \ \lambda = f(x, a),\ l(x, a) \le \gamma\right\}. \tag{30}$$

A UNBOUNDED. We define relaxed extended controls,

$$\mu(\cdot) \in \Gamma^r \doteq L^\infty(\mathbb{R}_+, \mathcal{P}(\overline{B(0,1)} \cap A),$$

$A^r \doteq \mathcal{P}(\overline{B(0,1)} \cap A)$ denotes now the set of Radon probability measures on the compact set $\overline{B(0,1)} \cap A$ endowed with the weak*-topology and we consider $\psi \in \{\overline{f}, \bar{l}\}$ extended to $\mathbb{R}^n \times A^r$ by setting

$$\psi^r(x, \mu) \doteq \int_{\overline{B(0,1)} \cap A} \psi(x, (1 - |w|^q)^{1/q}, w)\, d\mu \qquad \forall \mu \in A^r.$$

For any $x \in \mathcal{T}^c$ and $\mu \in \Gamma^r$, $\xi_x^r(s, \mu)$ is the relaxed trajectory, solution of

$$\dot{\xi}^r = \overline{f}^r(\xi^r, \mu) \qquad \text{for } s > 0, \quad \xi^r(0) = x. \tag{31}$$

In this case, $V^r(t, x)$ and $V^r(x)$ are given respectively by

$$V^r(t, x) \doteq \inf_{\{\mu \in \Gamma^r,\, \int_0^S (1 - |\mu(s)|^q)\, ds = t\}} J^r(S, x, \mu)$$

and

$$V^r(x) \doteq \inf_{\mu \in \Gamma^r} J^r(+\infty, x, \mu),$$

where

$$J^r(S, x, \mu) \doteq \int_0^S \bar{l}^r(\xi_x^r(s, \mu), \mu(s))\, ds \quad \text{for any } S \in]0, +\infty].$$

If A is unbounded, in order to have $V^r \equiv V$ we could again invoke a convexity condition analogous to (CV), for the extended problem. However, in view of the definitions of $\overline{f}$ and $\bar{l}$ this condition would be very difficult to be satisfied, since the control set $S(A)$ is not convex. Hence we introduce the weaker convexity condition (CV)′ below, where $S(A)$ is replaced by $[0,1] \times \left(\overline{B(0,1)} \cap A\right)$ and the space-time extended dynamics $(w_0^q, \overline{f})$ is considered. (CV)′ is verified, for instance, by a control-affine dynamics and a convex Lagrangian.

(CV)′ *Let A be a unbounded. For any $x \in \mathcal{T}^c$, the following set is convex:*

$$L(x) \doteq \Big\{(\lambda_0, \lambda, \gamma) \in \mathbb{R}^{1+n+1} : \exists (w_0, w) \in [0,1] \times \left(\overline{B(0,1)} \cap A\right), \\ s.t.\ (\lambda_0, \lambda) = (w_0^q, \overline{f}(x, w_0, w)), \quad \bar{l}(x, w_0, w) \le \gamma\Big\}. \tag{32}$$

Both for bounded and unbounded controls, the relaxed and the original finite horizon problems coincide.

Theorem 3.2. FINITE HORIZON. *For any $(t, x) \in]0, +\infty[\times \mathbb{R}^n$ we have that $V^r(t, x)$ is continuous, there exists an optimal relaxed control, and*

$$\mathcal{V}(t, x) \equiv V^r(t, x).$$

Moreover, assuming either (CV) *or* (CV)′, *there exists an optimal control $\alpha(\cdot)$ for the original problem in case either A is compact or $q > p$, and there exists an optimal extended control (w_0, w) for $p = q$.*

Proof. The equality, which could be proved directly, is a straightforward consequence of the uniqueness result in Theorem 5.1, since it is easy to show that $V^r(t, x)$ satisfies (52) in the viscosity sense. Moreover, it is continuous as $\mathcal{V}(t, x)$, since the relaxed data have the same properties of the original ones. The existence of an optimal control for the relaxed problem (which does not imply in general the existence of an optimal ordinary control) is well known.

If (CV) holds, an optimal control $\alpha(\cdot)$ for $\mathcal{V}(t,x)$ exists by standard arguments. When A is unbounded, in view of (CV)$'$, in correspondence to an optimal relaxed control μ_r for $V^r(t,x)$, there is a control $(w_0,w) \in \mathcal{B}(\mathbb{R}_+, [0,1] \times (\overline{B(0,1)} \cap A))$ such that $\xi(\cdot) \doteq \xi^r_x(\cdot,\mu) \equiv \xi_x(\cdot, w_0, w)$, $J^r(S,x,\mu) \geq J(S,x,w_0,w)$ and in addition

$$\int_0^S w_0^q(s)\,ds = \int_0^S (1 - |\mu(s)|^q)\,ds = t \tag{33}$$

for some $S > 0$. In general, $(w_0, w) \notin \Gamma$ since $w_0^q + |w|^q$ may differ from 1. Nevertheless, using the arc-length reparameterization Φ^{-1}, where $\Phi(\sigma) = \int_0^\sigma [w_0^q(s) + |w(s)|^q]\,ds$, the control (w_0,w) can be substituted by one taking values in $S(A)$, satisfying (33), and having the same cost and trajectory. This is possible since $\overline{f}$ and $\bar{l}$ are q-positively homogeneous in (w_0,w) (see also Proposition 1). Such a control is clearly the desired optimal extended control.

When $q > p$, we show that, in correspondence to any extended control $(w_0,w) \in \Gamma$ verifying (33) and $J(S,x,w_0,w) < +\infty$, there exists $\alpha(\cdot) \in \mathcal{A}$ such that

$$\mathcal{J}(t,x,\alpha) \leq J(S,x,w_0,w).$$

Suppose first that $w_0 = 0$ on an unique (bounded) interval $[s_1,s_2]$. Then the trajectory $\xi_x(s,w_0,w) \equiv \xi_x(s_1,w_0,w)$ for all $s \in [s_1,s_2]$ because of the definition of f^∞, while $l \geq 0$ implies that $\int_{s_1}^{s_2} \bar{l}(\xi_x(s,w_0,w),w_0,w)\,ds \geq 0$. Therefore $J(S,x,w_0,w) \geq J(S-(s_2-s_1),x,\tilde{w}_0,\tilde{w})$ if $(\tilde{w}_0,\tilde{w})(s) \doteq \chi_{[0,s_1[}(w_0,w)(s) + \chi_{[s_1,S-(s_2-s_1)]}(w_0,w)(s+s_2-s_1)$ for all $s \in [0, S-(s_2-s_1)]$. For the general case, set $\sigma = \sigma(s) \doteq \int_0^s \chi_{]0,1]}(w_0(s'))\,ds'$ and let $s = s(\sigma)$ be the right inverse of $\sigma(\cdot)$. It is easy to see that the control $(\tilde{w}_0,\tilde{w})(\sigma) \doteq (w_0,w)(s(\sigma))$ for all $\sigma \geq 0$ does the job. The above argument lets us immediately conclude in view of Proposition 1, since $(\tilde{w}_0,\tilde{w}) \in \Gamma^+$. $\square$

As it is well known, this relaxation property is no more true for the infinite horizon problem and $\mathcal{V}(x)$ does not coincide in general with $V^r(x)$, even in the simplest case of compact valued controls, as shown by Example 4 below. The following weaker results hold.

Theorem 3.3. Infinite Horizon.

(i) Assume either (CV) *or* (CV)$'$ *and $q > p$. Then for any $x \in \mathbb{R}^n$ we have*

$$\mathcal{V}(x) = V^r(x) \tag{34}$$

and there exists an optimal control $\alpha(\cdot) \in \mathcal{A}$ for the original problem.

(ii) Assume (CV)$'$ *and $q = p$. Then for any $x \in \mathbb{R}^n$,*

$$V(x) = V^r(x) \tag{35}$$

and there exists an optimal extended control, $(w_0,w) \in \Gamma$. If moreover (H2) *and* (H3) *hold for the same $\mathcal{T}$, then we have* (34).

Proof. Let us prove that, assuming (CV)$'$, $V(x) = V^r(x)$ for $q \geq p$. Let $x \in \mathbb{R}^n$ be such that $V^r(x) < +\infty$ (if $V^r(x) = +\infty$, $V(x) = +\infty$ too). In order to prove (35), let $\mu \in \Gamma^r$ be an optimal relaxed control, such that

$$J^r(+\infty,x,\mu) \doteq \int_0^{+\infty} \bar{l}^r(\xi^r_x(s,\mu),\mu)\,ds = V^r(x) < +\infty,$$

whose existence is proved in Theorem 4.1 below. Thanks to (CV)$'$, by standard arguments there exists a control $(w_0, w) \in \mathcal{B}(\mathbb{R}_+, [0,1] \times (\overline{B(0,1)} \cap A))$ such that $\xi^r_x(\cdot, \mu) \equiv \xi_x(\cdot, w_0, w)$, $J^r(+\infty, x, \mu) \geq J(+\infty, x, w_0, w)$ and

$$\int_0^\sigma w_0^q(s)\, ds = \int_0^\sigma (1 - |\mu(s)|^q)\, ds \qquad \forall \sigma \geq 0.$$

From the same arguments in Remark 4 applied to the relaxed problem, we have that $\int_0^{+\infty} w_0^q(s)\, ds = +\infty$. Now, $(w_0, w) \notin \Gamma$ in general, but by using the arc-lenght reparametrization and arguing as in the proof of Theorem 3.2, we can obtain an extended control in Γ with the same cost, and this proves (35). The last statement of (ii) follows from Theorem 3.1 (ii).

If A is compact, statement (i) can be proved by standard arguments. When A is unbounded, the equality $\mathcal{V}(x) = V^r(x)$ follows from the previous point together with Theorem 3.1 (ii). The existence of an optimal control $\alpha(\cdot)$ in the case $q > p$ can be recovered as in the last part of the proof of Theorem 3.2. ☐

Remark 5. In case A unbounded and $q = p$, even if $\mathcal{V} \equiv V^r$, both the original finite and infinite horizon problems may not have an optimal control.

4. **Finite-horizon approximation.** In this section we give a representation formula for the limit, as t tends to $+\infty$ of the finite horizon value functions

$$\mathcal{V}(t, x) \doteq \inf_{\alpha(\cdot) \in \mathcal{A}} \int_0^t l(y(\tau), \alpha(\tau))\, d\tau,$$

defined as

$$\Sigma(x) \doteq \lim_{t \to +\infty} \mathcal{V}(t, x) = \sup_{t > 0} \mathcal{V}(t, x) \qquad \forall x \in \mathbb{R}^n. \tag{36}$$

The following simple example describes what is expected to happen, for the compact control case.

Example 4. Let us consider the bi-dimensional control system

$$\begin{cases} \dot{y}_1 = \alpha(t) \\ \dot{y}_2 = |y_1(t)| \end{cases}$$

with $y(0) = x \in \mathbb{R}^2$, $\alpha(t) \in A \doteq \{\pm 1\}$, and define the cost function

$$J(t, x, \alpha) = \int_0^t |y(\tau)|^2\, d\tau.$$

Clearly, any trajectory issuing from $(0,0)$ has a strictly increasing second component, which gives immediately $\mathcal{V}(0,0) = +\infty$, while the relaxed value function $V^r(0,0) = 0$. V^r, indeed, coincides with the infinite horizon value function where controls $\alpha(t) \in [-1, 1]$ are allowed.

Now fix $t > 0$ and for every $n \in \mathbb{N}$, $n > 0$ let us set $h \doteq \frac{t}{n}$ and let us define the control

$$\alpha_n(\tau) \doteq (-1)^i \quad \forall \tau \in [ih, (i+1)h), \quad i = 0, \dots, n-1.$$

The trajectory issuing from $(0,0)$, relative to $\alpha_n(\cdot)$, has the first component such that $\sup_{[0,t]} |y_1(t, \alpha_n)| \leq \frac{t}{n}$ and for the second component $\sup_{[0,t]} |y_2(t, \alpha_n)| \leq \frac{t^2}{n}$ which gives

$$J(t, x, \alpha_n) = \int_0^t |y(\tau)|^2\, d\tau \leq \frac{t^3(1 + t^2)}{n^2},$$

and this yields $\mathcal{V}(t, (0,0)) = 0$ for every $t > 0$. Therefore, $\Sigma(0,0) = 0 = V^r(0,0)$.

The result suggested by the previous example can be extended to the case of unbounded controls as follows.

Theorem 4.1. *For any $x \in \mathbb{R}^n$, we have*

$$\Sigma(x) = V^r(x).$$

Moreover, V^r is l.s.c. and there exists an optimal relaxed control.

In case A unbounded, we use the following preliminary result, true thanks to hypothesis (5) and interesting in itself.

Proposition 3. *For any $x \in \mathbb{R}^n$,*

$$\Sigma(x) = \sup_{s>0} W(s,x),$$

where

$$W(s,x) \doteq \inf_{\mu \in \Gamma^r} \int_0^s \bar{l}^r(\xi_x^r(s,\mu), \mu(s))\, ds.$$

Proof. Let $x \in \mathbb{R}^n$. We recall that for any $t > 0$, $\mathcal{V}(t,x)$ coincides with the relaxed finite horizon value function $V^r(t,x)$ in view of Theorem 3.2. Hence $\Sigma(x) = \sup_{t>0} V^r(t,x)$. In order to conclude, it remains essentially to prove that the time constraint $\int_0^S (1 - |\mu(s)|^q)\, ds = t$ in the definition of $V^r(t,x)$ can be dropped, so that

$$\sup_{t>0} V^r(t,x) = \sup_{s>0} W(s,x).$$

Let us first show the simpler inequality

$$\Sigma(x) \geq \sup_{s>0} W(s,x), \tag{37}$$

true even in non coercive problems. By Theorem 3.2, for any $n \in \mathbb{N}$, there exists an optimal relaxed trajectory-control pair (ξ_n^r, μ_n) and some $s_n > 0$ such that

$$V^r(n,x) = \int_0^{s_n} \bar{l}^r(\xi_n^r(s,\mu_n), \mu_n(s))\, ds, \qquad \int_0^{s_n} (1 - |\mu_n(s)|^q)\, ds = n.$$

Hence

$$V^r(n,x) \geq W(s_n,x) \tag{38}$$

where $s_n \geq n$ by definition, so that (37) follows easily by passing to the limit as n tends to $+\infty$ in (38) (the $\lim_{s\to+\infty} W(s,x)$ exists and coincides with $\sup_{s>0} W(s,x)$ by monotonicity).

Now, by (37) the converse inequality is trivially satisfied if $\sup_{s>0} W(s,x) = +\infty$. Let us assume by contradiction that there is some $\eta > 0$ such that

$$\sup_{s>0} W(s,x) < \Sigma(x) - \eta. \tag{39}$$

Then for any $n \in \mathbb{N}$ there is some (ξ_n^r, μ_n) such that

$$\int_0^n \bar{l}^r(\xi_n^r(s,\mu_n), \mu_n(s))\, ds < \Sigma(x) - \eta.$$

Let $t_n \doteq \int_0^n (1 - |\mu_n(s)|^q)\, ds$ (≥ 0). If $\{t_n\}_n$ is unbounded, for some subsequence, still denoted by $\{t_n\}_n$, $t_n > 0$ for all n, $\lim_n t_n = +\infty$ and we get

$$V^r(t_n,x) \leq \int_0^n \bar{l}^r(\xi_n^r(s,\mu_n), \mu_n(s))\, ds < \Sigma(x) - \eta.$$

Thus letting n tend to $+\infty$ one obtains that $\Sigma(x) = \lim_n V^r(t_n, x) \leq \Sigma(x) - \eta$, which yields the desired contradiction.

If instead the sequence $\{t_n\}_n$ is bounded, so that $t_n \leq T$ for all n for some $T > 0$ by assumption (5) we get

$$C_2 n - (C_2 + C_1)T \leq C_2 \int_0^n |\mu_n(s)|^q \, ds - C_1 \int_0^n (1 - |\mu_n(s)|^q) \, ds < \sup_{s>0} W(s,x) < +\infty.$$

When n tends to $+\infty$, the l.h.s. tends to $+\infty$ and we get a contradiction also in this case. $\square$

Proof of Theorem 4.1. We consider only the case A unbounded, the proof for A compact being similar and actually simpler. By the previous proposition, $\Sigma(x) = \sup_{s>0} W(s,x) \leq V^r(x)$, being $\bar{l} \geq 0$. When $\Sigma(x) = +\infty$, we have trivially $\Sigma(x) = V^r(x)$. Let thus suppose $\Sigma(x) < +\infty$. For every $n \in \mathbb{N}$ there exists an optimal relaxed trajectory-control pair (ξ_n^r, μ_n) satisfying

$$\Sigma(x) = \lim_n W(n,x) = \lim_n \int_0^n \bar{l}^r(\xi_n^r(s), \mu_n(s)) \, ds. \tag{40}$$

Let $S > 0$. Owing to the compactness of the control set $\overline{B(0,1)} \cap A$, the set $\{\xi_n^r\}_n$ is uniformly bounded and equilipschitz on $[0,S]$. Moreover, for any $n \geq S$,

$$\int_0^S \bar{l}^r(\xi_n^r(s), \mu_n(s)) \, ds \leq \Sigma(x).$$

Therefore by Ascoli-Arzelà Theorem there exists a subsequence $\{\xi_{n'}^r\}_{n'}$, uniformly converging to some function $\bar{\xi}^r$ in $[0,S]$, such that, owing to (H0),

$$\int_0^S \bar{l}^r(\bar{\xi}^r(s), \mu_n(s)) \, ds \leq \Sigma(x) + \rho_S(n), \tag{41}$$

for some $\rho_S(n)$ with $\lim_n \rho_S(n) = 0$. Moreover, since $L^\infty([0,S], \mathcal{P}(\overline{B(0,1)} \cap A)$ is sequentially weakly*– compact (see [27], p. 272), there exists a subsequence $\{\mu_{n''}\}_{n''}$ of $\{\mu_{n'}\}_{n'}$ which converges weakly to some $\bar{\mu}$ in $[0,S]$. Therefore by a diagonal procedure we obtain a trajectory-control pair $(\bar{\xi}^r, \bar{\mu})$ defined on the whole interval $\mathbb{R}_+$ and such that for any $S > 0$ there is some subsequence $\{(\xi_n^r, \mu_n)\}_n$, where ξ_n^r converges uniformly to $\bar{\xi}^r$ and μ_n weakly to $\bar{\mu}$ in $[0,S]$.

For any $S > 0$, by the weak convergence, passing to the limit in (41) one has

$$\int_0^S \bar{l}^r(\bar{\xi}^r(s), \bar{\mu}(s)) \, ds \leq \Sigma(x).$$

Consequently, since $\bar{l}$ is nonnegative, $V^r(x) = \int_0^{+\infty} \bar{l}^r(\bar{\xi}^r(s), \bar{\mu}(s)) \, ds = \Sigma(x)$ (and $\bar{\mu}$ is the optimal relaxed control). $\square$

We are going now to discuss the relation of the previous approximation result with the original value function $\mathcal{V}$. A straightforward consequence of Theorems 3.3 and 4.1 is the following

Corollary 1. *Assume either* (CV) *or* (CV)$'$. *If A is unbounded and $q = p$ let* (H2) *and* (H3) *hold for the same $\mathcal{T}$. Then for any $x \in \mathbb{R}^n$ we have*

$$\Sigma(x) = \mathcal{V}(x),$$

where Σ is defined in (36).

If no convexity is assumed, we prove that $\Sigma(x) = \mathcal{V}_*(x)$, the l.s.c. envelope of $\mathcal{V}$, under some mild additional hypotheses $(\mathrm{H0})_1$ and $(\mathrm{H0})_2$. Let us remark that, since the boundary value problem associated to the infinite horizon value function considered here has not an unique solution, we have to prove this relaxation result directly.

$(\mathbf{H0})_1$ *(i) Hypothesis* (H0) *holds with the constants L_R, $M_R > 0$ and the modulus $\omega(\cdot) \doteq \omega(\cdot, R)$ independent of R and*

$$|f(x,a)| \leq M(1+|a|^p) \qquad \forall x \in \mathbb{R}^n,\ a \in A.$$

(ii) Moreover, $\int_0^1 (\omega(s)/s)\, ds < +\infty$.

$(\mathbf{H0})_2$ *(i) For every $x \in \mathbb{R}^n$ with $V^r(x) < +\infty$ there exists an* optimal *relaxed control μ such that, for some $\bar{R} > 0$,*

$$|\xi^r_x(s,\mu)| \leq \bar{R} \quad \forall s \in [0,+\infty[, \tag{42}$$

if A is unbounded $[|y^r_x(t,\mu)| \leq \bar{R} \quad \forall t \in [0,+\infty[$, if A is compact$]$.

(ii) Moreover $\int_0^1 (\omega(s, \bar{R}+3)/s)\, ds < +\infty$, where ω is the modulus of l introduced in (H0).

Hypothesis $(\mathrm{H0})_2$ (i) roughly says that relaxed trajectories going to infinity are not convenient. Both hypotheses (SC1) and (SC2) introduced in Section 3 yield $(\mathrm{H0})_2$ (i). Actually, we recall that condition (SC1) implies the UGAS property w.r.t. $\partial\mathcal{T}$ for the *relaxed* control system too. Therefore, *all* the relaxed trajectories approach the compact set $\partial\mathcal{T}$ asymptotically (see e.g. [5]). This easily implies $(\mathrm{H0})_2$ (i). (SC2) instead, implies (47) below, which we will show to be sufficient for $(\mathrm{H0})_2$ (i) in Proposition 4. Conditions $(\mathrm{H0})_1$ (ii) and $(\mathrm{H0})_2$ (ii) are fulfilled, e.g., if $\omega(r) = Lr^\gamma$ and $\gamma > 0$.

Theorem 4.2. *Assume either* $(\mathrm{H0})_1$ *or* $(\mathrm{H0})_2$.

(i) If either A is compact or $q > p$, then for any $x \in \mathbb{R}^n$,

$$\mathcal{V}_*(x) = V^r(x); \tag{43}$$

(ii) if A is unbounded and $q = p$, then for any $x \in \mathbb{R}^n$,

$$V_*(x) = V^r(x). \tag{44}$$

Moreover, if (H2) *and* (H3) *hold for the same $\mathcal{T}$, we have* (43).

Proof. We prove the theorem only for A unbounded, the proof for A compact being analogous and actually simpler. We show that (44) holds for any $x \in \mathbb{R}^n$. Both statement (i) for $q > p$ and the last part of (ii) for $q = p$ follow then from Theorem 3.1 (ii).

Since $V^r(x) \leq V(x)$ and V^r is l.s.c., then $V^r(x) \leq V_*(x)$ for any $x \in \mathbb{R}^n$. It remains to prove the converse inequality, where it is not restrictive to consider only $x \in \mathbb{R}^n$ with $V^r(x) < +\infty$.

Let us first assume $(\mathrm{H0})_1$. In this case it is easy to prove that $\overline{f}$ and $\bar{l}$ verify Assumption 3.1 of [1], so that (44) holds in view of Theorem 3.2 of the same paper. Actually, in [1] infinite horizon problems in L^∞ are considered, but for a nonnegative running cost $\bar{l}$, one has

$$\operatorname*{ess\,sup}_{s\in[0,+\infty[} \int_0^s \bar{l}(\xi(s), w_0(s), w(s))\, ds = \int_0^{+\infty} \bar{l}(\xi(s), w_0(s), w(s))\, ds.$$

Let now $(\mathrm{H0})_2$ be in force. Accordingly, let $(\xi^r(\cdot), \mu(\cdot))$, where $\xi^r(\cdot) \doteq \xi_x(\cdot, \mu)$, be a relaxed optimal trajectory-control pair satisfying (42) for some $\bar{R} > 0$. Let $\psi : \mathbb{R}^n \to [0, 1]$ be a C^∞ cut-off map such that for all $x \in \mathbb{R}^n$,

$$\psi(x) = 1 \quad \text{if} \quad |x| \le \bar{R} + 1; \qquad \psi(x) = 0 \quad \text{if} \quad |x| \ge \bar{R} + 3.$$

Now $\overline{f}_{\bar{R}} \doteq \psi \overline{f}$, $\bar{l}_{\bar{R}} \doteq \psi \bar{l}$ satisfy hypothesis $(\mathrm{H0})_1$ and thus Assumption 3.1 of [1]. Hence by the proof of Theorem 3.2 in [1], for any $\varepsilon > 0$ there exist an extended control $(w_0, w) \in \Gamma$ and an extended trajectory $\xi(\cdot)$ such that $\dot{\xi}(s) = \overline{f}_{\bar{R}}(\xi(s), w_0(s), w(s))$ for a.e. $s \in]0, +\infty[$ and

$$|\xi(s) - \xi^r(s)| \le 2\varepsilon\, e^{-2\bar{L}s} \qquad \text{for } s \in]0, +\infty[, \tag{45}$$

$$\int_0^{+\infty} \bar{l}_{\bar{R}}(\xi(s), w_0(s), w(s))\, ds \le J^r(+\infty, x, \mu) + \varepsilon + \int_0^{2\varepsilon} \frac{\omega(s, \bar{R} + 3)}{2\bar{L}s}\, ds, \tag{46}$$

where $\bar{L} > 0$ is the Lipschitz constant of $\overline{f}_{\bar{R}}$ (which can be assumed equal to $L_{\bar{R}+3}$) and ω is the same as in $(\mathrm{H0})_2$. Set $\bar{x} \doteq \xi(0)$. From (45) it follows that $|\xi(s)| < \bar{R}+1$ for all $s \ge 0$ as soon as $\varepsilon < 1/2$. Hence in view of the definition of $\overline{f}_{\bar{R}}$ and $\bar{l}_{\bar{R}}$, $\xi(\cdot)$ solves the original system (15) with initial condition $\bar{x}$ and (46) holds with $\bar{l}_{\bar{R}}$ replaced by $\bar{l}$. Taking the limit as ε tends to zero we conclude that $V_*(x) \le V^r(x)$.□

A sufficient condition to have $(\mathrm{H0})_2$ (i), is given in the next proposition. Let us remark that (47), even in the case A unbounded, involves only the original Lagrangian l and not the extended $\bar{l}$.

Proposition 4. *Let us assume that, for every $x \in \mathbb{R}^n$,*

$$\liminf_{|x| \to +\infty} \left(\inf_{a \in A} l(x, a) \right) > 0. \tag{47}$$

Then $(\mathrm{H0})_2$ (i) *holds.*

Proof. Let A be unbounded. Then condition (47) together with assumption (5) easily implies

$$\bar{l}(x, w_0, w) \ge \bar{C} \quad \forall x \ \text{ with } |x| \ge \bar{M} \text{ and } \ (w_0, w) \in S(A) \tag{48}$$

for some positive constants $\bar{M}$, $\bar{C}$, so that the same holds true for $\bar{l}^r$. Assume by contradiction that for some x with $V^r(x) < +\infty$, there exists some optimal relaxed control μ such that the corresponding trajectory $\xi^r(\cdot)$ satisfies $|\xi^r(s_n)| \ge n$ for some increasing, positive sequence s_n tending to $+\infty$. Then $\exists N > 0$ such that $|\xi^r(s_n)| > \bar{M}$ for all $n \ge N$. If $|\xi^r(s)| > \bar{M}$ for all $s \ge s_n$ for some n, then by (48) we should have an infinite cost, while $J^r(+\infty, x, \mu) = V^r(x) < +\infty$. Otherwise, we can suppose that for any $n > N$ there exists $s_{n+1} \ge c_n > s_n$ such that $|\xi^r(s)| > \bar{M}$ for $s \in [s_n, c_n[$ and $|\xi^r(c_n)| = \bar{M}$. Then by the estimate

$$c_n - s_n \ge \frac{1}{M} \log \left(1 + \frac{n - \bar{M}}{1 + \bar{M}} \right)$$

proved in Lemma 1, pag. 778 of [6], where M is the constant in (4), we get

$$\int_0^{+\infty} \bar{l}^r(\xi_x^r(s), \mu^r(s))\, ds \ge \sum_{n=N}^{+\infty} \bar{C}(c_n - s_n) \ge \frac{1}{M} \bar{C} \sum_{n=N}^{+\infty} \log \left(1 + \frac{n - \bar{M}}{1 + \bar{M}} \right) = +\infty,$$

that is, the same contradiction as above.

Wo omit the proof in the case A compact, since it is completely similar. □

In many applications (47) holds since for some $r > 0$, l satisfies the following stronger version of (5):

$$l(x,a) \geq C_2|a|^q + C_1|x|^r \qquad \forall (x,a) \in \mathbb{R}^n \times A \tag{49}$$

where $C_1,\ C_2 > 0$ and $q \geq p$ is the same as in (H0). Condition (49) holds, for instance, for in LQR problems, where $l(x,a) = x^T Qx + a^T Ra$ and the matrices Q and R are symmetric and positive definite.

5. **Maximal and minimal solutions and uniqueness.** In this section we give sufficient conditions in order to characterize $\mathcal{V}(x)$ as unique solution of the associated HJB equation introduced below. As a byproduct we also obtain the characterization of the limit function $\Sigma(x) = V^r(x)$. We start by recalling an uniqueness theorem for the finite horizon problem obtained in [26] (see also [24], where more general results, including second order PDEs, are obtained). We point out that these results cannot be derived by classical theorems within the viscosity theory, in view of the hypothesis $l \geq 0$ and of the growth of the data considered here. Then we derive from the results in [20] and [22] a uniqueness theorem for the infinite horizon case, generalizing that obtained for A compact in [23].

Let us define the Hamiltonian

$$\mathcal{H}(x,p) \doteq \sup_{a\in A} \{-\langle f(x,a),\, p\rangle - l(x,a)\} \qquad \forall (x,p) \in \mathbb{R}^{2n}. \tag{50}$$

Notice that in case A unbounded and $p = q$, $\mathcal{H}$ can be discontinuous and equal to $+\infty$ at some points. When A is unbounded and $q \geq p$, $\mathcal{H}$ can be replaced, as shown in [26] and [20], by the *extended* Hamiltonian

$$H(x,p) \doteq \max_{(w_0,w)\in S(A)} \left\{-\langle \overline{f}(x,w_0,w),\, p\rangle - \overline{l}(x,w_0,w)\right\} \qquad \forall (x,p) \in \mathbb{R}^{2n}, \tag{51}$$

which turns out to be continuous. Actually, considering H is useful even if $q > p$, since it allows to consider dynamics verifying $|f(x,a)| \leq M(1+|a|^p)(1+|x|)$ instead of the more restrictive hypothesis $|f(x,a)| \leq M(1 + |a|^p + |x|)$, assumed in most of the literature (see e.g. [8], [14], and more recently, [16] and the references therein). An analogous remark holds for l. Therefore in the sequel we will use H and, in order to unify the exposition, we will set $H \doteq \mathcal{H}$ when A is compact.

Example 5. In control-affine problems, or, more precisely, when A is unbounded, $q = p = 1$, and $\forall (x,a) \in \mathbb{R}^n \times A$ we have

$$f(x,a) = f_0(x) + \sum_{i=1}^m f_i(x)a_i, \qquad l(x,a) = l_0(x) + \sum_{i=1}^m l_i(x)a_i + l_\infty(x)|a|,$$

we showed in Section 5 of [24], that the evolutive PDE is equivalent to the following quasi-variational inequality:

$$\max\{u_t - \langle f_0(x),\, Du(x)\rangle - l_0(x),\ K(x, Du(x)) - l_\infty(x)\} = 0,$$

where

$$K(x,p) \doteq \max_{w\in A, |w|=1} \left\{ -\left\langle \sum_{i=1}^m f_i(x)w_i,\, p \right\rangle - \sum_{i=1}^m l_i(x)w_i \right\}.$$

An analogous equivalence holds for the stationary equation. This is the more usual formulation of the PDE associated to impulsive control problems.

For the finite horizon problem we recall what follows.

Theorem 5.1. [Corollary 2.1, RS] *We have $\mathcal{V}(t,x) = V(t,x)$ and it is continuous for any $(t,x) \in \mathbb{R}_+ \times \mathbb{R}^n$. Moreover, for every $T > 0$, it is the unique viscosity solution of the Cauchy problem*

$$\begin{cases} u_t + H(x, Du(x)) = 0 \qquad \forall (t,x) \in]0,T[\times\mathbb{R}^n \\ \\ u(0,x) = 0 \qquad \forall x \in \mathbb{R}^n \end{cases} \tag{52}$$

among the functions bounded from below and continuous on $(\{0\}\times\mathbb{R}^n)\cup(\{T\}\times\mathbb{R}^n)$.

The above uniqueness result, for the case A compact, can be found in [7]. For A unbounded, some comparison theorems in[8] (for the finite horizon problem) and in [14] (for the infinite horizon case), address just the coercive case $q > p$, as observed above, require stronger hypotheses on f and l, and imply uniqueness in the class of the locally Lipschitz functions. We refer to [18] for an uniqueness result among convex functions.

Let us now consider the infinite horizon problem with HJB equation

$$H(x, Du(x)) = 0. \tag{53}$$

In order to apply the results of [20], from now on we assume that

for any $R > 0$, there exists $\bar{L}_R > 0$ such that $\omega(r,R) = \bar{L}_R\, r$,

where ω is the modulus of continuity of l in (H0).[4] We recall

Theorem 5.2. [Theorem 4.5, M] *(i) $V \leq u$ for any nonnegative and continuous supersolution $u : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ to (53) in $\mathbb{R}^n$;*
(ii) $V^r (= \Sigma)$ is l.s.c and it is the minimal nonnegative supersolution to (53) in $\mathbb{R}^n$.[5]

Let us set

$$\mathcal{S} \doteq \{(u,\Omega), \quad \Omega \subset \mathbb{R}^n \text{ open, and } u : \mathbb{R}^n \to \mathbb{R}_+ \cup \{+\infty\}, \text{ supersolution of (53) in } \mathbb{R}^n, \text{ locally bounded subsolution of (53) in } \Omega, \text{ and } \lim_{x\to\bar{x}} u(x) = +\infty \quad \forall \bar{x} \in \partial\Omega.\}$$

The proof of the following theorem follows from Theorem 5.4 below.

Theorem 5.3. *Assume* (H2) *and* (H3) *for the same $\mathcal{T}$, and alternatively (i) or (ii) below.*

(i) Assume that either $(\text{H0})_1$ *or* $(\text{H0})_2$ *holds. Moreover, let $\mathcal{V}$ be continuous in $Dom(\mathcal{V})$ and satisfy the boundary condition*

$$\lim_{x\to\bar{x}} \mathcal{V}(x) = +\infty \quad \forall \bar{x} \in \partial Dom(\mathcal{V}); \tag{54}$$

(ii) assume that either (CV) *or* (CV)′ *holds.*

Then $\mathcal{V}$ $(\equiv V^r \equiv \Sigma)$ is the unique nonnegative viscosity solution to (53) in $Dom(\mathcal{V})$, among the pairs (u,Ω) in $\mathcal{S}$, where $\Omega \supset \mathcal{T}$, $u \equiv 0$ on $\mathcal{T}$. Moreover $\mathcal{V}$ is continuous.

If we drop $(\text{H0})_1$, $(\text{H0})_2$ *in (i), $\mathcal{V}$ (possibly $\neq V^r$) is the unique solution just among the continuous functions.*

[4] The sublinear growth of l assumed in [20] can be removed as in [16].

[5] A function $u : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ is a viscosity supersolution to (53) at x if either $u_*(x) = +\infty$ or, if $u_*(x) < +\infty$, it is a supersolution at x.

By the Kruzkov transform $\Psi(v) \doteq 1-e^{-v}$, the above free boundary problem, can be replaced by another boundary value problem in $\mathbb{R}^n \setminus \mathcal{T}$, whose solution, when unique, simultaneously gives both $\mathcal{V}$ and $Dom(\mathcal{V})$. More precisely, let

$$K(x,u,p) \doteq \max_{(w_0,w)\in S(A)}\{-\langle p, \overline{f}(x,w_0,w)\rangle - \bar{l}(x,w_0,w) + \bar{l}(x,w_0,w)u\}. \tag{55}$$

Theorem 5.4. *Under the same hypotheses of Theorem 5.3, there is an unique nonnegative viscosity solution $\mathcal{U}$ to*

$$\begin{cases} K(x,u(x),Du(x)) = 0 & in \ \ \mathbb{R}^n \setminus \mathcal{T} \\ u(x) = 0 & on \ \ \partial\mathcal{T}. \end{cases} \tag{56}$$

Moreover, $\mathcal{V} \equiv V^r \equiv \Sigma \equiv \Psi^{-1}(\mathcal{U}) = -\log(1-\mathcal{U})$ and $Dom(\mathcal{V}) = \{x : \ \mathcal{U}(x) < 1\}$.

If we drop $(\mathrm{H0})_1$, $(\mathrm{H0})_2$ *in (i), $\mathcal{U}$ (possibly $\neq \Psi(V^r)$) is the unique solution just among the continuous functions.*

Proof. Let us prove the theorem in case $(\mathrm{H0})_1$, $(\mathrm{H0})_2$ are not assumed. In order to apply the uniqueness result proved in Theorem 4.7 in [22], let us observe that, under hypotheses (H2) and (H3), the *asymptotic* and the *minimal exit-time value functions* $\mathcal{V}$ and $\mathcal{V}^m$, as well as their extended versions V and V^m there introduced, do all coincide. They also are equal to our infinite horizon value function $\mathcal{V}$ ($\equiv V$ by Theorem 3.1). Indeed, owing to (H2) and (H3), both original and extended nearly optimal trajectories have to approach at least asymptotically $\mathcal{T}$. In fact, since $V \equiv \mathcal{V}$, the conditions in hypothesis (H2) hold for V too, and as shown in the proof of Proposition 2, the liminf in (20) is zero also for the ε-optimal trajectories of the original system. Thanks to (5), the last statement follows now from (i) of Theorem 4.7 in [22], while the first statement is a consequence of (ii) of Theorem 4.7 in [22] together with either Theorem 3.3 when (ii) is assumed or Theorem 4.2, when (i) holds. □

Remark 6. Since when (H2) and (H3) hold for the same $\mathcal{T}$, the infinite horizon value function $\mathcal{V}$ coincides with the asymptotic exit-time value function considered in [22], sufficient conditions for its continuity can be found there (see (TPK)$'$ in [22]). In particular, when (H2) holds for $\mathcal{T}$, in view of Proposition 6.2 in [22], (SC1) or (SC2) for the same $\mathcal{T}$ imply not only (H3), but also the continuity of $\mathcal{V}$ and the boundary condition (54). Moreover, as already observed, they also yield $(\mathrm{H0})_2$ (i). Since in this section we suppose l locally Lipschitz continuous in x, condition $(\mathrm{H0})_2$ (ii) is trivially verified.

Therefore we have

Corollary 2. *Let $\mathcal{T} \times \{0\}$ be a viability set for (f,l). Assume the existence of a local MRF and either* (SC1) *or* (SC2) *for $\mathcal{T}$. Then*

(i) there is an unique nonnegative viscosity solution $\mathcal{U}$ to (56), which turns out to be continuous. Moreover, $\mathcal{V} \equiv V^r \equiv \Sigma \equiv \Psi^{-1}(\mathcal{U}) = -\log(1-\mathcal{U})$ and $Dom(\mathcal{V}) = \{x : \mathcal{U}(x) < 1\}$;

(ii) $\mathcal{V}$ ($\equiv V^r \equiv \Sigma$) is the unique nonnegative viscosity solution to (53) in $Dom(\mathcal{V})$ among the pairs (u,Ω) in $\mathcal{S}$. Moreover, $\mathcal{V}$ is continuous.

When A is unbounded, the case $q = p$ is the only one in which we could have $\mathcal{V}(x) > V(x)$ for some x. Since $\Sigma(x) = V^r(x)$, in order to characterize Σ, the well-posedness, that is the equality $\mathcal{V} \equiv V$, is not required. Hence in this whole section assumption (H2) could be weakened, by replacing in it the function $\mathcal{V}$ with

V. Accordingly, in Corollary 2 it would be enough to assume $\mathcal{T}\times\{0\}$ viable for $(\overline{f},\overline{l})$ and the existence of a MRF for the extended setting.

6. **Discounted infinite horizon approximations.** In this section we give a representation formula for the limit as δ tends to 0^+ of the infinite horizon value function with discount rate $\delta>0$:

$$\mathcal{V}_\delta(x) \doteq \inf_{\alpha(\cdot)\in\mathcal{A}} \int_0^{+\infty} e^{-\delta\, t} l(y(\tau),\alpha(\tau))\,d\tau.$$

To this aim, for any $\delta>0$, when A is unbounded, we also introduce the extended value function

$$V_\delta(x) \doteq \inf_{(w_0,w)\in\Gamma} \int_0^{+\infty} e^{-\delta\int_0^s w_0^q(s)\,ds}\,\overline{l}(\xi(s),w_0(s),w(s))\,ds,$$

and, agreeing with the notation of Subsection 3.2, if A is compact [resp., unbounded], we consider the relaxed version of $\mathcal{V}_\delta$, $\mathcal{V}_\delta^r$ [resp., of V_δ, V_δ^r].

As a first step, by Proposition 3.2 in [20] all these value functions are supersolutions to

$$\delta u + \mathcal{H}(x, Du(x)) = 0 \tag{57}$$

in $\mathbb{R}^n$. If they are locally bounded and with open domains, they also are subsolutions to (57) in their domains. Notice that, when A is unbounded, by Theorem 2.1 in [20], equation (57) can be replaced by

$$H_\delta(x,u(x),Du(x)) = 0 \qquad x\in\mathbb{R}^n,$$

where, for any $(x,r,p)\in\mathbb{R}^{2n+1}$, H_δ is the following continuous Hamiltonian

$$H_\delta(x,r,p) \doteq \max_{(w_0,w)\in S(A)} \left\{\delta r\, w_0^q - \langle \overline{f}(x,w_0,w),\, p\rangle - \overline{l}(x,w_0,w)\right\}. \tag{58}$$

By Corollary 4 in [24], for any $\delta>0$ we have what follows.

Theorem 6.1. *If $\mathcal{V}_\delta$ is bounded, then it is the unique bounded solution to (57) in $\mathbb{R}^n$ and it is continuous. Hence, if A is compact one has $\mathcal{V}_\delta\equiv\mathcal{V}_\delta^r$, and $\mathcal{V}_\delta\equiv V_\delta\equiv V_\delta^r$ otherwise.*

Remark 7. It is easy to see that, when A is unbounded, sufficient conditions in order to have $\mathcal{V}_\delta$ bounded are, for instance, either

$$|f(x,a)|\le \bar M + M(1+|x|)|a|^p \quad \text{and} \quad l(x,a)\le \bar M(1+|x|^r)+M_R|a|^q$$

or

$$l(x,a)\le \bar M + M_R|a|^q \qquad \forall (x,a)\in\mathbb{R}^n\times A \text{ with } |x|\le R,$$

for some $\bar M>0$, $r\ge 1$ (M_R is the same as in (4)). Formally, the same conditions with $a=0$ yield the boundedness of $\mathcal{V}_\delta$ for A bounded.

We refer to Corollary 4 in [24], for a characterization of $\mathcal{V}_\delta$ as unique solution to (57) in $\mathbb{R}^n$ in some classes of unbounded functions with prescribed growth at infinity.

Theorem 6.2. *Assume that each $\mathcal{V}_\delta$ is bounded. Then*

$$\lim_{\delta\to 0^+} \mathcal{V}_\delta(x) = V^r(x) \qquad \forall x\in\mathbb{R}^n.$$

Proof. We give the proof in the case A unbounded, being the other case similar. Taking into account that the sequence $\delta \to \mathcal{V}_\delta$ is monotone non increasing, by Theorem 6.1, we have

$$\Lambda(x) \doteq \lim_{\delta\to 0^+} \mathcal{V}_\delta(x) = \sup_{\delta>0} \mathcal{V}_\delta(x) = \sup_{\delta>0} V^r_\delta(x) \le V^r(x)$$

for every $x \in \mathbb{R}^n$. In view of Theorem 5.2 (ii), V^r is the minimal supersolution to (53) in $\mathbb{R}^n$, hence it is now sufficient to show that Λ $(= \Lambda_*)$ is a supersolution to (53) in $\mathbb{R}^n$ for any x such that $\Lambda(x) < +\infty$.

By the monotonicity of the sequence $\mathcal{V}_\delta$ and by the continuity of each $\mathcal{V}_\delta$, it is known that $\Lambda(x) = \Lambda_*(x) = \liminf_{\delta\to 0^+} {}_*\mathcal{V}_\delta(x)$ (see [7]). The claim follows now from stability results of viscosity solutions, taking into account the continuity of the $\mathcal{V}_\delta$ and the fact that we can consider the regular Hamiltonian in (58). ☐

In the above proof we used the upper optimality principle. Of course, it is also possible to obtain it by working directly on the control problem.

7. **Ergodic problem.**

In this section we briefly investigate the so-called ergodic problem, that is the convergence of the limits $\lim_{t\to+\infty} \mathcal{V}(t,x)/t$, $\lim_{\delta\to 0^+} \delta\,\mathcal{V}_\delta(x)$. Our goal here is just to describe how known hypotheses and proofs can be adapted to the case of unbounded controls. Hence in the sequel we consider A unbounded and assume f and l periodic in the state variable and global controllability. Our precise assumptions, together with (H1), are the following.

(H4) *(i) $T_i > 0$ $(i = 1,\dots,n)$ are real numbers and the functions $f(x,a)$, $l(x,a)$ are periodic in x_i with the period T_i $(i = 1,\dots,n)$. Moreover there are L and $M > 0$ such that $\forall x,\ x_1,\ x_2 \in \mathbb{T}^n$, $\forall a \in A$,*

$$\begin{aligned} |f(x_1,a) - f(x_2,a)| &\le L(1+|a|^p)|x_1 - x_2|, \\ |l(x_1,a) - l(x_2,a)| &\le L(1+|a|^q)|x_1 - x_2| \\ l(x,a) \le M(1+|a|^q), \quad |f(x,a)| &\le M(1+|a|^p), \end{aligned} \tag{59}$$

where $\mathbb{T}^n$ denotes the n–dimensional torus $\mathbb{R}^n / (\Pi_{i=1}^n T_i\mathbb{Z}) \sim \Pi_{i=1}^n [0,T_i]$.

(ii) There are C, $\gamma > 0$ such that for any pair x, $z \in \mathbb{T}^n$ there exist $S > 0$ and $\mu \in \Gamma^r$ such that $\xi^r_x(S,\mu) = z$ and $S \le C|x-z|^\gamma$.

A sufficient condition to have (H4) (ii) (with $\gamma = 1$) is the usual hypothesis that, for some $r > 0$, $B(0,r) \subset \overline{\text{co}}\,\overline{f}(x, S(A))$ for any $x \in \mathbb{R}^n$.

Remark 8. Owing to Theorems 4.1 and 6.2, at least when any $\mathcal{V}_\delta$ is bounded, $\lim_{t\to+\infty} \mathcal{V}(t,x) = \lim_{\delta\to 0^+} \mathcal{V}_\delta(x) = V^r(x)$ for every $x \in \mathbb{R}^n$. As a consequence, the limits $\lim_{t\to+\infty} \mathcal{V}(t,x)/t$ and $\lim_{\delta\to 0^+} \delta\,\mathcal{V}_\delta(x)$ converge obviously to zero when V^r is finite in $\mathbb{R}^n$. In fact, being $l \ge 0$ such a convergence is locally uniform.

When $l \le M(1+|a|^q)$ and (H4) (ii) is in force, V^r is finite as soon as $(f,l)(x,a) = (0,0)$ for some pair (x,a), or, more in general, if there exists a subset $\mathcal{T} \subset \mathcal{Z}$ such that $\mathcal{T} \times \{0\}$ is a viability set for (f,l). In this case indeed, for any $x \in \mathbb{R}^n$ it is possible to construct an admissible control $\alpha(\cdot)$ with finite cost, by concatenating a control steering x to $\mathcal{T}$ in time T, as in (H4) (ii), with a control keeping the trajectory inside $\mathcal{T}$ with null cost for all $t > T$. Such a control exists in view of the viability assumption.

Proposition 5. *Assume* (H3). *Then, for any* $x,\, z \in \mathbb{T}^n$,

$$0 \le \mathcal{V}_\delta(x) \le M/\delta, \qquad |\mathcal{V}_\delta(x) - \mathcal{V}_\delta(z)| \le MC\,|x-z|^\gamma. \tag{60}$$

Moreover, setting $\mathcal{W}_\delta(x) \doteq \mathcal{V}_\delta(x) - \mathcal{V}_\delta(0)$, *one also has*

$$|\mathcal{W}_\delta(x)| \le M_1, \qquad |\mathcal{W}_\delta(x) - \mathcal{W}_\delta(z)| \le MC\,|x-z|^\gamma, \tag{61}$$

where $M_1 \doteq MC(\sqrt{n}\max_{i=1,\dots,n} T_i)^\gamma$.

Proof. In view of Theorem 6.1, for any $x \in \mathbb{R}^n$ one has $\mathcal{V}_\delta(x) = V_\delta(x) \equiv V_\delta^r(x)$. Therefore the first estimate in (60) follows immediately from the fact that $\bar{l} \le M$, considering the relaxed control $\mu \equiv 0$. Assuming $V_\delta^r(x) - V_\delta^r(z) \ge 0$, as it is not restrictive, the second inequality in (60) can be obtained plugging in the DDP for $V_\delta^r(x)$ the control given by (H4) (see e.g. Theorem 2 in [2]). Both the estimates in (61) are easy consequence of (60). □

Theorem 7.1. *Assume* (H4). *Then there exists a constant* $\lambda \ge 0$ *such that*

$$\lim_{\delta\to 0^+} \delta\,\mathcal{V}_\delta(x) = \lambda, \qquad \lim_{t\to+\infty} \mathcal{V}(t,x)/t = \lambda \qquad \textit{uniformly in } \mathbb{R}^n.$$

Moreover, there exists some $\delta_n \to 0^+$ *such that*

$$\lim_{n\to+\infty} \mathcal{W}_{\delta_n}(x) = \mathcal{W}_0 \qquad \textit{uniformly in } \mathbb{R}^n,$$

and $\mathcal{W}_0 \in BUC(\mathbb{R}^n)$ *is a solution of*

$$\tilde{H}_\lambda(x, Du(x)) = 0 \qquad \textit{in } \mathbb{R}^n, \tag{62}$$

where

$$\tilde{H}_\lambda(x,p) \doteq \max_{(w_0,w)\in S(A)} \left\{ -\langle \overline{f}(x,w_0,w),\, p\rangle - \bar{l}(x,w_0,w) + \lambda w_0^q \right\}.$$

Proof. By Proposition 5, the Ascoli-Arzelà Theorem and the periodicity of the solutions imply that there exists a sequence $\delta_n \to 0^+$ such that $\lim_{n\to+\infty} \delta_n \mathcal{V}_{\delta_n} = \lambda \in \mathcal{C}(\mathbb{R}^n)$ and $\lim_{n\to+\infty} \mathcal{W}_{\delta_n} = \mathcal{W}_0 \in \mathcal{C}(\mathbb{R}^n)$. The second inequality in (60) implies that λ is a constant and consequently $\delta_n \mathcal{W}_{\delta_n} \to 0$ uniformly in $\mathbb{R}^n$. It is now easy to check that $\mathcal{W}_\delta$ satisfies

$$\max_{(w_0,w)\in S(A)} \left\{ \delta u\, w_0^q - \langle \overline{f}(x,w_0,w),\, Du\rangle - \bar{l}(x,w_0,w) + \delta\mathcal{V}_\delta(0) w_0^q \right\} = 0.$$

By the stability of the viscosity solutions and by the regularity of the above Hamiltonian, it follows that $(\lambda, \mathcal{W}_0)$ solves $\tilde{H}_\lambda(x, Du) = 0$. It remains to be proved that λ is uniquely determined and that the whole family $\delta\mathcal{V}_\delta$ converges to λ. The claim is that there exists an unique $\lambda \ge 0$ such that (62) has a bounded, uniformly continuous solution in $\mathbb{R}^n$. First let us prove that if there exist $\lambda_1,\, \lambda_2 \ge 0$, such that u_1 is a subsolution to $\tilde{H}_{\lambda_1}(x, Du) = 0$ and u_2 is a supersolution to $\tilde{H}_{\lambda_2}(x, Du) = 0$ then one must have $\lambda_1 \le \lambda_2$. Let us argue by contradiction and assume $\lambda_1 > \lambda_2$. We can suppose, eventually adding a constant, that $u_1 > u_2$. Let ε be small enough such that $\lambda_1 - \varepsilon u_1 > \lambda_2 - \varepsilon u_2$ in $\mathbb{R}^n$. Therefore u_2 is also a supersolution to

$$\max_{(w_0,w)\in S(A)} \left\{ \varepsilon u_2 w_0^q - \langle \overline{f}(x,w_0,w),\, Du_2\rangle - \bar{l}(x,w_0,w) + (\lambda_1 - \varepsilon u_1) w_0^q \right\} = 0$$

and u_1 is also a subsolution to

$$\max_{(w_0,w)\in S(A)} \left\{ \varepsilon u_1 w_0^q - \langle \overline{f}(x,w_0,w),\, Du_1\rangle - \bar{l}(x,w_0,w) + (\lambda_1 - \varepsilon u_1) w_0^q \right\} = 0$$

in $\mathbb{R}^n$. By the comparison principle underlying Theorem 6.1 we would get $u_1(x) \leq u_2(x)$, a contradiction. Therefore the claim is proved and one has $\lambda_1 \leq \lambda_2$.

Now let us assume that there exist $\lambda_1 = \lim_{\delta_n \to 0} \mathcal{V}_{\delta_n}$ and $\lambda_2 = \lim_{\bar{\delta}_n \to 0} \mathcal{V}_{\bar{\delta}_n}$. The above result yields that $\lambda_1 = \lambda_2$, so that the uniform limit $\lim_{\delta \to 0^+} \delta\, \mathcal{V}_\delta(x) = \lambda$ is proved.

In order to prove that $\lim_{t \to +\infty} \mathcal{V}(t,x)/t = \lambda$ uniformly, for the same λ as above, let us first introduce the function $v(t,x) \doteq C + \mathcal{W}_0(x) + \lambda t$ for all $(t,x) \in \mathbb{R}_+ \times \mathbb{R}^n$, where $\mathcal{W}_0$ is a solution to $\tilde{H}_\lambda(x, Du) = 0$ and $C > 0$ is chosen so that $C + \mathcal{W}_0 \geq 0$. Then v is a supersolution to (52) for any $T > 0$ and by the comparison principle underlying Theorem 5.1,

$$\mathcal{V}(t,x) \leq v(t,x) = C + \mathcal{W}_0(x) + \lambda t \qquad \forall (t,x) \in \mathbb{R}_+ \times \mathbb{R}^n.$$

Let us now consider the function $\tilde{v}(t,x) \doteq -C + \mathcal{W}_0(x) + \lambda t$ for all $(t,x) \in \mathbb{R}_+ \times \mathbb{R}^n$, where and $-C + \mathcal{W}_0 \leq 0$. Then v is a subsolution to (52) for any $T > 0$ and we get

$$\mathcal{V}(t,x) \geq \tilde{v}(t,x) = -C + \mathcal{W}_0(x) + \lambda t \qquad \forall (t,x) \in \mathbb{R}_+ \times \mathbb{R}^n,$$

arguing as above. By the last two inequalities, the proof follows. □

Remark 9. Let us observe that the effective Hamiltonian $\tilde{H}_\lambda$ really determines λ. This would not be the case, if there existed a function $\mathcal{W}_0 \in BUC(\mathbb{R}^n)$ such that the max in the definition of $\tilde{H}_\lambda$ was reached for every $x \in \mathbb{R}^n$ in a vector $(0, w) \in S(A)$. If fact, such a function would be a solution of

$$\max_{(0,w) \in S(A)} \left\{ -\langle \overline{f}(x, 0, w),\, Du \rangle - \bar{l}(x, 0, \bar{w}) \right\} = 0,$$

and then it would also solve $\tilde{H}_\lambda(x, Du) = 0$ for all λ. However, applying Theorem 5.2, such $\mathcal{W}_0$ would be greater than the value function of an infinite horizon problem with compact controls $(0, w) \in S(A)$ (where $|w|^q = 1$) and Lagrangian $\bar{l}(x, 0, \bar{w}) \geq C_2$, equal to $+\infty$. Again, hypothesis (5) plays a crucial role.

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